Combinatorics
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Lecture - 2
Pigeon hole principle - (Part 2)

This is the second lecture of combinatorics. In the last class you were looking at pigeonhole principle. So, now we will continue with the same thing in this class also, so more examples.

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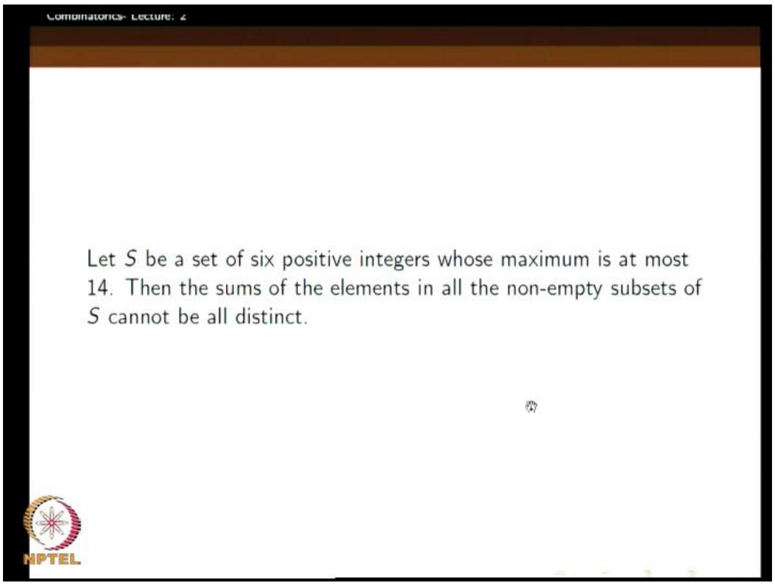

This was the last question we considered let S be a set of six positive integers whose maximum is at most 14. Then the sums of the elements in all the non-empty subsets of S cannot be all distinct.

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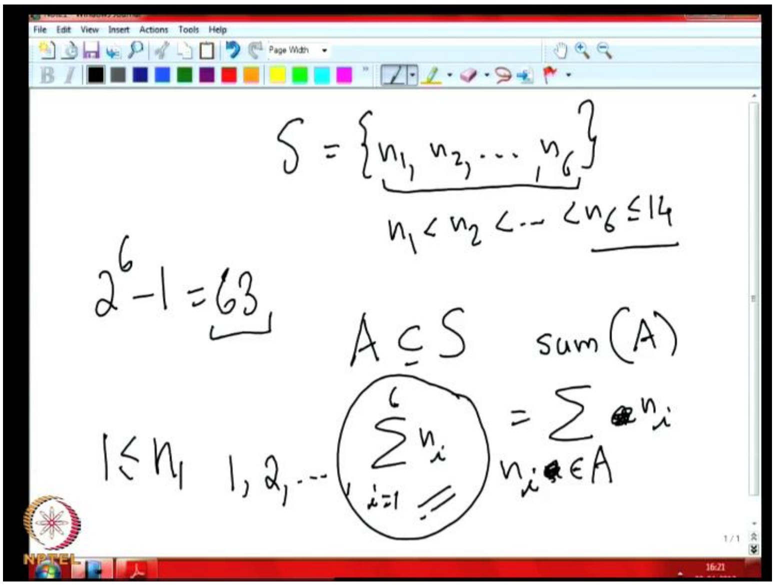

So we told that S, so we wrote it like n1, n2, say n 6 where n 1 is strictly less then n2 is strictly than say n 6, right and we know that n 6 is less then equal to 14, right. And now what we are interested in is the non-empty subsets of S, because there are six elements in S, there are 2 to the power 6 minus 1 is equal to 63 subsets of S. So, take a subsets of subset A of S say, then we are interested in sum of A; sum of A means sigma of a element of A, that means the members of a are summed up here because these are numbers, right. So sum, may be you can write it as n i element of A, so we sum up those n i's which are in A, and we will get one number corresponding to A, right, for each subset we will have a number of the sum.

The question is, is it possible that all these sums are distinct numbers different, different, different numbers, right. So, we will immediately ask because to apply pigeonhole principally we would like to know how many sums are there, how many pigeons are there? There are as many as like 63 sums because there are 63 non-empty subsets, so there are 63 sums, right. So, then the question is what are the holes? The holes will be the possible values these sums can take. Of course the smallest value is n 1, so let us say n 1 can be as small as 1, right. So, therefore it can take values starting from 1, 2 like that up to what? up to the sum of all these things can be a possible value, right; sigma i equal to 1 to 6 n i all those things can be added also but what will be the biggest possible values this can take because I only know that this biggest value here n 6 is at most 14. So, it can as well be 14.

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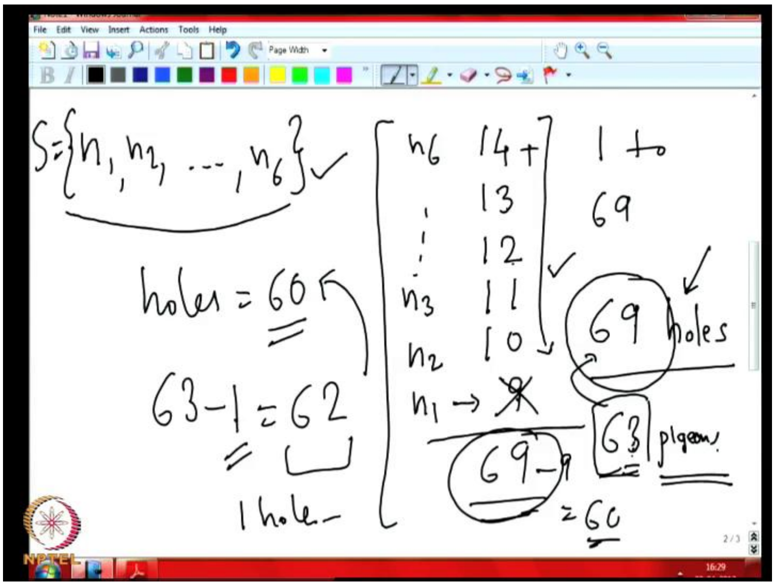

So now the biggest value it can take can be, say, 14 plus 13 plus 12 plus 11 plus 10 plus 9. So, if we add these things together you get 10 plus this is 19, right, this is the total possible values here is 69, right, 69 values are there possible. So, it can take values from 1 to 69. So, there are 69 holes because the possible values are considered as holes and there are 63 pigeons because 63 pigeons because there are 63 non-empty subsets, right, but of course there is no problem here because we need more than 69, 70 pigeons then only we can say that two pigeons will go into one of the holes, right. There will be one hole where two pigeons are there; that means there will be two subsets which will get the same value, right; here that is not possible. So, what can we do? There are two possible ways; one is how pigeonhole principle is not pigeon, pigeonhole idea is not working here because there are less pigeons than the number of holes.

There are 63 pigeons alone, because there are 63 non-empty subsets available and but then there are 63 possible values. One possible way is to reduce the number of pigeons because this number of pigeons is 63 here, right sorry not reduce the number of pigeons. To reduce the number of possible values, right, holes that is what we need to do, to bring below the number of pigeon. One possible way is to not that see if you do not allow six elements sets six elements subsets, then this value will go down drastically because we added all these numbers, right, to get 69 six numbers to get 69. Suppose we do not consider six elements subsets, we only consider up to five elements subset; that means the entire set that six element is there that entire set is we are not considering, remember then set was S was n1, n2, n6. Now we decide that we are not allowing this entire set S itself because that was adding up to 69. So, we will only take at most five members on this thing. Now two effects are there for this thing. One is this sum will go down because we can now cut off this thing, right, because we only have five members in any set we are considering.

So, this is only this will becomes 60, right. So, the possible holes now holes will become 60 if I do that, right but then the number of pigeons will also decrease because now we are not considering all subsets but luckily that decreases not that much because see when you say I am only taking subsets of cardinality either 1 or 2 or 3 or 4 or 5 not 6 itself. The only subset we are losing is the full set itself namely S because that is the only subset which contains six elements, right; that means 63 pigeons will now become 62 pigeons, right, become this was that one set.

This is S itself, this corresponded to the set S itself, right. Now we can say that there are 62 pigeons and then 60 holes. So therefore, there should be one hole where there are two pigeons which means there will be two of the subsets which we are considering now namely we are considering only subsets of cardinality utmost five now. Among them itself we will find two subsets whose values are equal because now possible values are only six.

So, that was one trick but of course we do not have to do it this way; our intension is just to reduce the number of possible values. So, recall that here the possible values go initially calculated like this but remember here itself we made a small, right, so then we can notice that for 69 possible values we will need n 1 to be as big as time because remember n 1 is the smallest here, n 2 is the second smallest, n 3 is the third smallest and this is n six 6.

Now if n 1 is nine then only you will get 69 total 69, right. But clearly if n 1 is 69 the possible values are only starting from, so in our previous side we have shown the possible values. So, the first possible value is n 1 itself, it can be as small of as one but if this is already 9, then it cannot be as small as one, right. So, it can be starting from 9 to this possible value, right in that case. So there will be some advantage here because if you here assuming that you are considering n 1 to be a little smaller we can see that we only have to start from there the count, right.

So, working on this also you can probably try to reduce the number of possible holes here, right. This the student can consider as an exercise but you have to do a little bit of work here because you know it is not that n 1 is 8, 9. In that case it is very clear because you can you can always see that up to 69 minus this 8 initially its values will not even come. So, we are already below the number of pigeons, right, 61 pigeons only will be there. But suppose n 1 goes little smaller then what will happen? Suppose n 1 was 7 but then you see the total sum will also be up to this plus 7, then you can minus the corresponding thing, right, so till 61 only will come. So, you have to formally prove it but I leave it to the student to work it out.

So, there is another way of looking at it also but the idea is only this thing and whatever you do here you show that the numbers of pigeons are considerably less then number of holes. So, now you carefully look at the problem and try to reduce the number of holes because we identified extra constraint or may be imposed extra constraints because the first approach was imposing extra constraints saying that I will not take six elements subsets I will only take up to five elements subsets. By imposing such an extra constraint I could bring down the number of holes; that is what there so that the number of pigeons becomes bigger than the number of holes, this is what we did right. So, here also this is also nice example to show and again let us see the next one.

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There is another question. Let m be an odd positive integer. Then there exist a positive integer n such that m divides 2 raise to n minus 1; this is what we have to show.

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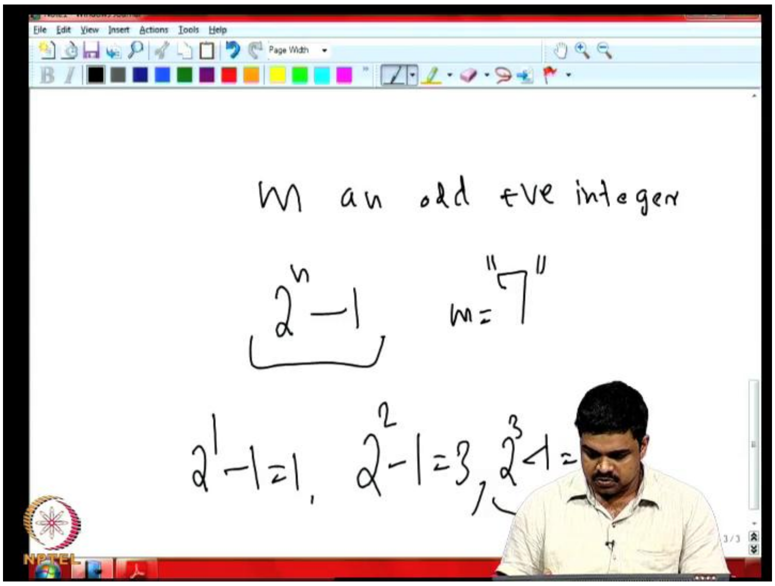

Here if we have m an odd integer odd positive integer, here asked to show that there exists a number of the sort 2 raise to n minus 1 such that m divides to raise to n and you can take some number, say, 7 m equal to 7, right. Now let us consider the possible numbers of this order. So, put m equal to 1 we will get 2 raise to 1 minus 1 equal to 1; definitely 7 does not divide 1. So, then 2 raise to 2 minus 1 this is three; 7 do not divide 3. So, next is 2 raise to 3 minus 1. This is 7 divide 7, right. So, there is a number here which 7 divide.

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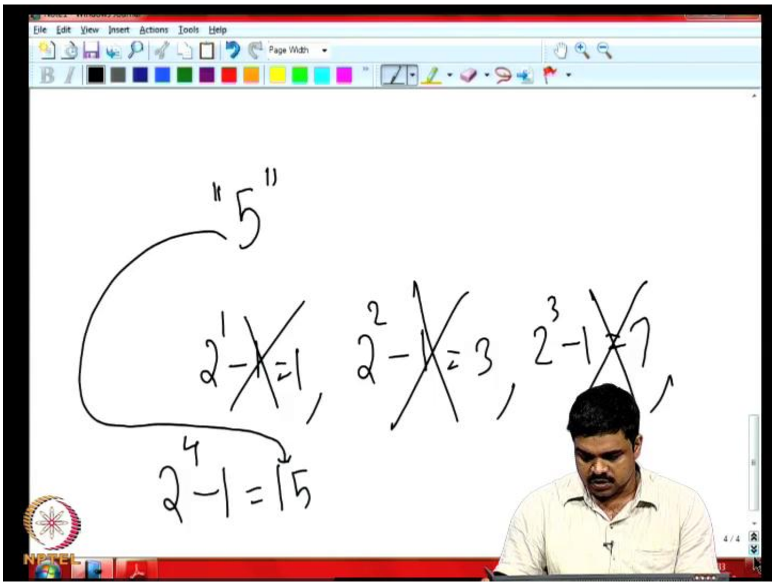

So, maybe you can take another example let us take another positive integer, say, 5 let us take. So, odd integer we are taking and now let us consider again numbers 2 raised to 1 minus 1. This is 1 no 5 does not divided it. Next is 2 raise to 2 minus 1 that is 3, no 5 does not divide it. Next is 2 raise to 3 minus 1. This is 7 no 5 does not divide it. Now 2 raise to 4 minus 1 that is fifteen, yes 5 divides it, right, 15 is divisible by 5.

So, you are able to always find a number of the form 2 raise to n minus 1 such that 5 divide it. Actually given any odd number m odd positive integer m you can find some 2 raise to power n minus 1 such that this m divides it, how do we prove this thing? So, it t it s a nice application of a pigeonhole principle, so happens that we do not have to keep going long to find that number which divides to which is divisible by m.

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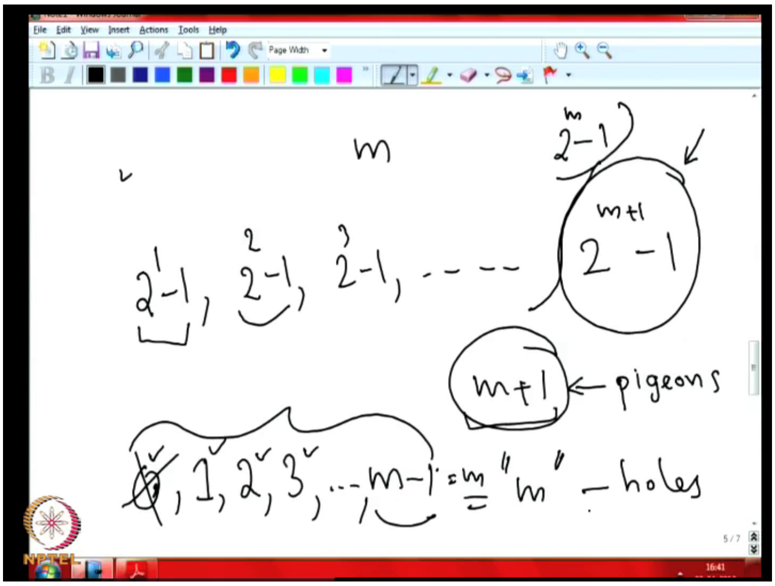

Actually we can consider this first 2 raise to 1 minus 1, say, 2 raise to 2 minus 1, 2 raise to 3 minus 1 first how many of them 2 raise to m plus 1 minus 1. These many numbers I have consider how many numbers? m plus 1 numbers starting from 2 raise to one minus one m plus 1 numbers, these are my pigeons now. Now you know I am trying to divide by m now. So, if I divide by m what are the reminders? The reminders are 0, 1, 2, 3, say, m minus 1. There are m reminders, these are my holes. So, there are m holes here m plus 1 this is m holes namely the reminders which I get when I divide these numbers by m, these are the holes; that means this number this pigeon will go in to the hole corresponding to the reminder which I get when I divide this number by m. So but there are m plus 1 numbers, so two of these pigeons two of these numbers will go the same hole; that means two of these numbers will give the same reminder.

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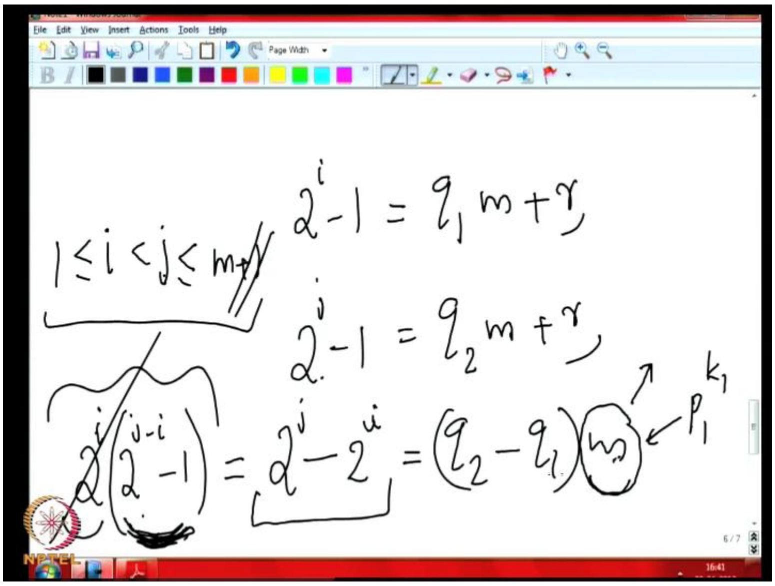

So, how will I write it. There will be some 2 raise to i minus 1 a 2 raise to j minus 1 such that your i and j. So, you can say i less then j this is at least 1 at least m plus utmost m plus 1 right, so that this will be, say, some q 1 into m plus some r 1 the reminder. This will be to q 2 into m plus r 2; r 2 is the reminder. Now you know the pigeonhole principle says that I can find one i and j; that means two pigeons two of these numbers such that their reminder is same; that means this will be equal to this; that means r 1 equal to r 2, right. Let us write it as just r then therefore just r, right, because r 1 equal to r 2, right; so that what I can do. Now what will happen if I subtract this from this? So, without loss of generality I assume that j is bigger then I, so this number is bigger.

So, subtract this from this, so I will get 2 raise to j minus 2 raise to i because this minus one cancels off when you subtract, right. This will be equal to q 2 minus q 1 into m; definitely this r will go away because r when you subtract that both they are the same, right, they will go away. So, this will be the thing; that means this number 2 raise to j minus 2 raise to i is divisible by m but that is not I want. What I want is that some 2 raise to n minus 1 some number of the form 2 raise to n minus 1 is divisible by it but this I can write as 2 raise to i because that is the smaller of this two into 2 raise to j minus i minus 1, right. This number can be factorized like this. So, m divide this number so that means but m is an odd number remember. So, m we can consider the prime factorization of m.

None of the prime factors of m can come from this 2 raise to i because 2 raise to i contains only 2’s and here in m there is no 2. If we consider the prime factorization on m they are all odd primes they are inserting in these things. So, how many of primes are coming they all are odd and which our powers of primes are there, right. None of them will be contributed by the 2 raise to i, so if this divides this number each of those prime factors should be available in this thing and at least as many times as it appears here; that means over instance some prime p 1 appears k 1 times here in this thing. Then defiantly p 1 raise to k 1 should be a divisor of this also, right, 2 raise to j minus i minus 1 also. Therefore, it follows that this m has to divide this alone. So, we can just strike this off, this is enough for me.

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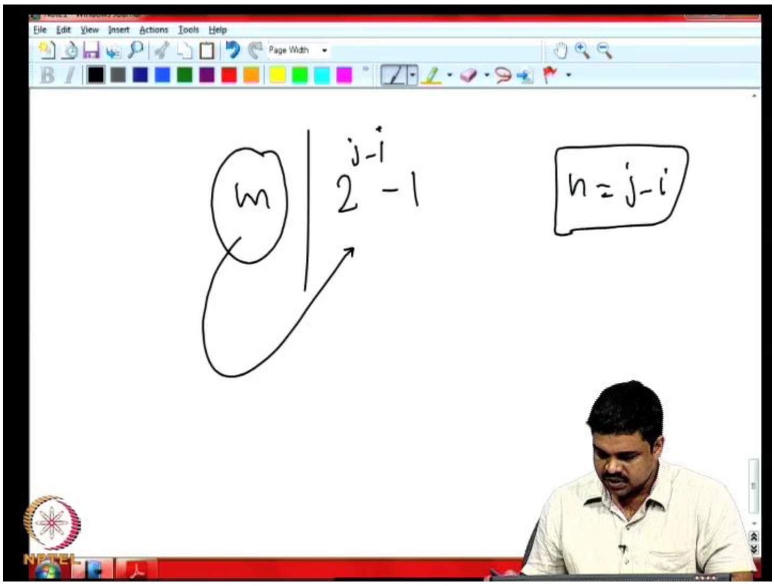

So we infer that m divides 2 raise to j minus i minus 1. So, we take n equal to j minus i that is all, right. So, we have shown that this m divides this 2 raise to j minus i minus 1, right. So, here it was crucial that I assumed m is an odd number; that is why I could infer in the previous step that this 2 raise to i cannot contribute any prime factor of m. So, every prime factor of m has to come from this part alone, right, this part alone. So, that is why this part alone; that is why we could infer that m divides this, right. Now going back we can also note that here we considered that 2 raise to m plus 1, right. So, these numbers we considered was 2 raise to 1 minus 1, 2 raise to 2 minus to it.

In fact we need not have taken this much. So, this is just a subtlety; we could have gone up to 2 raise to m minus 1; that means only the first m numbers of the form 2 raise to n minus 1 I could have considered and still got a contradiction, what was the crucial thing? The thing is here when I considered the reminders that I will get when I divide these numbers using m, I thought it can be 0, 1, 2, 3 or up to m minus 1; that means m possible reminders. In fact that is not true; if zero is a reminder then already it is like m is dividing one of these numbers, right, otherwise zero will not come. Suppose before contradiction if you assume that m does not divide any of these numbers then zero is not a reminder, right.

So therefore, we can we can assume that zero is not there is the list of reminders. So, only m minus 1 reminders are there. So, there are now m numbers are the pigeons this m minus 1, 1 to minus 1, these impossible reminders are the holes and then these m pigeons if you consider occupying the holes, then we can definitely say that two other pigeons should occupy the same hole and then the same argument this argument can be that means there will be a 2 raise to n minus 1, 2 raise j minus 1. This time I can say that i and j is in between 1 and m not m plus 1, right, and when you subtract it you will get the same thing and so the same argument holds good. So, this is how you should solve it. So, we can consider as similar problem now.

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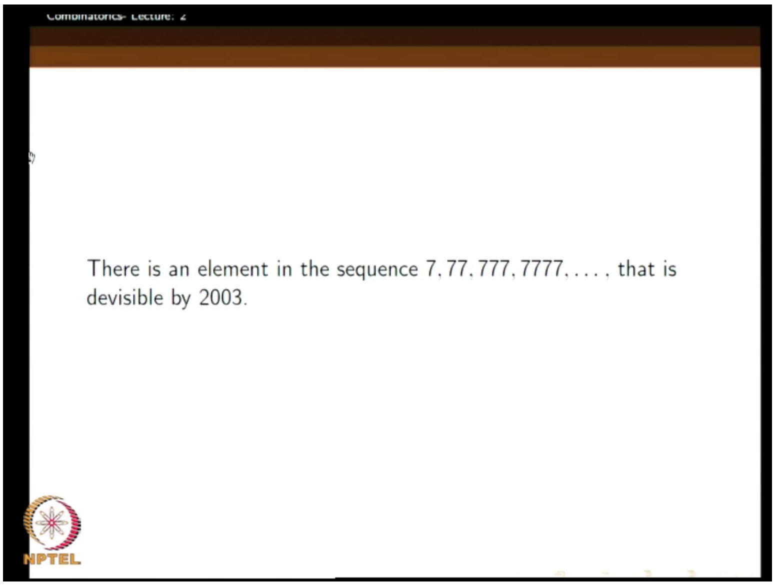

Yeah so similar problem, this is a so similar problem. There is an element in the sequence 7, 77, 777, 7777, etc that is divisible by 2003. Let us see how do we do this thing.

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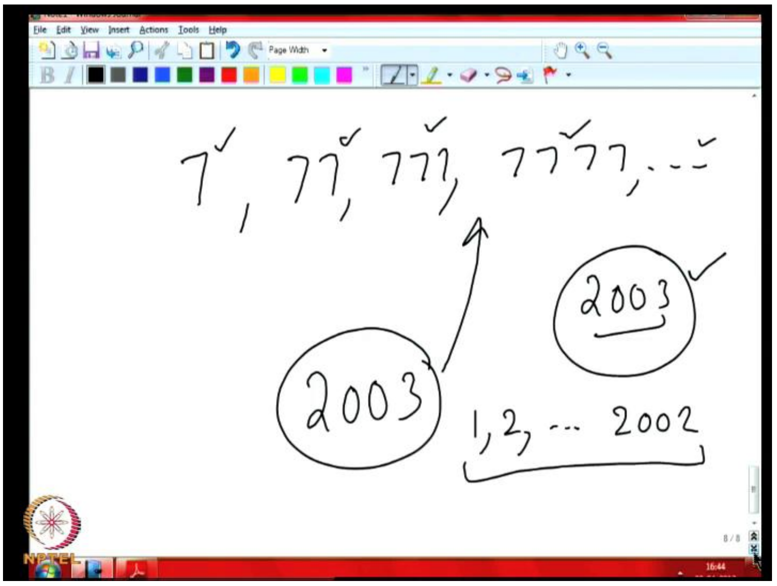

So, now the sequence first number is 7 then 77 then 777 then 7777 like that we can write a sequence like that. So, consider this number 2003. We claim that this will divide one of these numbers. So, not only that we will really claim that if we consider the first 2003 of these numbers seven I mean this is just one 7 here, there are 2 sevens here, there are 3 sevens here. So finally, if we count 2003 of them one of them should be divisible by this 2003 is what we are seeing interesting you know. So, how do we show that? So, this is like this. Again if we consider these first 2003 members of the sequence as pigeons, these are the pigeons. Now holes are the reminders again, you divide 7 by this, 77 by this, 777 by 2003 like that and collect the reminders.

This time again we can assume for contradiction that zero is not a reminder because if zero is a reminder then that means that 2003 already divides one of them, right. So, zero is not a reminder, assume for contradiction. Now so possible reminders are 1, 2, up to 2002. So, there are 2002 possible reminders. These are the holes reminders from the holes, right. So, that means there are 2003 numbers here one of the two numbers, say, the j th number and the i th number will be such that their reminder is same which means that like following the same way we did the last example.

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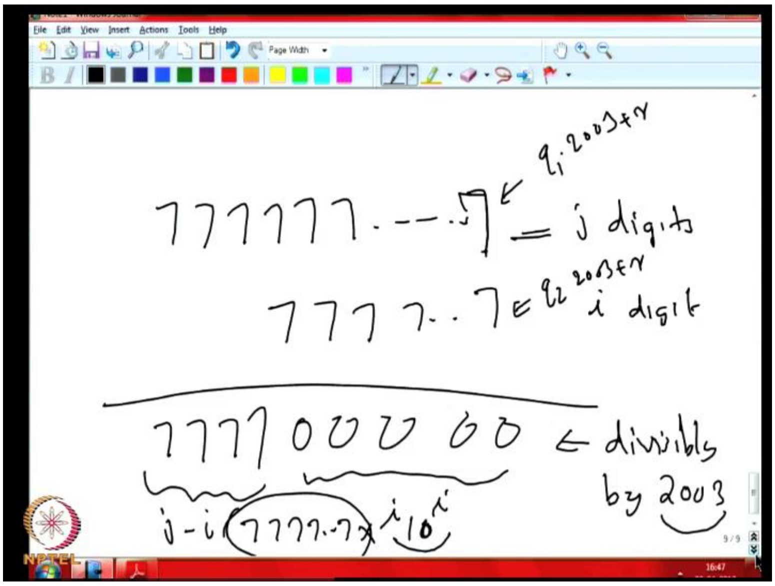

It means that there will be some 777777 say 7 this is j digits here. So, when I minus, say, some 7777 say 7 this is i digits, i being smaller then j. Both of them are such that when I divide by 2003 they give the same reminder then minus this form this; that means reminder will go away like what happened in the previous example. So, we will get something like 7777 then lot of zeros, right. This how many will be here? This is j minus i 7's here, then the remaining i 0 0 i 0 0, right. So, remaining i 0 0 but this number is divisible by 2003; that is what we see this one, right, because this is some q 1 times 2003 plus r, this is some q 2 times 2003 plus r when you minus this, this thing r goes away.

So, q 1 minus q 2 into 2003 is what I get and that is what this is but then this number is actually this 7777 into 10 raise to i. That is what it is because this i 0's you can see here j minus i 7's. This is 777 j minus i time 7's into 10 raise to I; same argument 2003 prime factors if we consider that has nothing in common with these 10 to the power i because this 10 to the power i is composed of just 2's and 5's. 10 has only 2 and 5, 10 to the power i will have i 2's and i 5's 2 to the power i into 5 to the power i but 2003 does not have a 2 because this is an odd number, it does not have 5 also s deviser. So therefore, all its prime factors should come from this part. So, what we can infer now is that 2003 divides the 7777 with j minus i digits, right. So that is the inference and it is interesting that we did not have to go a long distance to get this thing.

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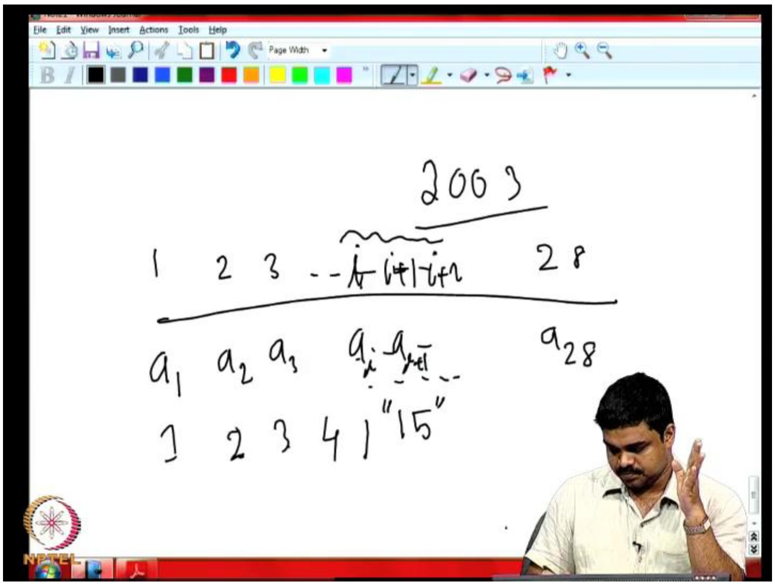

To show that 2003 divides this number we only needed to go only up to two 2003 members of the sequence because there we already see that the pigeonhole principle start working, right. So this is how it worked out.

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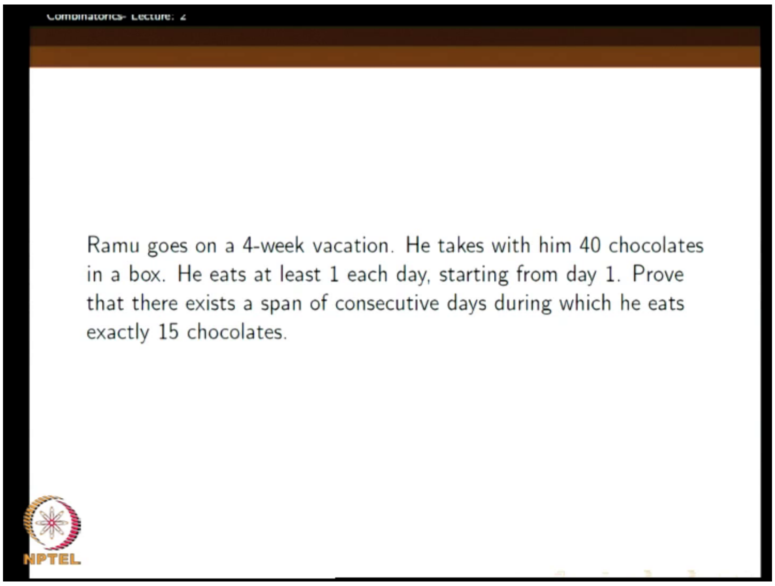

Now the next one we want to consider here is a problem. So, Ramu goes on a 4-week vacation; four week means 28 days, 7 into 4. So, what he does is while going maybe he is going to a place where he cannot buy chocolates may be he is into some forest or something. So, he takes in his box 40 chocolates; he takes 40 chocolates with him and he is in the habit of eating at least one chocolate every day, at least one chocolate he has to eat and what we now have to show is that there exist a span of consecutive days during his vacation, right, consecutive days during which he eats exactly 15 chocolates. So, what are we trying to prove now.

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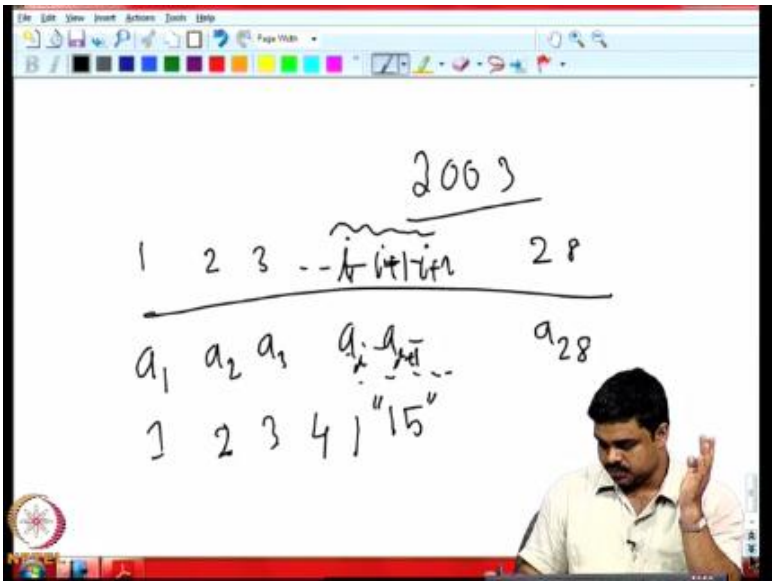

So, this Ramu's we can make a chart now. For instance this is day 1, this is day 2, this is day 3, so 28 days he has, right. So, let us write down the number of chocolates he has eaten; a 1 here, this is a 2 here, this is a 3 here and this is a 28. You know it is everyday he has eaten one chocolate; that means he has not finished off the chocolate, he did not eat his all the chocolates in the first day itself. He has some idea about how many remains; 28 days are there only but he would not eat too much. So, he will keep in mind that every day he has to eat one chocolate, right.

So, now what we want to show is so there will be some consecutives span of consecutive days the i th day, i plus 1 th day, i plus 2 th day; I do not how many days some span of consecutive days; it can be just one day also. If you count these numbers see this is a i plus a i plus 1. So, these numbers if you add up you will get exactly 15; that means the number of chocolates he has eaten is exactly 15. Such a span of consecutive days he will get is what I will say. Also we will wonder what is a so surprising about it is just that you need to get the exact number 15; for instance you may see there what is this, this can be 1, this can be 2, this can be 3.

So, you will add like 1 plus 2 plus 3 plus. So, now 3 plus 3 is 6 maybe 4, 15. Next if he eats 5 it will be done, maybe he will not eat 5 then. One, he will try to avoid getting a 15. So, then every time when he eats he will make sure that he will go backward and see what are the possible numbers which he can produce by eating this number. You just have to make sure that he never produces 15 but we are saying that it is not possible; at some point of time he will have to allow that one consecutive stretch of numbers. If somebody adds up the number of chocolates he has eaten it should be 15, why is it so? So, we will give an explanation of this thing using pigeonhole principle.

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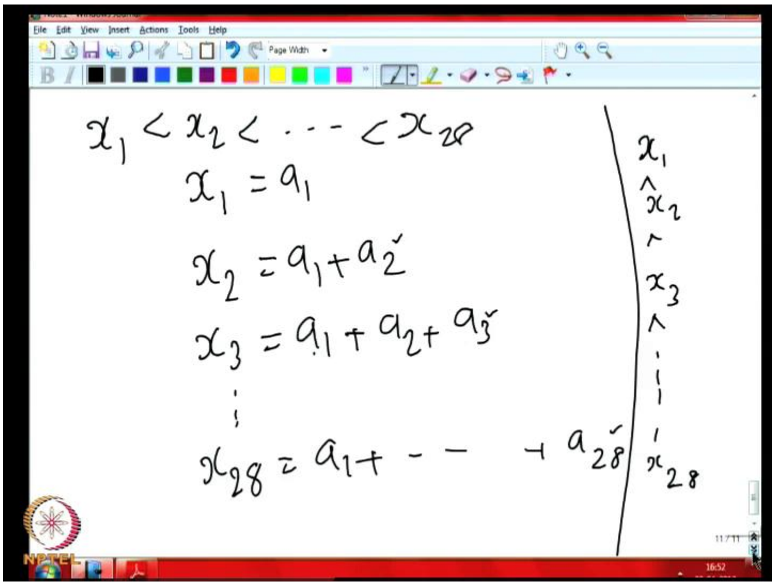

So, there are 28 days. Let us say we will define a variable x 1 so which is equal to a 1; that means the number of chocolates he ate during the first day. So, by the end of the evening of the day before he sleeps he will just count that, okay I have already finished this a 1 chocolates. x 2 means the second day night, so he has finished how many chocolates; that is a 1 plus a 2, right, two chocolates he finished. x 3 means the third day how many chocolates he has finished, a 1 plus a 2 plus a 3; that means the number of chocolates he has eaten. The first day is a 1, the second day he has eaten a 2 chocolates and third day he has eaten a 3 chocolates and so on; like that he will have 28 numbers like that, right.

This is a 1 plus 28. One thing you can observe about these numbers is that this x 1, x 2, x 3, they just keep increasing, right, like this x 28 because he is eating at least one chocolate because this number the last number is adding every time that is at least one. So, therefore that is increasing, right. So, that means we know that x 1 is strictly less than x 2 is strictly less than like that x 28, these are all different numbers. Now this does not help us to infer anything about consecutive 15 chocolates, right, consecutive span of days when he has eaten 15 chocolates, right.

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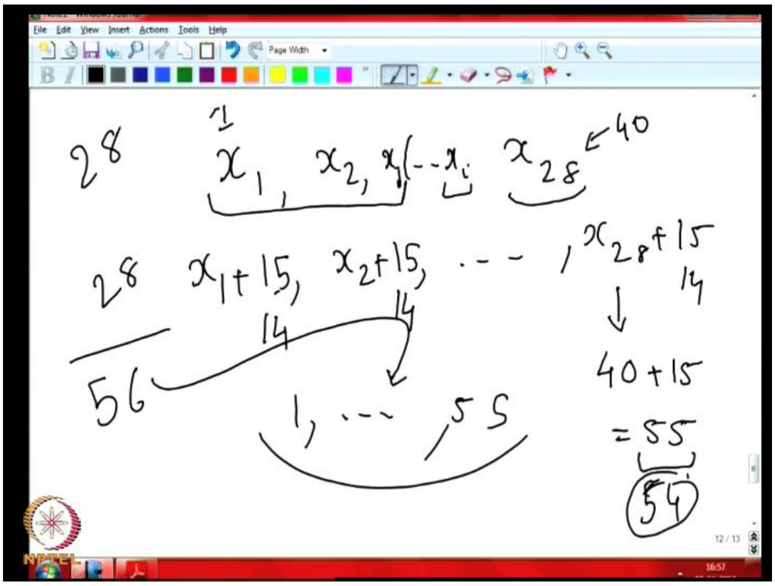

To do that, we will do a trick; this x 1, x 2, these are the numbers, here x 28. Now we create another list of 28 numbers like this, say, x 1 plus 15. This is one number, x 2 plus 15 is another number, like that x 28 plus 15. So, we just shifted each number by 15 added 15 more to that. Here also because these numbers where distinct, these numbers also will be distinct but we do not know whether sum of these numbers is equal to this or not. Now that is exactly what we are trying to show sum of the numbers have to be equal because there are 28 numbers in this line, 28 numbers here. Total we have 56 numbers but then what is the possible values maybe the smallest number can be one; x 1 can be one right, but the biggest number because this is only maximum 40 because here this is only maximum 40 x 28, right, because here definite he did not have more than 40 chocolates.

He can only eat 40 chocolates even in the 28th day by the end of the 28th day. So 40 plus 15 is 55. So, the values these numbers can take x 1, x 2, x 28 or x 1 plus 15, x 2 plus 15. The values these numbers can take are form 1 to 55 only. So, these are the holes the values it can take, these are the holes 1 to 55. The pigeons are these numbers themselves, there are 28 here, 28 here 56 pigeons are there. So, 56 pigeons and 55 holes; so there should be one hole where two pigeons are sitting; that means two of these numbers should have the same value.

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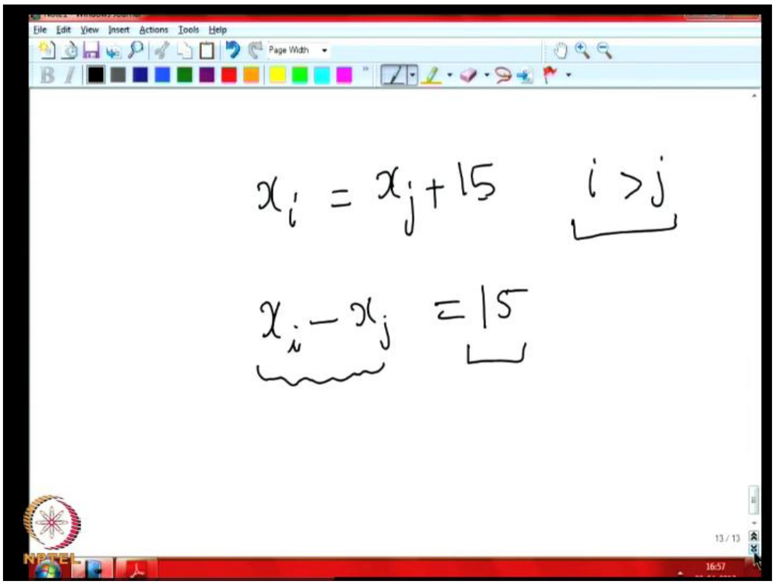

What does it mean? There should be some x i such that that is equal to some x j plus 15. See definitely x i has to bigger than x j; otherwise how can x i be equal to x j plus 15. So, that means i has to be greater than j, this is what we are seeing, right. So, what does it mean in this picture here y some x j and some x I, right, later much later some x i has come. So, this is the j th day j plus 1 eth day up to x j th day how many chocolates he has eaten because these is the number of chocolates he totally finished by the end of the i th day, say, x I, right and this x j is the number of chocolates he finished till j th day till the end of j th day; that means if I minus x j from x i what I get is the number of chocolates he ate on j plus 1 th day plus the number of chocolates he ate in j plus 2 th day till i th day, right.

So therefore, this x i minus x j will give me the number of chocolates he ate in the consecutive span of days starting form j plus 1 to i and that equal to 15; that is what we did now we took it to the side. So, this is what we have proved. Actually 15 is not really special here, we can see that in this proof if I had put 14 here, right, 14 here it will not make difference because you know these numbers are still different, we still get 56 numbers. Now here the possible numbers will be even smaller that is all 54, right. So therefore, there also the theorem is applicable. So, even 14 comes; so it is very interesting that not only 15 in consecutive span of days he has eaten, also some other span of days he has eaten 14 chocolates, right. So, it is not immediate that when you start thinking about this problem you would not immediately imagine that such a thing can be true, right. So, that is interesting that simple application of pigeon hole principle can show as this result, right.

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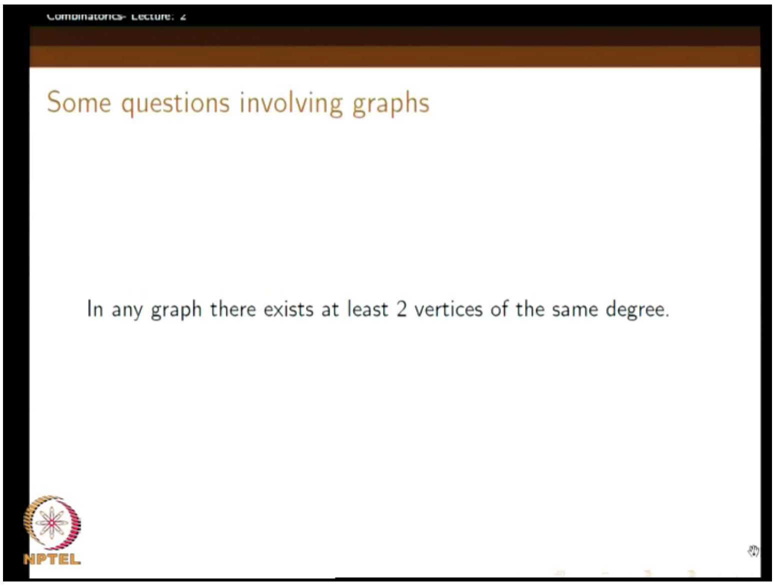

Now the next one, so we have seen several questions involving numbers where we can use pigeonhole principle and get some non-trivial statements. So, many are of course they are not theoretically very interesting many times but then they are interesting puzzles of course; otherwise you know in many places we have used this pigeon hole principle in clever ways, so we are bit surprise that okay, our result is coming out this way, right. So, now we will look at some question in graphs which are a little more interesting that way see more theoretically interesting. So, let us see not very deep but still but before getting into the questions we should remember what were this graphs. So, as I told in the beginning in the last class I am not going to get into very elementary details; I am expecting the student to know a little bit of definitions and all.

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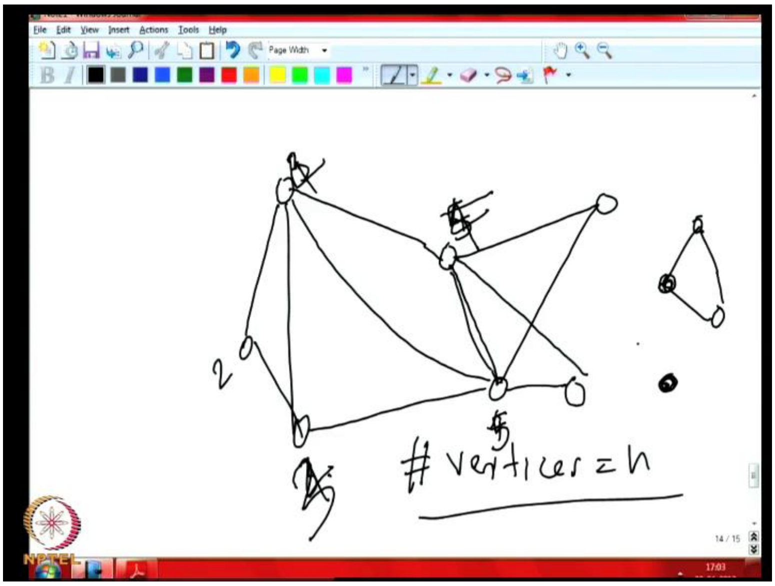

For instance, I would expect the students to know what a graph is; I will not get into the formal definition of the graph. So, there are vertices, there are edges, you know this is this is what we refer to the graph this kind of pictures, right, vertices on edges. So, formal definitions of what vertices are what edges are. Therefore, the students should go back to his some other very preliminary discrete Math books and read it; may be Grimaldi's some of the first chapters will give this details or some of those books where which I have mentioned in my last class, 'A walk through combinatorics.' It has a lot of graph theory or maybe any graph theory book will introduce you to the basic the first chapter of any graph theory book will introduce you into the basic definitions.

Now let us take a graph, right. Now our question is can we show that so it is an n vertex graph, number of vertices is equal to n. Can we show that here all the vertices cannot be of different degree; that means the degrees of all the vertices cannot be different and there should be two vertices on the graph with the same degree. For instance you can try; in this graph you look the degree is 3 here, here it is 2, here it is 3, may be will try to add one here to make it 4, so that it is different but then here it is again 4. So, what will I do; I will add one here to make it 5, right, so that now it is 2, 3, 4, 5 but here it is three. So, what can I do? So, maybe if I try to do it like this then it will be 4 but this is equal to here.

So, if I want to avoid that maybe I can try to connect it to this. This becomes 5 but then this becomes 4; this and this became equal. So, what is happening here? So, what I am trying to do is to redraw the graph, add more edges to the graph or we sometimes I try to delete the edges may be to beat this thing I can try to delete this edge but then this becomes 3 and this becomes 4. So, this and this became equal. So, always we have two vertices in the graph which are of equal degree, how do we explain this thing.

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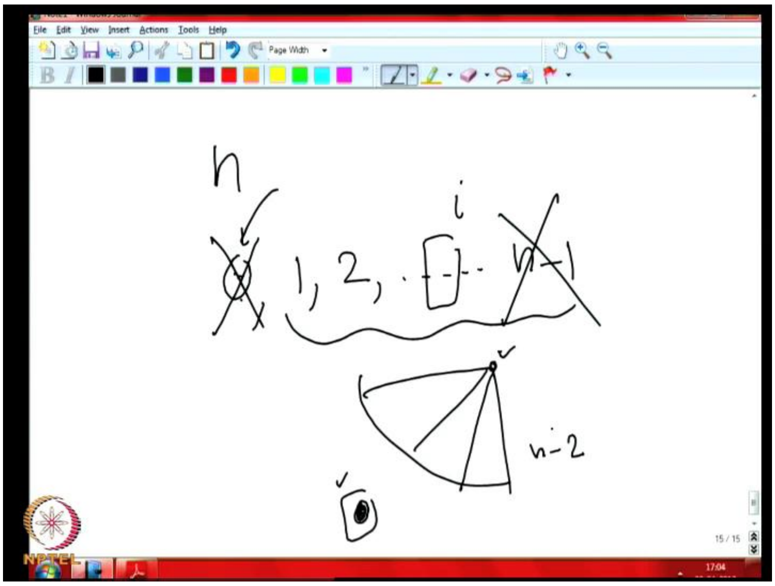

To see this thing it is not very difficult, the number of vertices and what are the possible degrees? The possible degrees are 0, 1, 2, n minus 1; there are n possible degrees, the vertices are n. So, let us take vertices as pigeons and the possible degrees as holes; that do not give any contradiction because there are n pigeons and n holes. If only we had one more pigeon than the number of available holes, then only we can apply the pigeonhole principle and say that there are two pigeons which occupy the same hole; that means there are two vertices having the same degree but one thing I can immediately notice. Suppose I am only talking about connected graph; connected graph means so this was a connected graph. For instance if I put just a vertex here which is not connected to any of them that is a disconnected graph, right, maybe even I drew something like this, right, this together is a disconnected graph, right, so this thing.

Let me say this is an isolated vertex because this is not connected to anything because that is a zero degree vertex. Suppose I assume that there are no zero degree vertices in the graphs, then I could have thought that is not there, then there are only this. Then immediately it is clear there are n vertices, there are n minus 1 holes, n minus 1 possible degrees, then there should be one hole where two pigeons are sittings; that means there should be two vertices with the same degree i, right understood, but then this is there what will you do. So, the point is suppose a zero degree vertex is there some isolated degree vertex is there in the graph, then it is clear that we cannot have a vertex whose degree is n minus 1. Why? Because now it is not connecting to this isolated this thing, this is gone, this is also gone.

So only n minus 2 other vertices are there to which it can connect to. So, this would not be there. If zero is there then this would not be there, either zero or n minus 1. One of them can be there but not together, right. If you have a zero degree vertex, then you cannot have a n minus 1 degree vertex because the vertex n minus 1 degree vertex is called universal, like that should connect to every other vertex but if it connects to this zero degree vertex also then how can this be a zero degree vertex. So therefore, interestingly we have either 0, 1, 2, 3 up to n minus 1, n minus 2 the possible values for the degrees these are the holes; that means n minus 1 holes or 1, 2, 3 up to n minus 1. These are the possible holes, right. So, not both ways, so we have therefore even when we allow isolated vertices when connected or disconnected does not matter.

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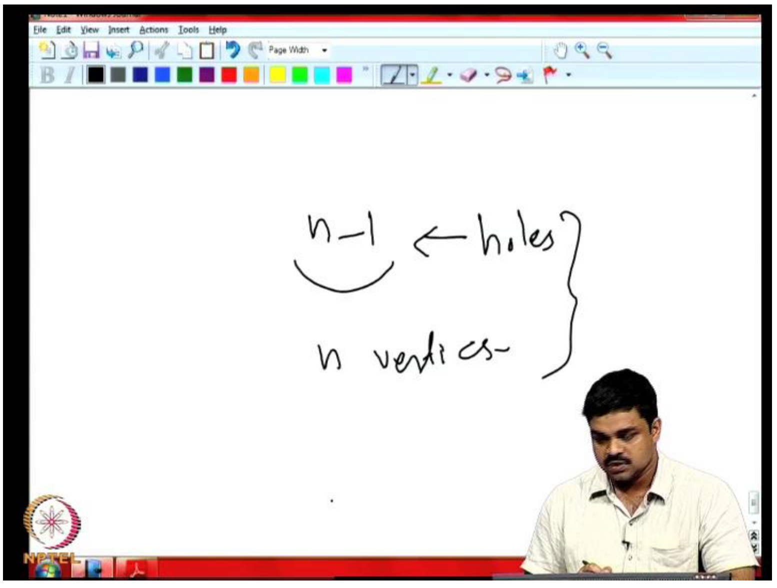

There are only n minus one possible value the degrees can take. It can be either zero to n minus 2 or it can be 1 to n minus 1 whichever way but there are only n minus 1 possible values it can take depending on the graph and these are the holes now and the pigeons are the n vertices. So, therefore there should be two pigeons; that means two vertices with the same degree which happens to be the same hole; this is what it is, right.

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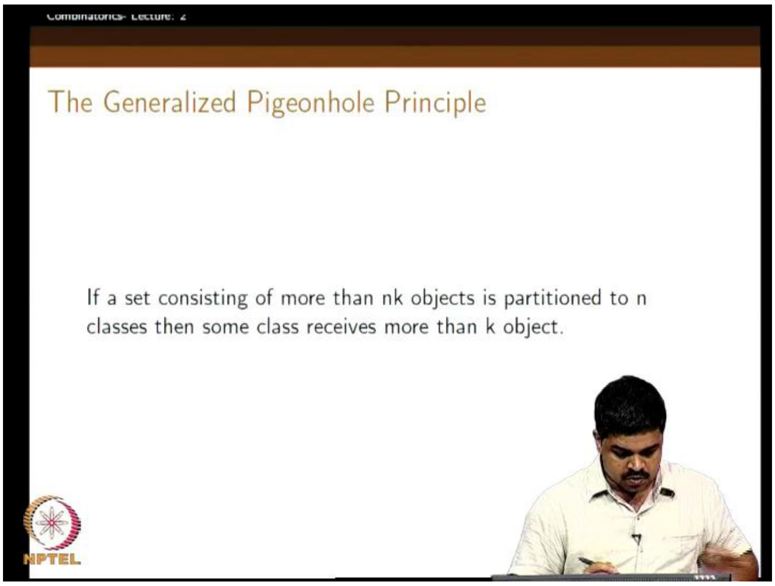

Now the next question we want to address is again a graph question. Before that let us mention the generalized pigeon hole principle; up to now we were always saying that there are more than n objects and these are partitioned into n classes. Therefore, there should be one class which contains at least two objects means more than one objects. Now let us say we have n k objects n times k objects and which are partitioned into n classes. Now it is easy to see that there is some class which receives more than k objects. Again the proof is simple, very intuitive. Suppose it is not true then every class gets only k objects but there are only n classes, we will have only n k objects in total but then we have assumed that there are more than n k objects. So, definitely there should be one class which gets more than k objects that is what exits.

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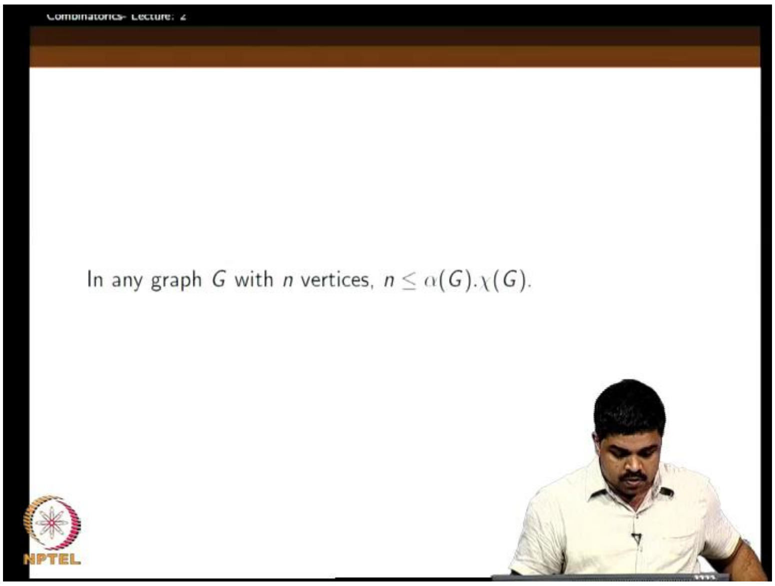

Now we will try to use this in a graph problem. So, of course this is like there are more pigeons now. There are n holes but there are n times k pigeons. Now we see that lot of pigeons should sit in the same hole; that means k plus 1 pigeons at least should go and sit in the hole. So it is just taking the average n k by n that is k. So, because numbers of pigeons are more than n k, so the average will be more than that. So, there should be at least one hole which contains more than the average is what it exits more than equal to the average more, sorry more than the average, because k is an integer, right.

So therefore, yeah, the average has gone above k. The number of pigeons is at least n k plus 1; n being the numbers of classes, the average is strictly greater than k, right. Therefore, the number of pigeons in the same class will be more than k. Now here we have a question. So, we are seeing that in any graph g with n vertices, n is less than equal to alpha g into chi of g. So, we have to remind you what is alpha, what is chi. So, in a graph alpha is the size of the maximum independent set.

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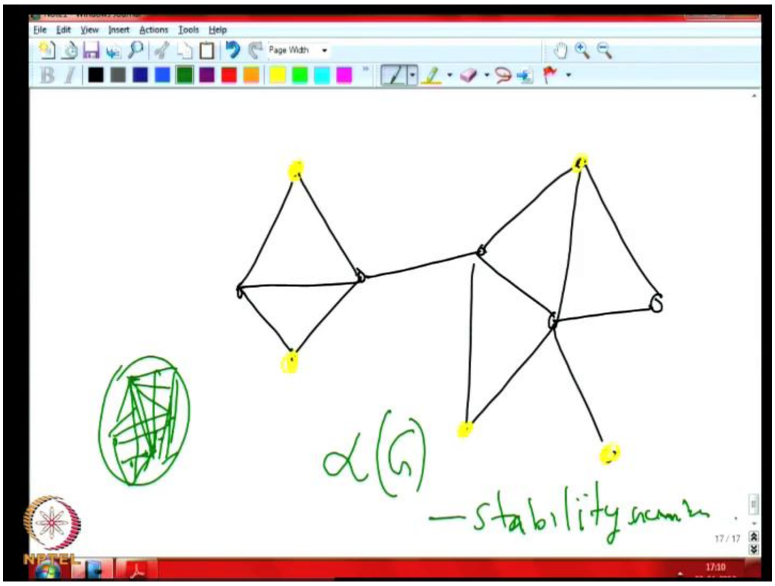

So, for instance let us draw one graph. So, this is one graph, yeah here. So, here I can mark an independent set like this; that means this is one member of the independence set. There would not be any edge between the members of the independent, this is another. So, this is another, say, this is another, this is another, right. This forms an independent set, this yellow vertices here if you look at them they do not have any edge between them. This kind of subset of vertices as such that no pair of them is connected or adjacent to each other, such a collection of vertices is called an independent set. Now for instance if you take a complete graph; complete graph means a graph where all the vertices are connected to each other, right.

The cardinality or the maximum independence, the biggest independence set is its size is going to be one because you can only take one; if you take one more there is a connection between them. Here what is the cardinality of the biggest independent set, did I get the maximum one. So, there I showed a five sized independent set. Now I could have done it in any bigger way, what does it look like for instance if I have taken. Now here from this part I can only take two and it looks like I cannot take more than you know five. So, anyway it does not matter. So, biggest means the biggest in number; we want to make the biggest largest independent set, that number is called alpha of g. It is also called stability number sometimes. So, this if you read any first chapter of any graph theory you will see all these definitions, right. For this thing I just want to introduce this problem, right.

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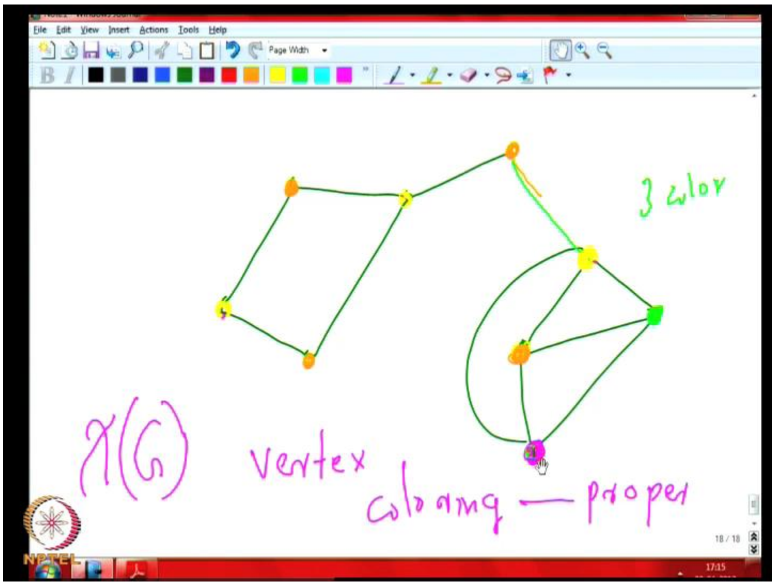

And there is another notation called coloring, graph coloring; coloring means so again you take a graph, so this time what we want is to give colors to the vertices. For instance I want to give a color to this thing and now when I give a color to this thing I should make sure that its adjacent vertices, the vertices which are connected by an edge should not get the same color. So, I should give, say, this one. So, here I can definitely give this, right, here I can give a different color. So, if you want to minimize the number of colors you can use maybe try to use this one.

So, now what we will do; I will use this one, right. Here so can I use red? No, I can maybe use, no it is not correct. I can use a green but I could have used a yellow, right. Here I can use red not green. So, here I can use no not red because now there is an edge here, maybe I can again try to use yellow. Here I can use green, so I have used one to see l one two three, three colors, right, but this is not the minimum I could do because I could have, not three, yes I have just used three colors. One will wonder whether I could have done with two colors, right. Before instance if I have tried yellow here, probably I could have shaded one color maybe but then here I have to change here I have to change. Here I will give red but then what will I do with this thing. Yeah, I need one more color; it is not possible, right.

So, here then I can give yellow here again, fine. No, it is not possible here. So, there is k 4 here; therefore, we will need one more color. So I probably made a mistake in the earlier coloring, so of course I have deleted it. Therefore, we cannot give same color to here. So therefore, we should give a different color here, right. So, we need these four colors because this is k four. So, because they are all connected to each other, they should get different colors. So, anyway I just wanted to introduce the notion of coloring, the vertex coloring and the idea is to get a proper vertex coloring; proper vertex coloring means we need to give colors to the vertices in such a way that the adjacent vertices; that means whenever there is an edge between two vertices they should get different colors. The question is to minimize the number of colors; the minimum number of colors by which you can do this is called chi of g, right. But an interesting thing you can notice here, if you take one color class.

So, here yellow, right, this yellow, this yellow, this yellow, right, this yellow, this yellow, this yellow, this yellow, they form an independent set interestingly, right. We studied what is an independent set just before there are no edges between them. Similarly, if you look at these red color vertices here, here, here, here, they form an independent set. There is no two of them with an edge between them, so it is an independent.

Similarly, this is anyway single term vertex one vertex that is an independent set by itself. Similarly, this one right, so the interesting thing about the color classes for instance if you pick up one color and put all the vertices which belongs to that color, I mean which has got that color as a color class that will form an independent set. So, if there are four colors we have actually four independent sets. Every vertex has gone into one of these independent set because every vertex has got a color, right.

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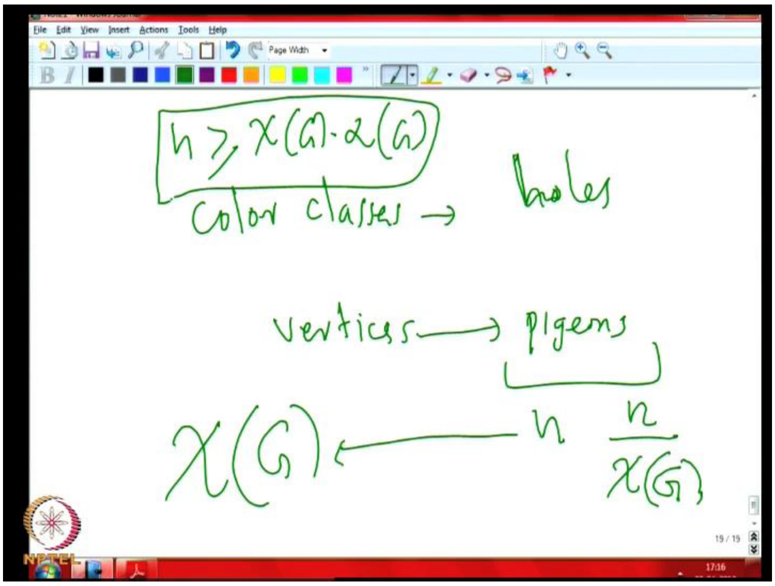

So, we see that here the color classes form the holes. Here the color classes form the colors, color classes or colors. Let us say colors form the holes, pigeons are the vertices. Vertices form the pigeons, color classes from the holes. Now here we have n pigeon's chi of G colors classes, right we have a coloring, right. So, because we have chi of G color classes and n pigeons, right, what does it mean? There should be one color class which contains at least n by chi of G pigeons, right. So, from this thing we will get n is greater than equal to chi of G into alpha of g as required, right; this is what it says. So, here we use a little bit of graph theory, so we will see in the next class.