Combinatorics
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Lecture - 21
Inclusion exclusion principle – Part (4)

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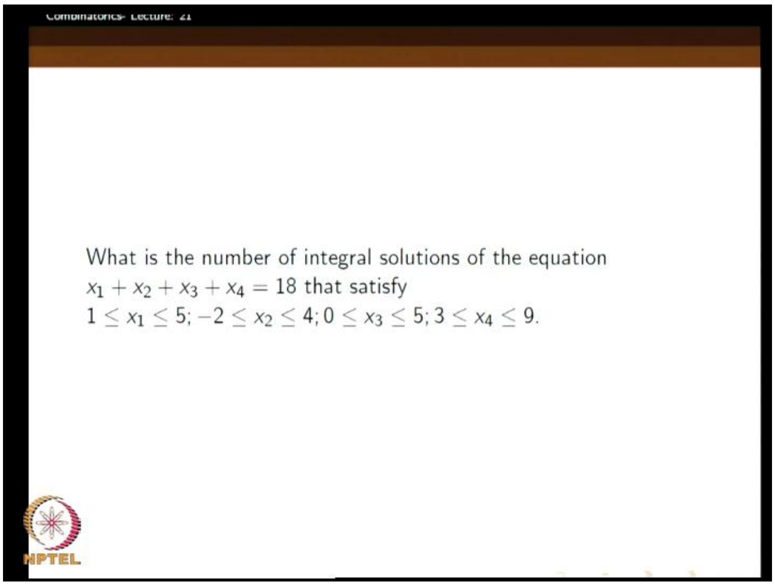

Welcome to the twenty first lecture of combinatorics. So, in the last class we were considering this problem namely x 1 plus x 2 plus x 3 plus x 4 equal to 18, and we want to find the integral solution of the equation, but there is a constraint for each variable; x 1 is in between 1 and 5, x 2 is in between minus 2 and 4, x 3 is in between 0 and 5 and x 4 is in between 3 and 9. So, how will we solve this thing, what we should notice?

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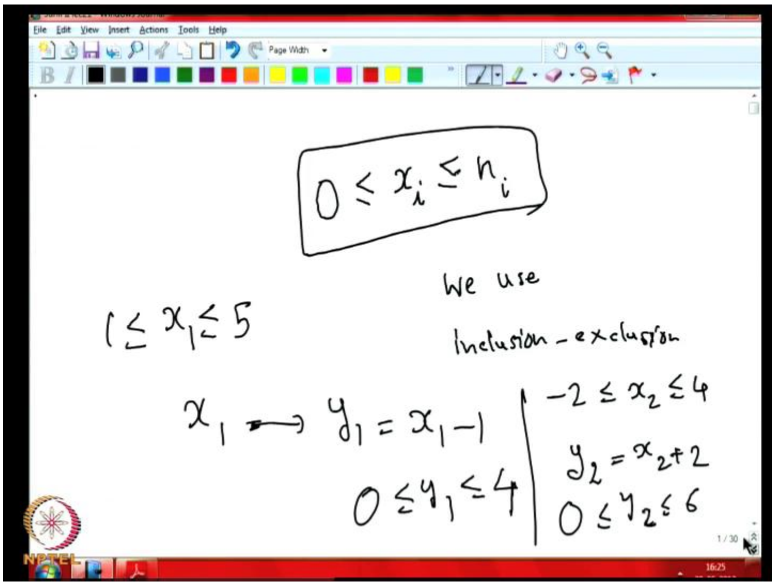

So, the kind of form we have described in the earlier formulation was something like this, each x i has to be in between 0 and some n i, then we will after that we will use inclusion-exclusion, inclusion-exclusion principle and solve, right. But then in this problem even this form is not there. So, for first is between 1 and 5; second variable is in between minus 2 and 4. So, we have to make the work on the lower bound for each variable first, right, we should make each of them 0. So, what we can do is we can replace x 1 by variable y 1, right which goes from 0 to 5. So, what we do first is we replace x 1 by a variable y 1, but then y 1 will be equal to x 1 minus 1, because our x 1 is in the range between 1 and 5.

So, y 1 will be in the range 0 and 4, right, 1 less then x 1. Similarly, you know x 2 as given in the problem is in the range minus 2 and 4; minus 2, say, x 2 is in between minus 2 and 4. So, now we replace x 2 by a variable y 2 which is x 2 plus 2 then what happens is this y 2 will be in the range 0, because minus 2 plus 2 is 0 and 6. Then whatever is x 2 we add two to it; we create another variable which ranges between 0 and 6, right. Similarly, so that is next x 3 is in between 0 and 5; that is fine and x 4 is in between 3 and 9.

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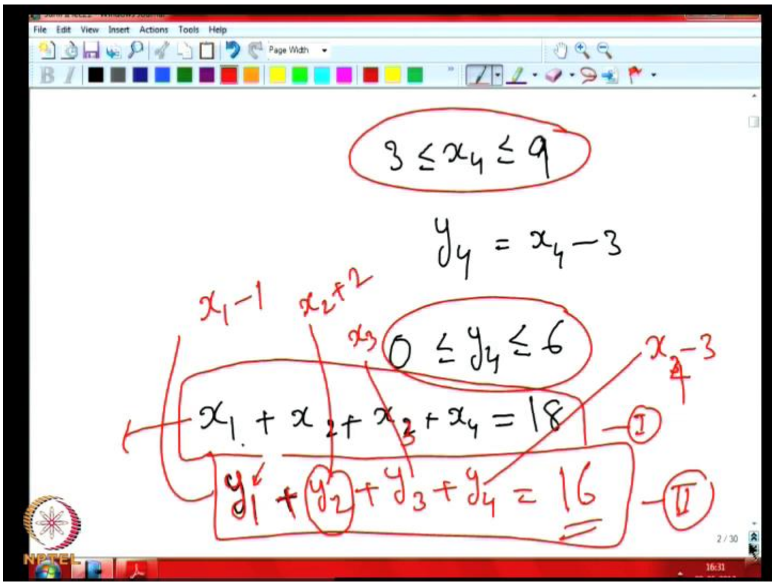

So, x 4 is in between 3 and 9. So, we will introduce y 4 which will be x 4 minus 3, so that y 4 will range between 0 and 6, right. Now we can write our equation. So, this was our equation x 1 plus x 2 plus x 3 plus x 4 is equal to 18, right. So, that we will rewrite as y 1 plus, so I will use a different color, y 1 plus y 2 plus, so this is 3, y 3 plus y 4 equal to something what is this. So, the change is that y 1 was x 1 minus 1, y 2 was x 2 plus 2, y 3 was x 3 only and y 4 is x 4 minus 3; you can just check it here. So, y 2 we define x 2 plus 2, y 1 we define as x 1 minus 1 and x 3 and y 3 are same; we did not change x 3, y 4 is x 4 minus 3, right.

So, how will it change when we add this thing? So, see this thing is x 1, x 2, x 3, x 4 is adding to 18. So, x 1 minus 1, x 2 plus 2 and x 3 and x 4 minus 3 will add up to minus 2 plus 2 the total 16, right. What we do is 18; this minus 1, we will make it 17, then plus 19 and this will become make it 16, right, this is the corresponding equation. So, formally we say that the solutions of this and solutions of this with the corresponding constraints of course when I say x 1 is in between 1 and 4, I am telling that y 1 is in between sorry; when I am saying that x 1 is in between 1 and 5, I am saying that y 1 is in between 0 and 4; when I am saying that x 2 is in between minus 2 and 4, I am saying that y 2 is in between 2 and 6, like that I have I am changing and the range of x 3 and y 3 is same and the range of x 4 was between 3 and 9, but y 4 will be between 0 and 6.

And our claim is that any solution integral solution of y 1, y 2, y 3, y 4; in other words the set of integral solutions of this second equation, this is second, this is first, right, the first equation may be and second equation. We will have a bijection with the set of solutions of the first equation. So, that is easy to see; I will not go very elaborately because I have to cover lot of other things. So, what we do is we just have to notice that. So, you get any solution here for this first equation, then what I can do is take the value of x 1 in that solution and then minus 1 from that and assign it to that value of x 1 minus 1, you assign to y 1 and whatever value of x 2 is there in a solution we add two to it and assign it to y 2. And similarly whatever value of x 3 is part of the solution x 3 whatever values x 3 gets we will give the same value to y 3, and similarly whatever value x 4 gets we will minus 3 from that value and give it to y 4.

It is clear that they add up; now the value is of y 1, y 2 y 3, y 4, add up to 16. So, it is a solution for this second equation and also it is clear that the way we have reduced the values the range will be correct because x 1 was in a certain range between 1 and 5. Now when you minus 1, lowest it can go is 0 and the biggest it can go is 5 minus 1 is 4, right. So, it will come in the corresponding range of y 1. So, similarly the reverse also; for instance if I get a solution for y 1 plus y 2 plus y 3 plus y 4 equal to 16 namely the second equation then we can cook up the corresponding solution for the first equation. For instance we will take the value of y 1, add 1 to it and give it to x 1.

Similarly you will take the value of y 2 and minus 2 from it and give it to x 2 and similarly the value of y 3 will be given directly to x 3 and value of y 3 plus 3 will be given to x 4 and it is easy to check that the corresponding if you add up now x 1, x 2, x 3, x 4’s you will get 18 and defiantly the ranges will be correct. So, there is bijection between the set of solutions of the first equation and the second equation respecting the corresponding constraints; that means we just have to deal with the second equation. So, we have to worry about the second equation only, right. So, now we can write the second equation.

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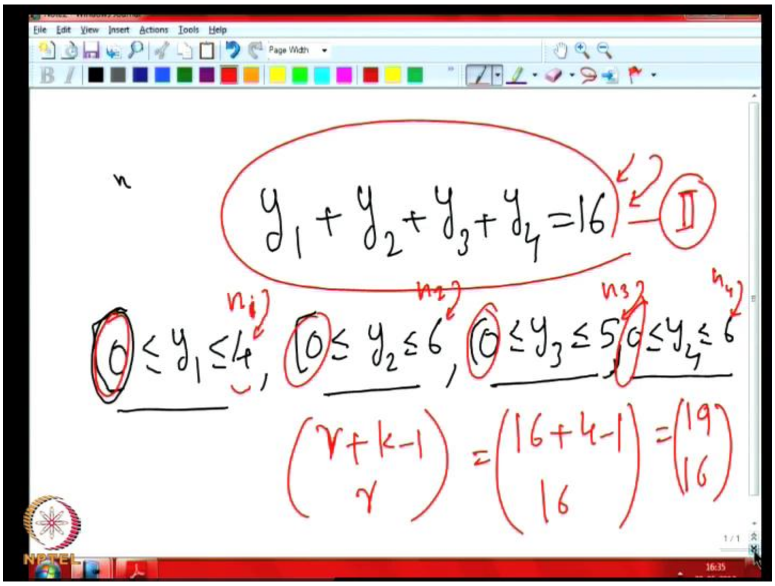

So, this is the second equation; this y 1 plus y 2 plus y 3 plus y 4 equal to 16, this is second equation. The constraints are y 1 is in the range 0 to 4 and y 2 is in the range 0 to 6 and y 3 is in the range whatever is given to x 1, right, that is 0 to 5 and y 4 is in the range 0 to 6, right. Each variable has a certain range. Now the good thing is the lower limit for each variable is 0 here see, the lower limit is 0; lower limit for each variable is 0. So, we are finding the form correct now. This n 1 is equal to 4, n 2 is equal to 6, n 3 equal to 5, n 4 is equal to 6; lower limit is 6. So, not that the corresponding r combination problem is that we want to create 16 combination using objects from a multiset containing four type of things where the first type has 4 copies in the multiset; that means you can only take that first type 4 times and the second type has only 6 copies and third type has only 5 copies and fourth type has only 6 copies, right.

So, this is the corresponding r combination problem. The corresponding balls and boxes problem is that we have 16 identical balls. So, we want to place them into label boxes, first box, second box, third box and fourth box. The first box can hold only utmost 4 balls, the second box can hold only utmost 6 balls and the third box can hold utmost 5 balls and the fourth box can hold only at most 6 balls, how many ways you can distribute 16 identical balls into this four labeled boxes with this constraints; it is the corresponding balls and boxes. Again we can try to solve it here. Now you know the next step is to use the inclusion-exclusion idea.

So, how will you do? So, first you define the universe as the set of solutions of this equation which we called equation two, right, without this upper limits; that means y 1 is at least 0, no upper bound on it. Similarly y 2 is at least 0, no upper bound on it; y 3 is at least 0, no upper bound on it and y 4 is at least 0, no upper bound. In that case we know the number of solutions namely we can use the r plus k minus 1 choose r formula where k is the number of variables namely this is 16 plus 4 minus 1 choose 16, namely that is 19 choose 16. This is the total number of solutions of this thing without the upper restrictions.

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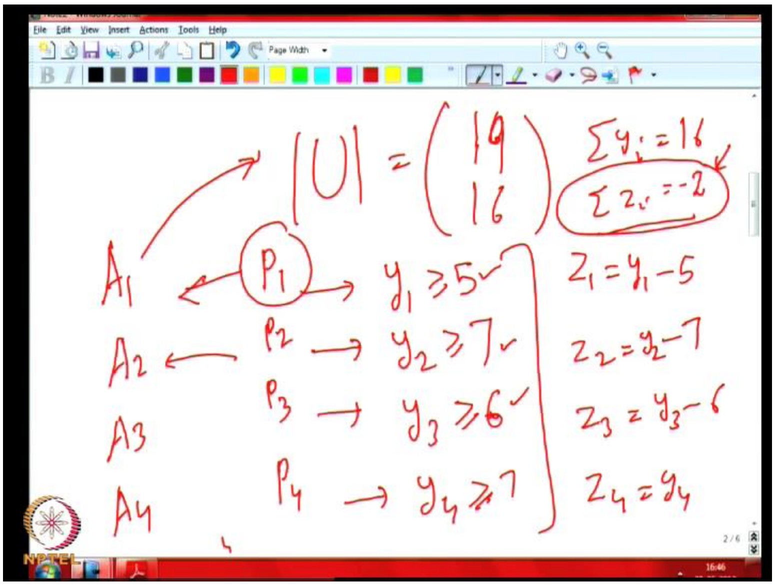

So, this will be the universe; these solutions will form the universe U will correspond to that 19 choose 16, right. Now we will define the undesirable properties as usual. So, there are four variables; with corresponding to the 4 variables we define four undesirable properties p 2, p 3 and p 4. So, undesirable properties are p 1, p 2, p 3, p 4 which are the undesirable property. First undesirable property correspond to the first variable y 1; that undesirable property says y 1 takes values at least 5 means because here we are seeing that desirable thing is utmost 4, so undesirable thing is at least 5. y 1 is greater than equal to 5 is the first undesirable property.

Second undesirable property concerned with y 2 but y 2 is greater than equal to here, utmost 6 is allowed. So, at least 7 is undesirable y 3 what is the undesirable thing; desirable thing it means it at least 6 because we are allowing only utmost 5. This is 6 and y 4 the undesirable property is at least 7, right, these are the undesirable properties. Now we have to find the set A 1 which is the set of solutions from the universe which has the undesirable property p 1. Similarly we want to find A 2 the set of solution from the universe which has the undesirable property p 2 and similarly A 3 and A 4.

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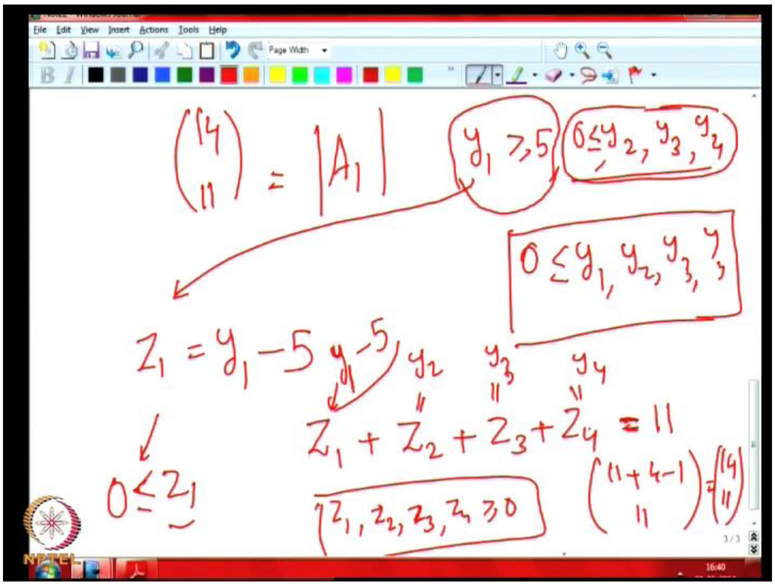

We can look at what is A 1; A 1 means y 1 is greater than equal to 5. A 1 is only that we have I 1 greater than equal to 5; for conditions on y 2, y 3, y 4 is just that it is greater than equal to 0, no upper bound nothing, right, because we are taking all solutions from universe because when we consider the solutions from the universe we were just putting these restrictions; y 1, y 2, y 3, y 4, all were at least 0, no upper bound. This was the way universe was defined.

Now we are defining a subset of the universe which is for these variables it is only as in the universe namely as we defined in the universe namely they have to be at least 0 that is all, but this first variable has to be 5. How will we calculate that? That is easy because you know we will usually define a new variable corresponding to set one namely y 1 minus 5, then what happens is now we will be looking for the solutions of z 1 plus z 2 plus z 3 plus z 4 equal to 16 where this z 2 is only y 2, z 3 is also y 3, there is no change here, z 4 is y 4, but this 1 is y 1 minus 5; z is y 1 minus 5 and of course the range here this restriction because for any value of y 1 we are taking z 1 as y 1 minus 5.

The restriction of z 1 is again just that it is greater than equal to 0 like others, right, because y 1 was at least 5. So, it is allowed to take value 5 or more. So, now z 1 is allowed to take value 0 or more, right. So, now the only thing is here it will not be 16, why? Because we have y 1 minus 5 means when you get a solution of y 1 plus y 2 plus y 3 plus y 4 which add up to 16 and now we are also putting a minus 5 here.

So, it has to be 16 minus 5, this will correspond to 11, right. So, now the solution of other, this is the condition; these are the condition now, z 2, z 3, z 4, all has to be greater than equal to 0, this is the condition. Now we know the solution for this thing. We can use the formula r plus k minus 1, choose r being 11, k being 4, four type of things minus 1, choose 11; that means 11 plus 3, 14 choose 11; that is a solution, right, 14 choose 11. So, we see that the cardinality of A 1 is 14 choose 11.

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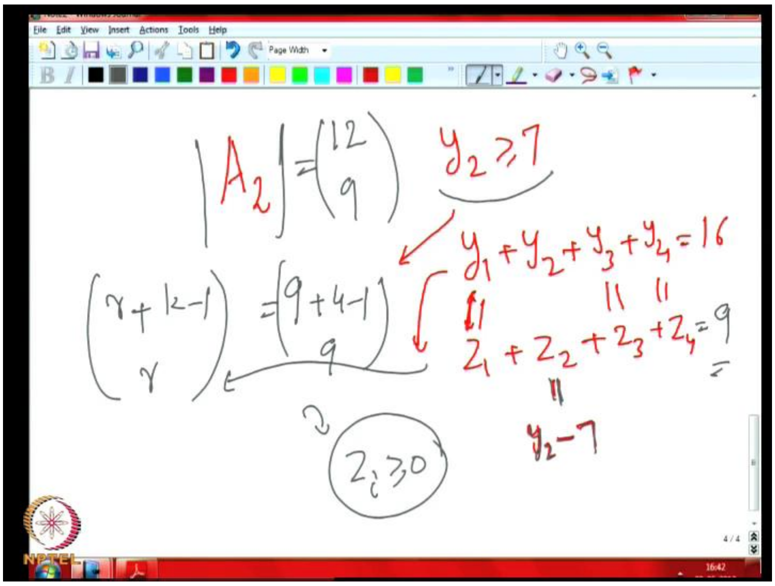

Now similarly we can define A 2 namely the set of solutions from the universe which has the undesirable property p 2 namely y 2 is greater than equal to 7 and this is also we can easily solve by converting this y 1 plus y 2 plus y 3 plus y 4 equal to 16 equation to corresponding z 1 plus z 2 plus z 3 plus z 4 equal to where y 1 and z 1 are equal, this is equal, this is equal, but here we will define z 2 as y 2 minus 7; that is the only difference, here y 2 will be defined as y 2 minus 7.

So, that means correspondingly this will become 9 because we are minusing 7 from this thing, when you add up to 16 this has to add up to 9 and the range of all z i's is now greater than to 0, right, because this was the only condition we have to deal with. So, we minused it then z 2 also is now greater than equal to 0 satisfying that, right. Now we can apply our known formula namely r plus k minus 1, choose r where r is equal to 9 and k is equal to 4; that is 12. So, that means cardinality of A 2 is equal to 12 choose 9. Similarly, we can deal with the third property and the fourth property also, right. Let uss not waste time repeating the same thing.

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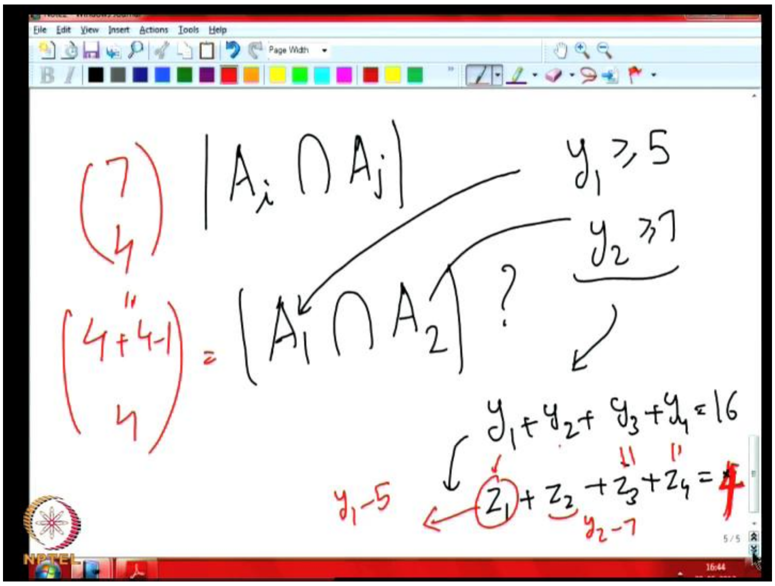

Now the thing we have to worry about is of course we have to worry about how do we find something like intersection of A i and A j; this also one example we can show. For instance A 1 intersection A 2, how will you find? What is A 1? A 1 means our y 1 the set of solutions from the universe where y 1 is greater than equal to 5, the A 2 is when y 2 is greater than equal to 7, right. Here we need both of them. Now problem, we will convert this y 1 plus y 2 plus y 3 plus y 4 is equal to 16, this is equation 2. As before z 1 plus z 2 plus z 3 plus z 4 equal to something, what is this something, right.

Here y 3 and z 3 will be equal; I will put this and this will be equal while here this one z 1 we have to define as y 1 minus 5 because you know this is at least 5, we want to bring z 1 to be at least 0. So, that means y 1 minus 5. Similarly z 2 we have to define as y 2 minus 7 and this will become 16 minus 5 we will make it 11 and another minus 7 we will make it 4, right. This will be 4 and r is equal to 4 now because we know how to solve this equation r equal to 4 and k equal to 4 we get this is 4 plus 4 minus 1 choose 4 namely 7 choose 4, correct. Similarly we can find this is a same technique for each of the other things, right.

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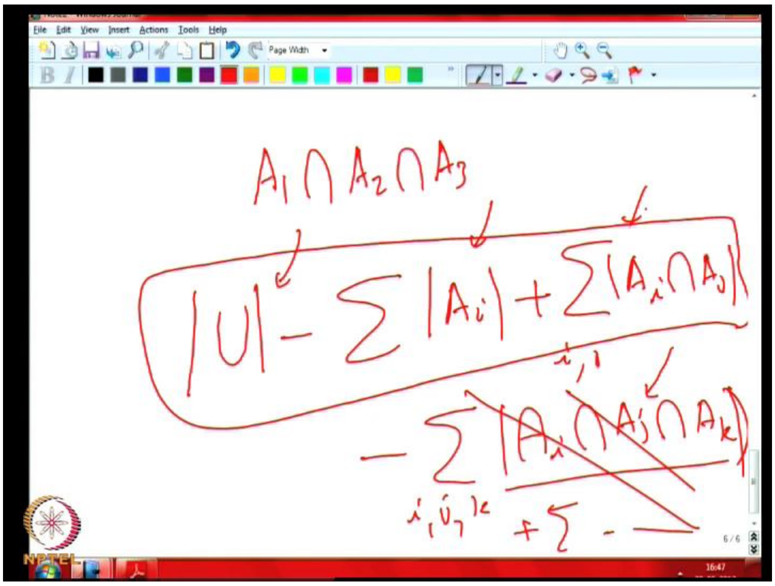

So, for instance in the next case suppose when we want to find A 1 intersection A 2 intersection A 3 or maybe we can consider for a change A 3. So, that means our condition here is that the undesirable property is this three, 5, 7 and 6. Corresponding z 1 will be defined as y 1 minus 5; z 2 will be defined as y 2 minus 7, right; z 3 will defined as y 3 minus 6 and then z 4 will be y 4, right. Then the corresponding sum sigma y i was 16 here and sigma z i will be what?

Because we have to minus 18 also, this will be minus 2, right, but we know that this is not going to happen. So, this solution will be 0, right. So, there in that case we can apply our knowledge that now all z i's are positive and it is an infeasible thing we are asking for because if these things are defined, then y i's are adding up to 16, then these things should add up to 16 minus 5 is 11 minus this, so minus 4, minus 6 will be minus 2, right.

This is what we will get, but then we know that if each z i is greater than equal to 0 by adding this z i's we cannot get minus i. So, the set of solutions we will have only cardinality 0. So, three things taken together, so definitely we can say other three things also will lead to a same thing; they are going to give us 0 the cardinality of the solution, four things taken together will be similarly so. So, now we can apply the inclusion-exclusion principle namely the final cardinality of this thing will be u minus sigma cardinality of A I; we have shown how to find this thing plus sigma for every i and j, we have to add up A i intersection A j's cardinality and minus sigma.

For every i j k, we have to add up the cardinality of A i intersection A j intersection A k and this as we have seen every time is becoming 0, right, and plus and this also will become this terms will go away; this is what we have told. And here we have a simpler formula here. Now it is simplifying, right, because the terms which are coming a little later in this sequence are all 0's as we can easily figure out from there and we have already calculated these things and these things. So, we can find out. So, that is enough for that this is general how to solve this. The number of solutions I have not solved, when we want to find the number of solution for that kind of equations with several possible conditions on the variables, right.

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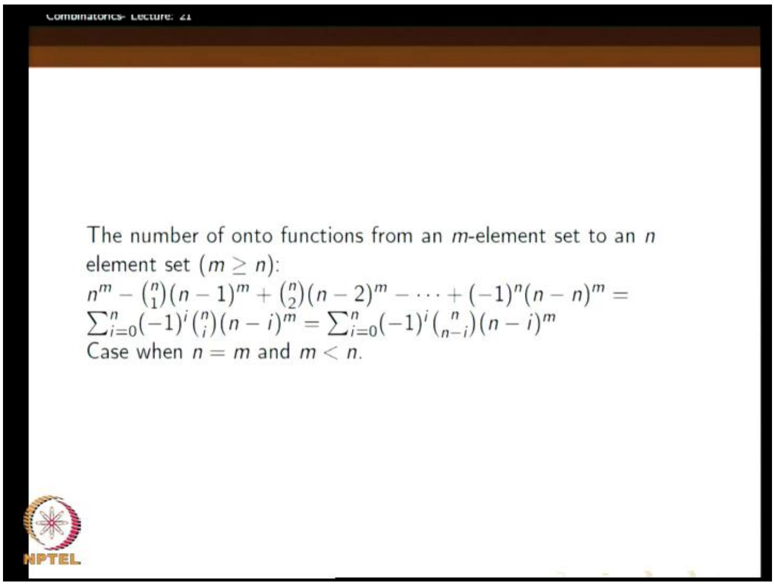

Now let us look at a more important problem of namely counting the number of onto functions from an m-element set to an n-element set where m is greater than equal to n. what are these onto functions?

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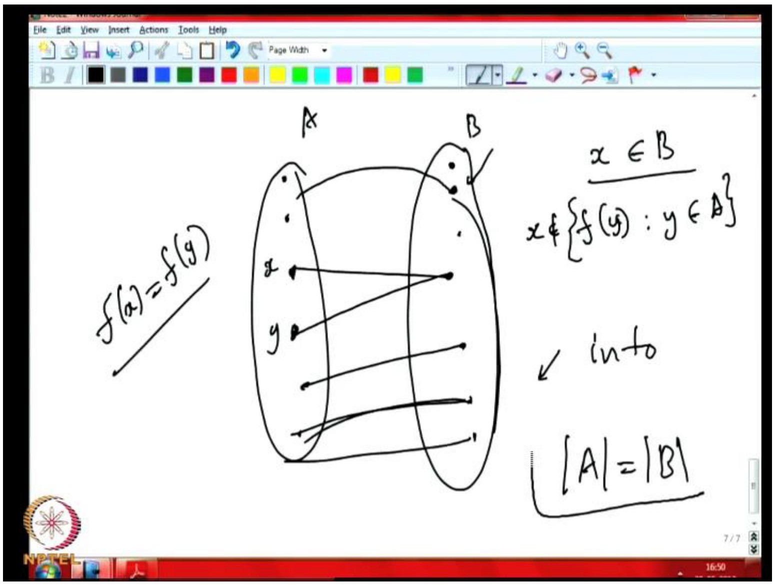

So, as I have already told in the beginning I am not defining functions and things. I assume that students are aware of what functions are and things like that but quickly a recap. So this is A, this is B; you know when I say onto function I mean that for each member of B we have a pre-image, this is onto function from A to B. So, it is not a rigid function; we have a pre-image, right. It is not that there is some element which did not get pre-image. So, there will not be no x element of B such that, right. So, if we consider the set of f of y's y element of a, this x will not be in this thing, right. There will be a pre-image for x. So, this is what we want two function

And we know what is an into function; into function means for two elements on the A side if you take two elements on the A side they will not have the same image. For instance this is x, this is y, they will not have f of x equal to f of y; this is what into function means. When a function is into as well as onto then what we can say is, so every member of B has a pre-image and that is a unique pre-image, in the sense that no two are coming to the same. So, for instance the pre-image will not be more than ones; there will be one. So, which means that the cardinality of A equal to cardinality of B and of course I hope you know all these things.

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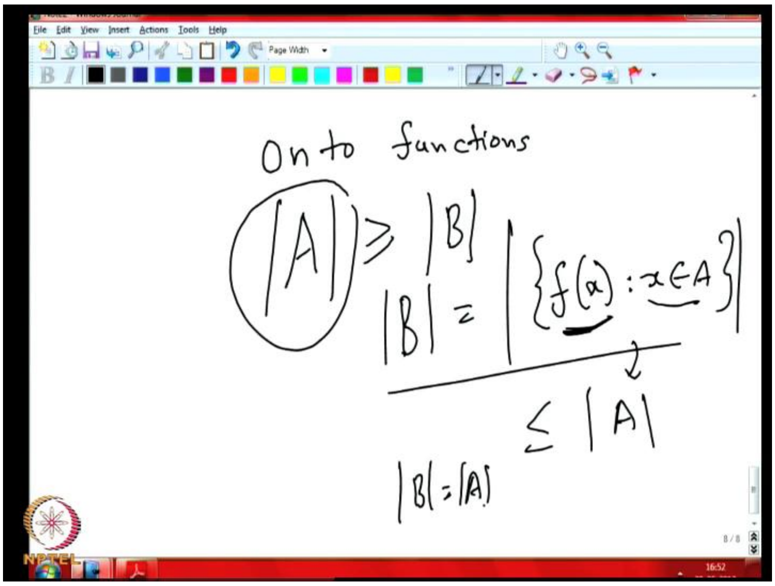

So, we are talking about the situation of onto functions, then it is clear that if it is a onto function there should be at least the number of elements in A should be at least the number of elements in B, because if every member of B gets a pre-image there should be at least as many members in A as there are in B; otherwise, the maximum number of images members in the image, right. For instance if you count this set x element of A cardinality, this can be utmost A right, because you know for each element of A we are getting one this thing.

For instance if it was not an into function, it is possible that two different x members of A can have same value f in the function. Then this will be even less then A, right, but if it is an onto function this will be equal to A, right, sorry if it is an into function this will be equal to A; if its onto f onto function we know that this has to be every member of B this has to be at least B, right. Every member of B has got one pre-image. Yeah, actually this is B is what I told, right, if every member has got, right, if it is an onto function. So, when it is an onto and into function then B equal to A is what we told.

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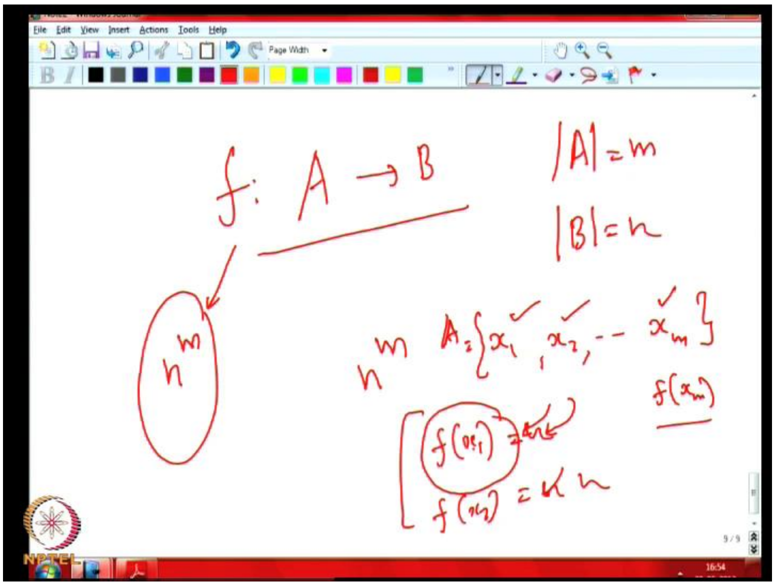

Now we want to count how many into functions can be there from A to B. So, let us assume that the cardinality of A equal to m, the cardinality of B is equal to n. If you were just asking that number of functions, we all know that it is n to the power m the number of functions why because you can take a member of x, you take, say, x 1, x 2, x m be A, right. Let A be x 1 and x to n then f of x 1. So, this can be any of the n values in B. So, there are n possible ways to assign values to f of x 1.

Similarly f of x 2, you can assign this value in n possible ways any of the ways because we are not restricting that once a value is taken that value should not be taken; we are not saying that it should be an into function or something like that, right, there is nothing like that. So, therefore, we do have. So, here there is n choice, here there is n choice, for this n choice, for this n choice, for this; for this means for deciding f of x m we have n choice. So, n to the power m is the number of functions total number of functions and of course we can ask how many.

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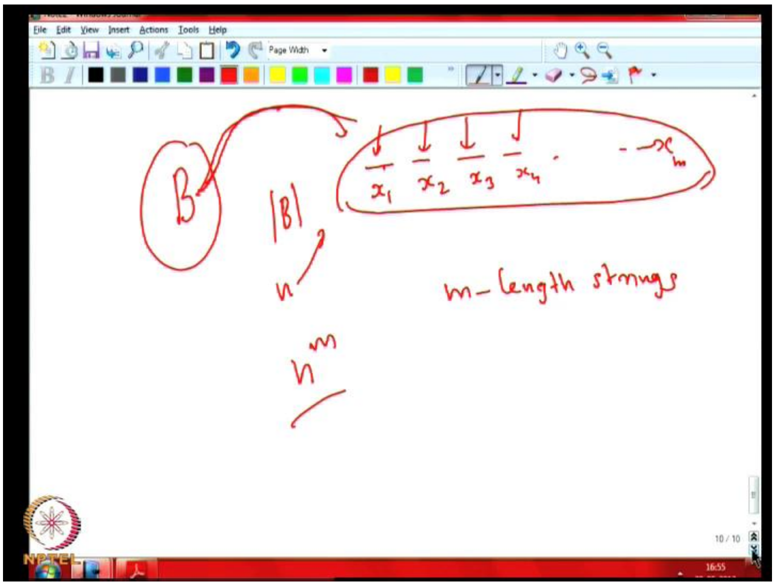

So, of course this is another way of saying it, so this is you know the members of A, we can write it as here x 1, x 2, x 3, x 4 right. So, the x m and we are assigning the members of B can be taught of as on letters we are making m length strings, right, using alphabet from the B being the alphabet, right, set of letters, right. We fill something here, fill something here, we do not have any constraint here. So, we can fill it in B ways cardinality of B ways; that means n here n ways; that is why it is n to the power m, right.

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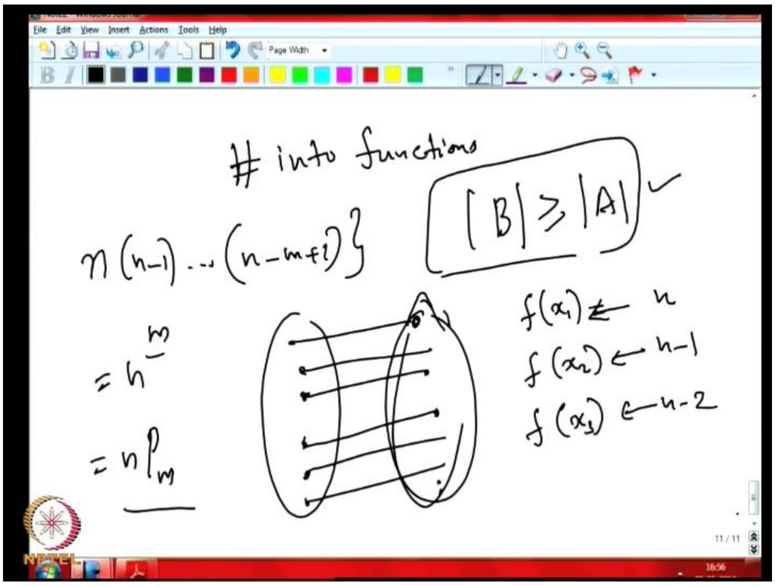

Now let us look at the number of into functions; this is also easy number of into functions. Note that when it is number of into functions, the condition is that the cardinality of B has to be greater than equal to A the other way, right. In the case of the number of onto function we wanted the cardinality of A to be less than equal to the cardinality of B, but here into function means you know any two things should go to two different things; I mean they should mapped to two different things in B. So, it is very clear that like here on this B side there should be at least as many things as there are on the A sides.

Otherwise we cannot assign separate separate things for different values here, right, and it is okay, f of x 1 you can assign in n possible ways because there are n things here and f of x 2 we can now assign in n minus 1 possible ways because once we assign a certain things to x 1, we cannot use the same thing now for x 2, right; that is the property of into, right, that is what the into demands into property demands. Similarly f of x 3 can now be given only n minus 2 values. So, naturally this is n into n minus 1 into up to finally, n minus m plus 1 which is n m following factorial or n P m, right. This is what the number of into functions, but both this counting the number of functions and number of into functions are easier than number of onto functions.

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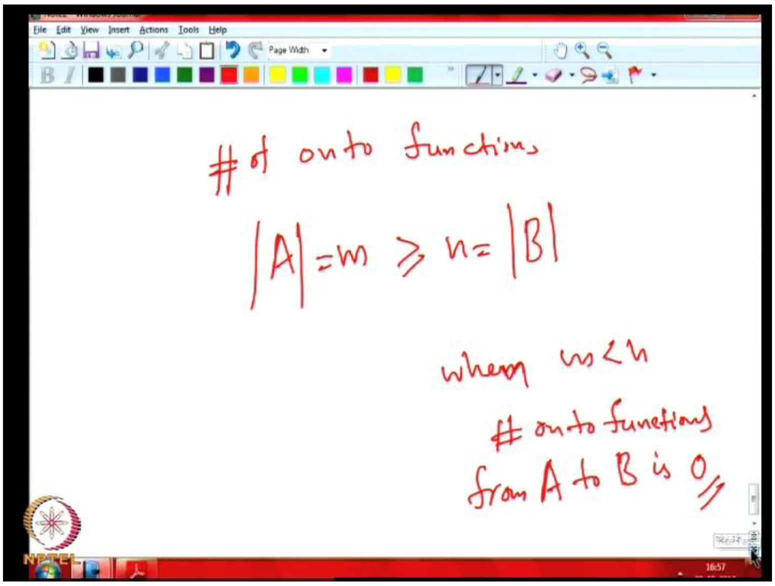

Now we look at the more difficult problem namely number of onto functions. As we told we are assuming that now cardinality of A equal to m greater than or equal to n the cardinality of B. Otherwise onto we cannot have onto functions; otherwise it is 0 when m is less then n, the number of onto functions from A to B is 0, right.

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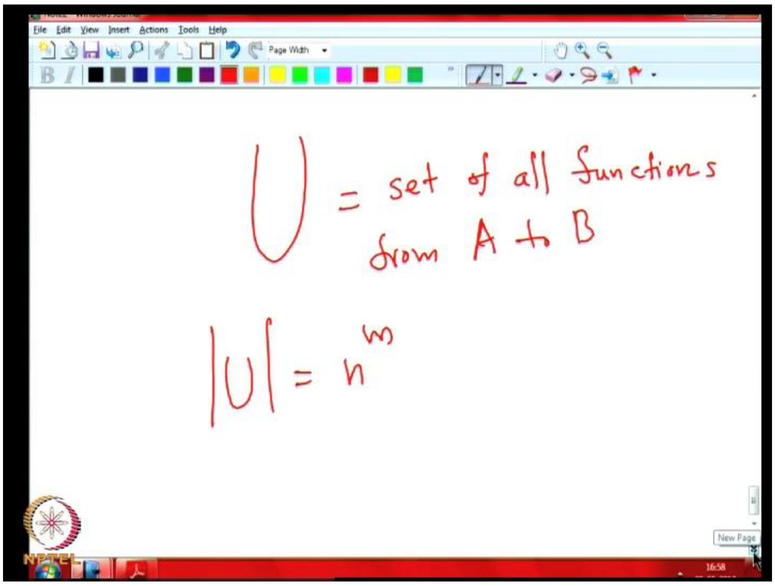

So, now to find the number of onto functions what we do is we define U the universe as the set of all functions, we are planning to use the inclusion-exclusion principle. So, we need the universe, set of all functions from A to B; we have already seen that the cardinality of the universe then is n to the power m.

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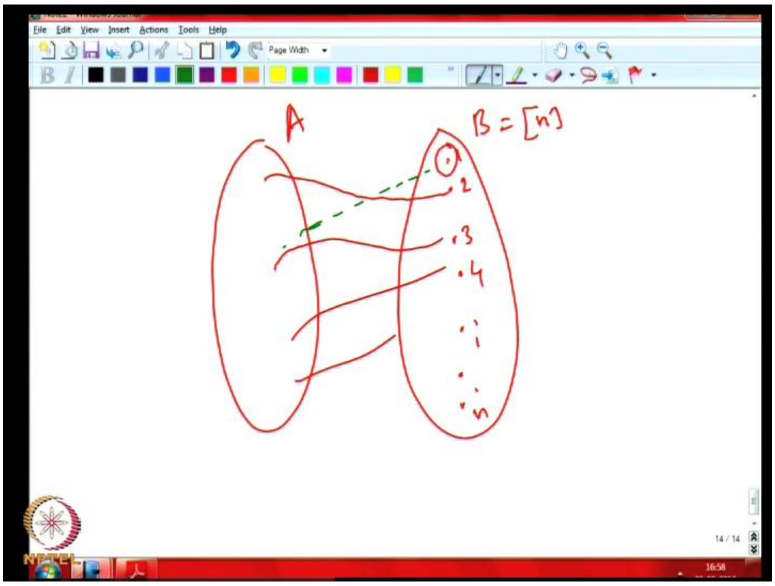

Now when we consider onto functions what are the desirable properties? This being A and this begin B say the members of B may be for the time being let us take B is equal to n. So, this is 1, 2, 3, 4 up to n. So, for instance one of the desirable properties is that a function is such that one does not have any pre-image; that means this one does not have any pre-image here, right.

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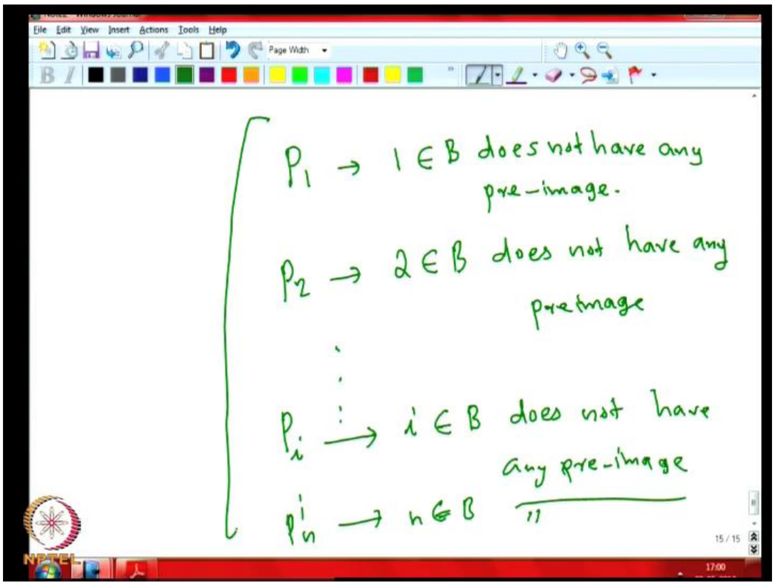

So, that undesirable property p 1 we can define as this; 1 in B does not have any pre-image. Similarly p 2 can be defined as 2, this is an undesirable property, 2 element of B does not have pre-image and similarly the i’th undesirable property namely i element of B does not have any pre-image. It is very clear how we decided these undesirable properties. So, finally we have p n and then n element of B, right, does not have any pre-image because if a particular member of B does not have a pre-image then the function we are talking about is not an onto function. So, therefore it is simple enough to decide this undesirable properties and the thing we want to count namely the number of onto functions is clearly.

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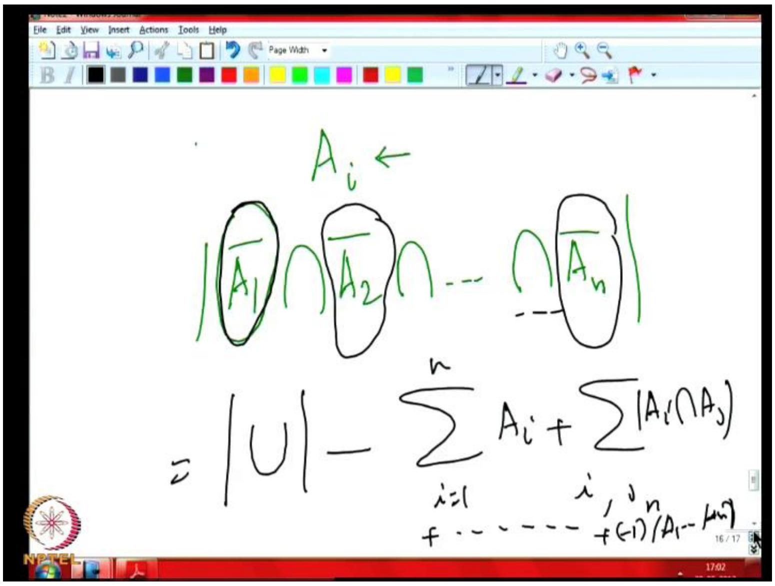

So, for instance if we define A i as the set of functions from A to B such that it has the undesirable property here namely this is the set of function such that i element of B does not have a pre-image; clearly we are counting this one, A 1 complement intersection A 2 complement intersection A n complement. So, this is A n complement, right, because here this means the functions in which one has a pre-image where this complement means the functions in which one has a pre-image, the functions in which two has a pre-image and the functions in which n has a pre-image. So, when you take the intersection the functions in which each of 1, 2, 3, up to n has a pre-image, right; this is what we are interested. This is by inclusion-exclusion is U minus sigma A i for i equal to 1 to n plus for every pair I, j if we consider A i intersection A j and so on so on, i plus till last minus 1 raise to n. So, you have to consider the intersections A 1 to m.

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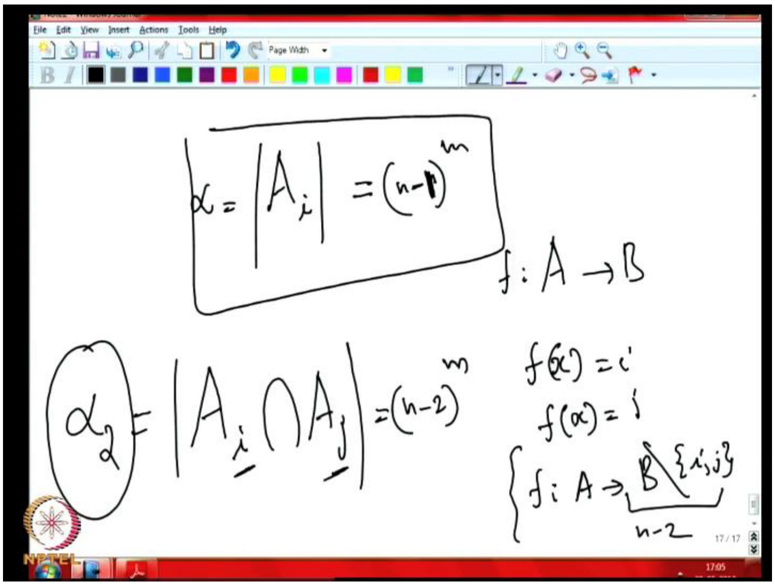

But we can use this simplified formula why because if you look at the pattern for instance what about A i? What is the cardinality of A i. The cardinality of A i is I mean the A i say is it is a function we are counting the number of functions such that i does not have a pre-image means no value no member of A is such that f of x equal to i; that means it is as good as defining functions from A to B bar I, right and you know number of functions here only we are talking about; that is n minus 1 is the cardinality of this set.

So, therefore the number of functions is n minus 1 raise to m. So, n minus 1 raise to m and we know that which i is unimportant because for any A i this is the number, right, number of functions whether it is A 1 or A 2 or A 3 or whatever. So, we can take this as alpha the cardinality of A I; this is the common cardinality for each A i, cardinality of A i is equal to n minus 1 raise to m.

Similarly if we want to find A i intersection A j cardinality, there also it does not matter which i and j I take. This is again alpha 2, why it does not take because you know that is the same value because here we say we are interested in the number of functions from A to B such that neither i nor j has a pre-image; that means there is no member in x such f of x equal to i. Similarly no member in a member in A such that f of x equal to j. So, it is as before we can say that it is as good as defining a function from A to B bar i, j; i and j are removed from B because here the cardinality of this set is n minus 2 and the number of such functions is n minus 2 to the power m. So, we can say that this is alpha 2 because it does not depend on i and j as long as we are taking two of those sets and taking the intersection, the cardinality is going to be n minus 2 raise to m because we are telling that the given two elements are not appearing as images at all; that means they do not have pre-images.

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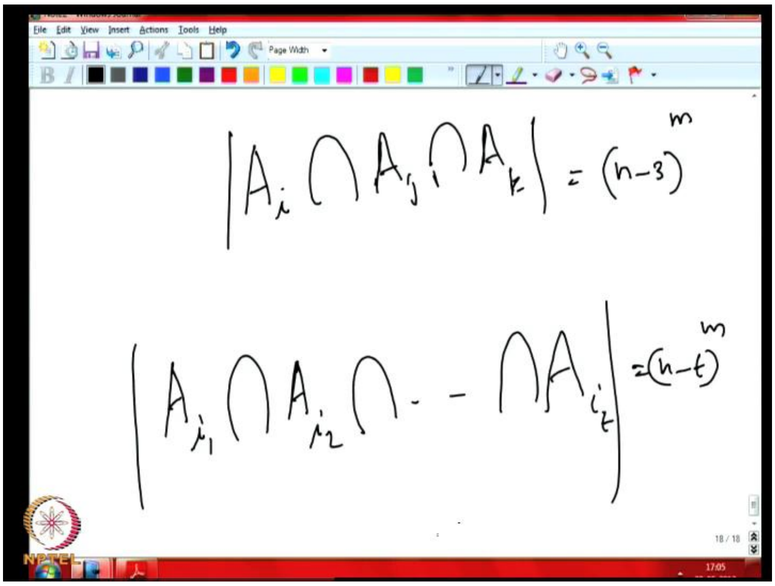

Similarly for instance three things if you want to consider, for instance A i intersection A j intersection A k; it is very clear that now three things are not getting pre-image. So, we can as well think that we are defining functions; they are counting functions from A to B minus i comma j comma k. So, that means there are n minus 3 possible values and n minus 3 to the power m is the total number of functions and in general when we are considering, say, A i 1 intersection A i 2 intersection A i t, this cardinality will be this t things, right, n minus t to the power as we want to know.

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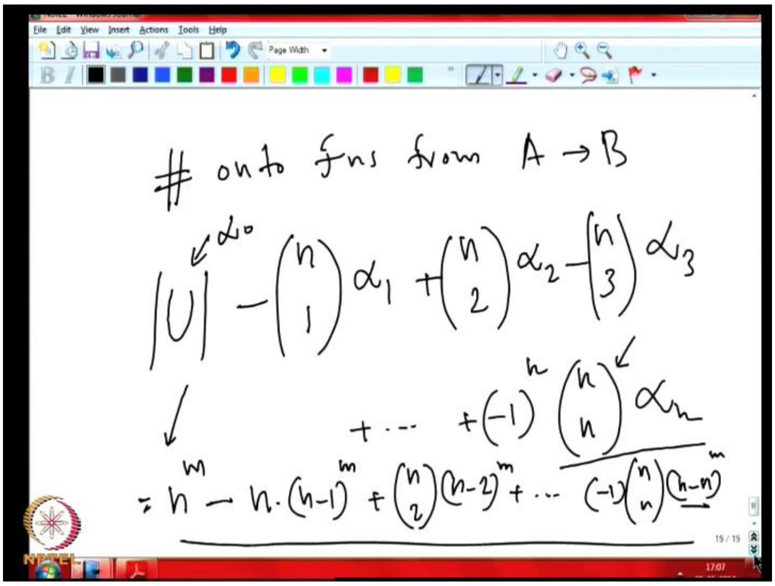

So, therefore the number of onto functions from A to B now can be found by the inclusion-exclusion principle by this thing. So, if you call this as A alpha 0. So, this is n choose 1; this n begin the cardinality of B, right, alpha 1 plus n choose 2 alpha 2 minus n choose 3 alpha 3 and minus 1 raise to n n choose n alpha n; this is what we will see, this is what n raise to m, this is what n into n minus 1 raise to m and this is what n choose 2 into n minus 2 raise to m and finally minus 1 raise to n choose n into n minus n raise to m which is 0 is the raise to m, it is clear for instance if all the n elements in B says we are such that they do not have a pre-image, then there are no such functions because the members of A has to get mapped to something in B, right.

So, therefore this last term is going be 0 clearly but anyway n minus n raise to m we are writing it, right, this is the number of onto functions. Of case this is just a formula; if you want the values we have to evaluate it by putting the value of n and m but of course it is useful enough. When we really need it we can compute it or another thing we should consider.

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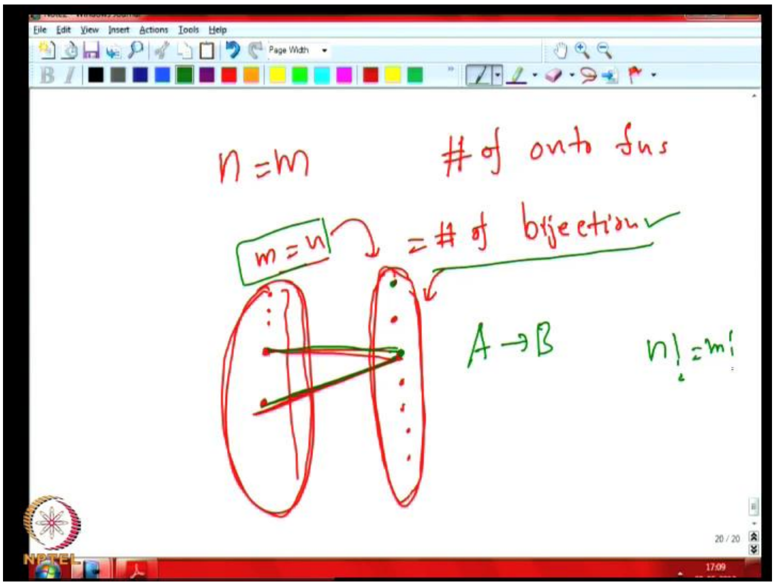

Two special cases we can consider here namely when n equal to m what is the situation. When n equal to m we know that as we have discussed the number of onto functions is actually the number of bijections, right, why because we have shown to every member of B has a pre-image. So, that means no two members of the A side can get mapped to the same member in B, why is it so? Because if two members in A got mapped to the same member in B, now we know this number of members here is only equal to the number of members in B. If two of them got mapped to the same person here, then definitely the total images we see here total the function values if you count, right, that will be straightly less than the cardinality of this thing, the cardinality of this thing; that means the cardinality of this thing because this are equal, right; that means it will not be onto.

There will be some value here which will not get a pre-image because we unnecessarily did not use the resources properly. So, they are exactly equal; equal means we should have mapped one for one, right. So, there will be one left here. So, that means it will not be an onto function. So, when m equal to n and the function is onto we can also infer that it is into; that means it is onto and into together; that means bijection and we know when m equal to n the number of bijections from A to B, right, from n to n, right, that is n factorial; that is n factorial equal to m factorial, right.

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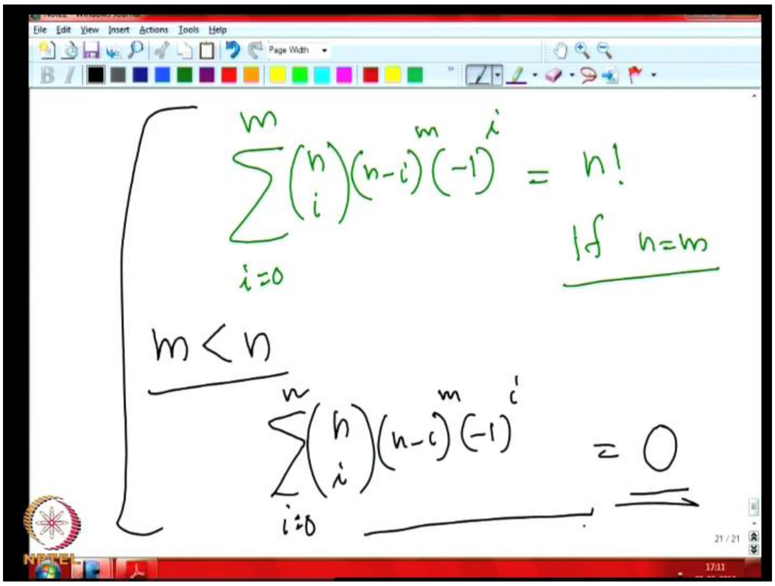

So, that means this formula we wrote sigma n choose 0 into n choose i n minus i raise to m; this is the way i can write it for 0 to m, right, with minus 1 raise to i. So, when i equal to 0 that is n choose 0 namely 1 into n raise to m itself and i was equal to 1 that is n into n choose 1 into n minus 1 raise to m; it is the negative thing and so on. This will become n factorial if n equal to m; this is what we told. Now the same situation we can consider what will happen if m was less than n, then defiantly we cannot have a onto function, right, because the number of members in A is strictly less than the number of members in B, how can every member in B get a pre-image? So, the number of onto functions has to be 0 but this formula still works, right.

So, this is n choose i into n minus i raise to m minis 1 raise to i, right, i equal to 0 to n, this has to evaluate to 0 somehow. So, that is what; that is two special cases. So, we can think. So, we can try to come up; so one can try to see whether one can this is kind of a combinatorial proof we have shown that this from when m is less then n this will add up to 0. So, by manipulating the terms can you get this same result; that you can try. So, we have already discussed the onto functions; now we will look at another problem.

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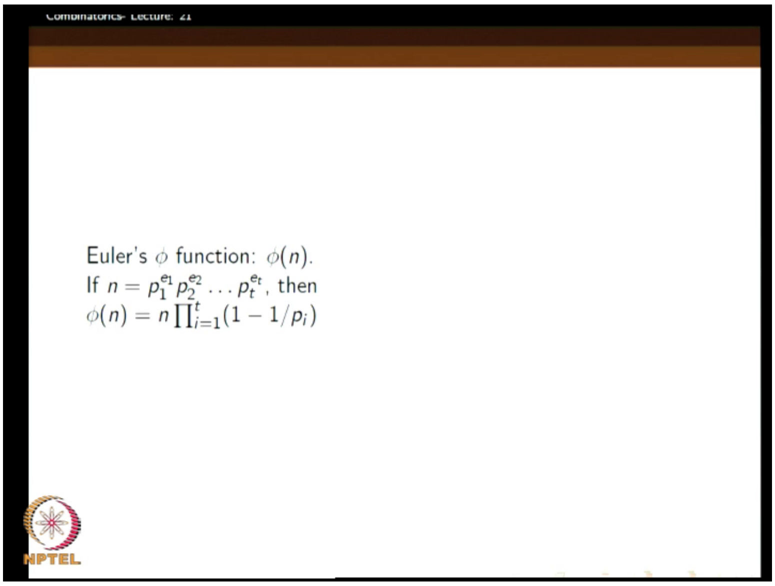

So, this is also very interesting and important question. There is something called Euler's phi function, right, what is Euler's phi function? So, we have to remember something from the elementary number theory. So, one thing is the notion of relatively prime.

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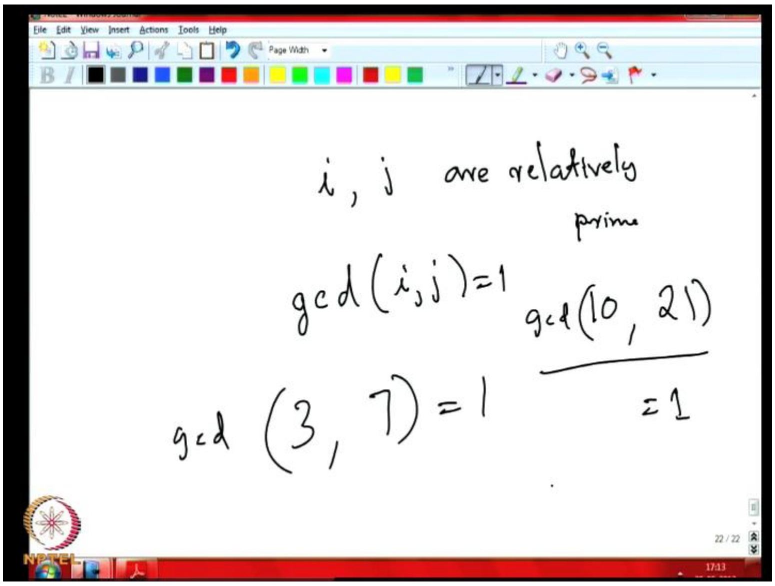

So, when do we say two numbers i and j are relatively prime or co-prime feature, they are relatively prime. So, that is when g c d of i, j equal to 1. So, for instance 3 and 7 they are relatively prime because g c d of 3, 7 is 1; for 10 and 21 g c d of 10 and 21 is 1. So, they do not have any common factors other than 1; that is what it says, right, these are relatively prime.

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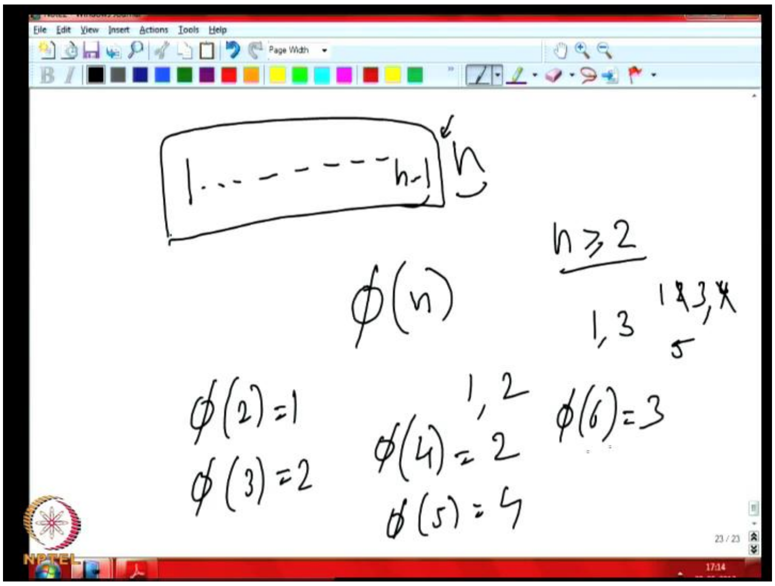

So, one parameter which is interesting in number theory is the number of, so for given n how many numbers are there from 1 to n minus 1 which is relatively prime to n. For instance for 1 it is 1 itself, sorry so we will say greater than 1. So, from 1 to n we will consider, for numbers and greater than equal to 2 we will consider and for 2, 1 is readily belong to 2. So, it is equal to 1 only. So, this numbers from 1 to n minus 1 which is relatively prime to n here.

So, below n plus or we can even say less than equal to n because n anyway is not going to be relatively prime. So, that is denoted by phi of n, this is the Euler's phi function, right. So, phi of 2 is equal to 1 and phi of 3 is equal to 2 why because 1 is relatively prime, 2 is also relatively prime to 3 and what is phi of 4? Phi of 4 is equal to 1 as relatively prime, 2 is not relatively prime but 3 is relatively prime, this is 2. So, phi of 5 is 4, one, two, three, four and phi of 6 is what? Phi of 6 is 1 is 2 is not 3 is 4 is not 5 is one, two, three, right. So, like that.

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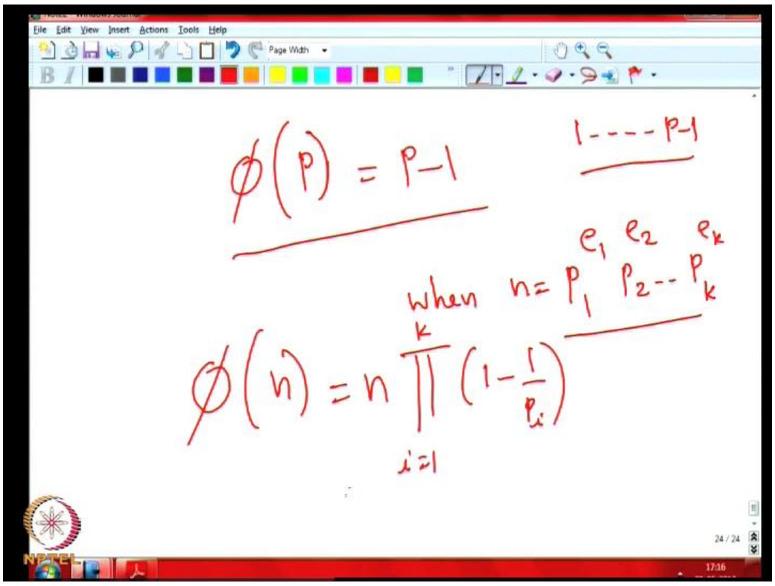

So, now we can look at more general cases. For instance what about phi of a prime number p; it is very easy to see that it is p minus 1 because all the numbers below p from 1 to p minus 1 will be relatively prime to p, p being a prime number, right. There is nothing common, like no common factor will be there between p and a number between 1 to p minus 1, phi of p is p minus 1 and now we want to derive a general formula for phi of n when n equal to p 1 raise to e 1 p 2 raise to e 2, say, p k raise to e k, right; this is what we want. We will show that this is equal to n into product of i equal to 1 to k 1 minus 1 by p i.

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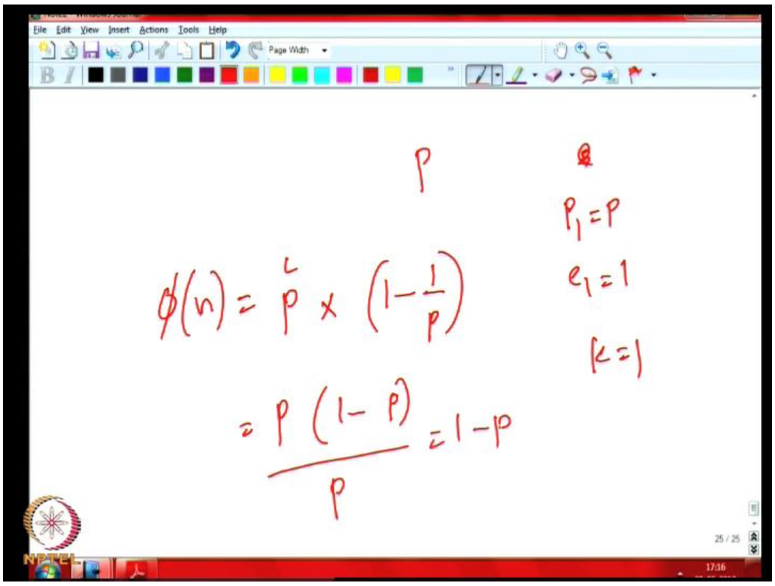

For instance you can put suppose it was a just a prime number; that means p 1 is equal to p, e 1 is equal to 1 and so that is what, so that k equal to 1 case. In that case you can see that phi on n is equal to n into, n here is p, p into product of i equal to 1 to 1, right only one term will be there 1 minus 1 by p, what is this. So, then we do not have to put the product at all, right. So, this is p into 1 minus p by p which is 1 minus p as we have seen before, right.

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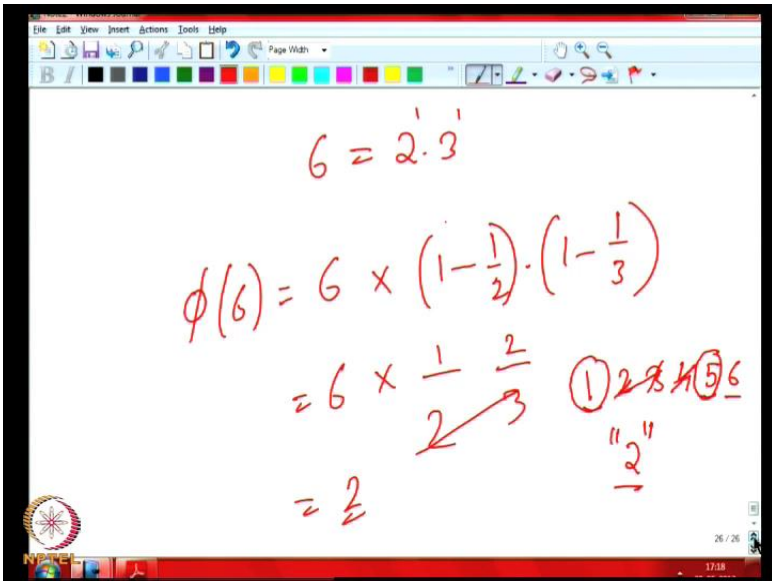

So, similarly we can we can do the calculation and find out for instance what is for 6; 6 equal to 2 into 3. So, e 1 and e 2 both are 1 here. So, then phi of 6 is equal to 6 into this is what; 1 minus 1 by 2 into 1 minus 1 by 3, right. So, this is 6 into, so 2 minus 1 that is 1 by 2, this is 2 by 3, so we get 2. So, what was six 1, 2, 3, 4, 5, 6, correct. So, I wrote it wrongly before it is 2. So, this 2 is not relatively prime, 3 is not relatively prime, 4 is not relative prime. So, 1 and 5 are the relatively prime to 6. So, 2 is the answer. So, we get 2. So, here I have written phi of this is wrong. So, this is 2; I think I thought 3 also is relatively prime. So, that is a mistake of course. So, this is 2. Anyway, this is the formula. So, we want to derive this formula using inclusion-exclusion principle.

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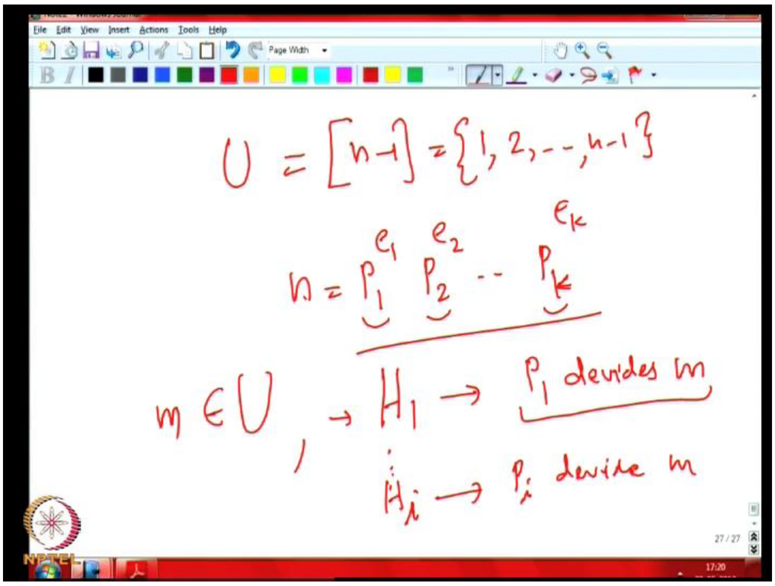

The technique is very similar; what we do is we will take n minus 1 as the universe means n minus 1 means from 1, 2 n minus 1 and then we define the undesirable properties, what are the undesirable properties. So, because our n is p 1 raise to e 1 p 2 raise to e 2 simply k raise to e k. So, undesirable means a given k in the universe that is not relatively prime to n, this is desirable, right. Why is not relatively prime; there is some common factor and if there is some common factor there is one of this p 1 or p 2 or p k as a common factor, right. If there is a common factor, there is a common prime factor, right. So, it is either p 1 or p 2 or p k. So, you can define the property undesirable property. So, here I have used p 1, p 2, etc for undesirable properties.

So, let us say H 1 is the first undesirable property; that is p 1 divides k, of course p 1 divides n. If p 1 divides k also then p 1 is a common factor for n and k. So, this is indeed in undesirable property. Similarly H 2 would mean p 2 divides k and similarly H i would mean p I, the i'th prime number divides k. The properties are defined of H i of k; that is what it divide k, right, k has the property H i is what we are saying. So, like that we can divide. So, we goofed up by saying that this k; I should have taken a different name for the number may be we will make m, m for the number p divides m because k I am using for that number of prime numbers here. So, this was k. So, this lets say m. So, we will say H 1 of m p i of m, these are the undesirable properties.

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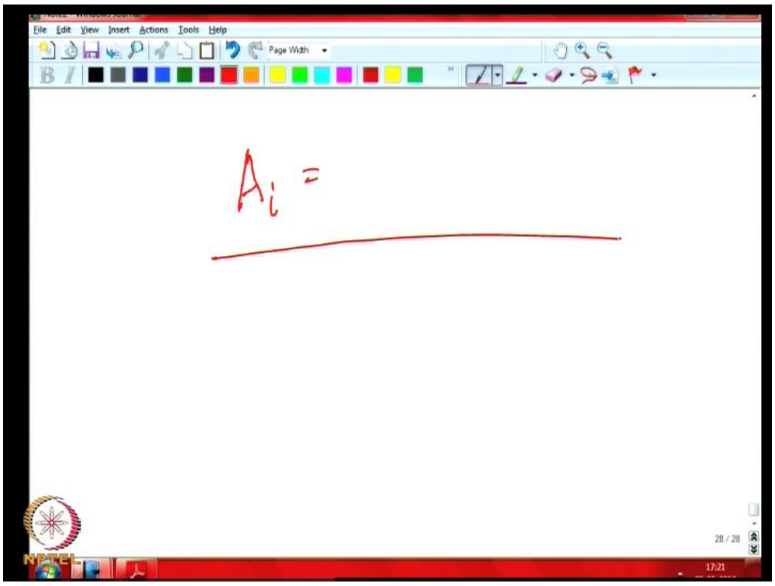

Now we also define A i is the set of numbers from the universe f 1 to n minus 1 the numbers from 1 to n minus 1 such that it has property H i. So, we will continue in the next class.