Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 20
Inclusion Exclusion Principle - Part (3)

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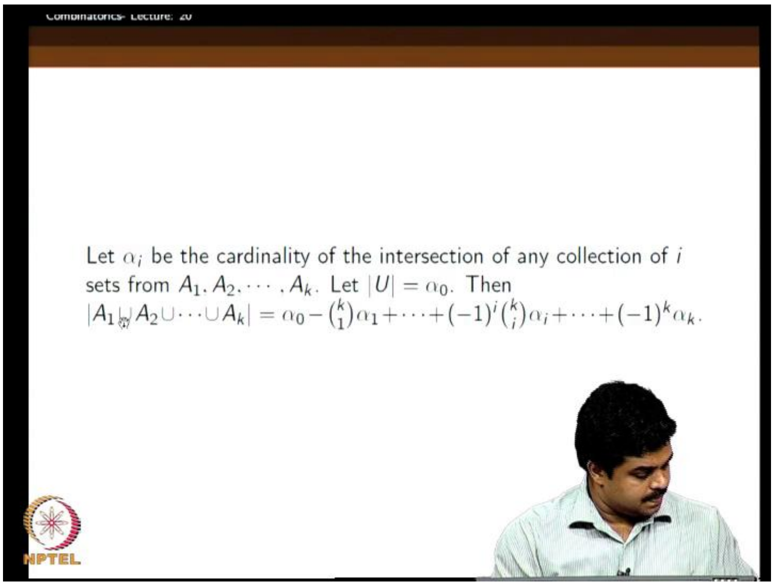

Welcome to the 20th lecture of combinatorics. In the last class, we stopped at this simplified situation where we can apply the inclusion exclusion principle. The general case was this. Now, the simplification is that suppose you universe is of cardinality alpha 0 and any pair of, sorry any singleton set here and singleton set is of cardinality alpha irrespective of which set all singleton any a i is of cardinality alpha 1, and any pair has cardinality alpha 2 and any intersection of any pair.

That means, for instance a i and a j, if we take irrespective of which i and j their intersection has cardinality alpha 2 and any triple, for instance i j k. If I take alpha irrespective of which is this i, j and k, A i intersection, a j intersection, a k always have cardinality a alpha 3 and so on, right. In that k is the formula simplifies. Here in this case I have made a mistake in this slide. So, what we want is the compliment of this, the cardinality or the compliment of one union, two unions a k. This is what we meant.

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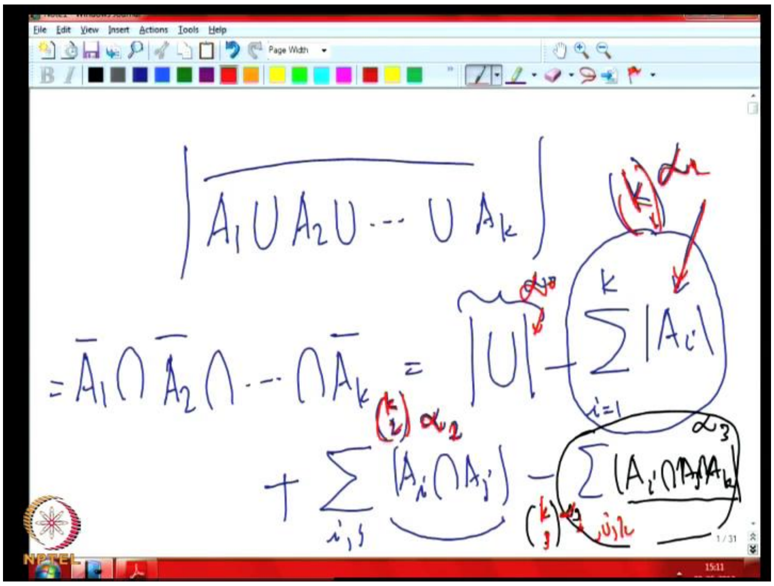

So, of course I will write correctly. We want a one union, two unions a k, the compliment of this. What is the cardinality of this? So, we have seen that this is same as a one compliment intersection, a two compliment intersection. So, a k compliment, right. If this is cardinality of u minus sigma i equal to 1 to k, the cardinality of a i 2 minus, sorry plus sigma for pair i, j. If I take a i intersection, a j cardinalities and minus like that, we had written the formula, but here we know each of look at this term. Each a i here has cardinality alpha 1. So, this is cardinalities alpha 0. This each has cardinality alpha 1. Therefore, total here is k, sorry k chose 1 into alpha 1 k k times alpha 1, we can say k chose 1 times alpha 1.

Similarly, here every pair, each of this irrespective of which i and j, I am talking about the cardinality carries alpha 2. So, total here will be k chose 2 times alpha. So, you can use a different color, k chose 2 times alpha 2 for this thing, right. So, this is k chose 1 times alpha 1 for this and this is alpha 0 from here. The next term, when i j k are considered and a i intersection, say so a i intersection, a j intersection, a k is considered. So, always this alpha 3 irrespective which i j k, I am talking about.

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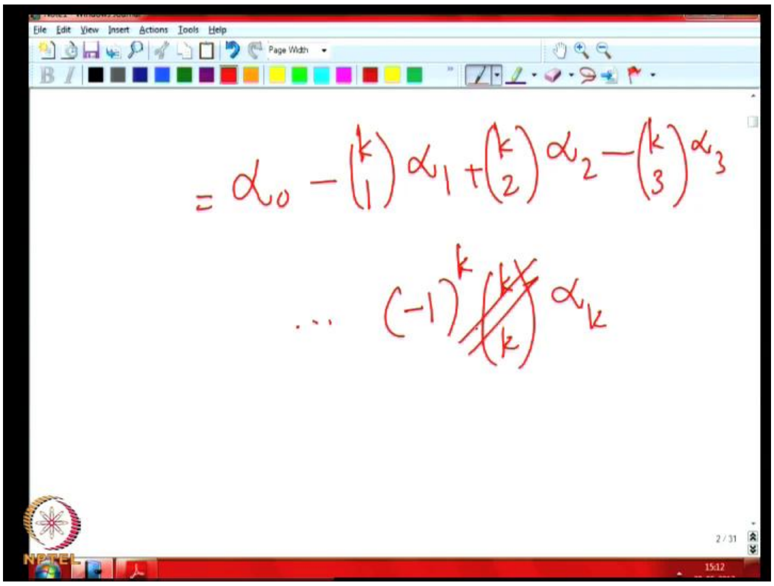

So, therefore, this total, this sum will be how much? So, k chose 3 into alpha 3, right, k chose 3 k chose 3 into alpha 3. So, we can rewrite this as alpha 0 minus k chose 1 into alpha 1 plus k chose 2 into alpha 2 minus k chose 3 into alpha 3 and so on. Total minus 1 raise to k. See, k chose k into alpha k, right. This is not equate because we know this is just one. So, we can remove this thing. So, we can just say into alpha k, right. This is the formula.

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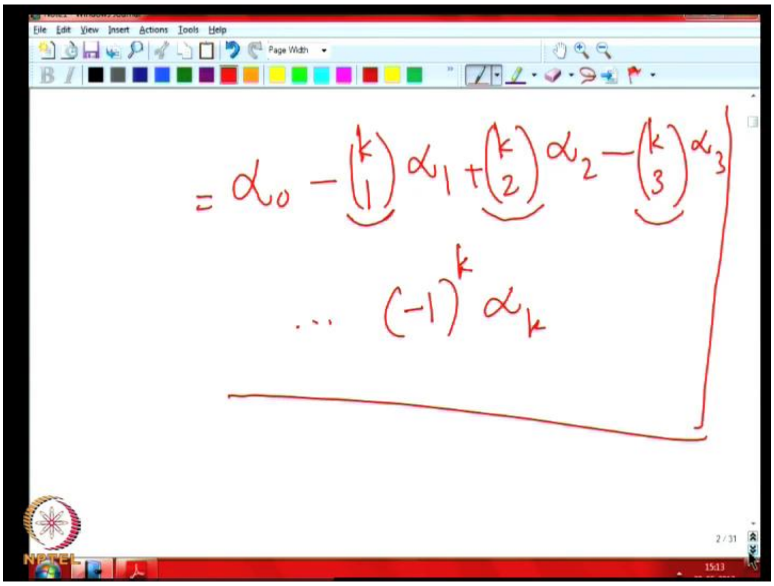

This is not nothing non-trivial. Definitely it is quite easy thing. You just have to observe that when in a summation when every term that is summed up everything, that is summed up is of equal value. Then we can just multiply it by the number of terms in the sum, right. That is what for instance, in the first sum, we have k chose one terms. In the second term, we have sum, we have k chose two terms. In the third sum, we have k chose three terms and all. That is all, right.

Now, we see, why do we write it as a special case. We could have anyways remember the earlier formula and definitely, we could have applied it whenever we want that because we would have identified that is true, but again many cases like the situation is like this because the cardinality of say some t sets taken out of the k sets, we have their intersection. Cardinality will only depend on what is this t naught on which t sets are selected. For instance, in the next example we will illustrate it.

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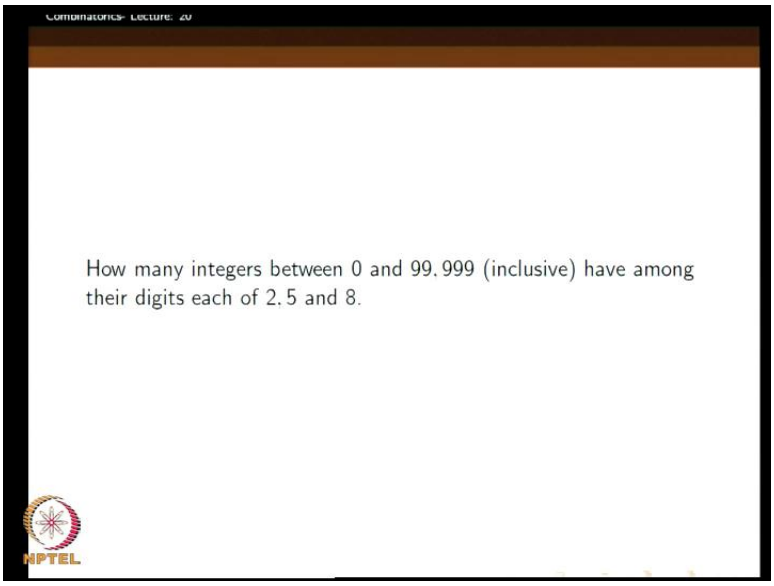

Let us see, so let us look how many integers between 0 and 99,999 inclusive both 0 is also included and 99,999 is also included among their digits each of 2, 5 and 8, right.

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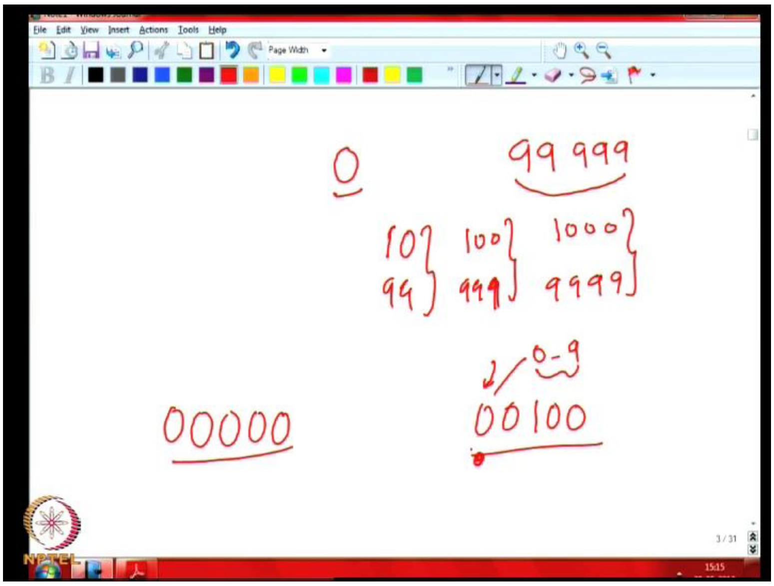

So, we are considering numbers from 0 to 99,999. See of course, one may worry that there are like first, we have one digit numbers and then we have two digit numbers, right. 10 to 99, there are two digit numbers and then from 100 to 999, there are 999. They are all three digit numbers. Then we have 1000 to 9999. We have four digit numbers and so on, but then we do not have to worry about all those things. We can think that the first number is just 00 by pairing like 0, 0, 0, 0, 0, five zeros. We put for the 0. For instance, a number like 100 can be paired with two 0's in the front and we can make it five digit number because that we can say that anywhere there is no restriction in each position. I can use 0 to in any of these digits, 0 to 9. 10 digits we can take, we can use this. Therefore, if you ask how many numbers are there, right. So, it is just easy to say it is because there is a five digit number and then we can fill here anything from 0 to 9.

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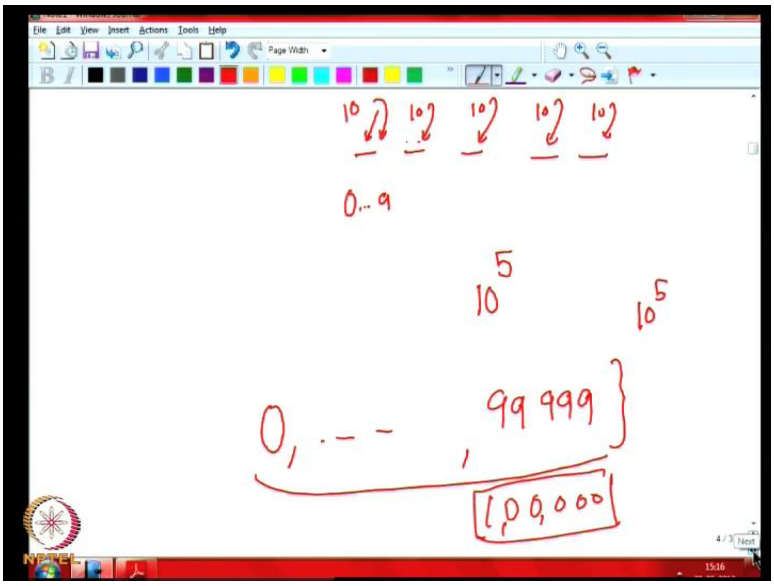

So, there are 10 possibilities to fill this thing and then here there are again 10 possibilities to fill and there are another 10 possibilities to fill here, another 10 possibilities to fill here. Therefore, the total numbers of possibilities are 10 raise to 5. It is quite easy to verify also because starting from 0, we reach 9999. So, total number we have is 10 raise to 5. Of course, 10,000, right sorry, 10 raise of 5. Sorry, 1,00,000, right. This will be the number.

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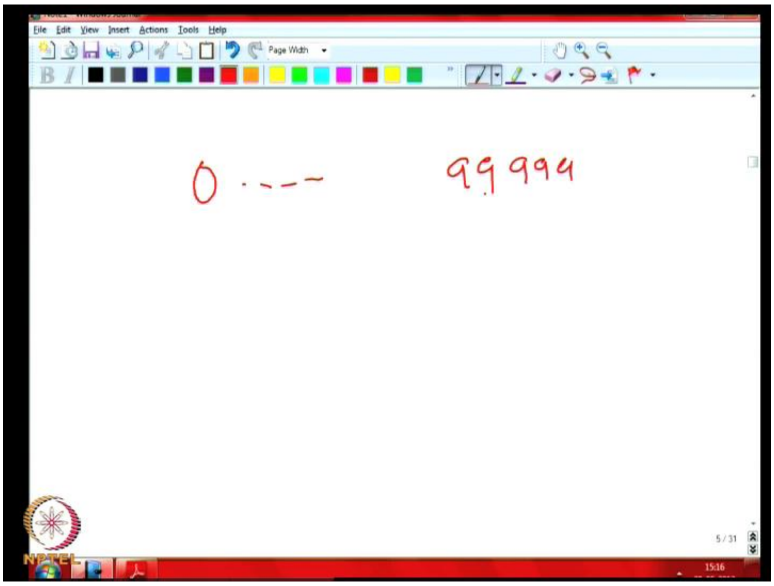

So, these many numbers are there, right. Now, the question is not how many numbers are there. That is not the question. We want then number from 0 to 999 satisfying this following property, which is the following property, namely we should have 2, 3 and 8 in the numbers, right.

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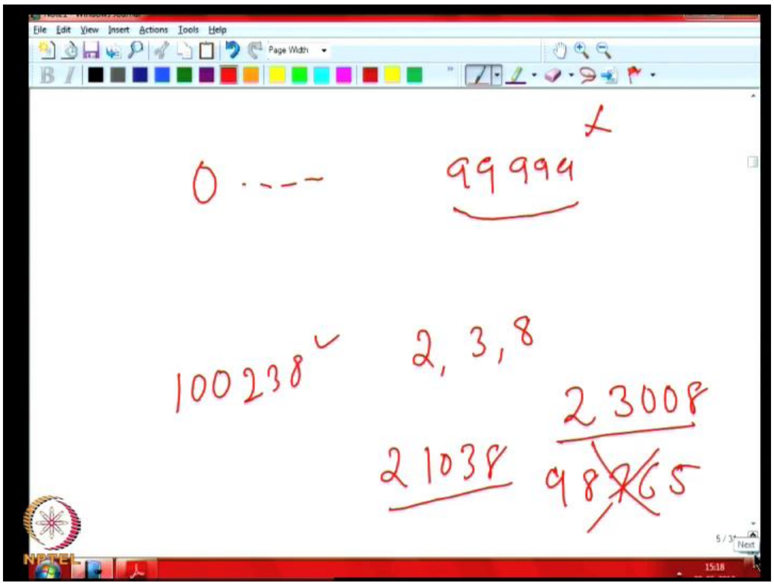

For instance, a number like this is not acceptable. We do not want it. We want 2, 3 and 8. The numbers for instance, if I will consider a number like 2, 3, 8. This is fine for 2, 1, 0, 2, 3, 8, sorry 2, 1, 0, 3, 8 or 2, 3, 0, 0, 8 is fine, right, but not a number like 9, 8, 7, 6, 5. This is not fine, right. So, how numbers are there? This is the question, right. So, is it enough to find out in how many like how we will solve this thing.

So, a direct approach would be to count those numbers which have this thing, but then we will find out. Now, in the first position, two appears in or in the second position two appears, but then three can also appear in the first position and the issue is that they all should appear together, right. So, one can try to solve it directly, but again it is a bit complicated, but I would say before trying inclusion exclusion principle to solve this thing, the student can make an attempt to solve this thing whether map, see whether his answer is matching with the answer which we get by inclusion exclusion principle. I do not spend time explaining like why it is difficult and things like that.

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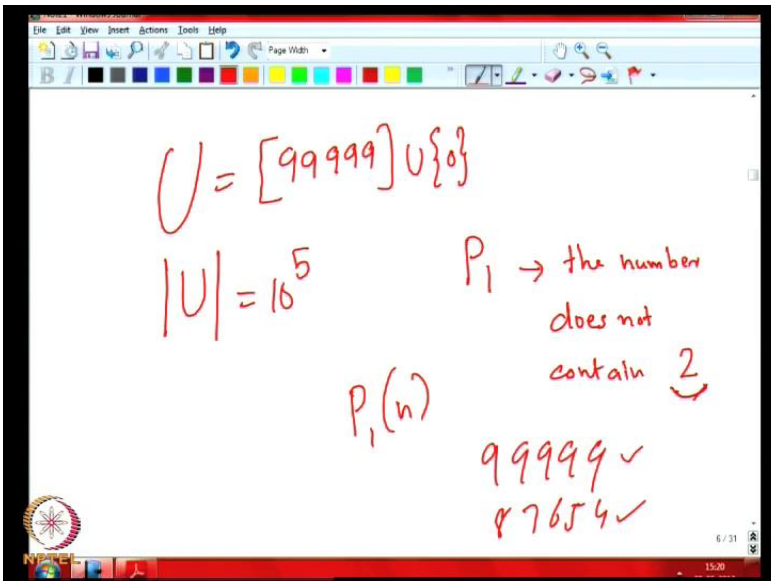

So, it is up to you. You can try, but our intension here is to see how we can apply the inclusion exclusion principle to solve this thing. So, as usual we take the universe as these numbers from 0 to 9999. So, this set, this is our universe. So, there are cardinality of the universe is then so this is not just because according to our notation 0 is not included here.

Therefore, we will include 0 also and therefore, the cardinality of u is equal to 10 raise to 5 as we already inverted and now, we will define certain properties. P 1 is the property that the number does not contain. So, for instance when I say P1 of n, that means number n when you write it out, it does not have the digit 2 in it, right. For instance, P1 of n will be satisfied by say this number or may be this number, right. All these numbers will satisfy P1 of n because those numbers do not have 2 in it, right. We have only considering numbers from the universe, say 0 to 9999.

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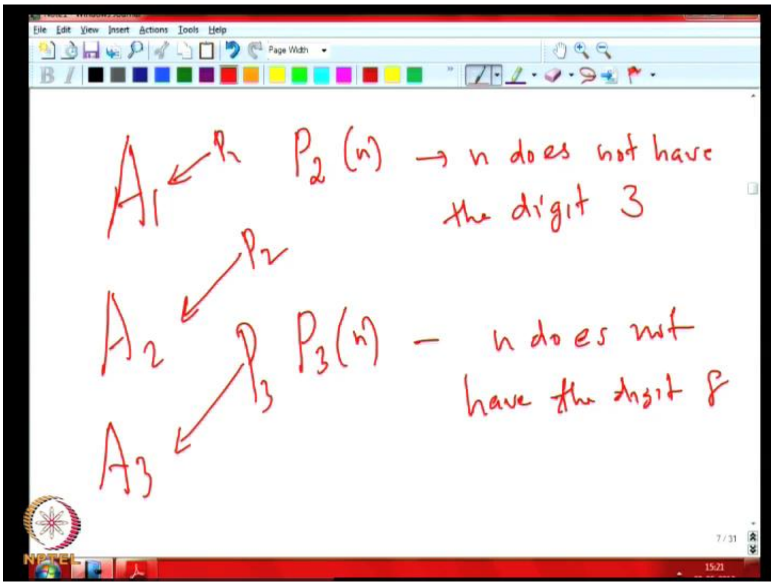

Similarly, we will define P2 of n to be the property that the number n does not have the digit 3, right and similarly, P3 of n can be defined as n does not have the digit 8, right.

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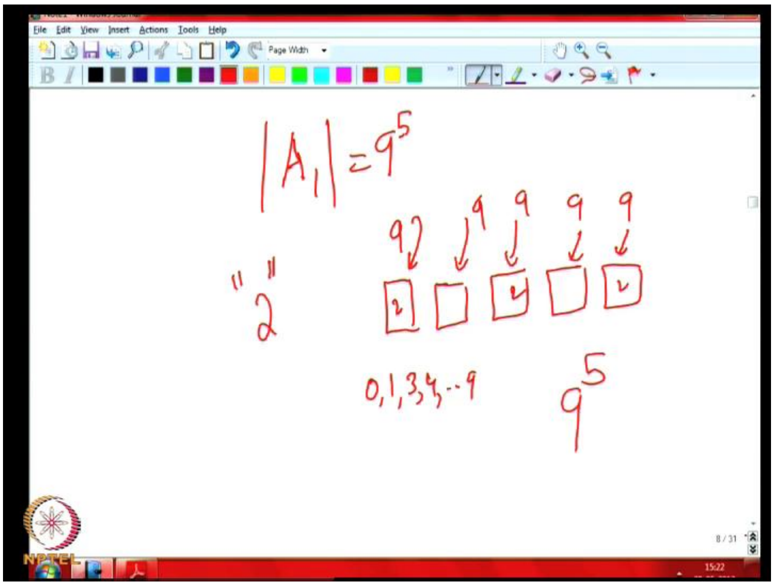

So, now we can also define the corresponding sets as we are always saying A1, A2, A3. One is the set of numbers which has property P1, and this will correspond to the set of numbers from the universe 0 to 99,999 having property P2. Similarly, a 3 will be the set of numbers in that range and from that universe having property P3, namely the numbers which does not have the digit 8. So, the cardinality of A1 is easily calculated and similarly, A2 and A3.

So, how do you calculate the cardinality of A1? Now, that is easy because you just need to make sure that we do not have two among the digits or the numbers we have. We are talking about five digit numbers. That means, in the first position, we have only nine choices. Now, earlier we had ten choices. Now, 2 is not allowed. Now, we have see choices are 0, 1. 2 is not there, 3, 4 up to 9. So, there are nine choices here.

Similarly, for second digit position also, we have nine choices. Third digit position also we have nine choices, fourth digit position also we have nine choices and fifth digit position also we have nine choices clearly because you know 2 is not allowed. So, we can never write 2 here or here or here. Anyway, none of the five digits we can write 2. We can only use 0, 1, 3, 4, 5, 6, 7, 8, 9, only nine of them.

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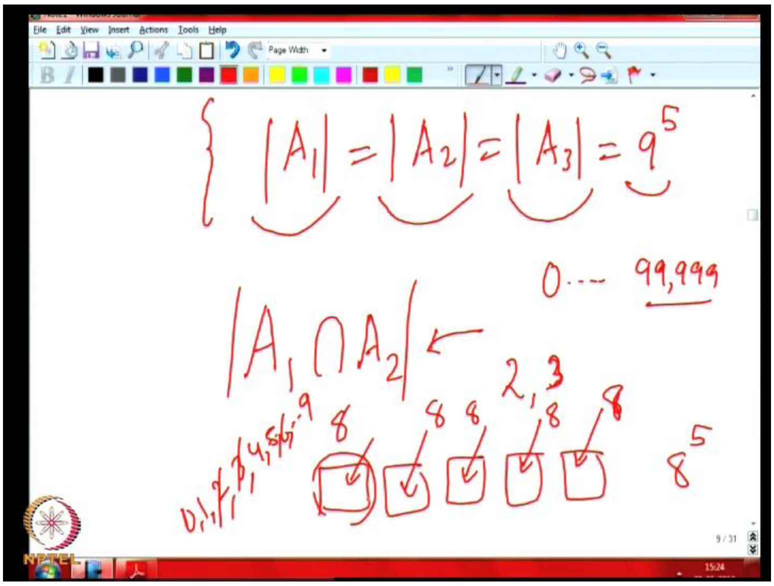

So, the total number is 9 raise to 5, right. So, the cardinality of A1 is 9 raise to 5. So, this easily is the same technique will tell us that the cardinality of A1 is equal to cardinality of A2 is equal to cardinality of A3 is equal to 9 raise to 5, right because it does not matter whether 2 is not allowed or 3 is not allowed or 8 is not allowed. This one of the digit is not allowed as what is this stuff, this argument here. Therefore, each of those cardinalities when we are not allowing 2 among the digits or when you are not allowing the 3 among the digits or when we are not allowing 8 among the digits, the total number is always 9 raise to 5. Here is one example, where all Ai's have equal cardinality, right and similarly, let us see whether it will extend to two pairs of sets. For instance, if I take A1 intersection, what does it mean? We are talking about the numbers from the universe. So, that means from 0 to 99,999 such that it does not have 2 among its digits and it does not have 3 among. That means 2 and 3 are not allowed.

Now, clearly again we use that these are the five digits, the positions of the five digits. The first position now can be filled in only eight ways. Why? Because 1 is allowed, 0 is allowed, 1 is allowed, but 2 is not allowed, 3 is not allowed, right. Now, 4, 5, 6 up to 9 is allowed. So, total eight possibilities. Similarly, here also eight possibilities are there, here also eight possibilities are there, here also eight possibilities, here also eight possibilities because 2 and 3 are not allowed.

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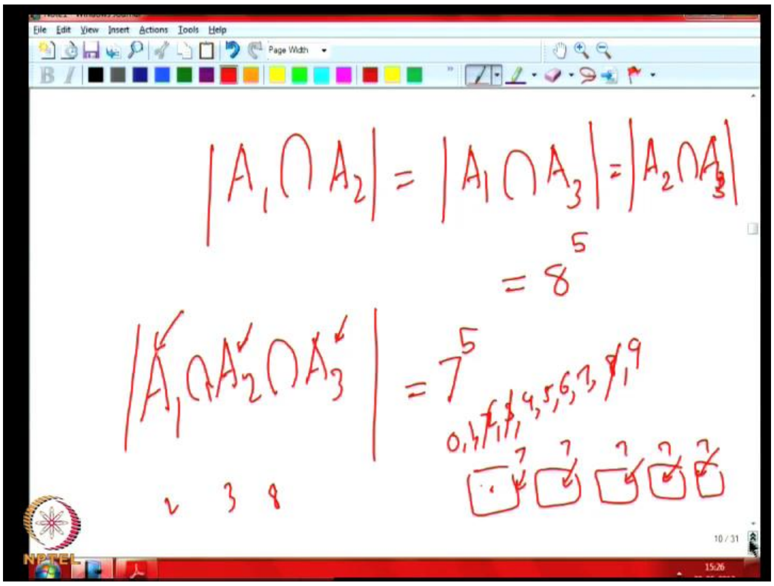

So, A to the power 5 is the cardinality of A1 intersection A2 and of course, the same is the cardinality of A1 intersection A3 because if you say the digits 2 and 8 are not allowed, that is a same argument, right because you say at this digit position, we have 2. Two of the digits from 0 to 9 are not allowed. That is all we have to worry about. Therefore, we get that A1 intersection A2 cardinality is equal to A1 intersection A8, sorry A1 intersection, A3 is equal to A2 intersection A3. All the three, two selections of two sets, right. There cardinalities are all seen. These are all A to the power 5, right.

Now, you say what the third one is. A1, A2, A3, there are all the three together. Their intersection is this easy to find of course because this is again we see that five digit numbers is to be made and this says that I do not allow digit 2 and A2. That means, we do not allow digit 3 and this says, digit 8 is not allowed here. We have only now 0, 1, 2 is not allowed, 3 is not allowed, 4, 5, 6, 7, 8 is not allowed. 9 is allowed. Now, you know three digits gone out of ten. Therefore, seven possibilities only are there here. Seven are the possibilities for here, here, here, here and here. So, this cardinality will be 7 raise to 5, right.

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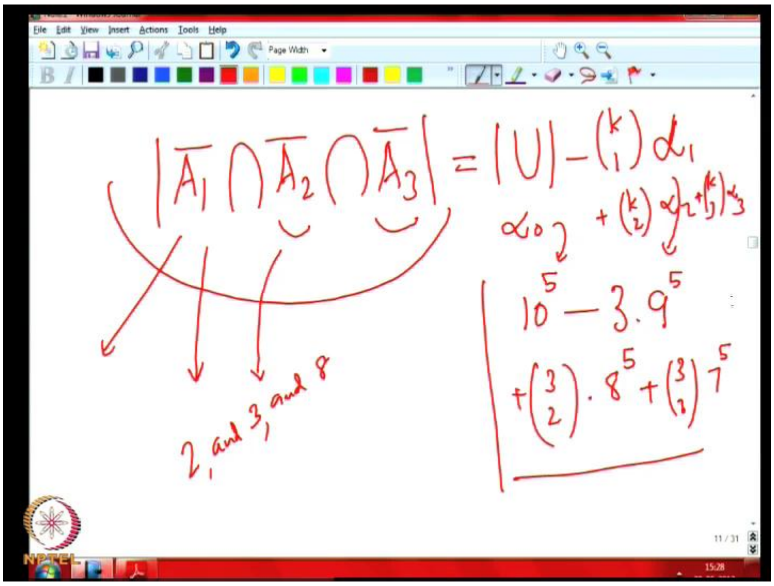

Now, we can substitute in that formula, namely what is the cardinality of A1 compliment intersection, A2 compliment intersection, A3 compliment. Clearly, students have already understood that this is what we are looking for. Why? Because this corresponds to those numbers which have A2 in among their digits, right because A1 was those numbers, those set numbers which does not have two units. So, A1 compliment will be those numbers which have definitely have A2 unit. Similarly, A2 compliment will be the set of those numbers from the universe which have f3 unit among the digits, right. Similarly, A3 compliment will be the set of those numbers from the universe which have an 8 in it. So, this intersection will mean these are the numbers from the universe which have 2, 3 and 8 among their digits, right and this we know how to calculate.

Now, this is universe minus universe is also called alpha 0 which is 10 raise to 5 minus this alpha 0, right minus k chose 1 k is 3 chose here. K is equal to 3, 1, 2, 3. So, that is 3 into what was the cardinality of each Ai which is essentially alpha 1. We have written alpha 1. Alpha 1 is as we have seen before, it is 9 raise to 5, right. So, the next is plus k chose 2 into alpha 2, every pair, right. K chose 2 is 3 chose 2, 3 chose 2, 3 chose 2 is again 3, right. We can write it as 3 into 8 raise to 5 because alpha 2 is 8 raise to 5. We have already seen that and finally, we will get 3 chose 3 k chose k, right. So, k chose 3. There k equal to 3. So, that is 3 chose 3 into alpha 3, right. Alpha 3 is 7 raise to 5 as we have seen, right. So, this will be the answer, right.

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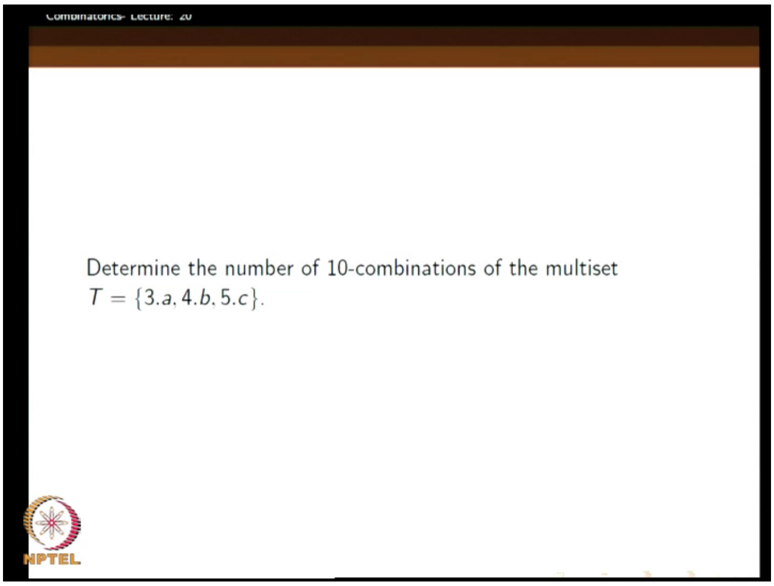

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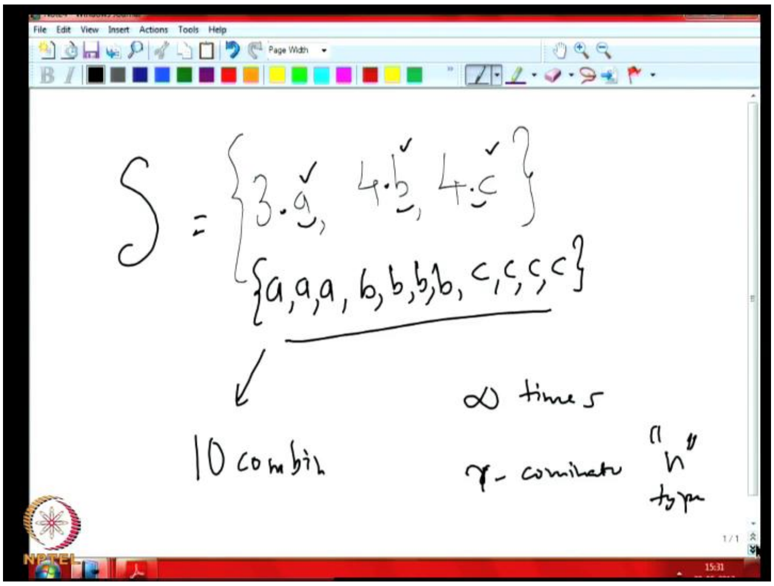

So, here is method. There is a method to find this one of course. So, we have used inclusion exclusion to solve this stuff. Now, what we do is look at the next problem. Now, we take a general question whichever in the number of 10 combinations of the multi set, t equal to 3A, 4B, 5C, 3A, 4B, 5C, 5C. So, this is a multi set, right. So, let us consider this multi set S. A3, this is this notation. We had seen that 3A means we have A three times here and then B four times here, right and C four times here. This multi set from this thing we need to create ten combinations. Ten combination means we need to select ten things out of this, right. We had considered this problem before and then we know that the issue is that at the time we were saying that if each of this A, B , C whole were repeating infinite times, right and we want to make r combinations, right. That means, there is no restriction on how many times I can take a certain type of thing.

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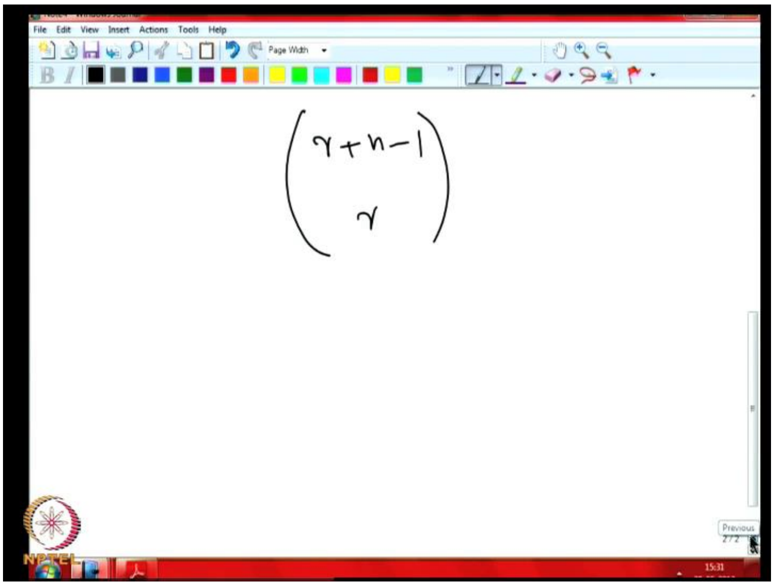

Then, there we want to make r combinations and then there are n types of things. Then the answer is r plus n minus 1 chose r. This is what we did elaborately in when we studied combinations with repetition or combinations of the multisets, right, but here what is the difference? The difference is that of course these A's are not, we cannot take A's any number of times for instance if you want to make a ten combination.

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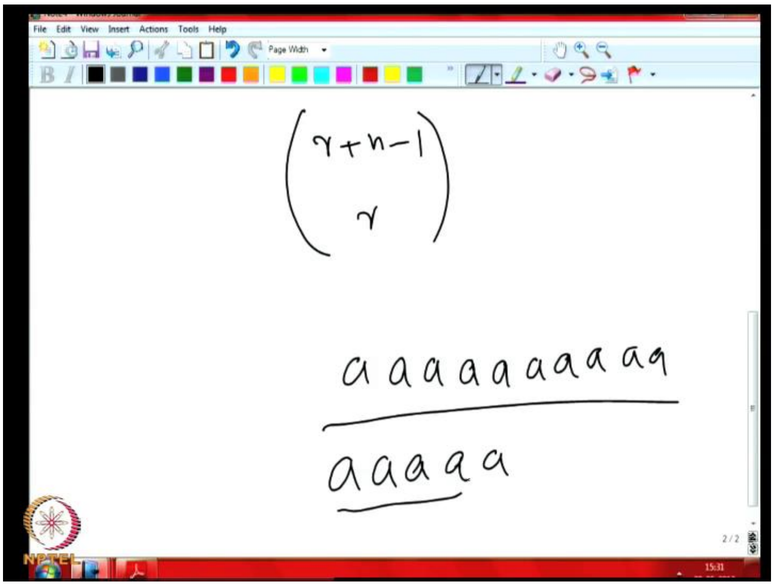

The combinations of the form a a a a a. This ten times a is not allowed. Ten a's are not allowed, not even five a's are allowed here because we have only three a, sorry or not even four a's are allowed, right. This kind of combination, these kinds of the selections are also not allowed, right.

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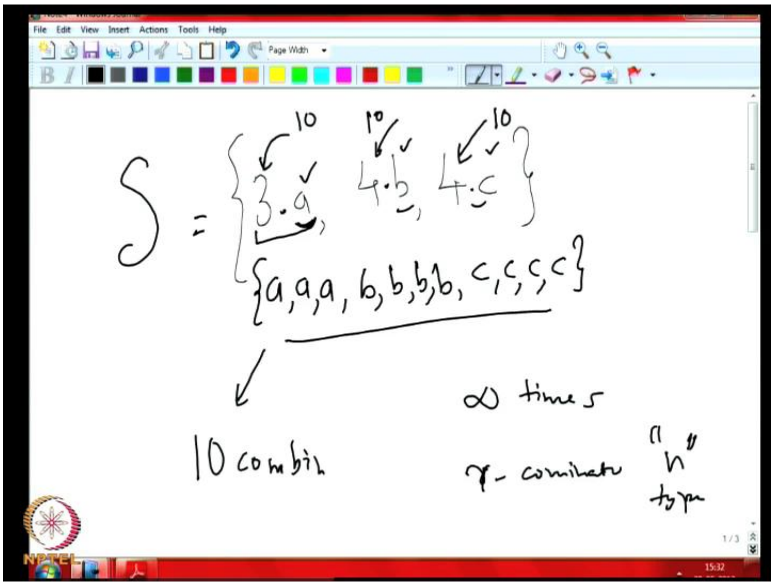

So, this is one issue, these repetition numbers now is limited. Of course, we have infinity to use the other one because when we are using, when we are talking about ten combinations, we just need ten a's. If you had everything, ten at least, then we would not have noticed that repetition number is not infinite because see whenever we want to take an a, there will be a because we are only taking ten things out of it and as long as ten things are available in the multiset, ten a's are available in the multiset. A's will never run out when we try to make a selection.

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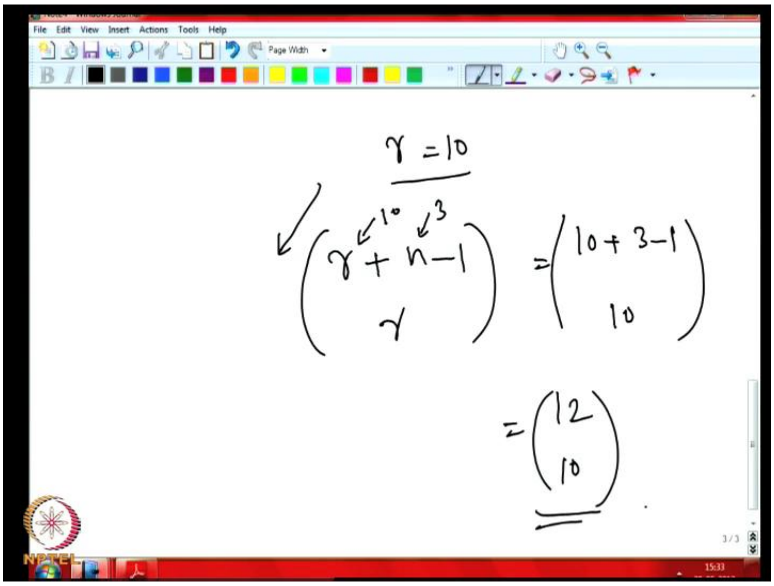

So, we can make all possible selections the way we want. So, if all type of things in the multisets had at least r. Repetition number is r. In this case, r equal to 10. Then we could have used the earlier formula, namely r plus n minus 1 chose r. In this case, it would be r being 10 and n being 3. It would be 10 plus 3 minus 1 chose 10, right. That means, 10, 12 chose 10, this would have been the answer, right but unfortunately, here we do not have that facility. We have only 3 a's. In case, if you try to take more than 3 a's, multiset will not have that many a's. So, what will we do now?

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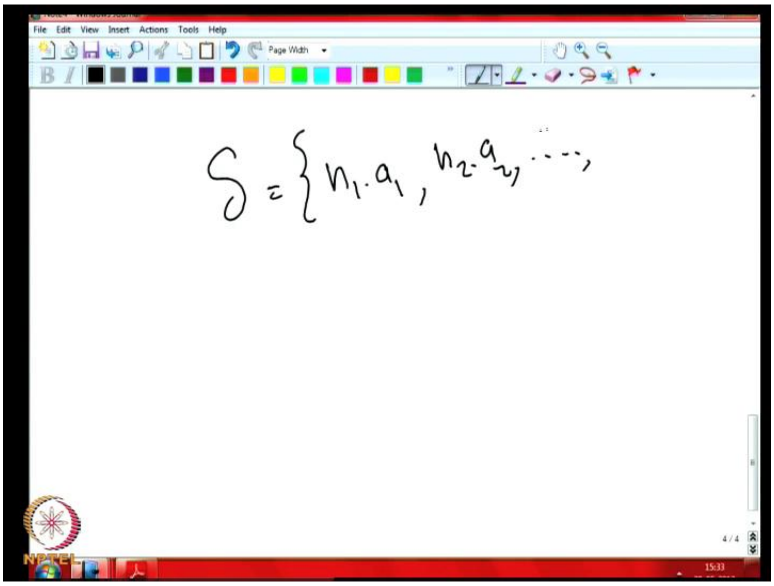

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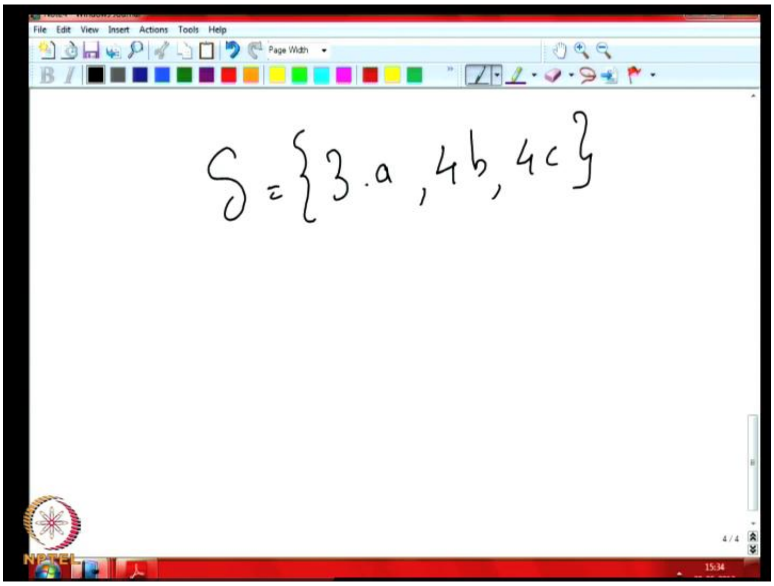

Now, this is a new type of question. So, instead of looking at the most general question where we say that we have a multi set with a n1, a's, n1 a1's and then n2 a2. You can look at this general situation of this, but instead of discussing it first, we will just take examples for 3 a's are there and 4 b's are there. So, it is a special case. This is enough to illustrate the point and then 4 c's are there. So, this is more quite enough to illustrate the point. So, later we will discuss what will happen in the general case, right.

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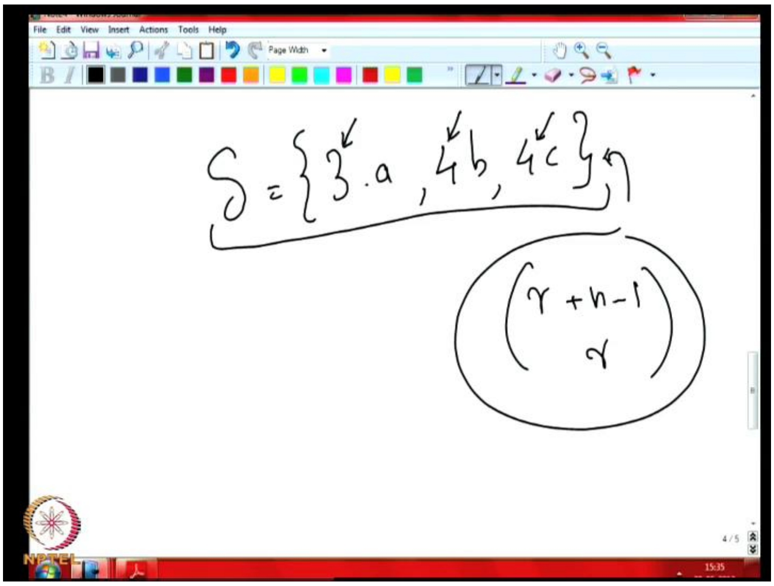

So, the key thing we want to tell here is that in this situation, we can use the inclusion exclusion principles to get an answer, right. So, of course this will give you rather complicated formula which is split out by the inclusion exclusion idea. So, not very neat formula like the one wrote in the repetition number equal to infinity case, namely something like this may not come. We will use this formula to get the answer for this thing. So, how will you formulate this question. Of course, the universe now is considered like this. So, the universe is that of suppose we had infinite repetition numbers. There is no restriction that the only 3 a's are there, only 4 b's are there, only 4 c's are where we just assume that there are 10 a's, 10 b's, 10 c's, right. In infinite number of a's and b's are c's also you assume does not matter.

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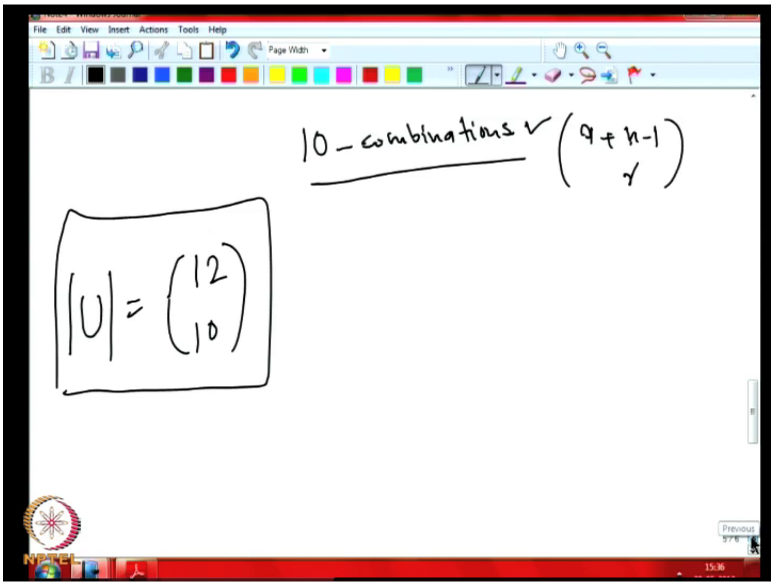

So, what we do is we take that as the universe. For instance, all possible ten combinations. Ten combinations when there is no restriction on the repetition numbers, right that is s. We have seen that is 12 chose 10. How did we got it? We applied the formula r plus n minus 1 chose r, where r equal to 10 n equal to 3. So, this is the universe cardinality. This will be the cardinality of the universe.

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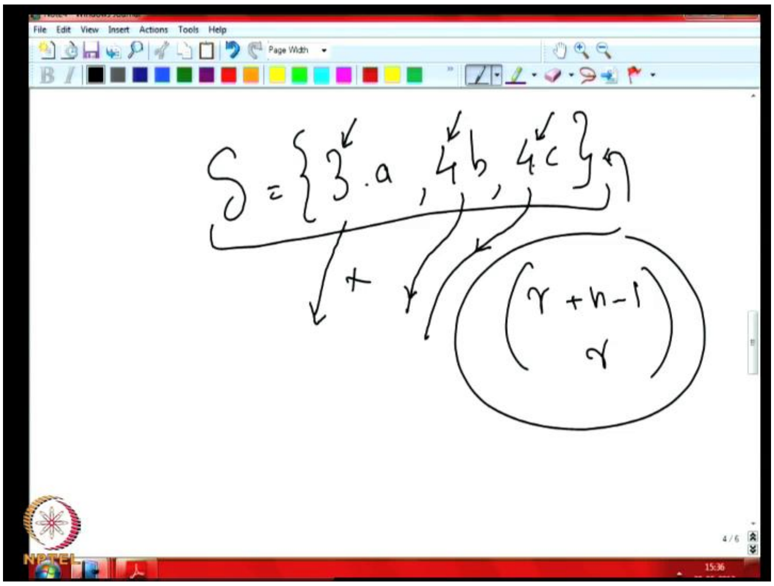

Now, what is the cardinality of, I mean what we do is we want to get rid of the bad things from the universe, right. We want to only select from the universe, those ten combinations which satisfies our requirement, namely that is ten combinations which can be made out of this multi set. That means, any ten combination which has more than three a's. It should be rejected. Any ten combinations which has more than four b's in it should be rejected. Any ten combinations which have more than four c's in it should be rejected. This is what we have to say.

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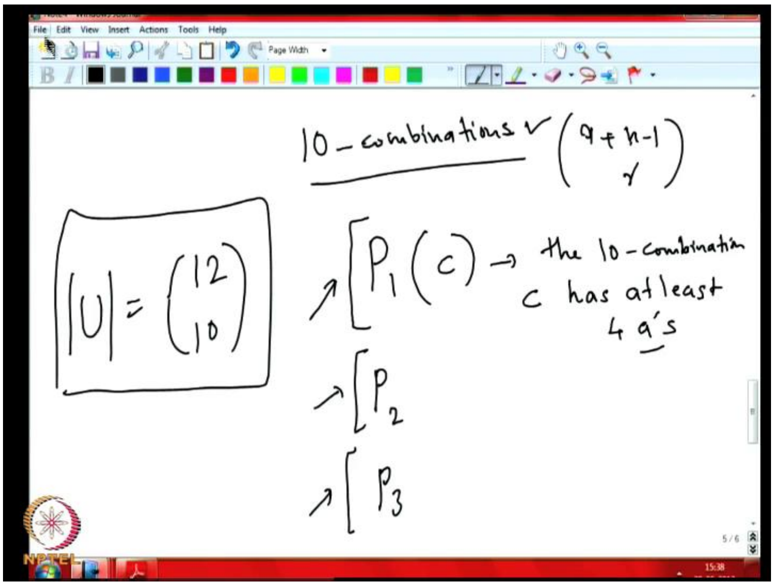

So, to capture this thing, we will gain define properties P1, P2, P3. P1 is what there the property is. So, the properties are defined on members of the universe. That means, let say we will take it a combination c, right, P1 of c. That means, the ten combinations c has at least four a's, right because we have only three a's available in the multi set given to us. Four a's are not allowed, but we are defining property that this is an undesirable property. We are defining the undesirable property, right. We usually define the undesirable properties, say when we want to apply the exclusion inclusion principle, right. This is the first undesirable property, namely the ten combinations, say c being a ten combination. It does not have, sorry it has at least four a's. So, it does not satisfy the required property.

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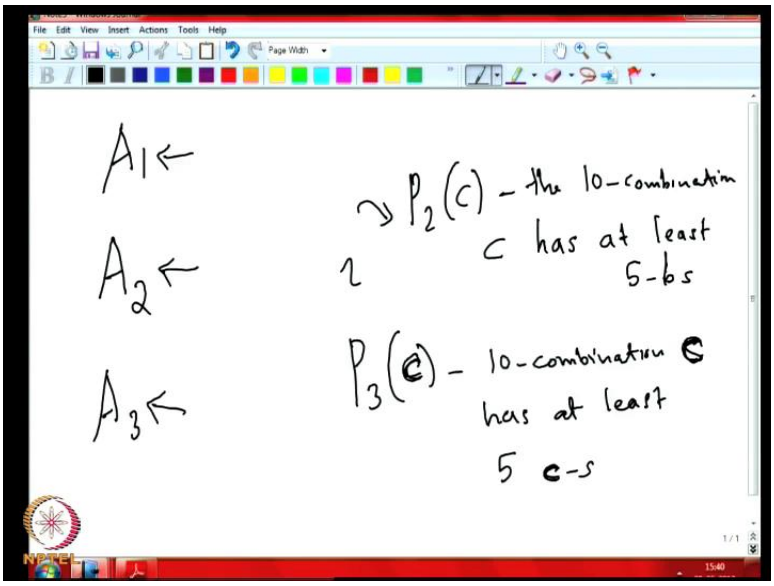

Similarly, P 2 will be the property that the ten combination at least five b's recall that in the multiset given to us. There are only four b's. Therefore, if we take five b's, then if ten combinations has five b's, we have to reject those ten combinations, right. Therefore, this is the second undesirable property and the last third and last undesirable property is that the ten combination c has at least five c's, right. This is for in this c, this should not be considered. There is a capital C here. The ten combinations and this is just that letter c, right. So, these are the three and of course, as usual we will define A1, 2 be the set of ten combinations from the universe which has property P1, namely those ten combinations from the universe, such that they have at least four a's in it. Similarly, A2 is the set of those ten combinations which have property P2. That means, those ten combinations from the universe which has at least five b's in it and A3 is the set of ten combinations from the universe that has property P3, right.

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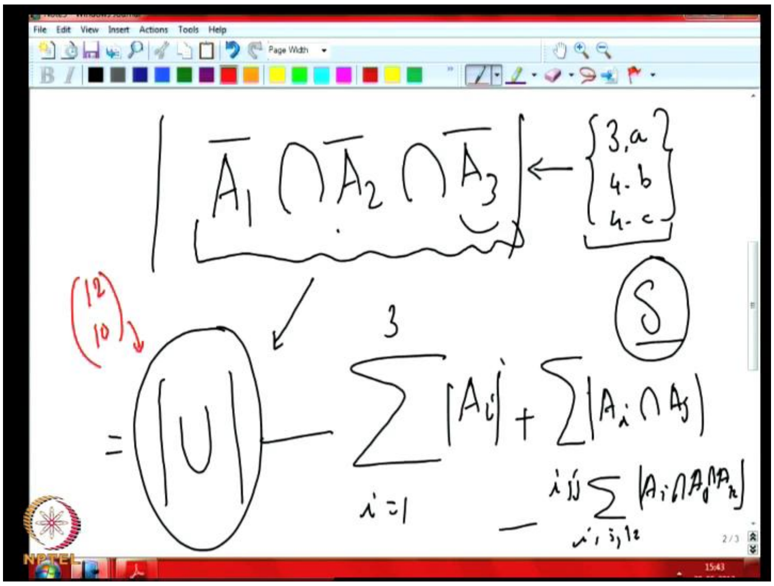

So, now it is very clear that what we are looking for is this A1 compliment, A2 compliment intersection, A3 compliment because since A1 means those ten combinations from the universe which has at least four a's. A1 compliment means those ten compliments, ten combinations from the universe which has three, at most three a's. Similarly, A2 means the ten combinations from the universe which has at least five b's. So, A2 compliment means the ten combinations from the universe which has at most four b's. Similarly, A3 compliment would mean the ten combinations from the universe which has at most four c's.

So, those ten combinations in the intersection of A1 compliment and A2 compliment and A3 compliment, there will be the ones we are looking for, namely those ten combinations which has at most three a's, four b's and four c's, right. This clearly is coming from the multiset s, right given to us that would be valid ten combinations which can be formed from the multiset given to us, right. How do we estimate this thing? We apply the inclusion exclusion principle and we know the cardinality of this thing is given by the cardinality of u minus sigma i equal to 1, 2, 3 here and its k equal to 3. So, Ai cardinality, then we have 2 plus for every i j. So, Ai intersection, Aj we have to find. So, there are 3 chose 2 of them, namely three of them. So, we have to add it and then finally, minus for every, i, j, k here in this because there are only three of them.

We have only one set here, right namely one intersection A2, intersection A3, right. We just have to now have to verify whether we can find out the values of this and values of each of cardinality of Ai intersection A j for every pair h, i and j. That is all we have to check, right. So, let us say how we do this thing. So, the cardinality if you already is known, namely that was 12 chose 10. As we calculated, it is just assuming that there is infinite repetition number we just found out the value, right. So, by applying the formula, we studied earlier, namely r plus n minus 1 chose r. Now, to find the cardinality of Ai, say A1 let us take A1. A1 is what? A1 is the set of ten combinations, such that there is at least four a's in it.

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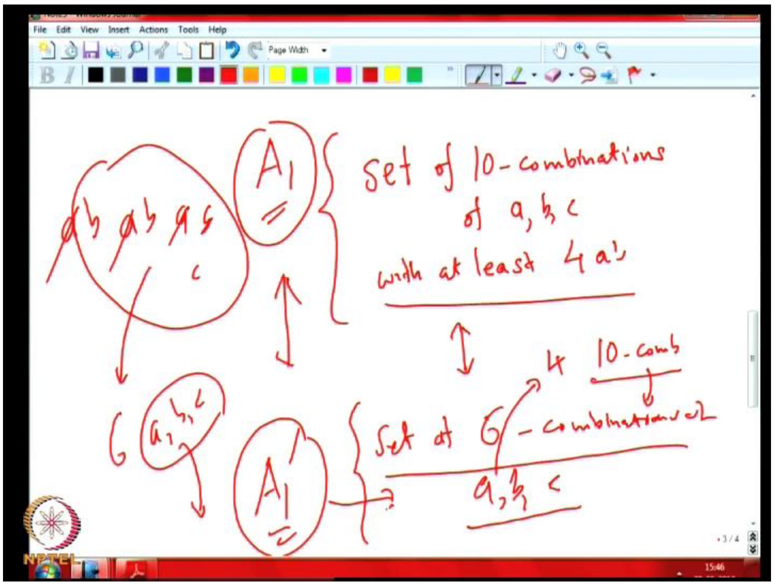

So, what we do is, we set a bijection between the set often combinations, right. Combinations of a, b, c with at least four a's and the set often because four a's we remove. That means, six combinations. So, a b c without any restriction here. We do not keep any restriction. We do not say that there should be, there is a bijection between these two sets. This set is A1 only. Why is it so? This you can say, right as A1 dash is this same because you know if you get a ten combination with at least four a's in it, say some four a's in it with some b's and all. So, what we do is we just remove those four a's and take the remaining. Just remove four a's from that and take the remaining that will be a six combination using a b c.

So, now there can be b's, a's and c's. Sometimes, a's may be 0 what so ever, right. So, we will get a six combination. Now, therefore, for every ten combination, we can create one unique six combination like this by throwing away four a's from that because there are already four a's in this kind of combinations in ten combinations in A1 conversely. Suppose, you get a six combination of a b c's, where there are a's and b's and c's and the repetitions the number of times. Repeats may be 0 or 1 or whatever. What we can do is we can add four a's to it, right. So, what we will get is ten combinations because this is six combinations. We add four more a's to it and we get a ten combinations where there are at least four a's, right. We make sure by adding four a's into it that becomes a ten combinations with at least four a.

So, we have pointed out a bijection between the two sets. So, instead of counting this, we can count this A1 dash, namely and this is easy to count. Why? Because we know the formula for that. Because here we do not have any restriction on how many a's can be there and all. So, this is just that any repetition number is allowed. Therefore, that is we can use r plus n minus 1 chose r formula here. R being 6, right, this being 3, n being 3 of course.

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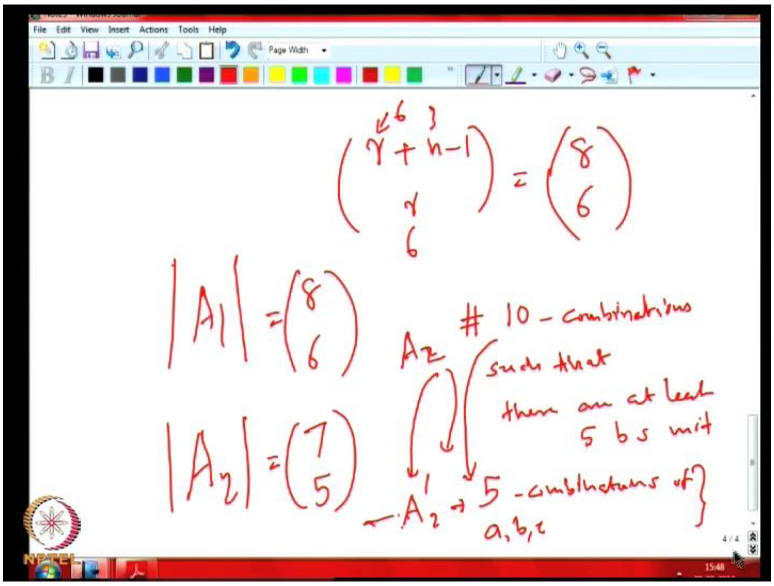

So, that is 6 plus 2 is 8 chose 6. Similarly, A1. Cardinality of A1 is A chose 6. Now, if you want to find A2 cardinality, A2 cardinality is what we are talking about. The number often combinations in from the universe ten combinations, right, such that there are at least five b's in it. By following this same argument as before, what we can do is this is A2, right. We can set up a bijection between A2 and say, A2 dash, where A2 dash is the ten combinations of a, b, c, not ten combinations here we take because five b's are there.

So, five combinations of a, b, c what we do when for any ten combination from A2. That means, any ten combinations which had at least five b's, we can delete those five b's from it. We can remove five b's from unit from it and we get a five combination and that does not have any restriction. That means, it can have even 0 b's, right. So, we are not saying that there should be this many a's or this many b's. Anyway, similarly, if we have some five combination of a b c's, whatever we can add five more b's to it and make a ten combination.

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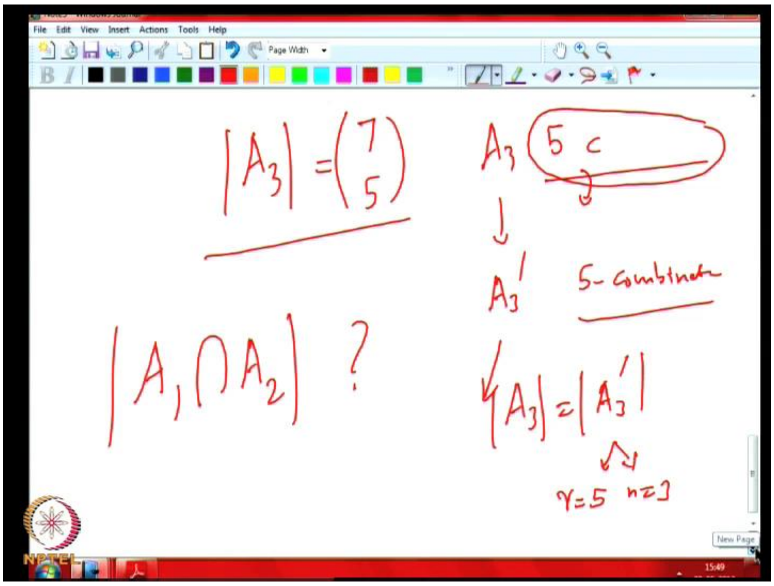

So, definitely with r equal to 5, we will get the answer to be 5 plus 2 chose 5. That means 7 chose 5, right putting r equal to 5. Similarly, if you want to find A3 cardinality, that is also easy because now again the restriction is that they need at least five c's in it. So, we set up a bijection between that set A3 and A3 dash, where A3 dash is the five combinations of a, b, c, right. So, as before what we can do is when we have for any ten combinations in A3 which has of course at least five c's. We remove five c's from it and we get a normal five combination of a, b, c and this. Therefore, A3 cardinality can be found by finding A3 dash cardinality here. We can apply the formula with r equal to 5 and n equal to 3 and we again get from this thing five plus 3 minus 1, that is 7 chose 5, right.

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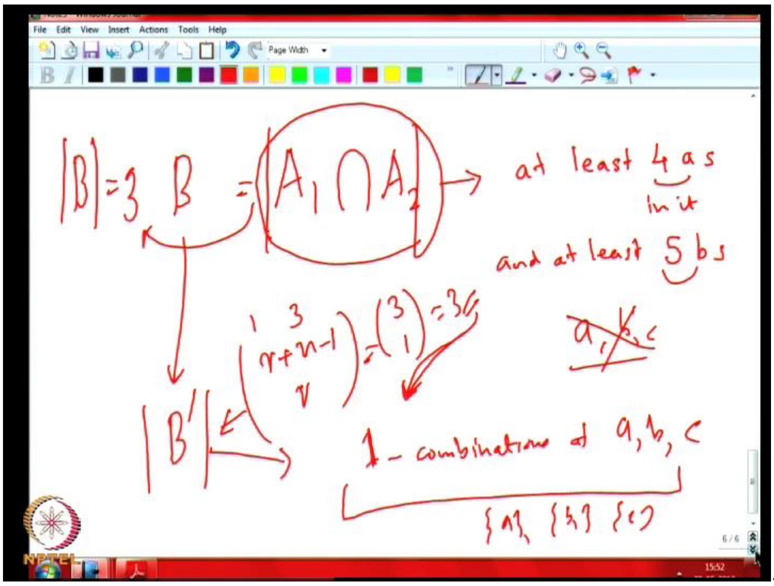

So, we can easily find the cardinalities of A1, A2 and A3, right. Now, we want to see whether we can find the cardinality of A1 intersection A2, right. Is it possible to find the cardinality of A1 intersection? What is this A2 intersection? A2 is the set of ten combinations which has at least four a’s in it and at least five b’s in it. Now, what we can do is we can let this p called p. We can set a bijection between b and b dash, where b dash is defined as the one combination of a b c. Why is it so? Because you know you take any ten combinations with at least four a’s and five b’s and from that, you just remove four a’s and five b’s and we end up because ten things, where there nine things are removed.

Now, just remaining one we will take that that will correspond to the one combination in b dash, right. That is the bijection we are setting up and reversely, if you take anyone combination of a b c that can be a, or b or c, whatever we can add four a’s and five b’s on top of it and we will get a ten combination which satisfy the property that there are at least four a’s and five b’s, right. This is indeed a bijection. Therefore, bijection between we just have to count this and this number is easy to find using the formula, namely this is r plus n minus 1 chose r with r equal to 1 n equal to 3.

So, this is just 3 chose 1 equal to 3. So, even one without formula we can easily find this thing because these are just one combinations of a b c. So, only one combination, either a, or b or c, right, there are three such things. We can see that here also we could without going through this bijective argument also, we could have easily inferred that A1 intersection A2 cardinality e is indeed 3 because we need four a’s, five b’s. We already finished of nine things, right by compulsion. We forcefully have to take nine of them, right, four a’s and four b’s. The last thing only we had a freedom to select.

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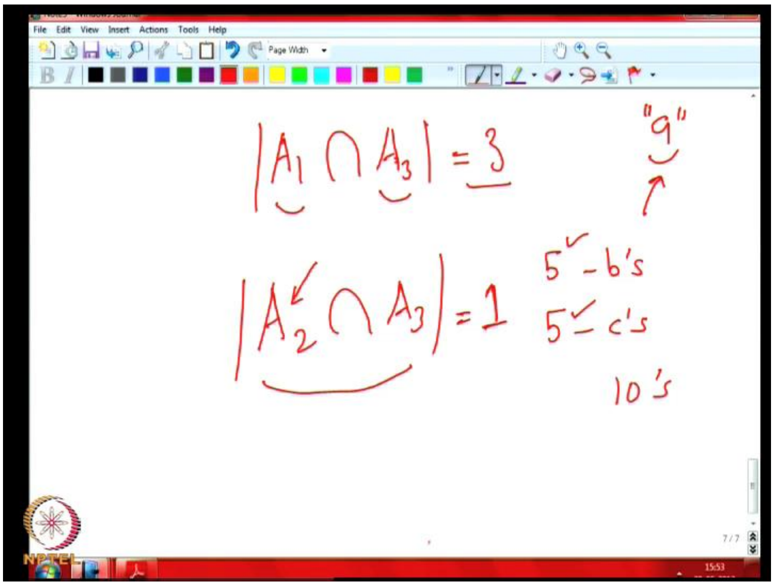

So, we can select either a or b or c. There was three ways to do it. Therefore, it was not at all difficult to infer that this cardinality of b i equal to c. That means, this is 3, right. Similarly, we can find the cardinality of A1 intersection A3. What is it? There should be at least four a’s and five c’s. So, that means nine things are already taken in the ten combinations. Only one can be taken. The same argument as before we can do it in only three ways and if you want to find A2 intersection A3 cardinality, this is what this set is. This set is the set of ten combinations, where there are five b’s and five c’s that can be done only in one way because no five b’s, five c’s, already ten things we have selected, right. At least five b’s and five c’s should be there. Then this should be 1, right. There we cannot do anything, right. We can only leave it as such, so that is only 1, right.

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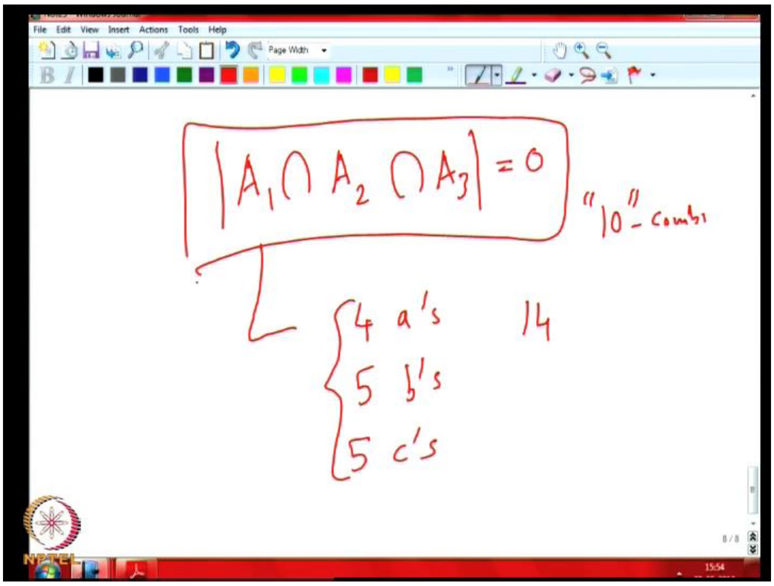

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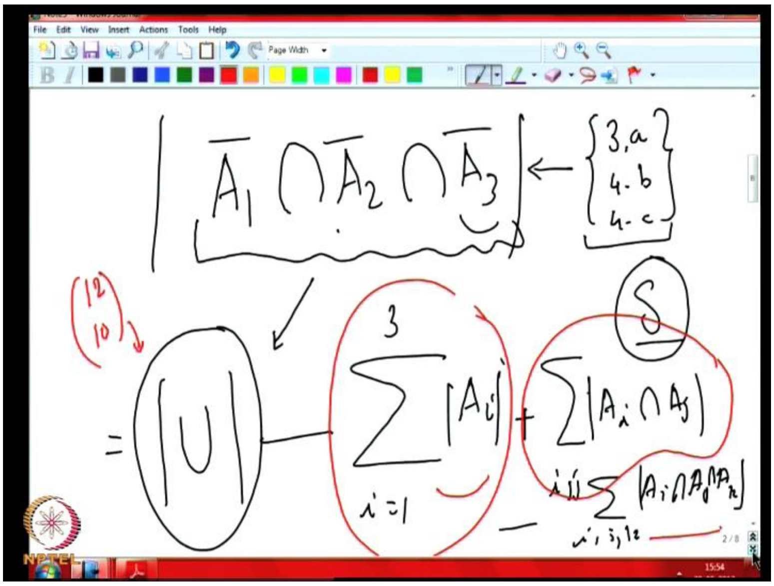

So, A2 intersection A3 is 1. Now, if you want to look at the last thing, so we got all the terms required except the last term, namely A1 intersection A2 intersection A3. How do we find the cardinality of this thing? This means we need four a's in the ten combination, five b's in the ten combination and five c's in the ten combination. So, this already ask to take 4 plus 5 plus 5 is 14 things, but we are only talking about ten combinations. That is not possible, right. So, this cardinality has to be 0, right. Therefore, we can substitute whatever we calculated up to now in the formula. We set up using the inclusion exclusion, namely this formula here. We have A12 chose 10 here. So, all the numbers we had calculated and here for all the pairs we had calculated the cardinality and the final thing was 0, right.

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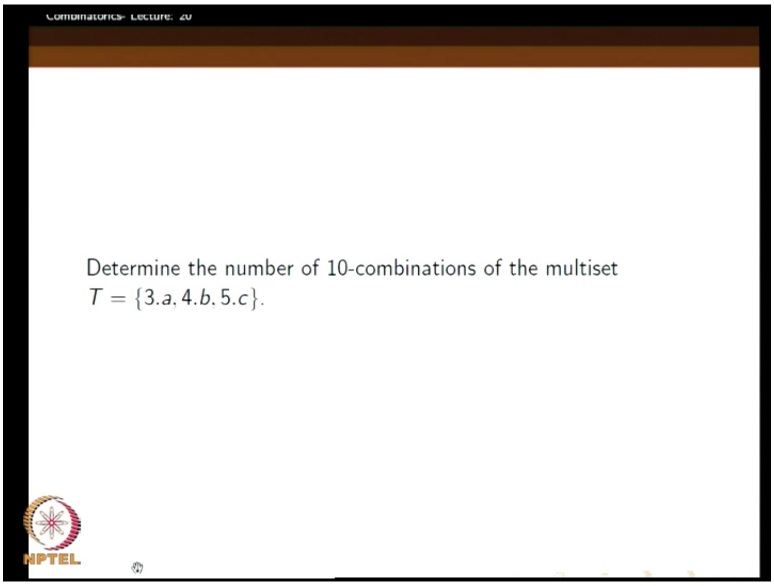

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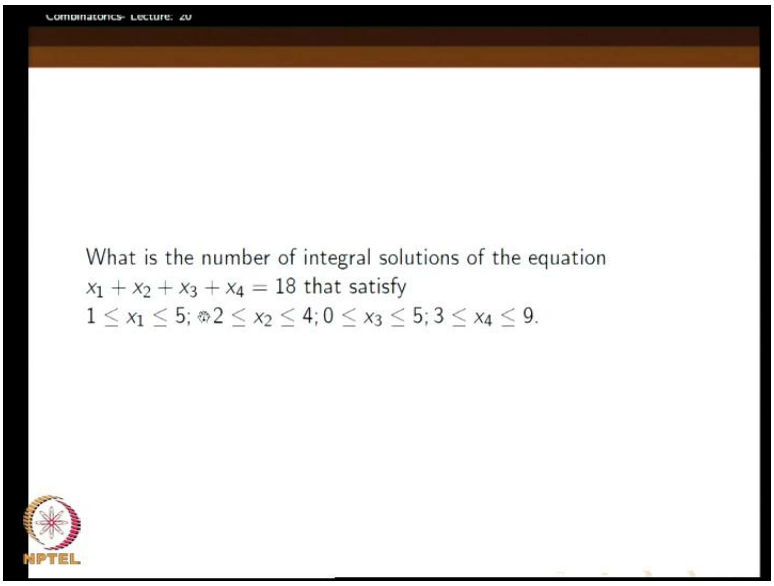

So, we can estimate it. It is not important to do the computation. So, we leave it to this today and if that is definitely not important, so we just need to know the technique, right. Now, we will look at the next question because this was an important thing. Therefore, we will look at the generalized form as was telling in the beginning. So, the number of r combinations of the multiset n 1 A1, n2 A2, sorry this is a small mistake here and so n2

A2. So, this we discard this portion, so that is n2 A2 and n 3 A3 and nk ak, right equals the number of integral. Fine. So, I will explain it using. See, this is the multiset we are interested in n1 A1, n2 A2 and this is nk ak. There are k type of things and then each of them are coming n i times. A i is coming n i times in the multiset.

Now, we want to make r combinations, right. So, as we have mentioned before or all I mean is many times we have mentioned. Now, if each n i is greater than equal to r, then it is as good as each saying that each type has infinite repetition number because we are only making r combination. For instance, if I want to take some A i, we will get r copies of it. So, we have no restriction, no constraint. Therefore, we can use that formula r plus here k types k minus 1 chose r. The other extreme also is easy. For instance, name this n 1 equal to 1, n 2 equal to 1, then k equal to 1. Suppose, each of them, all of them are 1, then also is easy. It is just saying that because though we call it multiset, then it is just a set A1, A2, Ak, no repetition, right.

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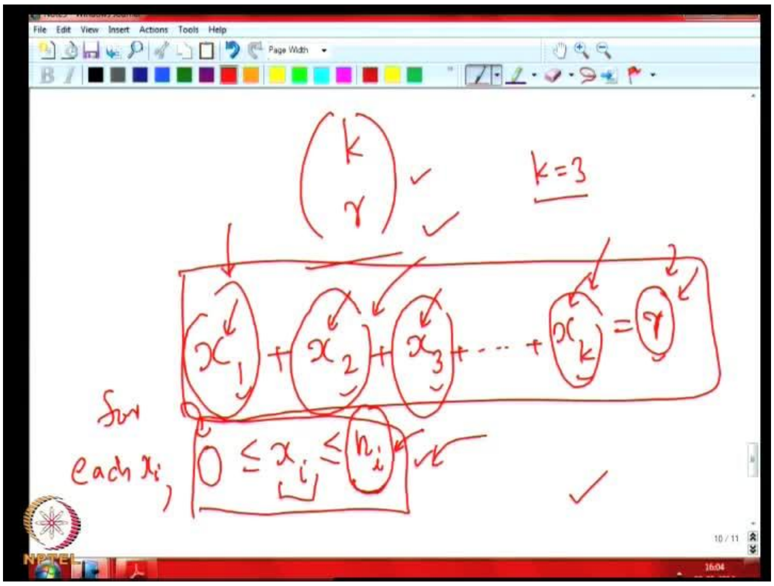

So, how many r combinations we can make out of s? That is simple. K things are there out of that r combinations are to be made, that is n k chose r. That is very easy. So, the problem is when some n i's are in between r and 1, it is bigger. I mean, some n i's are bigger than 1 or smaller than r, right. Not all of them are bigger, not all of them are equal to 1, right. In this case only we are worried, we have to worry. So, that is the situation we explained elaborately using k equal to 3 k's we took and then showed how to apply the inclusion exclusion principle, but only thing is when k is big, for instance not in our example. It was 3. Suppose, k was much bigger. We have to write a very long formula using inclusion exclusion and each individual term has to be estimated. Definitely, it is not very neat. It may not be useful, but if by a computer probably you can program it because of course, every step is implementable. It is not that we have to be clever at some point.

We know what to do at each point, but we will finally, we can do if we have a computer, but that technique definitely helps if you really desperately want to find the number, right. We do not have any very nice elegant formula for that like in the extreme case, right. Now, what we want to look at is something which we have already seen, namely it is equivalence. So, this x1, this solution of integral solutions of x 1 plus x 2 plus x k equal to r, right. So, that is what we will say. So, we will again consider that this equation x 1 plus x 2 plus x 3 plus x k, that is equal to r. See, we are interested in the situation, where each x i is greater than equal to 0, each x i, right. For each x i, we have this condition, namely 0 greater than x i. Then that is as good as saying that we have already discussed it. It is as good as saying that we want to find the r combinations of k type of things, where each thing can be taken infinite times, right.

So, that is we had described it before what corresponds between these things, right and this k type of things, right. That is that we had discussed in the earlier class and we had also told that this was equivalent to balls and beans problem, where we wanted to distribute r identical balls into k distinct boxes. That means, boxes which can be differentiated from each other. That means, labeled boxes, right, the boxes which can be distinguished from each other, right. First box, second box, third box and k box.

So, that is also where we allow some boxes to be empty, but this r ball should go into those boxes how many ways we can distribute, but we never considered the situation where we also have an upper bound on this x i, namely before each x i is upper bounded by n i sum n i, where this n i is not r or more, right. It this n i being less than r. So, as we can easily see this corresponds to the earlier situation, namely when r balls are to be distributed and sorry, r. So, we have to make r combinations, but we have a restriction that i-th type of thing can be selected only at most n i times, right. That is what this is same because you remember the bijection was that the value which x 1 gets will correspond to the number of times the four type of object is selected to make the r combination, namely x 2 would correspond to the number of times the second type of object is selected to make the r combination. Similarly, x k would correspond to the number of times k-th object is selected to make the r combination, right.

Now, this extra restriction means that the x i is less than equal to n i. That means, x i, i-th type of thing cannot be taken more than n i times, a maximum n i times only. It can be taken. That is what we considered in the last example how we can use the inclusion exclusion principle to do that, right. So, in the balls and beans case, this x i would correspond to the number of balls that we place in i-th box, right. So, when I say x i is at most n i, it would correspond to saying that in that, in the i-th bin, i-th box, we can place at most n i balls. We cannot put more than n i balls in i-th box. That is what it mean.

So, we should remember these three equivalent problems because when a particular situation comes, we may be able to model with the most convenient model or we may be able to identify. These are all equivalent thing, either identical balls being placed in k labeled boxes, right with a various restrictions, either boxes has to be non empty, empty and all these things we have discussed before. Now, we are telling the extra restriction that a particular box, i-th box can hold only maximum n i balls, right. So, that is what in the equation form. This is what is happening here. X i is at most n i is what it would correspond. Similarly, in the r combination case, it is equivalent to saying that r-th type of object can be taken at most n i times to make the r-th combination, right.

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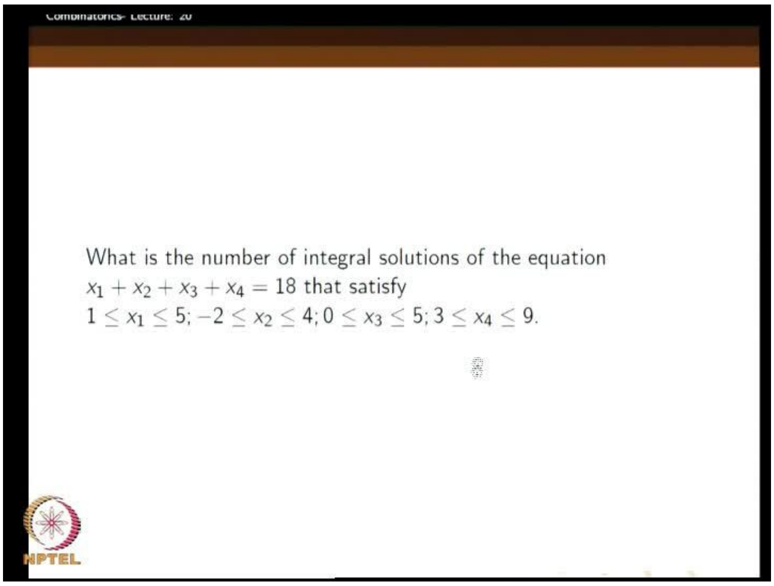

So, we just describe this correspondence. The next question we want to consider is in this line. So, we put more restrictions and then see whether we can solve it. We are stretching ourselves further. So, what is the number of integral solutions or the equation x 1 plus x 2 plus x 3 plus x 4 is equal to 18 that satisfy 1 less than equal to x 1 less than equal to 5. Here, instead of zero less than equal to x n, we are putting one upper limit is also added, right. So, here we are saying minus 2 less than equal to x 2 less than equal to 4 up to now. We were always saying 0 less than equal to x 2 less than equal to something, but here allowing minus value and the third variable has the correct range, 0 less than equal to x 3 less than equal to some five upper bound is introduced.

Similarly, the forth one has upper and lower bond. The lower bond is not 0. So, how will you manage? We know mostly how to do this thing. The technique we have already discussed. The only thing is the upper limit. Actually, the upper limit is the new one that we have to take care of by the method we discussed in the previous problem, namely that inclusion exclusion principle. This one we will discuss in the next class.