Combinatorics
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Lecture - 19
Inclusion exclusion principle-Part (2)

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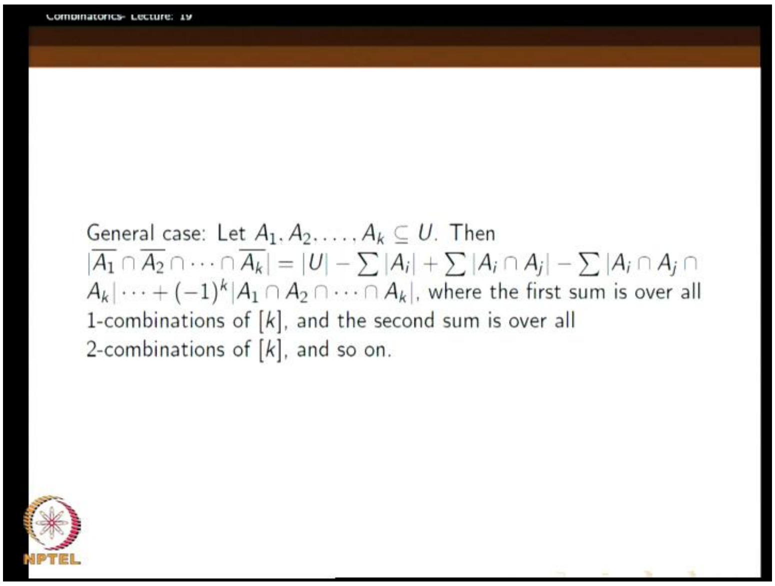

Welcome to the 19th lecture of Combinatorics. In the last class, we were considering this problem namely, we have universe U and A 1 A 2 A k etcetera. So, this up to A k, k subsets of universe of the universe, you are given. So, we are considering finite U and so what we are interested in is the number of elements in U minus A 1 union A 2 union A 3 union up to A k, but in another words, in the complement of A 1 union A 2 union up to A k, how many elements are there, how many elements are there. In the complement of A 1 union A 2 union up to A k. In again the we can easily see that, this is equal to by Demorgan rules, we can easily see that that is equivalent to asking the cardinality of the complement of A 1 intersection the complement of A 2 intersection the complement of A k.

How many members are there, we have discussed it out carefully, in the last class considering 2 sets k and 3 sets k when k equal to 2, k equal to 3. Now, we are saying that the general formula for this thing can be given like this, it is equal to U minus here, U minus U minus some of the cardinality of the A i's, you sum up the cardinality of the each set given. And plus the some of the cardinality of the A i intersection A j where, A i A j, A i not equal to j any each 2 element subset of 1 to k you take and those corresponding 2 subsets, if you take and find the intersection and those cardinalities you add together.

There are not, there are k chose 2 such things, you have to add them together and minus all possible 3 subsets of 1 to k, you take and the corresponding A i intersection A j intersection A k, you consider right and then we add. So, just plus or minus plus or minus, you do until the last last term will be minus or plus depending on whether k is A odd number or even number as you can easily see because it is alternative plus and minus. So, that is the formula how did we get the formula, the the formula we biggest. In fact, the we looked at the pattern of the formula when k equal to 2 that means, when only 2 sets are there. In the collection that means, A 1 and A 2 alone then we saw that there is U minus A minus A minus B that is minus of sum of A plus B cardinalities of A and B.

So, A 1 may be, we can say A 1 and A 2 and plus A 1 intersection A 2 that was only 1 2, 2 members means 2 and then 3 numbers also between the 3 number case also, we considered the formula that that, your 1 more term namely, the last term being the intersection of all the 3 sets that also for all the pattern. And actually, we we gave 2 kinds of arguments to show that this formula is correct. One was done as we wrote down the formula means, we we kept on arguing and wrote down the formula and then therefore, it was correct in the 2 cases using the venn diagram, we were arguing. Now, it was like, we want to find the members in you, which are not in any of the A i's. So, you first take all the members in U i sorry U, so that is cardinality of U right.

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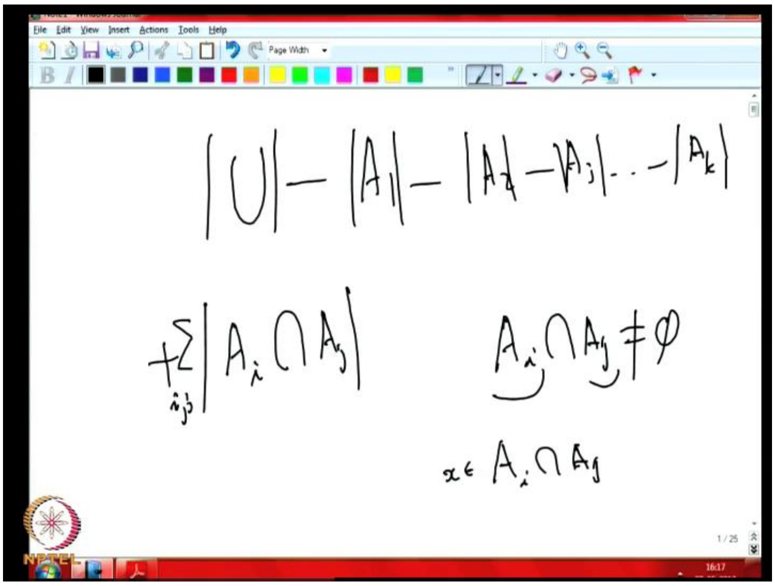

And then that is why, how why we wrote u then in minus of the members in A 1 minus of the members in A 2. So, like this A 1 minus of the members in A 2 minus of the members in A 3 and so on. So, up to A k, we can minus of so this would indeed give the answer, if each this A 1 A 2 A 3 A k were, pair wise disjoint where, A i intersection A j was an empty set for every pair of i and j in 1 2 k. This would have simply given because here subtracting things, which we do not want and then nothing is subtracted more than required, but unfortunately that is not the case, we will because there will be several things this A intersection A j would not be empty many cases, there will be nonempty intersection.

And those members in A i intersection A j when it is not empty, we get subtracted twice that ones for this and one for this. So, that is why we told for instance, if there is a member in A i intersection A j, they got subtracted twice therefore, I have to re add it, they got it, got subtracted twice, so we have to re add it. So, we just re added it, we decided to re add this thing and we did digit for every pair i j, that was the that was the next type of things every pair i j right selected. But, then you noticed that, if a member was in more than 2 sets means, let us say 3 sets then.

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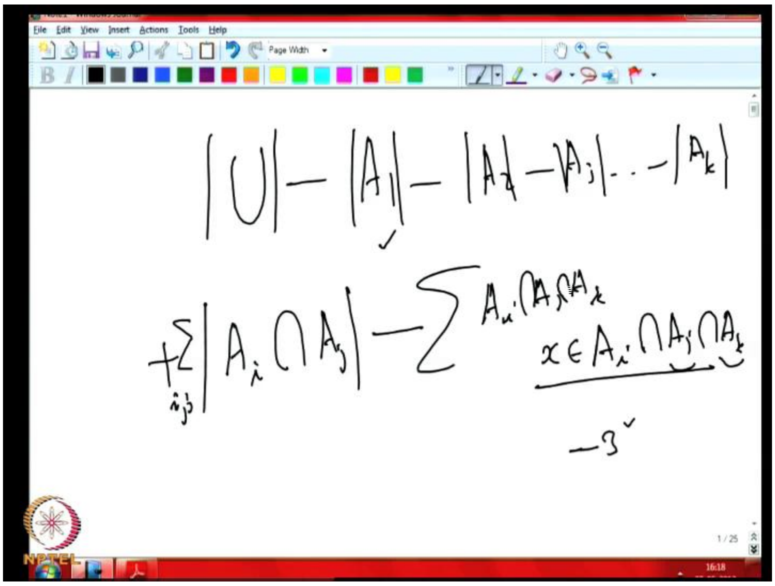

Initially they got subtracted 3 times once in say, if a number was in A i intersection A j intersection A k, that member was subtracted for this 1 A i when it was as part of A i that is member contributed to the minus term in A j for A j, contributed to the minus term for A k also that means, minus 3, this subtracted. But, then we re added also, but then again 3 times it was re added. So, 3 times subtracted and 3 times re added. So, we would think that, it was not subtracted at all but it has to be subtracted. So, we subtract A such terms again all A i A j A k intersection things. So, we found out all triples, we found out and then there are k chose 3 triples for k chose 3 triples of 1 k.

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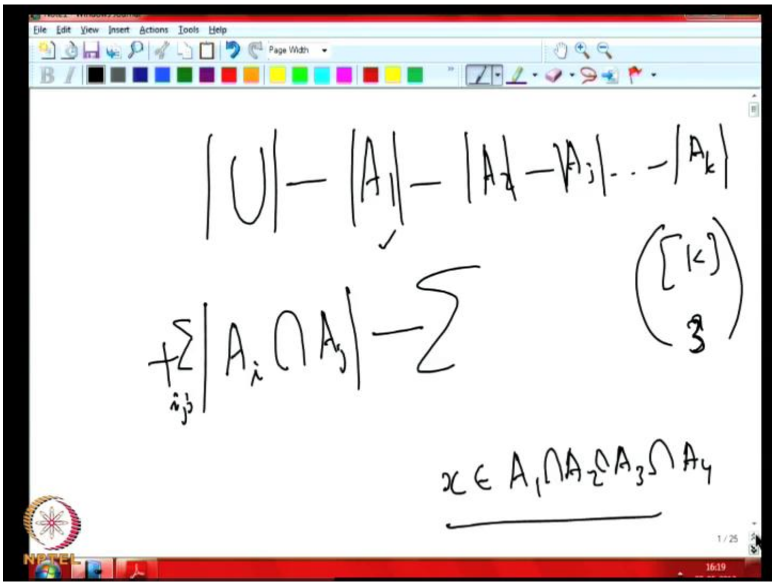

So, this is k chose 3 for every triple in this set k chose 3. So, we find out the corresponding A i intersection A j intersection A k and then that cardinality subtracted off right. Then we see that suppose, there is A x, which is in 4 elements A A 1 A 2 A 3 and A 4, the intersection of 4 elements, they got subtracted too many times. So, we have to re add it, this was the thing, we can carefully argue it and probably make it. So, the convince yourself that this will be the pattern of the formula, but again we need a regress proof though, the proof is kind of in this direction only.

So, we need neat proof for this thing that is why, we were following a counting argument namely, how much each member of the universe is contributing to the sum on the r h s. Because, we want those members in U, which are not in any of the A i's to contribute exactly once while those members in U, the universe, which are in at least to 1 A i should contribute only 0 then only.

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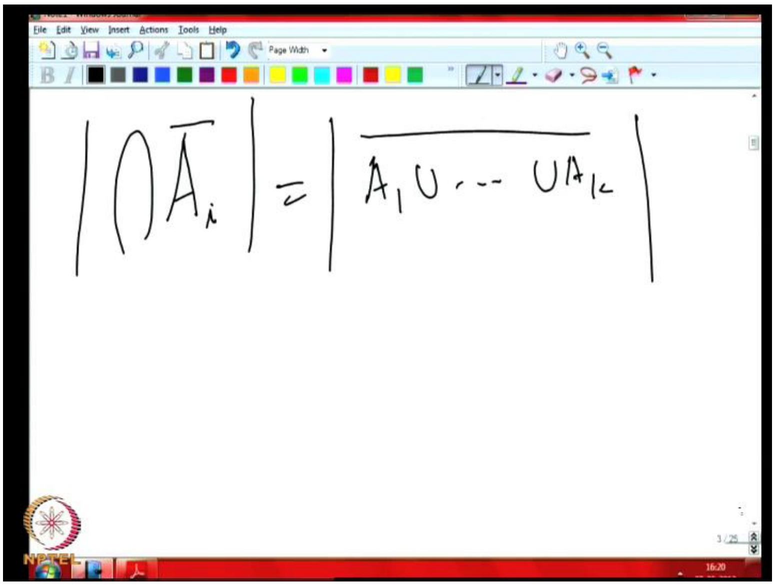

We will get the correct count, we are looking for namely the cardinality of this intersection that means, the cardinality of the complement of this right. Then only, we will get it. So, now we will carefully look at this counting argument and prove that the formula, we wrote down in the slide here is correct.

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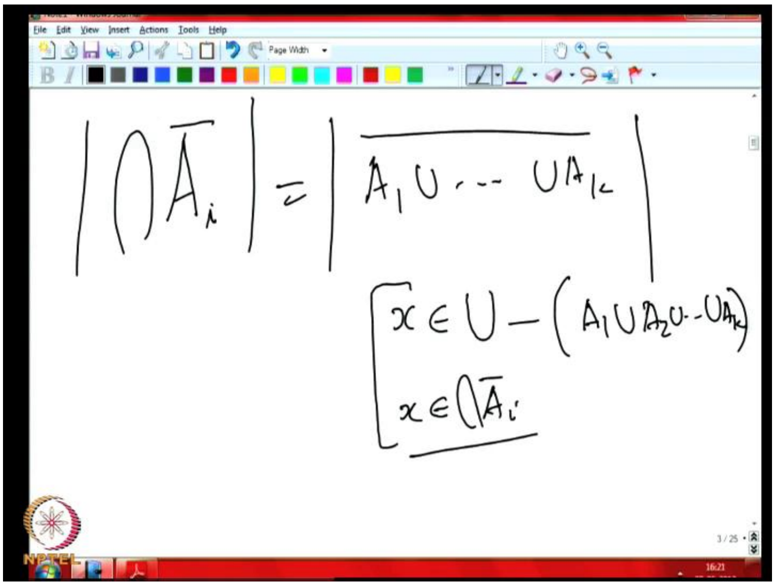

So, he do we do this thing. So, we will consider just 2 cases, first case is when x element of U bar. So, as we wanted A 1 union A 2 union A k; that means, x when x is in the component of x is in the required set that means, this set. How this contribute to the total sum. So, we can go back to the slide here.

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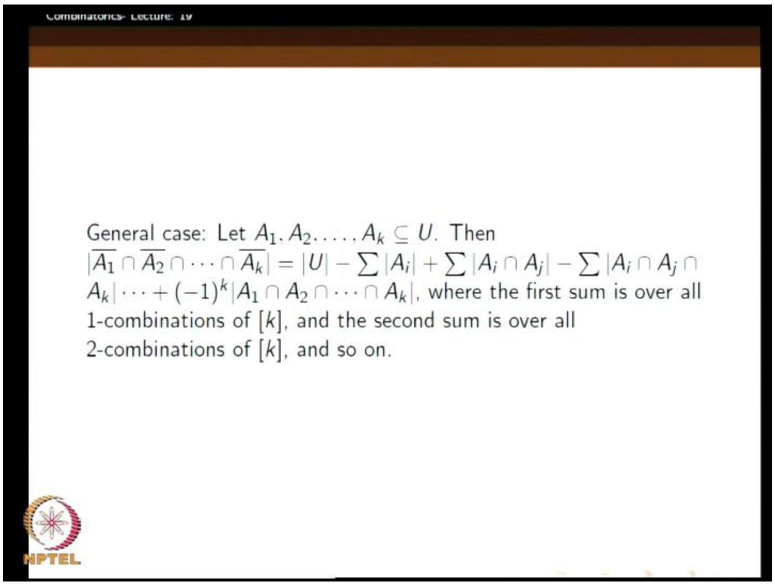

In the first term U, it is contributing 1, in the second terms onwards it is not contributing at all. So, we can write it on here.

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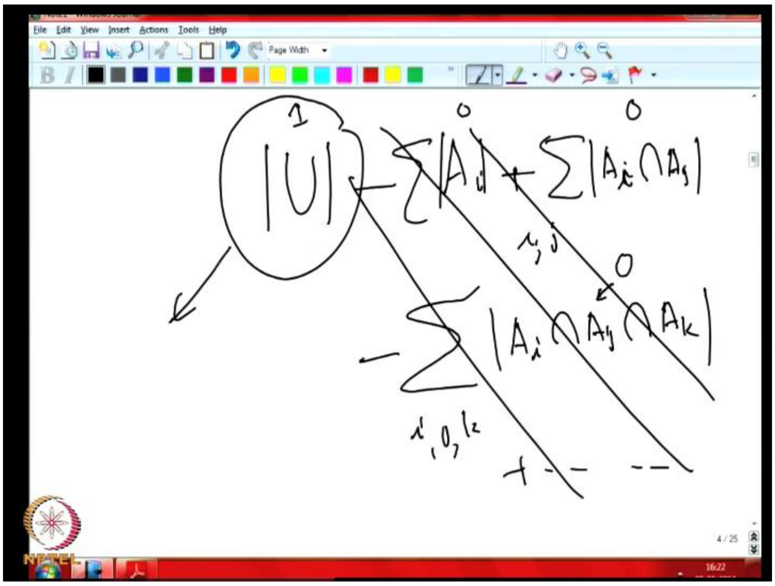

So, in U it will contribute, but then remaining terms, if in non of the A i's, it will contribute, because in A i's it is not there. Sigma A i cardinalities, it would not contribute because here, it will contribute 1, each of them it will contribute 0 only. Similarly, this kind of terms where, i and j are considered and then A i intersection A j is considered right, here it will only contribute 0, because it is not in A i or A j right.

Similarly, i j k in 3 of them. So, for instances none of A i A j k contains that member. So, this will be also I mean, it will be contributing this sum also 0, I mean any of the later comings terms, it will be contributing 0 only. So, this term would not be it wont be contributing to this terms. So, it will only contributing to the first term namely 1 and that is contribution is just 1 as we discussed. So, that is fine.

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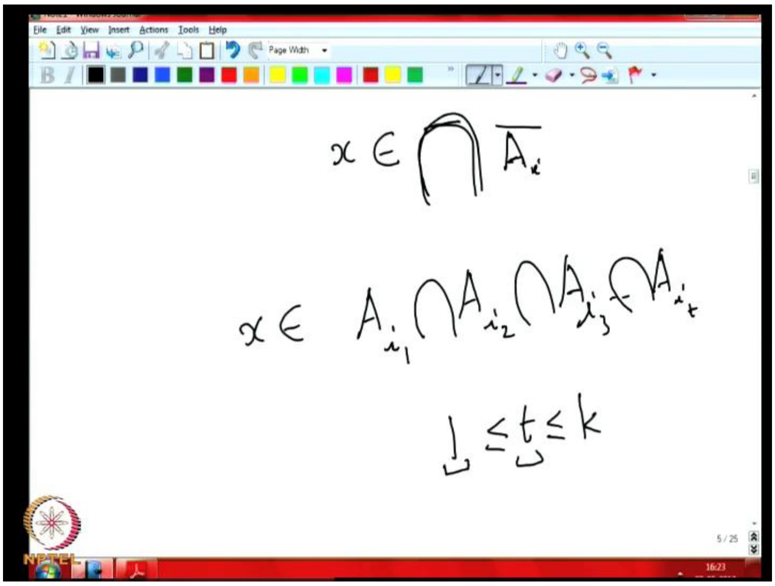

Now the second case is suppose x is not in the required set that means, it not in A i complement union, so that means, x is in some A i's. So, we will say that x belongs to exactly, which A i's, it belongs to let us say it belongs to A i 1, it belongs to A i 2 and it belongs to A i 3 say it belongs to A i t, not that t is anything between 1 and k, because there are total k sets and we know that it belongs to at least 1. For t, we will take the general case of it it belongs to t. So, this is and it does not belongs to any of the other thing.

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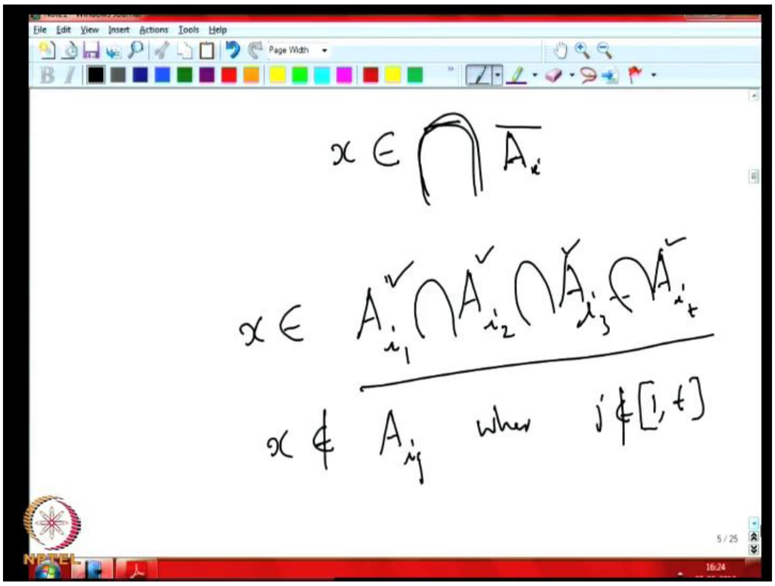

Actually, we can write it does not belongs to any of the other things x does not belongs to A i j where, j is not in say 1 to t. So, no other only in this sets it is belongs to right, exactly t sets it belongs to and these are the t sets in, which A x is present, remaining sets does not contains in that case, what will happen to that x, clearly x will contribute 1 to this term x will contribute one to this term.

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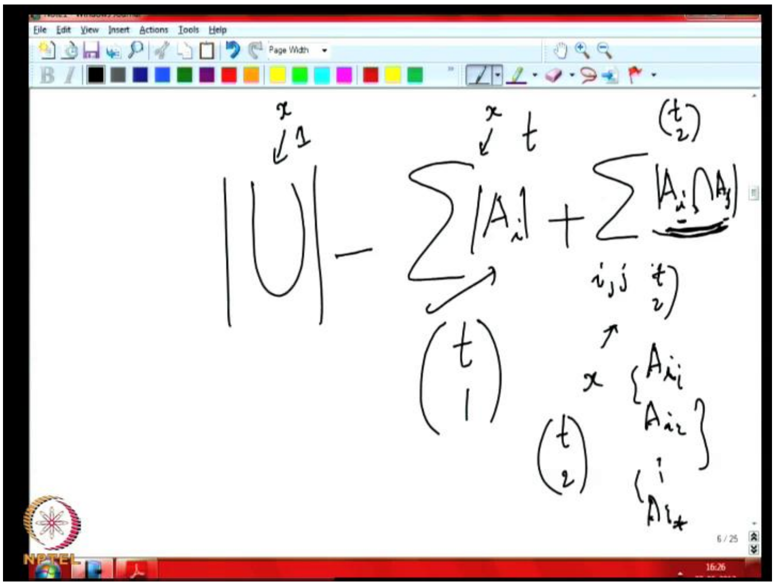

Because, it is there in U and minus sigma A i's, all the A i's does not have x. So, but in exactly t A i's, it is there. So, this t A i's it will contribute say it is t chose 1, because out of those t things, if you take any 1 set it is in this sum it is contributing. And the next term of this sort, i j any pair of sets, if we take A i intersection A j kind of sets for any pair from pair i j from 1 to k, that will x will contribute how many times, you know if this i and j are say A i and A j are such that it does not belongs to either A i or A j definitely it is not contributing to this A i intersection A j, because they wont be there in it.

So, only the pair of sets from the collection of sets a A i 1 A i 1 A i 2 A i t in, which x is x belongs to right any pair, if you take from this thing then that will contributed how many pairs are there, definitely t sets are there any t, t chose 2 sets, which which we can take from here, if we take the intersection of those such pairs, we will we we will have have x in it. So, it will be contributing. So, the total contribution here is t chose 2 right, t chose 2 will be the total contribution.

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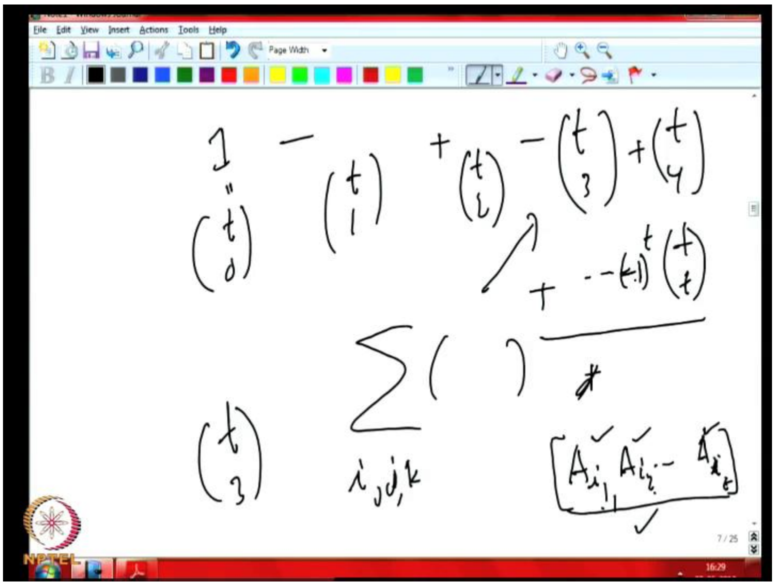

So, the first it will contribute 1 to this thing 1 is you can take it us t chose 0, then the next term, it will contributed minus t chose 1 and then next term, it will contributed t chose 2. And the third term it will contributed t chose 3 minus t chose 3, why because that is sets of this form i j k A i j k, so for instance. So, yeah…

So, 3 3 3 members of the collection is taken and their intersection cardinality is put here, but then x will belong to only those intersections of 3 tuple 3 3 tuples of sets where, this sets are coming from i this collection A i 1 A i 2 A i t A i t. Because, these are the only sets in, which x belongs to and then x, if it does x does not belongs to all the 3 sets, we are taking the intersection then x will not be there in the intersection.

Therefore, how many 3 sets, we can select from this things that is only, we should take there are t chose 3 possible ways to select k subsets in right and each of them will contribute 1 4 x, because x belongs to each of them.

So, this will be t choose 3, in the second term, third term in the fourth term, it will be A plus this is this is A plus term, the next term it will be A sorry, this is A minus term, this is a minus term. Next term will be A plus term that that should be t chose 4 by the same argument and so on, all the way to t chose t when, we considered the sums over every t positive subsets or the k sets.

Only 1 collection of t subsets will contribute to it, because the entire A i 1 A i 2 A A i t comes then only their intersection will contain x, because if any other t subsets intersection of any other t subsets from A 1 to A k is considered then there will be 1 subset in it, which does not contain x. Because, we know that only this t subsets contain x and no other.

Therefore, they will not contribute only this can contribute from now on in the next term onwards, there would not be any contribution from x, because we are selecting t plus 1 or more subsets. And there, because all the t plus 1 subsets, there in any term cannot contain x, because x is only contained in this t subsets like when, you consider t plus 1 subsets and take the intersection x, it would not be there in one of them and in the intersection it would not come. So, it wont contribute, so up to here.

The total contribution due to x is this, t chose 0 minus t chose 1 plus t chose 2 minus t chose 3 plus t chose 4 and all the way up to t chose t and the sign depending on whether it t was an odd number or even number. So, minus 1 raise to t, we know that this sum is going to be 0, why because we have studied it earlier that this identity we had studied right this identity we had studied.

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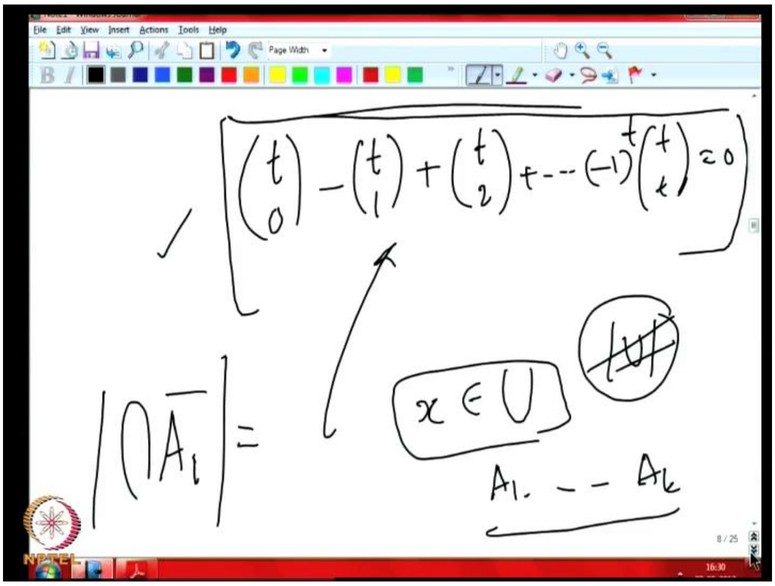

So, we mentioned that this will easily follow from the binomial theorem 1 plus x raise to t expand it and put x equal to minus 1 anyway, we do not have to redo it. Therefore, the contribution of an element x of the universe, which belongs to exactly t sets in A 1 to A k will be this and this turns out to be 0. And A member of the universe x of the universe member x of the universe, which belongs to none of this A 1 to A k will contribute exactly 1 namely, this first term nothing else.

Therefore, the total sum will have contributions only from the members of U, which does not belongs to any of A 1 to A k namely the members of x in A i complement intersection. And therefore, we are counting the cardinality of this thing by the by the procedure.

So, this completes the proof of that. So, this is the proof now what you can do is we just remember, what remember the formula and of course. So, we can apply it even that way, but it is good to know the proof of course otherwise, if you forget it then we can reproduce it very easily. It is not a difficult proof just have to understand, how we count it the counting was based on rather than counting directly that how many are there in U and then (( )) how many are then each of A i and summing and then (( )). So, rather than that, we asked took an element of the universe and ask how much is this 1 contributing, we saw that what should contribute is contributing exactly 1 what should not contribute is not contributing at all, because of that identity. So, that was the proof.

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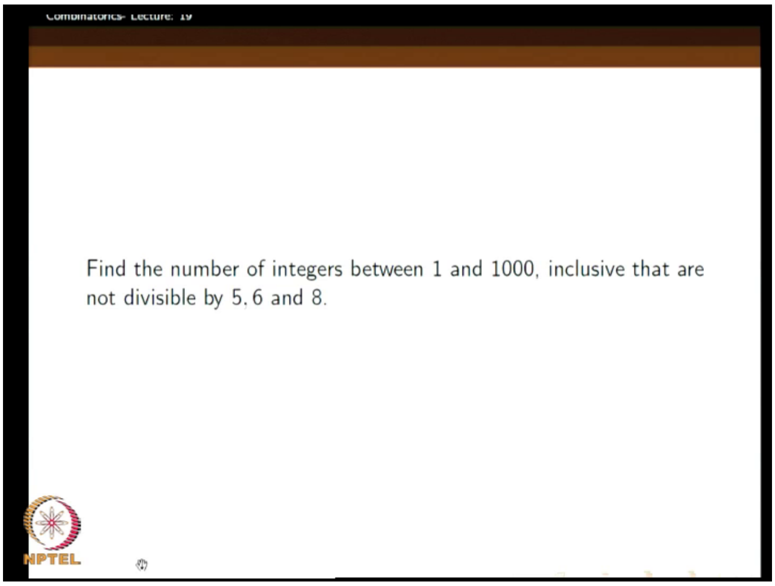

Now, we will consider some applications of this. Here is a simple problem find the number of integers between 1 and 1000 inclusive, inclusive means including 1 and 1000 that are not divisible by 5 6 and 8.

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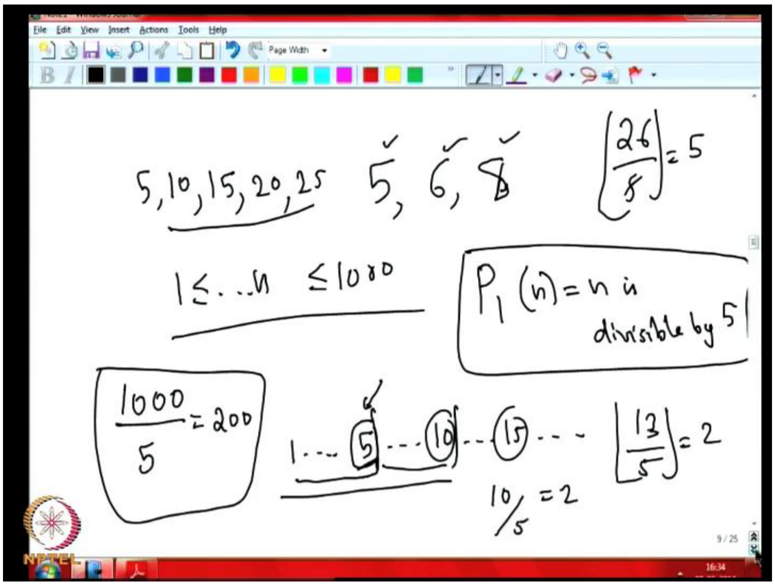

So, we want to find the number of integers in between 1 and 1000. So, we are considering numbers in the range 1 and 1000, which are not divisible by 5, 6, and 8, usually the sets, we can say properties. The first property, we are interested in is that the how will you find it out, we can say that the property P 1 corresponds to the property of a number P 1 of n that means, n is say n is divisible by by 5.

What kind of numbers are divisible by 5, for instance 0, we are not interested after 1 5 is divisible then the next 10 is divisible next 15 is divisible like that, every 5 numbers, we have 1 numbers, which is divisible by 5. How many numbers are there from 1 to 1000, which has this property for instance 1 to 10, if you consider there are 2 of them. So, easy way to do is divide 10 by 5, we get 2 up to 13, if you consider, you can divide by 5 and take the floor. So, that is 2 up to 26, if you consider you can divide it by 5 take the floor, you get 5 for instance up to 24 26, if you go 5 10 15 20 25 5 numbers.

It is not very surprising what you do, what you just observe is every consecutive 5 numbers, if you take there is 1 divisible by 5 and next consecutive 5 like that. So, it is it make sense to divide by 5 and take the floor, you will get the number of numbers divisible by 5. So, the number of numbers, which have property 1 P 1 from 1 to 1000 is clearly 1000 divided by 5 namely 200.

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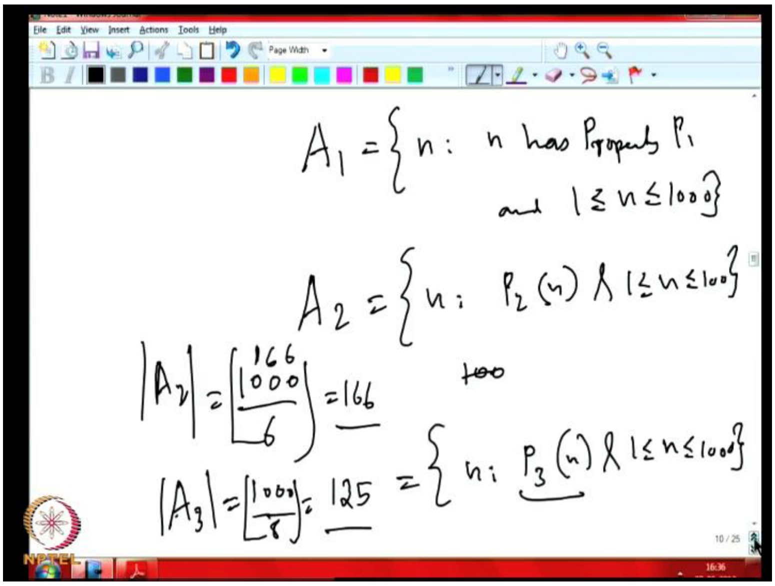

So, we can say that A 1 is the set of numbers, such that n has P 1 property P 1 property P 1 and of case1 less than equal to 1 less than equal to 1000. Similarly, A 2 we can define as those numbers n having property P 2 and P 2 in 1000, because we are interested in 5 6 and 8, this P 2 is the property that and is divisible by 6.

How many of them are there 1000, this cardinality of A 2 is easily calculated as 1000 by 6 floor 1000 by 6 floor, we can calculate to be 6 166 4 and 36 166 and then A 3 is the property that A 3, we can define as the property that P 3 of n and sorry, A 3 is the set of number between 1 and 1000, which have property 3. Namely P 3 is defined as the property that n is divisible by 8 that also, we can find by dividing by 8 and finding the floor this is 20 yes 125 right 125. So, we can easily calculate the 3 sets.

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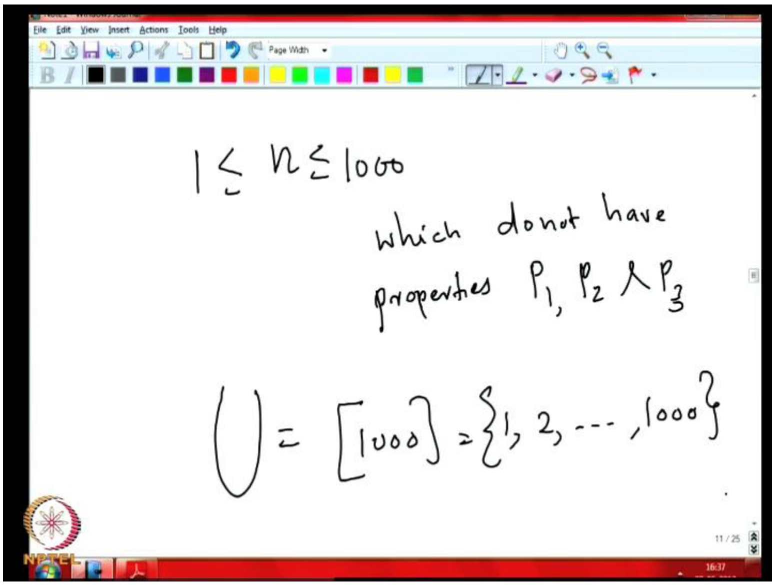

So, but why are you calculating these 3 sets, because we calculating these 3 sets. Because, we are ask to find those n between 1 and 1000, which do not have properties P 1 P 2 and P 3. It is easy to see that, we are if you consider U as this set that means, 1 to 1000, the universe is 1 to 1000.

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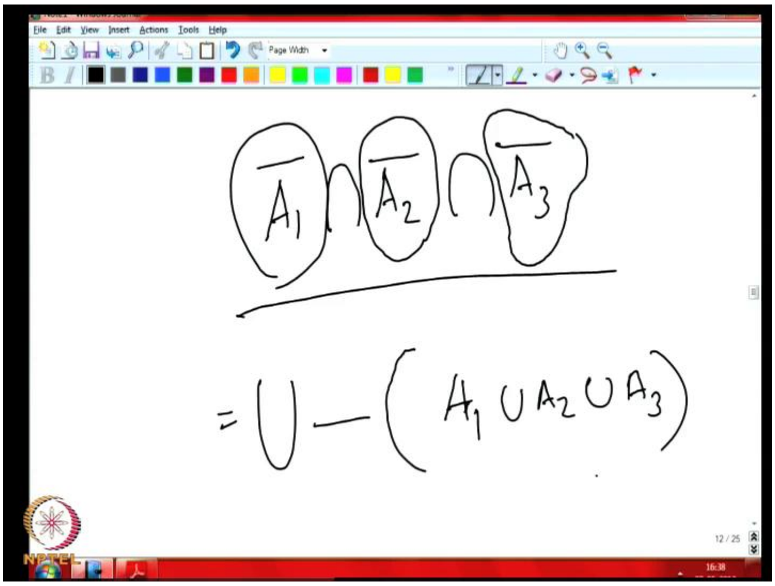

We are interested in A 1 complement intersection A 2 complement intersection A 3 complement that means, the numbers this correspond to the numbers, which are not divisible by 5. This correspond to the numbers, which are not divisible by 6, which are these correspond to the numbers, which are not divisible by 8 and we are interested in those numbers, which are not divisible at 5 and not divisible by 6 and not divisible by 8. So, we are interested in this thing, so which is essentially U minus A 1 union A 2 union A 3.

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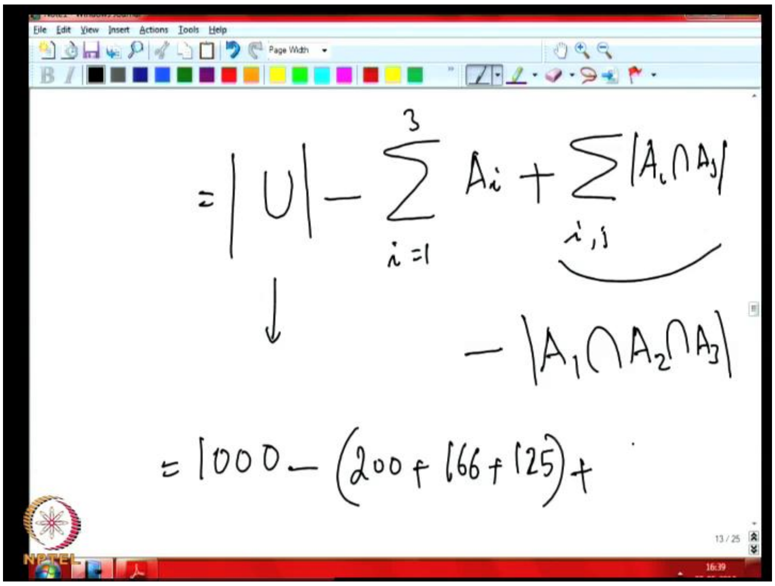

So, we know the formula for this thing this is by the formula, it is cardinality of U minus sigma A i, here I is only 1 2 3 and plus i j all pairs, if you want to consider, you have to consider. A i intersection A j this thing and finally, this will be 1 set A 1 intersection A 2 intersection A 3 cardinality. This is what 1000 minus this is what 3 sets, 3 sets are you know, we have already calculated this cardinality. So, this 3 sets namely first was 200 second was 166 and then 125. So, this is 200 plus 166 and 125, this is what we have then we are interested in.

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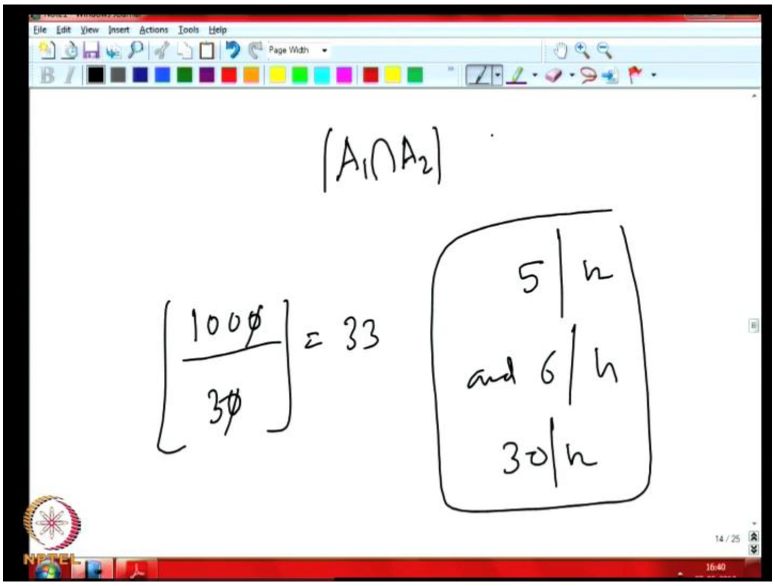

We have to re add this for every pair every pair A i intersection A j, we have to add here, we have A 1, we have to do A 1 intersection A 2 for instance what is that that means, it is divisible by 5 and it is divisible by 6. So, how it divisible by 5 and 6, it is divisible by 5 and 6, if it is divisible by 30 sorry, see if a number is divisible by 5 and sorry, 5 divides suppose 5 divide n and 6 divides n, it is clear that 30 divides n and any number, which is divisible by both 5 and 6 is divisible by 30. So, it is section essentially equivalent to counting the numbers, which are divisible by 30 between 1 and n and that is easy to see, this is this 1. So, this is 33, you have to take floor.

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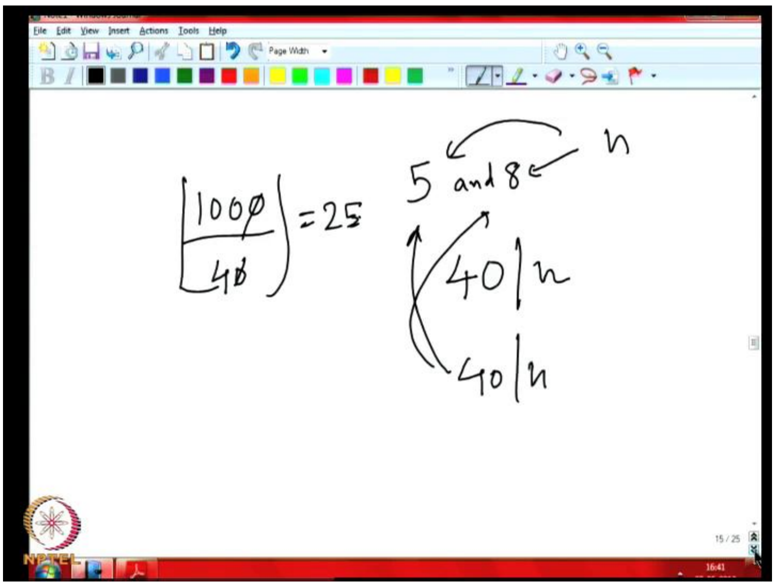

So, similarly, we can find the numbers, which are divisible by 5 and 8 those essentially, it is very easy to see that, if a number is divisible by both 5 and 8 then 40 has to divide it and the other way, if 40 divides that number then both 5 and 8 divides it. So, we can find the number of numbers divisible by both 5 and 8 by dividing by 40 taking the floor that is 25.

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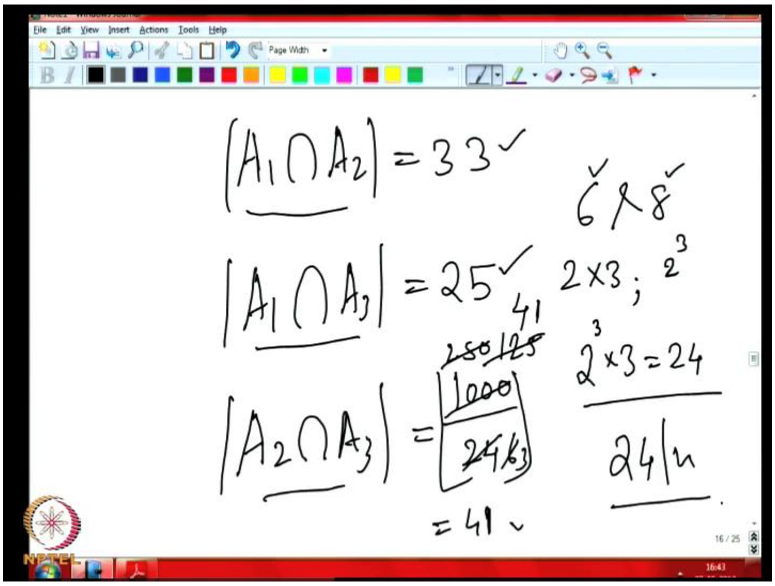

Now, if A number has to be divided by both. So, what we have calculated now is A 1 intersection A 2 is equal to 33 and A 1 intersection A 3 is equal to 25 and then now finally, we are interested in A 2 intersection A 3 also, how much is that, this is the numbers, which are divisible by both 6 and 8. So, the both 6 and 8 means, here we have 2 3 here we have 2 cube. So, its its little thought will reveal that what we are interested in is this number 24.

So, for instance if A number is divisible by 6 and 8, it is definitely divisible by 24 why because because it is divisible by 3 6, there is a 3 in it, because it is A it is divisible by there is A 8 in it. So, 8 into 3 24 in the on the other hand, if A number is divisible by 24 then of course 6 also divides here, because 24 is divisible by 6 8 also divides it. So, you can count by asking how many numbers are divisible by 24, so this is this right. So, how many numbers are divisible by 24 256, so 25 by 3. So, that is 441, so 125 divided by 3 is 41. So, we get 41 here right.

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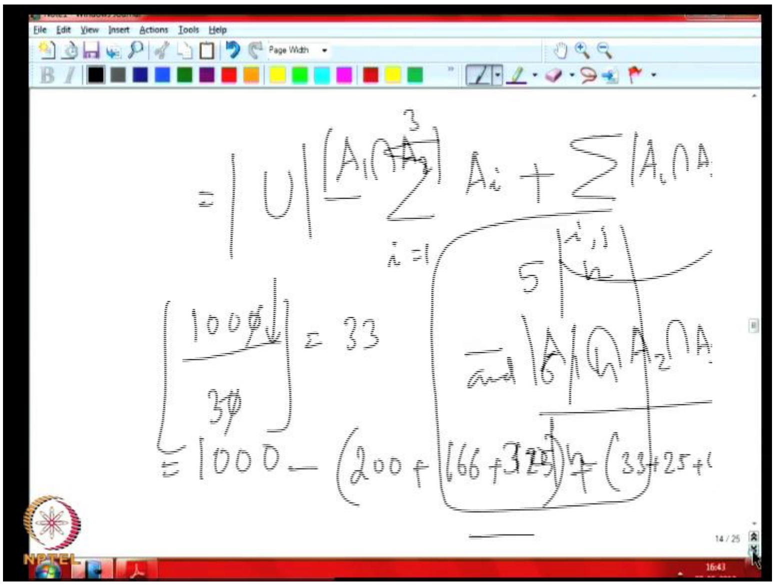

So, we get all these numbers, which are see the in cardinalities of the intersection of any pair, now we substitute in it 33 25 and 41 in the formula in 33 25 and 41 in the formula. And the final term, we have to minus this term.

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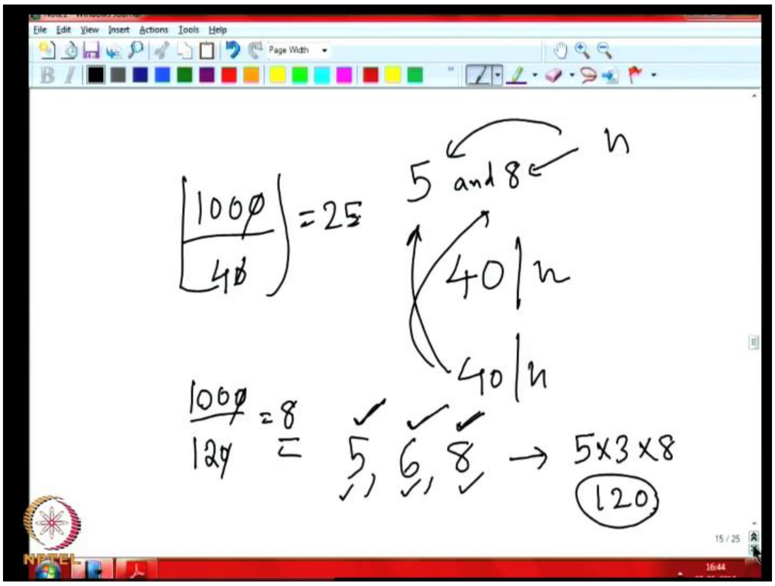

That means, the numbers, which are divisible by all the 3 of them right, it should be divisible by 5, it should be divisible by 6, it should be divisible by 8 of course. So, that is equivalent to the number, which are divisible by 5 into 3 into 8 that is 120. Why is it so because if A number is divisible by 5 6 and 8 then definitely, it is divisible by 120, because it is divisible by 5, there is three in it, there is also 8 in it, so 8 into 3 into 5. So, 120 on on the other hand, if A number is divisible by 120, it is definitely divisible by 5 6 as well as 8, therefore we can just find out the numbers by dividing dividing by 120 and then we get it, this is how much, so 8. So, we can minus it of there the final term.

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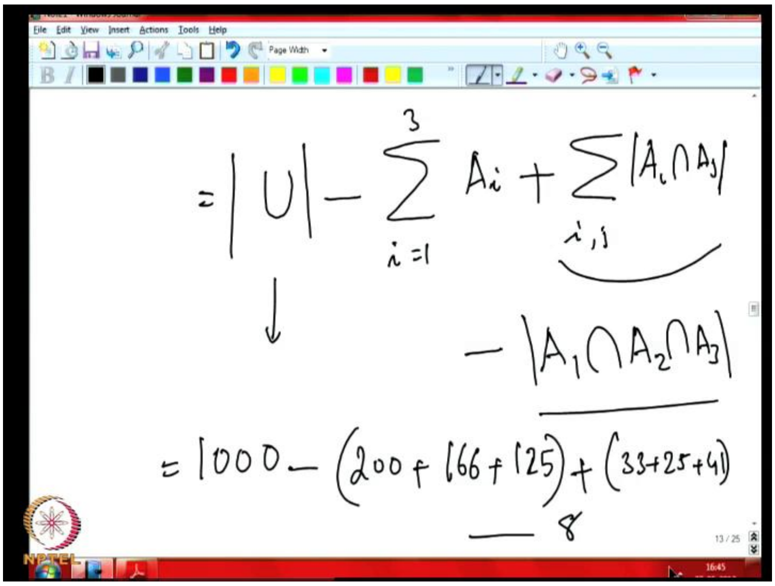

We can minus it of 8. So, the. So, adding up these things, we will get the answer. So, this kind of question, we can you know inclusion exclusion, typically we can apply this inclusion exclusion consequence.

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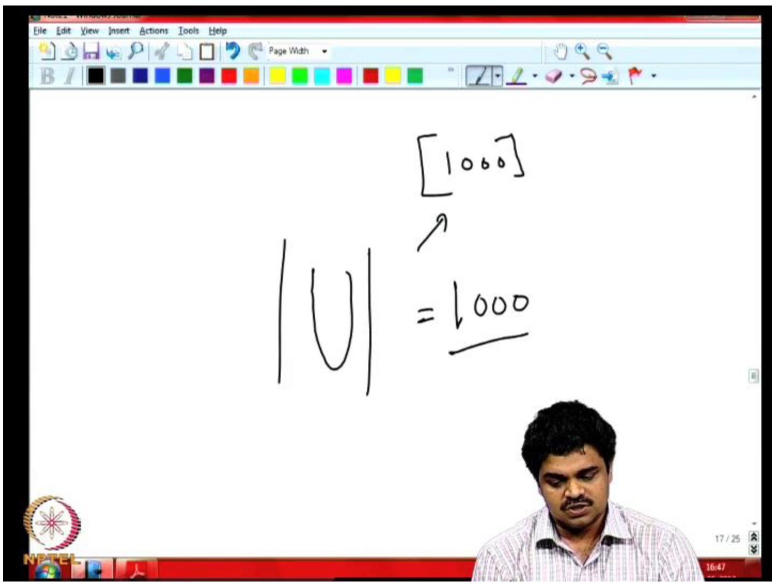

I will just summarize it once again. So, we were interested in the numbers in 1 to 1000, which are not divisible by 5, not divisible by 6 and not divisible by 8. So, how model the problem that was the most important thing, we model the problem such that, we can use inclusion exclusion right idea principle. So, first we noted that so this U can be considered this as this 1000 and it is there are 1000 members in the universe, now in this universe there are it is easy to find how many members are divisible by 5. How many numbers are divisible by 6 or how many divisible numbers are divisible by 8 or any of the combinations. That means, how many numbers are divisible by 6 and 8, this kind of question are easier to answer, because it is just a division by asserting number and taking the floor.

So, now if we can easily find the cardinalities of the intersections then we know we can apply the inclusion exclusion principles. Because, inclusion exclusion principle says if you want to find the members, which does not satisfy any other properties then you can use it, I means it is first cardinality of universe minus of the cardinality of the members satisfying individual properties. First property, second property the then 2 properties together with alternate sign and 3 properties together and so on. So, that is what we did here.

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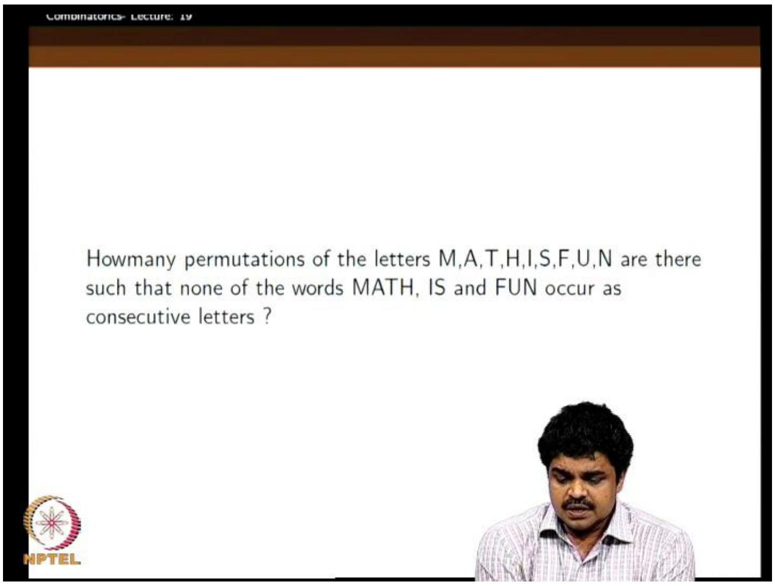

So, now we will see the another example, here is a question about permutations, how many permutation of the letters math is fun are there, such that none of the words math is and fun occurs as consecutive letters math is and fun occur as consecutive letters.

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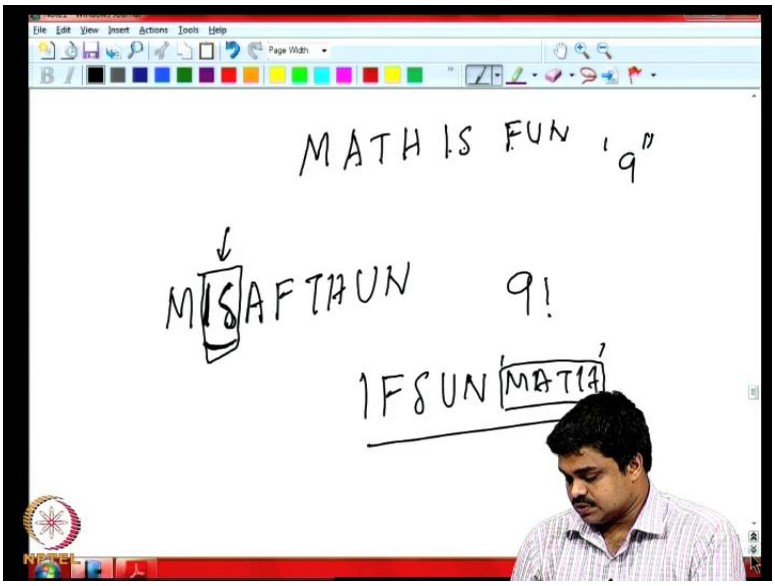

So, if you ask this is the question math is fun. So, we have 1 2 3 4 5 6 7 8 9 letters here and these are all distinct letters right is not, it right now nothing is repeated here. M M is not there A is only once, T is only once, H is only once, I is only once, S is only once, F is only once, U is only once, and the N is only once. If we were just asking how many sentence can be formed using this things, I mean we will how many permutation of this letters can be formed, it would have been easy nine factorial is the answer.

But, the question here is a little twister, it is saying that you can, I am asking for the permutation of this letters, such that the word math does not come together that means, for instance you can make you can make you cannot make a permutation ISFUN MATH is a permutation of math is fun but we are not allowed to use this thing. Why, because this word math is coming together here or nor I can I cannot form the permutation of this is also A permutation of math is fun. This is not allowed, because is is coming as a word here.

So, you can you the only, I am only interested in those permutations where is will not come together, I mean like you cannot read is or math or fun in consecutive positions right. How many ways, you can do this as what you are asking. So, now again we want to model this as an it is a problem where, we can apply the inclusion exclusion principles.

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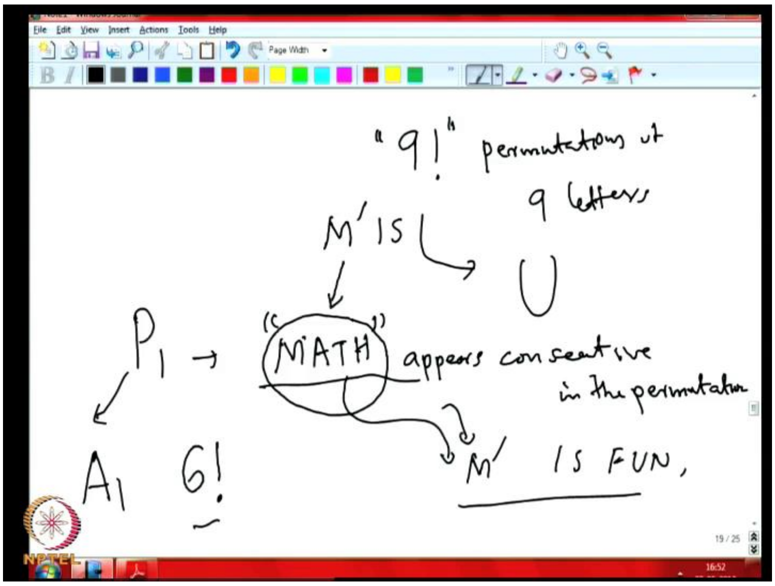

Let us say this 9 factorial permutations form our universe 9 factorial permutations of the 9 letters 9 letters in math is fun from the universe, this are universe. Now you see the properties, we are interested in I mean what, we want as those permutations, which do not have those properties. First property P 1 would be that the word math is 4 letter word math come consecutive appears consecutive consecutive in the permutation unfortunately sorry, we can we can easily find this thing.

This is easy to find how many are there, let us say A 1 is the set of permutations, this is the subset of the universe such that math appears consecutive. Because, what do, we do we have seen this kind of problems before what we do is we think of this as 1 letter say M dash. And then the remaining letters, how many are there out of 9 letters, we compress 4 into 1, so that means, we have 6 letters.

So, we have 6 factorial ways of doing this, because what we do is where ever we insert. So, where ever, we see M dash, we insert this entire math that is what I will do right, what we do is. So, M dash is fun. So, total 6 letters, we take any permutation of this thing will insert that. So, you now we can insert M dash can be substituted by math and the other way suppose, if we get A permutation where, math is coming consecutive, I can substitute by M dash, there we will get A corresponding permutation here. So, there are 6 factorial permutation, which where math appear consecutive right.

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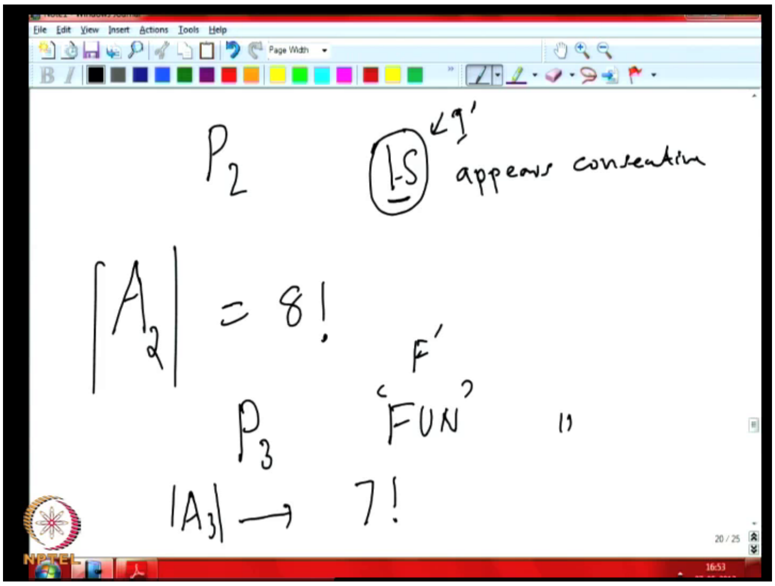

Similarly, the second property, we do not like is that is appears consecutive in the permutation. Lets say the permutation that means, the subset of that 9 factorial permutation, which have this undesirables property P 2 is A 2. The cardinality of A 2 is also easily found, because there are 2 letters, which should not comes from is should should come consecutive appear consecutive, because if it satisfy property, it appears consecutive, we can proceed into 1.

So, let us say this is I dash right, then it is out of 9 letters 2 the 2 of the letters is compressed into 1 tied them together, I safe right then we see that only 8 letters are there right. So, there are 8 factorial possibility is do that. Similarly, property 3 is that fun appears consecutive and the corresponding set A 3 that means, those permutations, which has the word fun consecutive. How many are there, because the fun can be now compressed in to 1 say F dash, now out of 9 3 has become 1, now 7 letters. So, 7 factorial.

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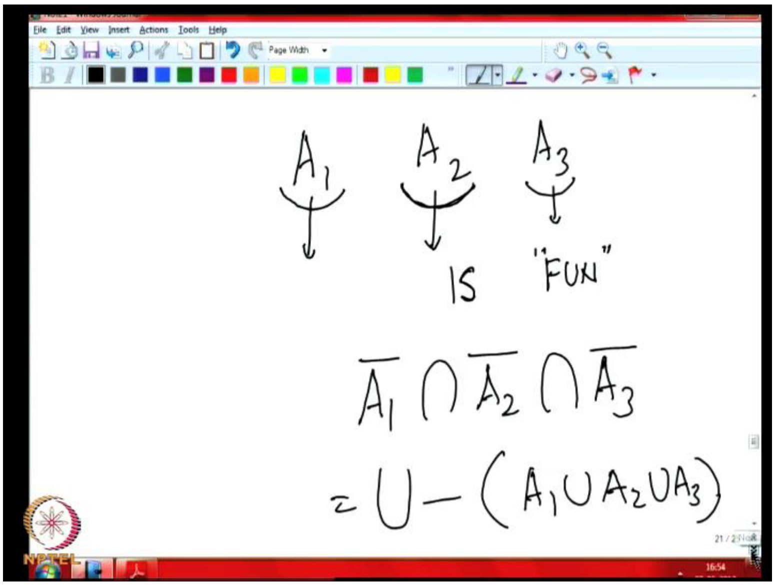

So, now you see that our A 1 A 2 A 3 corresponds to the permutation, which half the property, which do not desire. So, A 1 has the property that math comes consecutive A positions in consecutive positions, A 2 has is A set of permutations where is appears in consecutive positions and A 3 is the set of permutations where, fun appears in consecutive position. Now we know that the question ask for those permutations, which does not have any of these properties neither math should come in consecutive positions is should come in consecutive positions nor FUN should come in consecutive positions.

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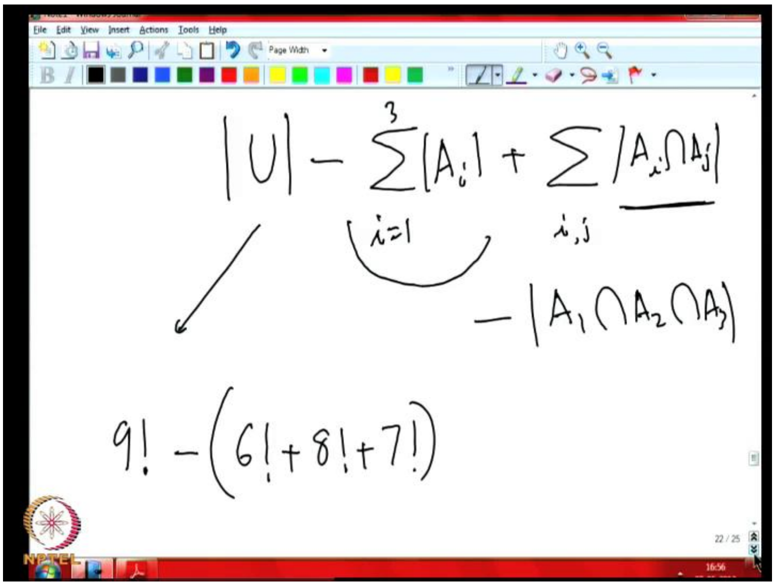

So, if that is what we want what we are asking for is this A 1 complement intersection A 2 complement intersection A 3 complement, this is what we are looking, which is essentially U minus A 1 union A 2 union A 3, this is what, we are looking for. If that is what we are looking for if that is what we are looking for then we can apply the inclusion exclusion principle by this formula by using this formula for every pair i j, we have to find the cardinalities of A i intersection A j and then minus 3 cardinality. Now, we have already seen that this cardinalities, we know if this, we already know this is what 9 factorial and here, we know that this is 6 factorial plus 8 factorial plus 7 factorial. We have calculated it here, so 8 factorial 7 factorial and 6 factorial. Now, the next question is how will you find this kind of cardinalities.

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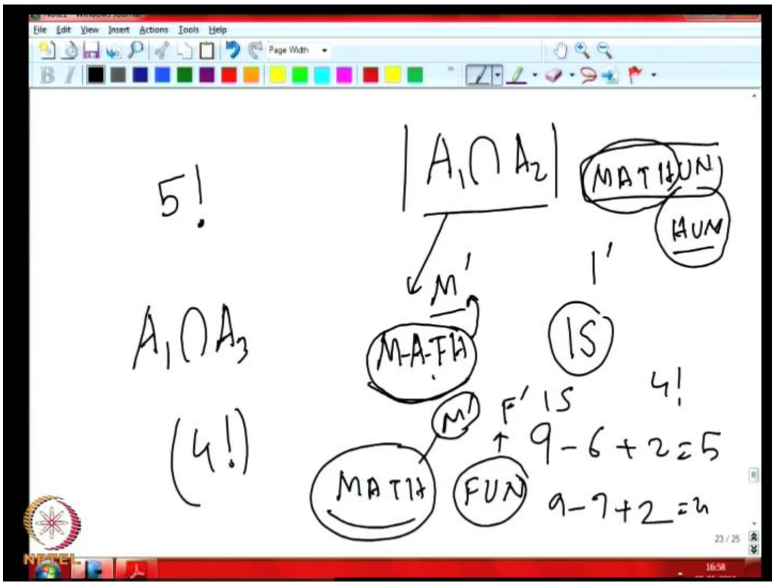

So, what is this this says say what about A 1 intersection A 2 cardinality of A 1 intersection A 2, A 1 says math appears consecutively, A 2 says is appears consecutively what we do is we substitute, A math with say M dash 1 letter. We just tie them together as 1 word 1, we will represent it is just 1 symbol and is with 1 symbol I dash say. Now, we have instead of 9 letters, we have right and the 3 plus 2 5 letters right, here we minus 6 from this thing plus 2, because here, we substituted M dash n i dash n total 5 letters we have. So, this 5 letters can be permitted in 5 factorial ways, now is it is 5 factorial then whenever, we see M dash, we just re substitute math there and whenever you see I dash, we will re substitute is there right. So, we will get those permutation where, both math and is appears in consecutive position, that is what this is A 1 intersection A 2.

Similarly, to find A 1 intersection A 3 is easy, because math has to be consecutive position and fun has to be in consecutive position, now 4 plus 3 7. So, 9 minus 7 plus 2 how much is that that is 4, 4 factorial ways, we can do this thing, because we what we do is we just replace this 4 letters with M dash and then this fun with say F dash. Now, M dash F dash and I S, we have 4 factorial ways of permitting them. Now, wherever we see M dash, we can replace it with math and wherever, we see F dash, we can replace it with fun.

So, that is not that here, it what was I mean this this thing, we should not that math and fun, they have different letters in it M A T H for instance, if we had math and H U N, we will have more complication, because you know it can come in 2 different ways either. Now of course there also, we have to be careful then this argument will not work of case [of case/of course,,] then also will say that math hun.

So, after M A T H H U N has to come therefore, M A T H H U N has to be a continues word like that right about you can explore the other possibly of case [of case/of course,,] instead of M A T H is F U N. Some other 4 letter words or that some few words, you can select and suppose, if you want to avoid them what will what do you have done. Similarly, this argument this method is saying when I want to consider see A 1, we have consider A 1 intersection A 2 A 1 intersection A 3.

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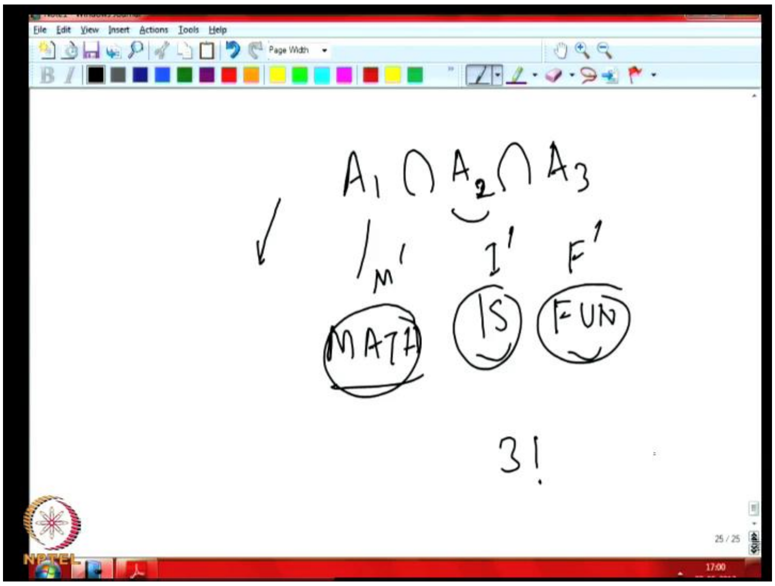

And finally, A 2 intersection A 3, this also we can see, because is should come consequently and fun should come consequently the same technique. So, what we do is we some substitute it with say I dash this with F dash. So, that is 9 minus 5 plus 2 total 6 6 factorial ways, we can and finally so that we can substitute in the formula here, so each of them and then we sum it up. And here final term is that the cardinality of A 1 intersection A 2 intersection A then how how will I do this A 2 to find the cardinality of A 2 intersection A 3 intersection A 6, we sorry, A 1 intersection A 2 intersection A 3 that means, math should come consecutively. So, this we can take us M dash is should come consecutively. So, I dash and then fun should come consecutively. So, there are only 3 letters now right, because every letters is block together. So, 3 factorial is of permuting this so we can substitute that math is fun wherever we see that. So, all of them will be consecutive in this things right.

Now, we can put it there formula gives us the cardinality of U minus A 1 union A 2 union and A 3 therefore, that will that will give us the final answer namely the number of permutation, which has the property that math does not come in consecutive positions is does not come in consecutive positions and fun does not come in consecutive positions right. This is what it will give and a routine technique, if you say.

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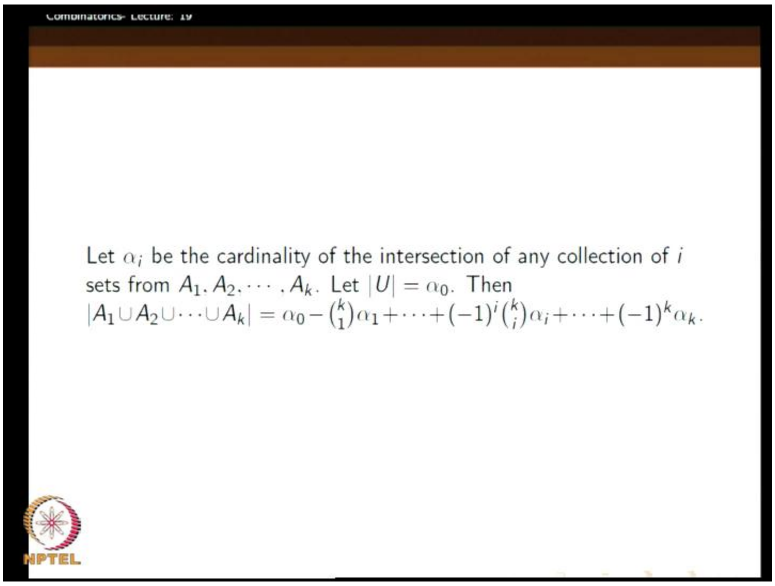

Now I consider 1 2 example we considered. So, here what we can do is we can simplify the formula, this formula seems to be very tedious to right. So, we can simplify the formula in some cases. So, it is not very surprising that, we can simplify the formula in some Cases.

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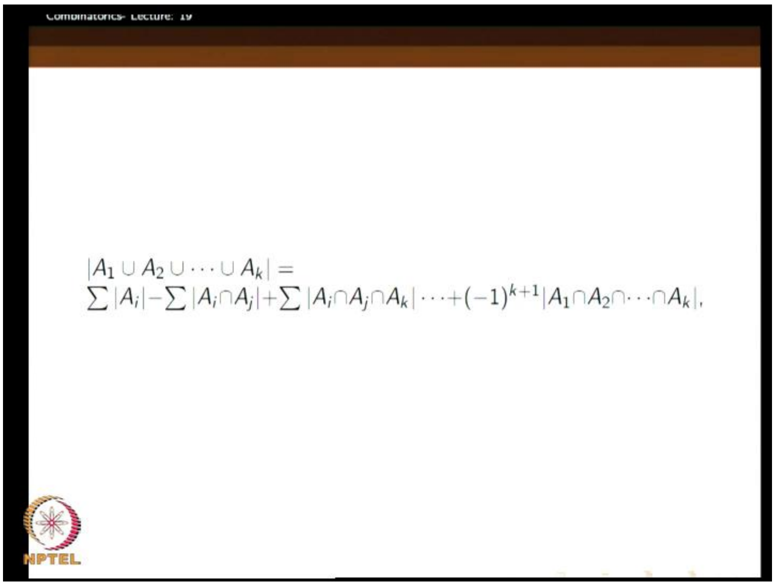

For instance, we can go back to the formula sorry, I i just go to mention 1 more thing here. So, suppose you wanted A 1 union A 2 union A k only.

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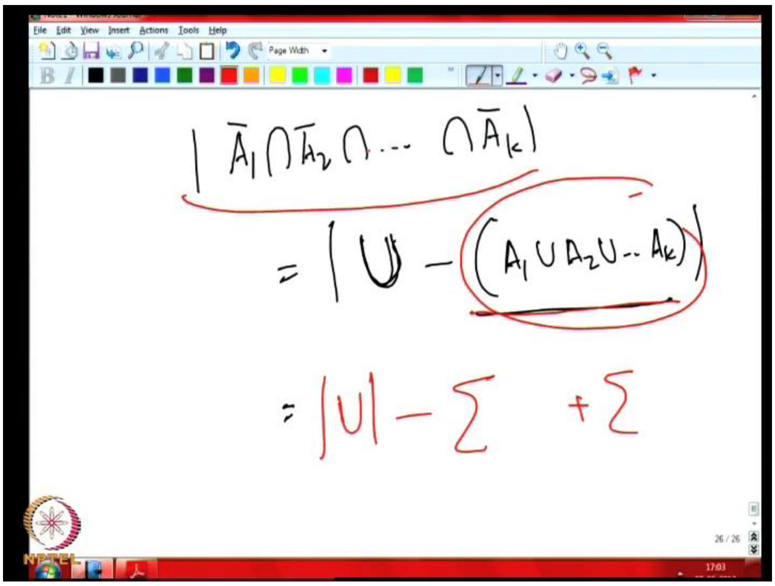

For instance up to know, we are saying that I I am interested in A 1 bar intersection A 2 bar intersection A k bar, which is essentially the cardinality of U minus union of U minus say A 1 union A 2 union A k right. Now of course suppose, if I am interested in the cardinality here this 1. So, it is not not it at all difficult, because you no this is U minus something plus something minus something like that. So, at this you see U minus this is what I want right. This what is this? If I want this what I can do is I can take the remaining right, because U minus this this actually this. So, this is the complement of this right.

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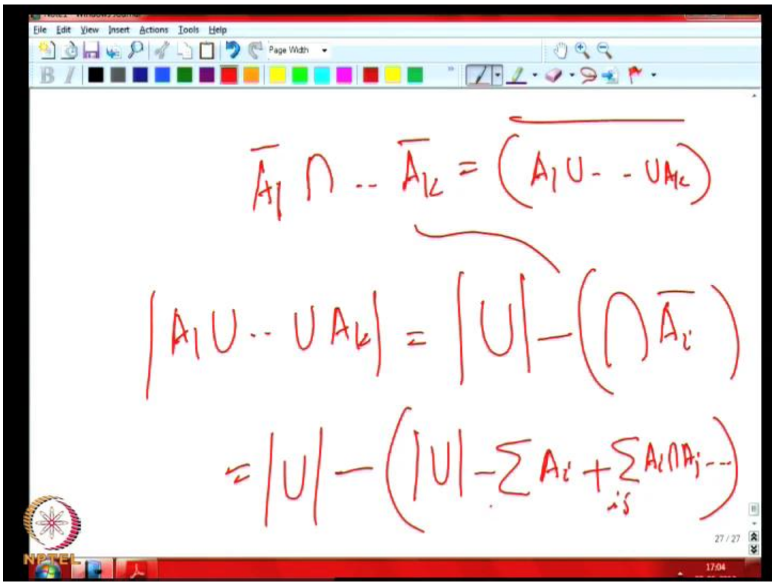

So, this we know A 1 bar intersection A k bar by de morgan rules is essentially A 1 union A k complement. So, if you want to find A 1 union A k cardinality. So, we are just finding U minus this cardinality intersection of A i complement, which will be by up substituting for this in the formula, this will be U minus sigma A i plus sigma i j, A i j's sorry, A i intersection and so on. So, this U and U cancel, but each of the remaining terms change in the sign.

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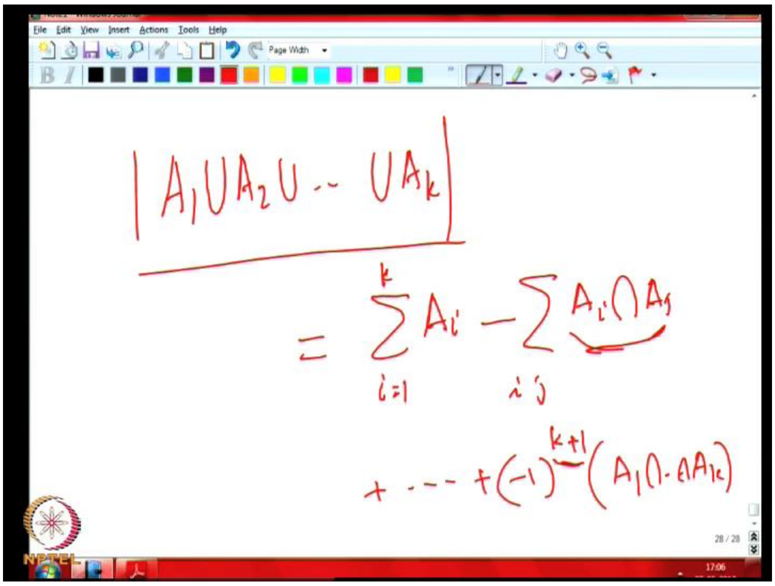

So, we can write the formula like this A 1 union A 2 union A k cardinality is is sigma A i. So, this is i equal to 1 to k minus sigma i j, the signs are change, now that is what and if like that finally, we get minus 1 raise to k plus 1, because this is all the terms are taken its becomes k plus 1. So, A 1 intersection A k, because you know this k plus 1 only means that, we have change the sign to the next.

So, earlier it was minus then it will become plus 1 earlier it was 1 that it will become plus 1 it is every where, the sign has change earlier it was a negative sign. Here that has become plus this was A positive sign here, it is become minus and so on. All for instance when ever, we were considering odd number of things together that was given A negative sign in the earlier formula and when ever, we where considering even number of sets and taking the intersection, we where giving positive, now we change for even we give minus and odd give positive right. It is not nothing, it is just that, we will notice that what ever earlier, we where calculating was the complement what we want here, this 1 and just minusing of from U therefore, the signs change and U disappeared.

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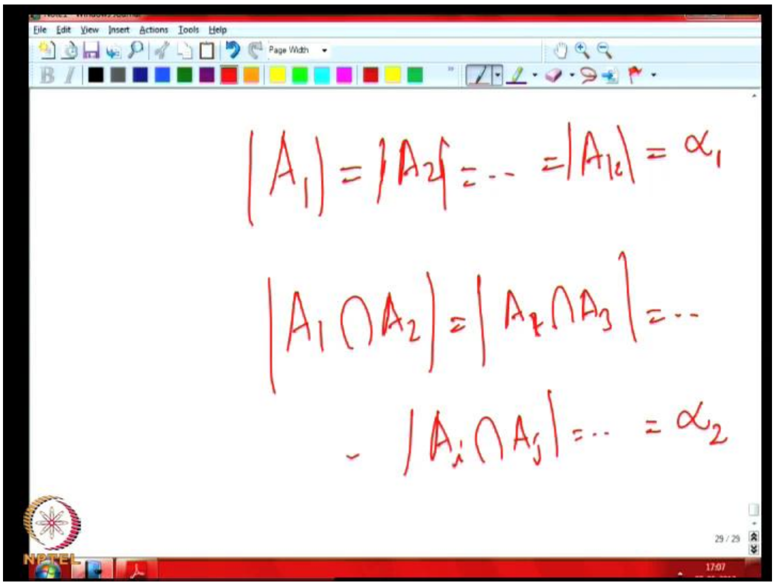

So, and coming back to the discussion where, we just stopped suppose, we know that A 1 cardinality is equal to A 2 cardinality is equal to A k cardinality that means, all set are of equal cardinality lets say this is alpha 1, just give A. So, it is A alpha 1 uniform family, we where talking about all cardinality are same A k, A 1 to A k. Now A 1 intersection A 2, this is equal to say A 2 intersection A 1 intersection A 2 A 3 and so on.

All A i A intersection A j, if you take any pair, if you take the intersection and this happens to be same alpha 2 means some problems, because of the structure. It may so happen that which ever pair of sets, you take there intersection is going to have the same cardinality, it may not depend on 2 sets. it will only Depend on the factor there are 2 sets whose intersection, you are considering.

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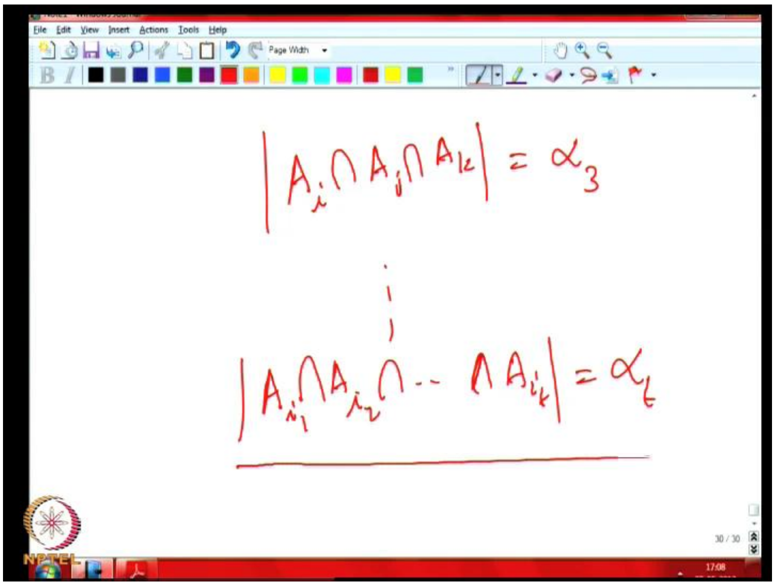

Similarly, you say for any i A i A j A k, if I take and suppose, they cardinality is alpha 3 is not that always it like that but suppose it is like that. And in general, suppose when I consider, I sets A 1 or t sets intersection A i 2 intersection A i t, this is alpha t then we can substitute the earlier formula by this formula.

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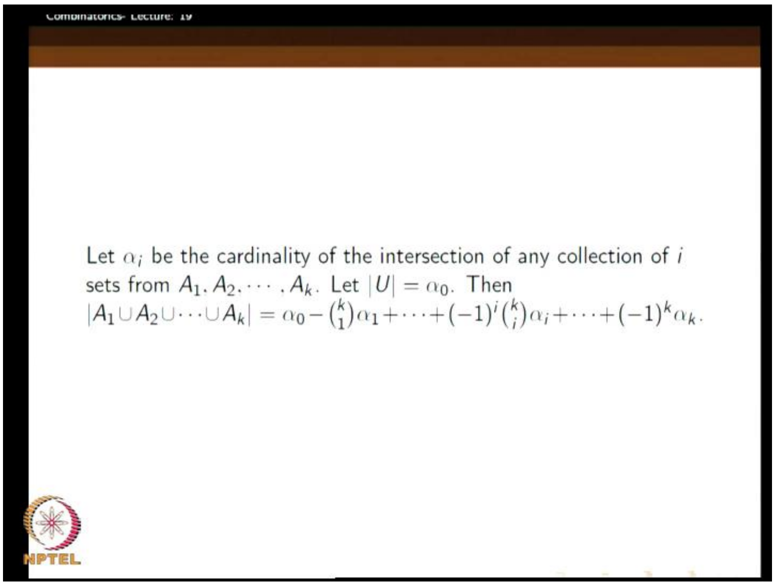

So, we can substitute the earlier formula by this formula alpha 0 alpha 0 being the cardinality of U minus k chose 1 alpha 1 plus k chose 2 alpha 2 plus k chose 3 alpha 3 and so not plus. This are minus and plus alternating and for the typical 8th term will be minus 1 raise to i, k chose i alpha i plus final term will be minus 1 raise to k alpha k. Why if it so because you know as we have seen when, we are first term alpha 1 it is A size of the single term sets and all single term sets are of the same size namely alpha 1. So, there k chose 1 of them, here we where.

So, we just add them together, we just k chose 1 times alpha 1, similarly, the second term any pair of sets, if you take irrespective of, which sets, if you if you take 2 sets and take the intersection is always the cardinality alpha 2. Then we just have to bother how many pairs, we can take that is k chose 2. Therefore, k chose 2 into alpha 2 is the total sum there would, we have got right plus and next is negative sign minus term and any triple, if you take any 3 sets, if you take from the k given sets.

So, they there intersection is always going to have alpha 3 elements in it therefore, when you sum it up what we can get is k chose 3, because there k chose 3 possible ways of selecting 3 sets k chose 3 is the same to alpha 3. So, this goes on right this formula would. So, this is definitely trail but just that the the, we will have cases where, this this can be easily applied.

So, we do not have to always remember the other formula. Many cases it may so happen that this property that when you talk about the cardinality of the intersection of t sets that only depend on t. It does not depend on, which t sets the sets how many sets intersection here intersection of how many sets we are taking matter. So, we can apply this formula more conveniently, this formula means, we can the simplified version. So, we can remember it easier. So, we will continue in the next class.