Combinatorics
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Lecture - 18
Hall's Theorem for regular bipartite graphs; Inclusion-exclusion principle – Part (1)

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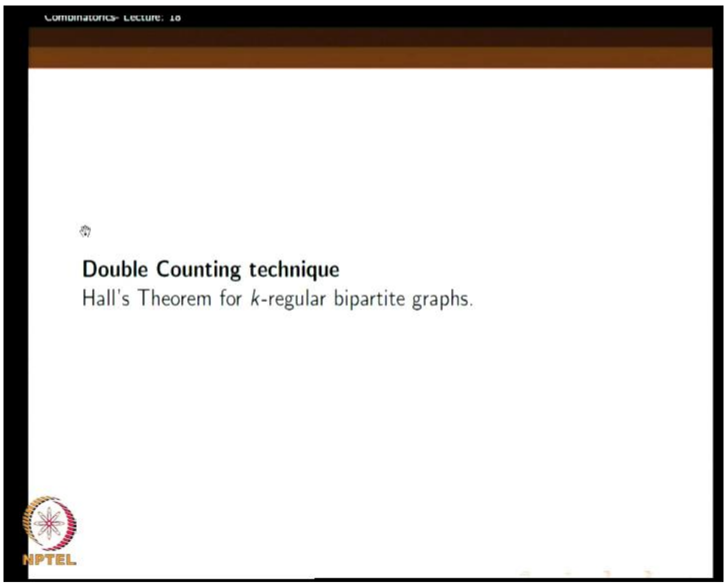

Welcome to the 18 lecture of combinatorics. In the last class, we were considering Hall's theorem. So, we were doing it as a part of considering third one example. So, double counting technique and counting in two different ways in comparing and then inferring something from that, right.

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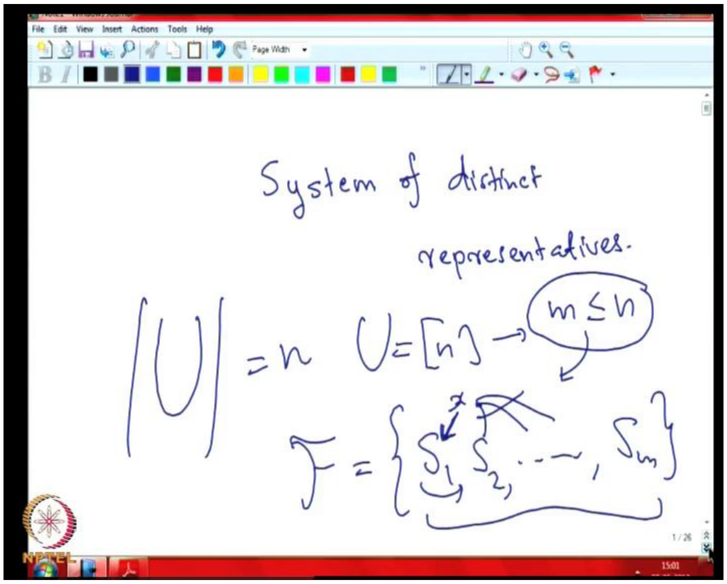

Hall's theorem is in itself an important theorem. It is about the system of distinct representatives. In this problem, we have a universe. So, let us say it is cardinality and we can take it as n for instance for example u is equal to n, right. Now, we have a family f of subsets of U. Let us say, it is S1, S2, Sm. So, this m can be less than equal to n or greater to, n, but our interest is that we should get a representative. So, we have to select a few elements actually, m elements. So, we want to select m elements from u. So, that means m has to be less than and equal to n, otherwise how will we select it, right, such that one of them can be a representative as to S1 and another one can be a representative of S2, and another one can be, finally another one of them can be a representative of Sm, but no person can become a representative of more than one sets, a sub in this collection, right.

So, that it is of course it is ok that if a particular x is a representative of x n x s 1, but member of S2, S3, everything, it is not a problem, but we are allowing one person to represent only one set and that person has to come from that set, right. That element has to come from that set. So, this is now Hall's theorem gives a necessary and sufficient condition for the existence of such a distinct representative as we have elaborately discussed in the last class.

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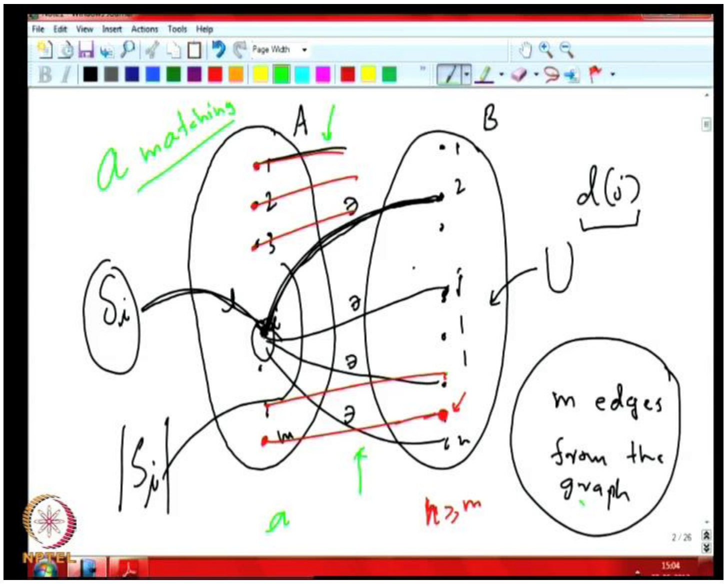

This problem can be recast as a bipartite graph problem. So, for instance, here is a bipartite graph problem. This is the A side, this is the B side and on the A side, the vertices, there are m vertices, right. So, 1, 2, 3 up to m and each vertex correspond to say, the ith vertex corresponds to the set S i. That means, it is the S i, the set S i is like represented as this vertex here, right and on the B side, we have the elements of the universe. That means, 1, 2 up to n, u, the elements of the universe, right. Now, the edges as put like this, the S i if it contains the few elements here, so now they are those members of S i from here will be connected to S S i, right.

So, this is membership actually, right. So, any edge from i to j here means that j-th element of the universe belong to the i-th set, right. Then of course the degree of the, a vertex on this side is the cardinality of the set S i, right and then the degree of the vertex j here is essentially d of j. That means in how many sets it is part of. Now, that is the way it corresponds to here. Now, the question of selecting distinct representatives corresponds to selecting m edges from this graph, such that these m edges have the n points of this m.

Edges are all distinct. That means no two edges in this m edges share an n point, right. So, m edges are there. All the edges are there going from this side to this side, right. So, that means one edge should be like this, one edge should be like this, one edge should be like this, one like this. So, every vertex here should be part and point of the one of the edges because I am taking m edges. So, there are only m things here on their side that is always go from A to the B side here. There are n elements and n, it is greater than and equal to m. So, therefore they can be elements, say which are not touched by edges, but then if a particular vertex here is touched by only one edge, it is not possible that we have something here and then something else is coming here. That is what will never happen. So, this kind of collection of edges is called a matching. It is called a matching. The graph theory, it is called matching, right and in this specific case, we say that we seek a matching of the A side. That means, we want every vertex on the A side to be matched, right.

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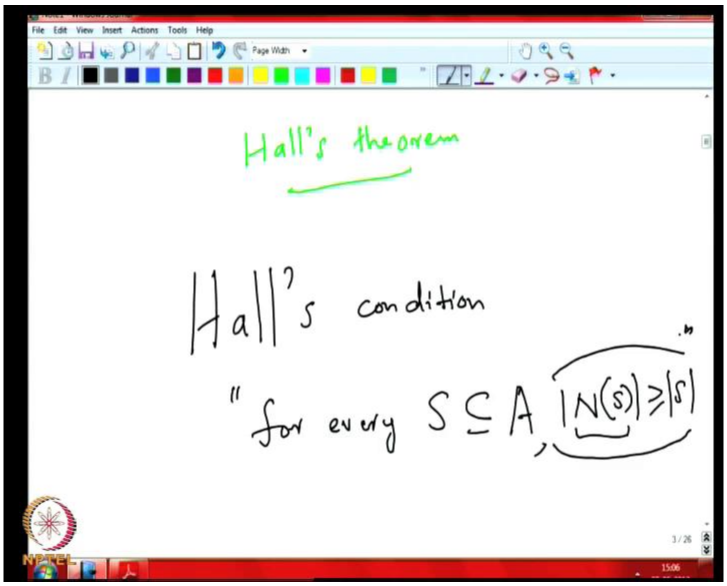

Then, matching Hall's theorem is about the existence of, such a matching is about the existence of such a matching and the important thing is something called Hall's condition. So, this theorem gives a condition and if this condition is met, the Hall's theorem says the system of distinct representative access or the matching of the access. What is the condition? The condition says for every S subset of A. That means, if you consider any subset from the subset of what is going to be of a side and then if you count the neighborhood of S, you see the neighborhood be all on the B side. This has to be at least as much as the cardinal to S as big as the cardinality of S n of the cardinality should be and greater than equal to S cardinality.

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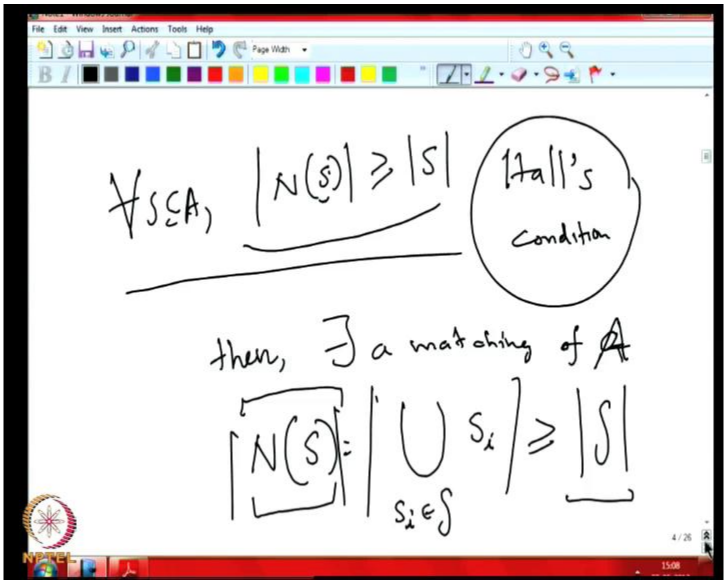

N of S cardinality should be greater than equal to cardinality of S for all S subset of A. This is the Hall's condition, say if Hall's condition is met, and then if Hall's condition is met, then there exit a matching of A. So, the language of the distinct representative is this. So, what do we mean by the cardinality? What is N of S? N of S is essentially the union of all the members of filled S, sorry for instance, this S we should here it is A will be some set, some S i in S, right and we take the union of S i's, right. This will play because this is essentially the vertices on the A side corresponds to some subset of the universe and then this S is a subset of S on the A side. That means, the collection of vertices from the A side, they are certain subsets from the family and then you take union of all those subsets.

They consist of some elements, right. So, those elements of the universe are to down from the other side, B side. They constitute n of s, right. so we say that N of S has to be greater than and equal to S, which means that if you take this union, this union has to be greater than equal to how many s i's are there on this. That is what we are saying, how many subsets we are saying? It is all the family should be such that we take A 1 subset and then each subset should contain at least one element. If two subsets are there, any two subsets are there. If you take the union of that, there should be at least two elements in the union and then if you take any three subsets, then there should be at least three elements in the union and so on, right.

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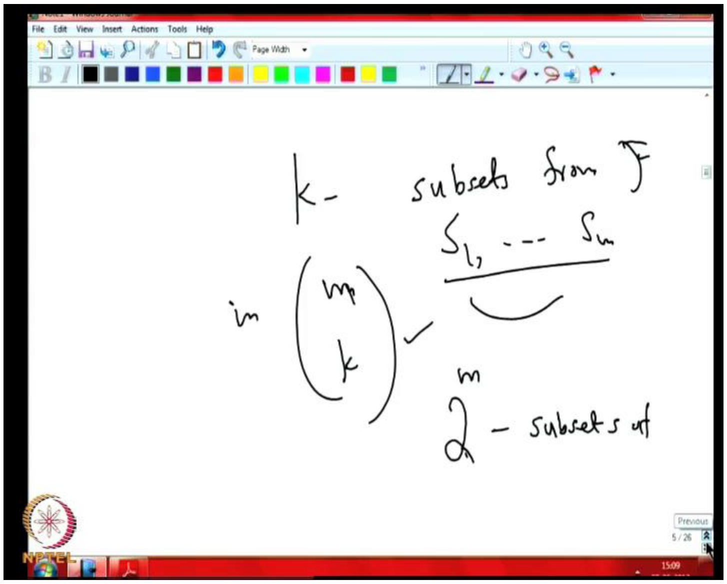

So, in other words, you can select k subsets from the subsets, k subset, not k subsets. K subsets from F in definitely m choose k ways, right because they are m total m subsets s 1 s m. So, from these subsets, you can tell it is a two way and if you take any k subset of this collection, if you take the union and the number of elements in the union should be at least k is what we are saying and there should not be any subset with less than m, less than k members in the union. When you select k things, not that they are of that Hall's conditions says for all successors, that means any possible subsets you should consider and there are 2 raise to m possible subsets of that subsets of S, right. So, for every subset, it should work including F i. In F i, it is trivially true because n of F i is just empty. So, 0 is greater than 0. So, therefore it is correct, right.

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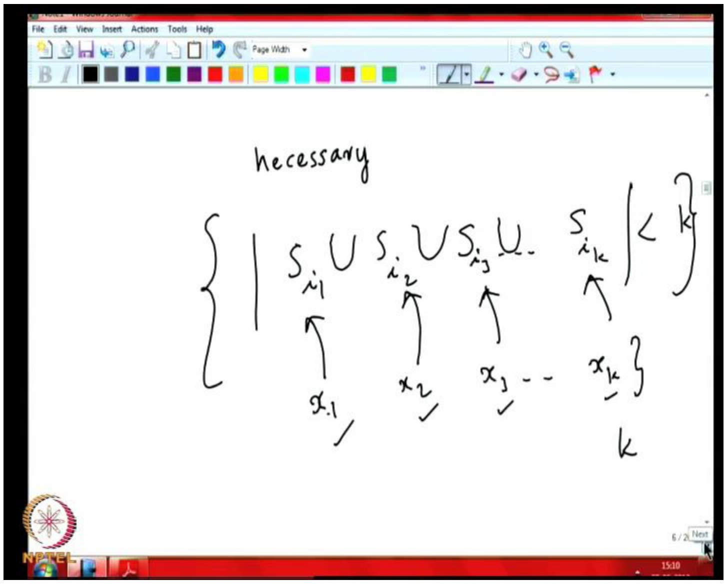

So, now this is a necessary condition, very simple. Why? Because if this is not met, there will be some k element, k subsets, k elements subset, right. S i 1, i 2, S i 3, S i k, such that if I take the union of them, the cardinality is strictly less than k, right. What does it mean? Then definitely you cannot find x 1, x 2, x 3, x k such that this belongs to this belongs to this, this belongs to this, this belongs to this and there are all different because then they will be actually x 1, x 2, x 3, x k. They themselves form k elements, they come from the unions, but by our assumptions, it is less than the union contains less than k things, right.

So, therefore, this condition that the union of si 1, si 2, si 3, si k for any selected i 1, i 2, i 3, i k should be greater than equal to k. It is definitely a necessary condition for distinct representatives to exist. So, the sufficiency is the non-trivial aspect. We won’t get into the proof of this. As I told, what I want to means this is available in any graph theory book or you can consider the graph theory person or learn it from the graph theory course.

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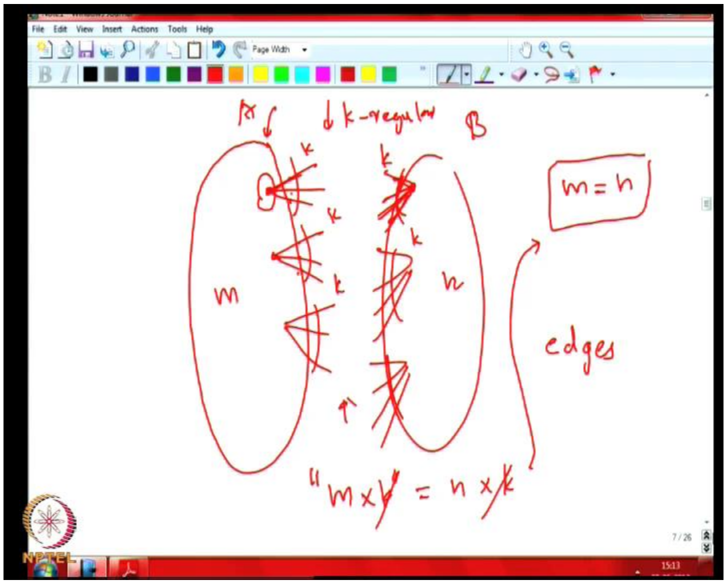

So, what I want to discuss here is a special case of it newly when the bi parted graph, we consider is k regular, right. So, k regular bi parted graph. So, this is A side, this is B side. So, you see k regular bi parted graph will be like this, right. Every vertex here, k edges will be going out from every vertex and k adjust will be coming out from every vertex, right. Now, we want to show that the Hall's condition is met by this thing.

So, first let us know that trivially the number of vertices here n and n, if there are n m. As we know, m vertices are here. These n vertices are here and if vertices are here, so we should have m equal to n. This is what we first claim. Why is it so? Because we do the double counting here. So, now we count the edges, right. We count the edges in two different ways. One from this side. That means, we count the edges like, ok here this is the first vertex, 1, 2, 3 and this is the second vertex from 4, 5, 6. Like that we count, right. So, since each vertex here has k edges incident on it, then I have definitely m vertices here. So, m into k edges are here counting from this side, that A side, right.

So, not that every edge we have counted. Why? Because each edge is incident on exactly one vertex incident. On the A side, it is an incident on one vertex, exactly one vertex because the other end points on the B side and the point is on the B side. Definitely the same number of edges we can count on the B side which vertex it is k edges. So, since there are n vertices on the

B side that is n into n into k n into k, so what we can say is because this is we are counting the same things, the number of edges in the graph that will be equal, right. So, k cancels and then we get m equal to n. That is what it is. So, the same kind of strategy we will use to illustrate, to prove that the Hall's theorem is valid for k regular edges.

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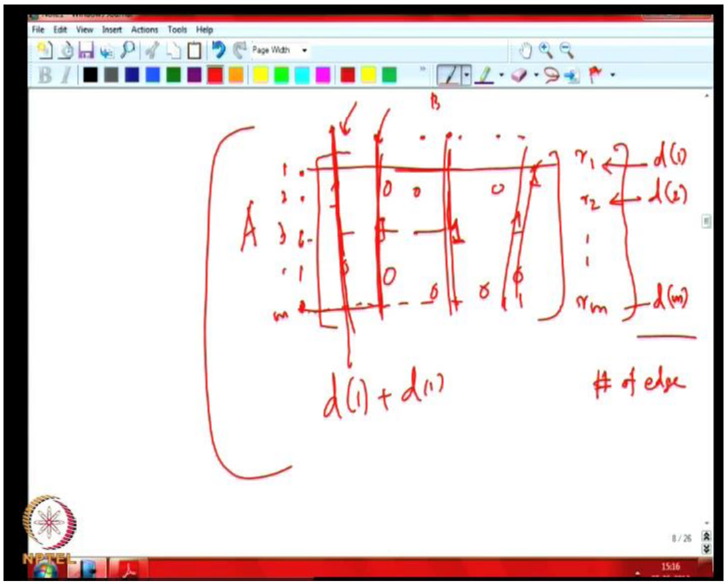

So, in say if you want to say, it in the matrix form which we were discussing before. So, here I am taking the A side, the vertices of the A side as the rows. So, this is 1, 2, 3 up to m, but the vertices of the B side are taken as columns, right. Now, the graph of some kind where the matrix representation for the graph will be like this. So, if i j-th entry will be 1, if there is an edge from i to j i on the A side, j on the B side, so of course edges are always going from A to B side.

So, we do not have to worry about like why there is no edge inside A, there is no edge inside B. So, therefore all the edges will be of this form that it is between i and j, where i is from the A and j is from B, ok. So, this is the situation we will get. Of course if you want to count the number of edges, we want to count the number of one's in the matrix, as I told one way to count the number of edges is by counting the number of one's in the rows, r 1, r 2, r m, right. So, this r 1 is essentially the degree of vertex. One on the A side. R 2 is essentially the degree of vertex 2 on the A side and r n is the degree of vertex n on the B side, right because see essentially given a vertex, if you scan through this thing, you see so many one's there. That means, it is connected, this vertex is connected to this, this, this and this, right. So, this will give you the total number of edges. This is the number of edge, number of edges in the graph. Similarly, the number of one's we could have counted by counting along the columns.

So, in the column if we count here, this is degree of vertex one on the B side. So, we can say, yeah the degree of vertex one on the B side and then because we have numbered the vertices on A side as 1,2, 3 up to n, the vertices of B side as 1, 2, 3 up to m. There they are using the same number, but we should understand that the columns count the degree of what is in the B side. If you count the number of one's in acceptance of eight columns, so you see the degree of the eight vertexes on the B side, right.

So, therefore some of those numbers we will get the total number of one's here. So, by using we can match them, we can equate them because they are equal, right. That is what is that the situation and then of course I just put a thing that previous using the notation of the previous matrix form. So, these are the things that are quite simple and just wanted to give a frame work in case some student is finding it a little difficult to think about this. How do I double count that matrix picture is very nice.

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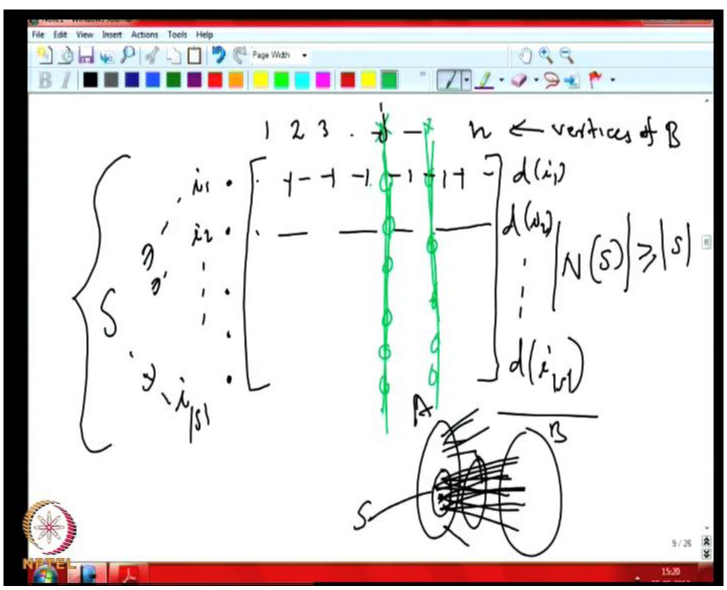

Now, again coming back to the Hall's condition, how do I say that the number of, for neighbors for any subset s on the A side, N of S has to be greater than and equal to cardinality of S for accurate (( )). So, from the previous matrix, we can collect those vertices i 1, i 2, i cardinality of S, right. These are the vertices in the S, right. So, this all belongs to S, this belongs to S, each of them, right. So, now if we scan this row, this again, the columns are again 1, 2, 3 up to n. The vertices of B, the vertices of these are vertices of B, these are just selected rows. That means the vertices of the A side, but those vertices which are in S, right.

Now, we create a sub matrix of that actually. So, we selected few rows. That is all. Now, we can scan again for the 1's in the first row. So, what will we say is this will be the degree of i 1, right. Similarly, here if I scan through the 1's and add counter number of 1's, I get degree of i 2, right and here I will get degree of i cardinality s because for each, the degree of each vertex in the set S, we will get, if I sum it up, I will get the number of edges in the graph which are incident on some vertex in S. In other words, those edges which are going out of the vertices of S, we will get, right. So, in the graph picture, we have some S here, those edges which like this, right. So, this is what we will get, right, the other edges and we are not counting it out. So, these are the edges, right. This being S, this being A, and this being B, right.

What are the neighbors of S here? So, you know certain columns. So, for instance, say the j-th column here, that will that correspond to the j-th vertex. How will I say that vertex is a neighbor or not. So, if it is all 0 here, that means j is not a neighbor of this. No vertex in S sense an edge to j. That is why we see 0's here, right. We can kind of imagine that we are discarding those rows. We can remove those, sorry discarding those columns, where all are 0's.

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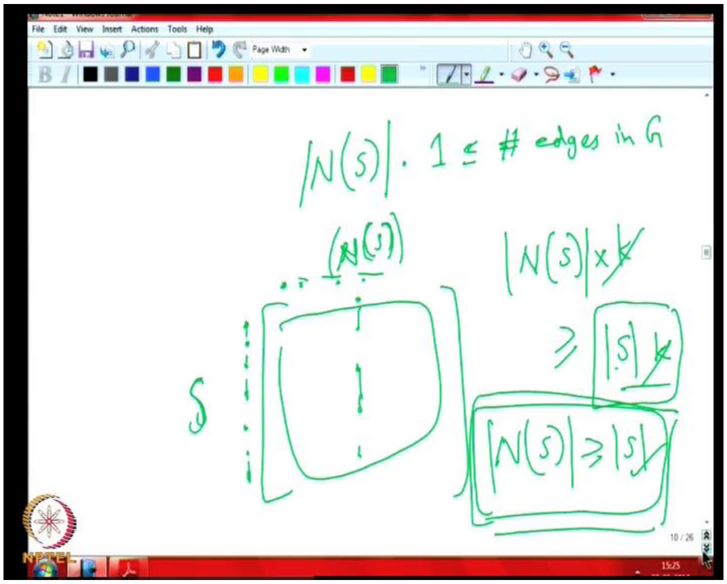

So, the remaining rows, remaining columns correspond to N of S and these columns are such that there is at least one edge in it. So, the number of edges as I say the number of, edges in number of edges in G, right. So, I am counting. So, what I did was I created a sub matrix here and then I removed some of these column which are not in N of S because they were all 0's in there columns. That means they are not in the remaining columns. All have at least one in it. Therefore, they are in N of S and then now the matrix is slightly different.

The matrix is S and the vertices of S is listed here and then the vertices of N of S is listed here now, right because those vertices of B which are not in N of S now remove it, right. So, here I know that the number of edges going out of S will constitute actually the number, the 1's in the matrix. Now, we know this is at least N of S into 1 because you know every column now has at least one N in it, right and we know this is less than equal to because the number of 1's we have already calculated S into K, right because that is we are summing up the degrees of the vertices and they all have of K degree vertices, K degree.

So, this is not an upper bond. So, what we want is a lower bond. So, what we say is at least have one such (( )) instance only get, but still we know that this cannot have more than K. So, while this is at least one of s, definitely how big can it go? This cannot go into k, right because each column can have at most K 1's in it. So, maximum n of k into this thing has to be greater than equal to S into K, right and in the worst K's, it can be N of S case into 1, but in the maximum K's, it can be N of S into K. Therefore, the number of 1's here cannot be more than N of S into K, but you know the number of 1's in this thing actually is S into K. So, it is N of S into k is greater than equal S into K. So, we can cancel it off and we get N of S is greater than equal to S as we want. This is the Hall's condition and it seems we are talking about any subset S.

So, when S is pi, it is trivial, otherwise we can make this matrix, this argument make work, right. So, this proves that the Hall's condition is trivially true. Now, trivially true we need to put some little effort to show that. So, for the K regular case, so for in what is the k regular case. It means that every subset has equal cardinality. The k uniform hyper graph, we consider as we use the terminology we introduced in the last class. K uniform hyper graph, the subsets, the subset f. The family of subset f is such that each subset in f has the same cardinality K uniform hyper graph, right and also each vertex in the universe has equal degree. That means, there are all in the each vertex is an exactly K sets. In this special case, this turns out to be K regular bi parted graph from the point of view of the sets system. It may look a little too much to ask for both this condition, namely your subsets are K uniform and also, your each vertex has degree equal to K, right. So, of course we are only asking for degree at most K, right.

So, the double counting is very clear. What we have done is we cooked up a matrix like this. So, S and N of S verses N of S being the set of, we are considering some arbitraries subset. Those rows are picked up and N of S is the neighborhood of S, right. Those columns are picked up and then we look at this matrix. In this matrix, the number of 1's is counted. When we add up the number of 1's row wise, that will give us sum of degrees of the vertices in S, that is S into K. This is the number of S into K, while the number of 1's can also be counted column wise, but in a column, maximum K 1's can be seen. So, this number of 1's in the matrix cannot be more than N of S into K. So, N of S into K is greater than equal to S into K. Therefore, N of S is greater. We cancel off K and we get N of S is greater than equal to S. This is what we mean, right.

So, these kinds of arguments are very common in graph theory combinatorics. So, this much is enough for the table counting technique. The last problem we discussed was a little, was interesting enough for spending sometime on it, right because this system of distinct representatives that is in itself an important topic. So, now though the most of the things we consider during this course are taken as examples should not like it is. Not many times these examples can be. They themselves can be important, right. They may be general enough problems that it may appear in some other applications which you consider in some other context, right. Therefore, one has to take note of these examples beyond like the technique that we are studying, right. It is not just the technique; also sometimes these problems themselves are important because they have a certain generality about them.

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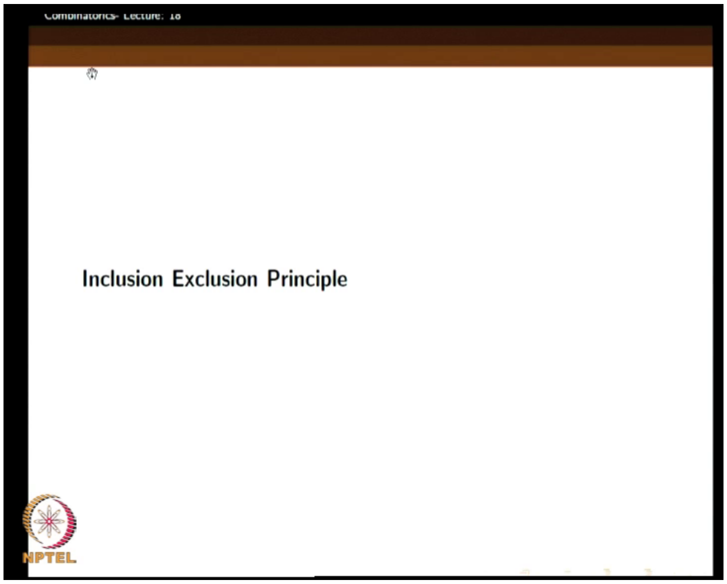

The next topic we want to consider is inclusion exclusion principle. This is another technique which is quite important. So, what is this inclusion exclusion principle?

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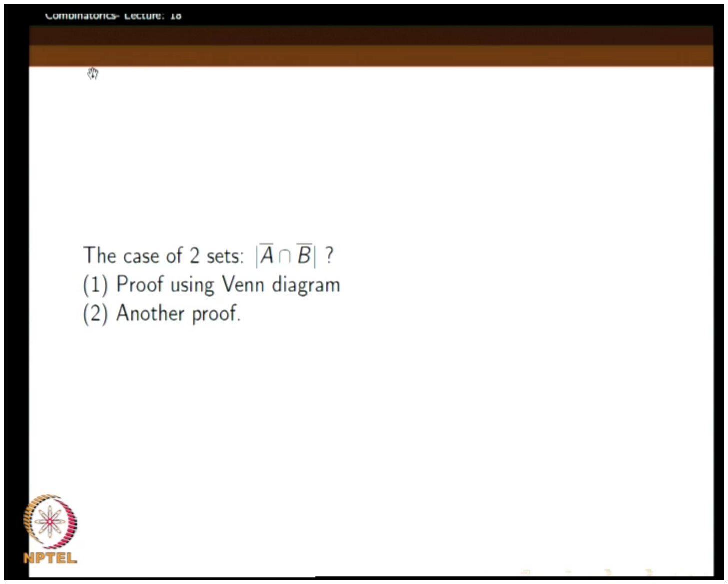

So, we consider this question and we have two sets A and B, some universe is there of course and then we have two subsets. So, A and B, we can assume that universe is, say n or 1 to n. As usual, it is a finite numbers are there in the universe and what we are interested in is finding the cardinality of a compliment intersection B compliment, right. So, I will draw a picture to illustrate it.

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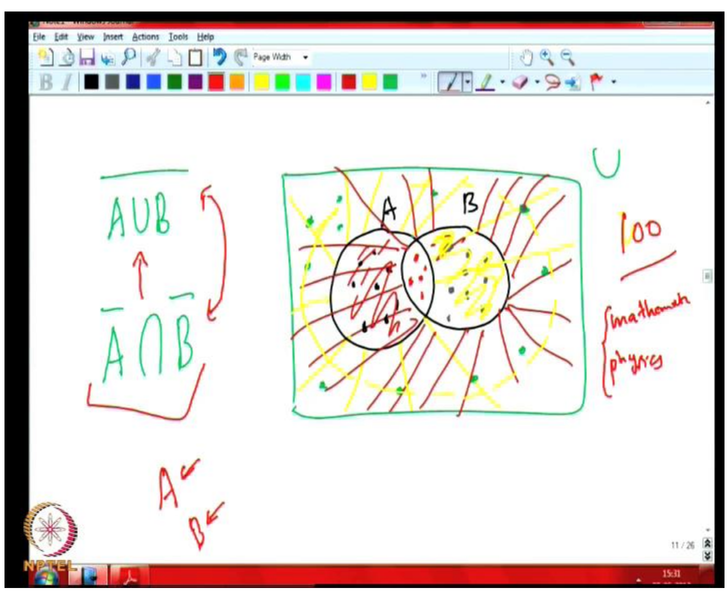

So, let this be the universe U, right. Now, you can draw two sets. Typically, it can be like this AB, right. So, now these are all members of A, these are members of A and A minus B, namely which are in A, but not in B. These are members of B and A together B and intersection B. So, these members of B and then there are members outside it also, right. So, what we are interested in is the green type of members. That means, which are outside A, outside B. That means, members in A union B compliment or in other words, the members in a intersection A bar, intersection B bar, A compliment intersection B compliment.

So, we should note that a compliment is something I can mark it like this. This is bright color. This is yellow color. Yes, may be it can be, it is this, right. So, in the Venn diagram, this yellow region which we are marking is the A bar and B bar or say, you can use this brown color to see this is B bar, right. So, now in the intersection, what is there? So, everything which is not in a union, so in this part is in brown, this part is in brown, but definitely not in yellow. Similarly, this part is in yellow, this part is in yellow, but not in brown, but brown yellow both comes in here. That is where we put the green dots, right. The green type that is what we are interested in.

So, you can take several examples. For instance, this can be a class of a hundred students. This can be class of hundred students and then out of that A, be the set of students who go to the Mathematics class, mathematics lectures and B may be the set of students who goes for the physics lectures. There are some people who go for both and there are some people who do not go for mathematics, who do not go for mathematics or for physics, neither mathematics nor physics, right. They are not interested in both. They can be people who go for mathematics, but not physics and that can be people who go for physics, but not mathematics. So, when suppose we want to estimate or we want to find the number of students who attend, would not attend mathematics or physics. None of them, right.

So, then that will be this because being the students who attend the maths classes, being the students who attend the physics classes, A compliment intersection, B compliment will be the students who do not attend the mathematics class and do not attend the physics class which is essentially you are familiar with the mode. That means, U and all. So, that is A union B compliment, but these are all may be 11, 12 standard material before we don’t ellobrate more than this.

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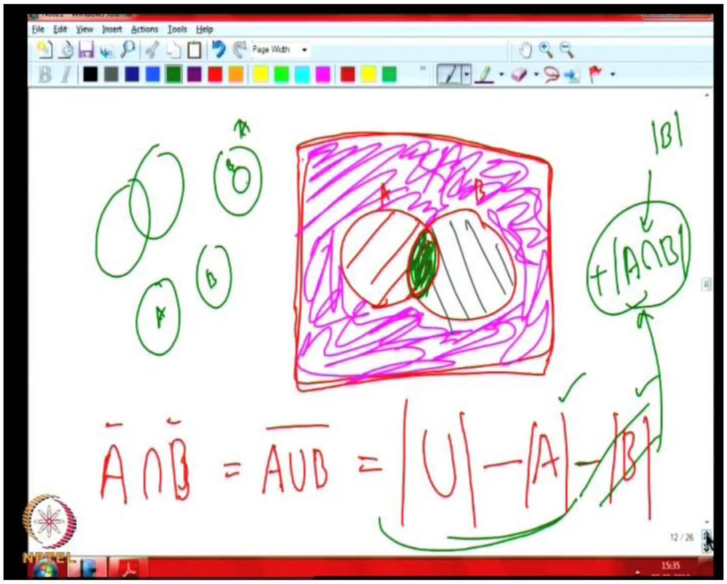

So, then we see if we want to find out the number from the Venn diagram, it is very clear what we should do. So, this is A, this is B. Now, if I want to find a A union B compliment which is essentially A compliment intersection B compliment, what we do is we consider the cardinality of U. That means, this one, this entire thing, right. From that U minus A cardinality, this will be minused off. Then minus B minus A cardinality, this will be minused off, then minus B. So, this will be minused off, but then we see that this portion, namely this portion is minused off twice because finally we are interested in this region. So, this region which means marking like this, right. This region if you are interested in, so it is natural that we took the entire thing U and minused of A and the minused of B and we expected we get this answer and the number here, but the problem is that this green part which I marked here, this is the intersection of A and B which got minused off, A as well as B, both two times. So, before I have to readd it, plus cardinality for intersection B, that is what we need to do.

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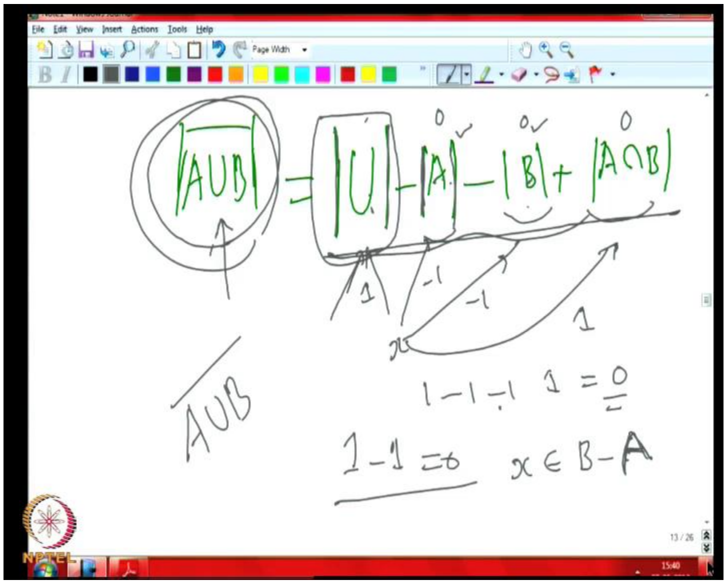

So, the final formula will be A union B compliment cardinality is equal to cardinality of A union B compliment and is equal to U, the cardinality of the universe minus cardinality of A minus the cardinality of B plus the cardinality of A intersection B. So, one may worry whether this formula is correct or not. For instance, it is not very vigorously done. What are the situations? I drew a Venn diagram. Is it always true that A intersection B has to like this or sometimes can be like this, right, sometimes can be like this.

So, am I considering all cases or because I gave clues based on the picture, am I taking all the cases carefully. So, in this case, it is not very difficult to consider all the cases very carefully and see that we are actually considering all possibilities. For instance, when A and B were disjoint or A intersection B will be empty, then you know A union minus A minus B is the correct answer. This is empty. Therefore, it is like adding a 0 to it. It is not a problem. Similarly, if this was the case, A and B, so this itself would have been A is inside and B is inside A.

So, this U minus A would have been enough to give you the answer because B anyway goes away when I remove A from U, but then I have removed B also, but then readded A intersection B, but A intersection B will be the cardinality of B. So, it is like minusing B and then adding B back, so that it cancels off and gives you back B minus A. So, in all these cases, this is correct, right, but we can do one thing. We can argue it in a slighlty different way to make it a little vigorous, may be in this case if it is not more vigorous and then they are more sets to consider more genrealized cases. We will need definetly a different approach. Then just saying that minus or add something back, right. Therefore, we will carefully consider it.

So, what are the members in it, right? So, here for instance, I want to count these members here, this region, right in U A union B compliment. I mean which are not in A, not in B, right, but in U. So, definitely these are the kind of members we want to count, but those members which we want to count will actually come here in U. So, it is counted in this part once, right. On the other hand, those members are never counted in this part or this part because they are not in A. Even then for a particular x, how much is it contributing to this?

Entire sum on the one hand side of this x is contributing 1 to the sum this U cardinality because x is contrbuting 1 to this count, but nothing to this count because A does not contain x at all because when you minus A, some other members in A are contributing to this con which is similarly it does not belong to. Therefore, when I am minusing cardinality of B, some other members in the universe is contributing to this substracted number, right and they are not in A, not in B. They definitely not in A intersection B. There such access are not contributing to this also.

So, if I consider the contribution of x in this region, so x in this region, that means A compliment intersection B or x in the compliment of A and B, then those things contribute to this sum in the right hand side only once, namely in the first term, right. The other term, they don’t contribute here. So, for instance, for each member here we contribute once to U. So, therefore this count is as same as their contribution to here, but then what about the members which are inside this or inside this. You can see inside A alone, right. A minus and B part.

So, suppose particular x belong to A minus B part. See this belongs to A part, sorry A, then definitely here it is contributing one, whereas x belongs to U and here, it is contributing 1 minus 1, right. This is minused off. Here, it is not contributing anything. That means 0. Here, it is contributing 0 because it is not a part of this. So, it is 1 minus 1. 0 is the total contribution coming for that particular x, right. So, similarly, we can consider a particular x in B minus A. That means, those members which belongs to B, but does not belong to A. They will contribute

1's to U, 1 minus 1 to 1 to minus 1 to B and 0 to A and 0 to this term and 0 to this thing. So, therefore 1 minus 1 total contribution which is 0.

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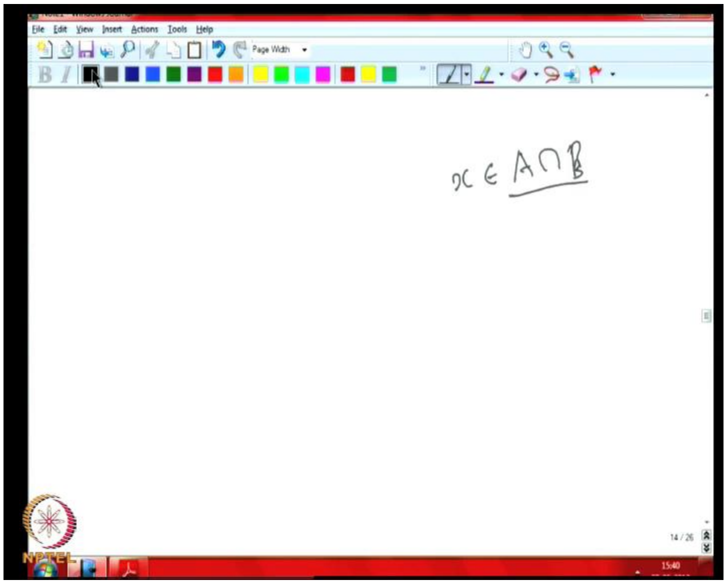

Finally, those vertices which are in, say in x element of A intersection B which are in both A and B, they contribute 1 to U. So, they contribute. Such x's will contribute 1 to U minus 1 to this A and minus 1 to this B and definitely 1 to this. So, this will be 1 minus 1 minus 1 plus 1, right. Total we will get a 0, right. So, what we mean is any x which does not belong to A union B bar does not contribute to the sum on the right house. Also, only things which contribute to the sum in right hand side are the members in A union B. Therefore, our formula is correct. Of course, this is just two element set. We can always consider a slightly bigger example, namely with three elements. There may be question like this.

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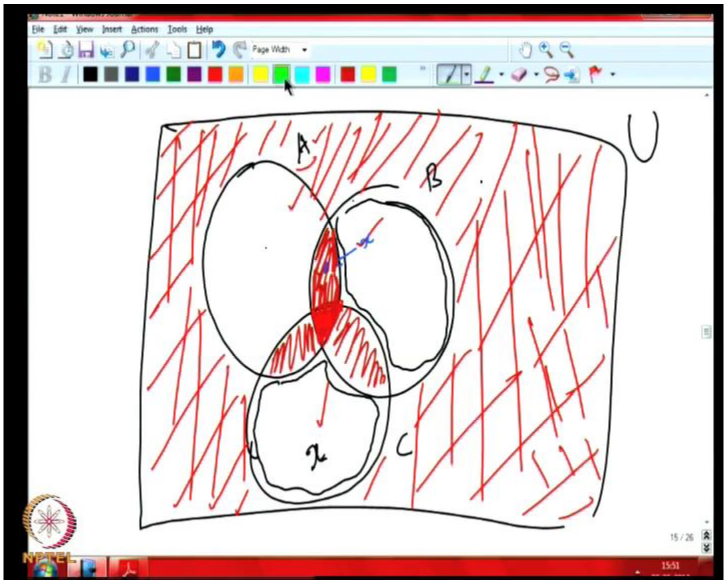

So, that is a big universe. So, it is U, right and then let us say A, B, C. So, the Venn diagram is only indicative. It can be in a different way also.

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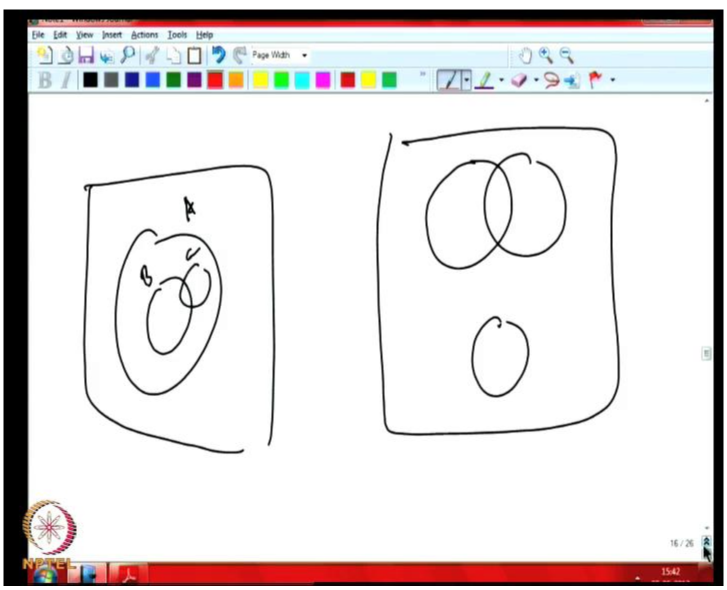

We could have drawn A B C like this, right or this is also A B C. So, several possibilities are if we just take one example case, right. So, here in totally what we were interested in is this region, right. For instance, A may be the students who go to the maths class lectures, B may be the students who go for the physics lectures and C may be the set of students who go for the chemistry lectures. What we are interested in is the students in this red region. That means, those students who do not attend the maths classes, if students who do not attend the maths classes and the physics classes, do not attend the physics classes and do not attend the chemistry classes, they don't go to any of physics, chemistry, maths, right. So, how many are there that I was asking, right. So, of course we can apply the earlier technique. This is like this.

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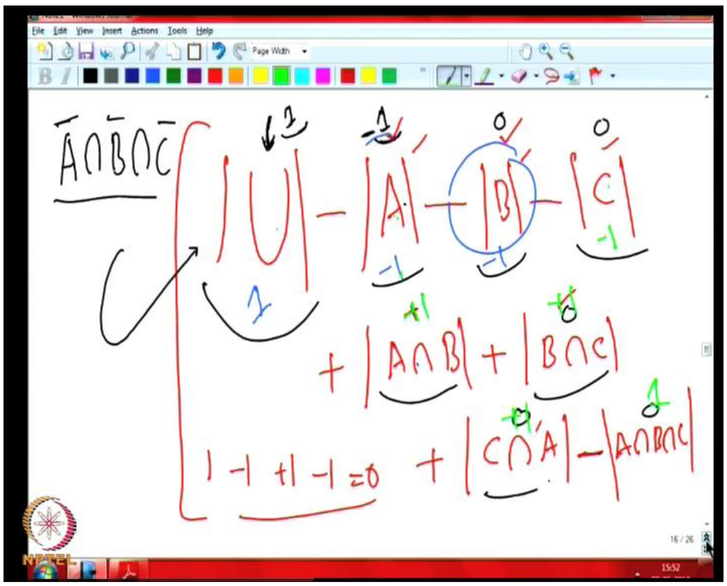

So, U cardinality of U, the total, then minus of A, then minus of B, then minus of C. This is the initial try. So, for insitance if A B C, all of them were dischurn, we would have become successful by now because you know we just have to minus the cardinality of A and the cardinality of B and the cardinality of C. From the cardinality of U, we would have got those, the cardinality of A compliment intersection B compliment intersection C compliment, namely the cardinality compliment of A union B and C. We would have easily got it, but then we have intersections here, n minus A, n minus B. We have actally minused, say this portion which portion, say this portion, this brown portion two times, right because from A part of this, we have minused it once, as part of B we have minused it once and as part of C we have minused it once, right.

So, again all these portions are minused two times. What if the this portion, the middle portion? So, yeah got minus three times. Isn't it? Because it as part of this, this is part of this as part of this also, right. So, what we do is that this is the strategy. So, what we do is first we readd the cardinality of A intersection B and then B intersection C intersection A because we noticed that the members in A intersection B was minused of twice, once part of S as part of A, once part of as part of B. So, we readded it, but then noticed that the members in A intersection B intersection C were minused of three times because once here, once here, once here, but now we readded it three times.

So, we three times minused it and three times readded it. Here also, here also and here also. That means, its contribution is not those members in A intersecton B intersection C are not minused at all. You minused it three times and then added it back three times. That means, we have not minused it off. So, therefore we have to minus it once again. Minus A intersection B intersection C cardinatlity. Once again this is the formula, right.

So, I thinks it is convincing. Now, what we did is to find the number of members in A union B union C compliment. That means, the members in U minus A will be the union C, right or in other words, the members in A compliment intersection B compliment intersection C compliment. That is what we are interested in, right. So, what we can do is we first consider everything in U and then minus of the members in A, then minus of the members in B and then minus of member in C, but then if A,B, C are all distinct, we are all definetly through by this procedure, but then there are members in a intesection B possibly. There are members in A intersection B possibly and there are members in C intersection A possibly. Each of these members were minused off two times.

Once these members of A intersection B were minused off, once a part of A and once as part of A, so we added all of them because we want to minus only once, not two times, but when we did it, what happened to the members in A intersection B intersection C is they were minused of three times because once a part of A, once a part of B, once a part of C, but readded also three times with once as part of A intersection B, once as part of B intersection C and once part of C intersection A. So, we have to minus it once, right because we want only things which are not in A, not in B, not in C, but the things in A intersection B intersection C are in all of them.

So, we have to minus it once again. So, that is how this formula works. So, that is correct because we have carefully argued it. So, now we can also consider the trick, the technique, the argument. We consider in the last thing. For instance, we take the member in this region, some x of the universe from which is coming from this region, namely that x we have to actually count because that belongs to the compliment of A union B union C. In other words, that belongs to A compliment intersection B compliment intersection C compliment.

So, how does this x contribute to the total count in this? So, definitely contribute once to 1 to U and definitely 0 to this one, 0 to this one, 0 to this one, 0 to this one, 0 to this one, 0 to this one because that particular x is not part of A, not part of B, not part of C. Therefore, this minus cardinality of A minus cardinality of B minus cardinality of C excetra are not effected by that particular x. Similarly, A intersection B intersection C intersecion or A intersection B intersection B does not contain that x. Therefore, all the contribution of that x to each of this term is 0. So, overall it is contributing one for this entire sum, right.

So, that is correct. They are contributing correctly, but suppose if you take x from here for instance which is in A, not in B or C, only in A, right. So, then such A contributes definitely one to because (( )) is part of U and 1 to A and it is not contributing to B or C definitely this 0 because it is not a part of A, not part of B and not part of C. Similarly, it won’t contribute to A intersection B intersection C because it is only in A. It is not in intersection B because it is not in B. Similarly, here everything is 0. So, its total contribution is 1 minus 1, 1 for 2 and 1 minus 1 to A, sorry minus 1 to this minus A, right. So, total contribution is 0. So, similar argument shows that the members which are coming from this region, that means only from B. That means, those members which belongs to B, but does not belong to A, does not belong to C.

Similarly, those members which are coming from this region, namely graph to see, but does not belong to A or B, they also contribute only actually 1 minus 1 to U and minus 1 to at minus cardinality of A term, right. No other term will contain the contribution. So, therefore they are proper and now, if you take a member from here for instance, some x from here which belongs to A intersection B. That means, it belongs to A and B, but not to C, but then exactly two sets, they belong to. What will be the contribution? There contribution will be 1 to U minus 1 to A minus this term, the second term. So, minus 1 to this term and then of course this term will contribute 0 because it does not contains in 0 here. On the other hand, it will contribute plus 1, right because it is being added, right. So, the total contribution is 1 minus 1 plus 1 minus 1, that is total 0, right. So, that is true of any element which is coming just two sets out of three sets it belongs to two of them, but not the third and now, if it comes to, if you consider any member which belongs to all the three of them, that means any member which is coming from this portion, say something like this, right.

So, such members will contribute to all the terms here, right. How will they contribute? They will contribute 1 to minus 1 to this, may be k term minus A and similarly, minus 1 to this cardinality minus cardinality B. Similarly, this one to this one and then plus 1 to this plus 1 to this and plus 1 to this and minus 1 to this. Now, how will you go calculate it how much is there because here, it is 1. How many A's are there? Three A's are there, right.

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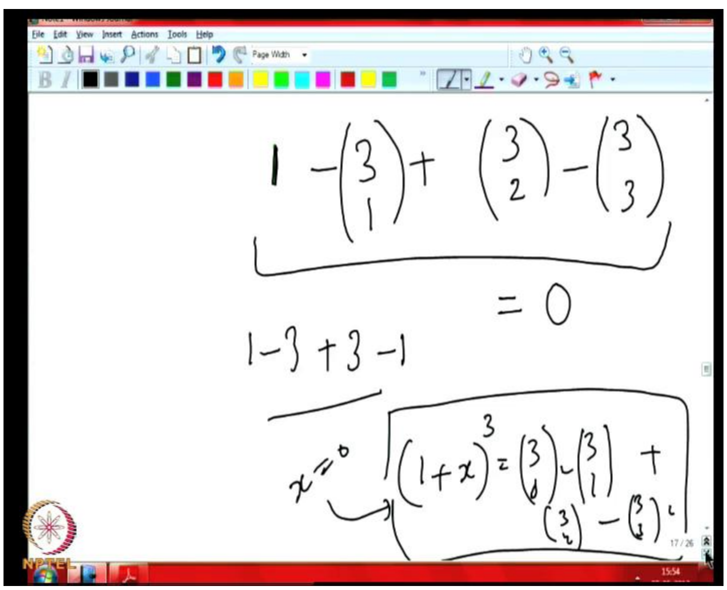

So, the total contribution is 1 2 1 2 and minus 3 because 1's 2, A's 1 to B 1 to 3 or minus, then this actually is 3 choose 1. Each set is selected one times out of three sets. Not of each of the selection of one set minus 2, then two elements sets 3 choose 2 possible ways. We can select A and B, B and C, A and C. For each of them, we have contribution from that, such as x is one's. Finally, 3 choose 3, right. That means, all the three sets, it is contributing minus 1.

So, this as we know is adding to 0, right because 1 minus 3 plus 3 minus 1, but this is adding to 0 and you know this is as we can see, this identity we have seen earlier. If you do 1 plus x cube, so this will be input x equal to you expand it and this will be n-th 3 choose 0. 3 choose 0 is 1 minus 3 choose 1 x plus 3 choose to x square minus 3 choose 3 x cube and you put x is equal to 1, sorry this was plus 1. This was plus 1, put x equal to 1 minus 1 and then we will get this alternate minus 1's. Minus 1's is required, right. This is what we will get putting x equal to 0 in this thing, but anyway, we have seen it before. So, therefore it is not very surprising. Otherwise, in this case, definitely you can evaluate it seperatly or also, we can just write down the values and sum.

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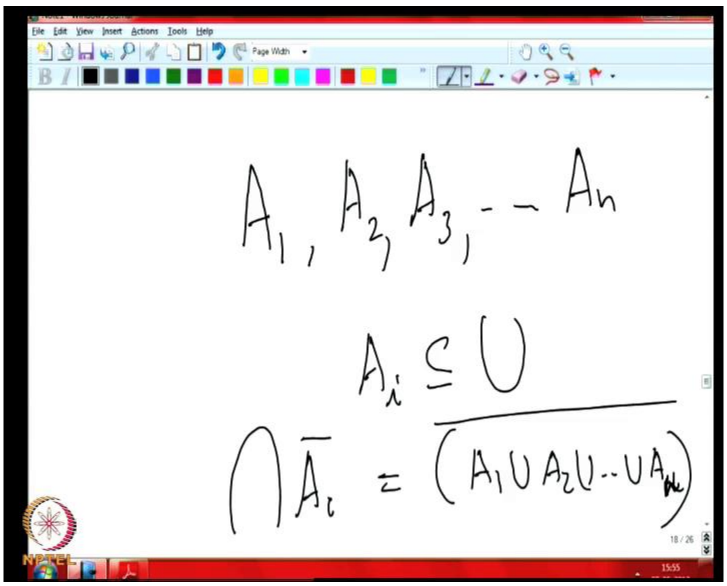

In the next case, we consider the most general situation when we have n sets in A1, A2, A3, An and these are which Ai is the subset of the universe U, right. Now, we are interested in the intersection of Ai compliments which is essential. As we know, it is A1 union and A2 union An all compliment, right. This is what we want to estimate.

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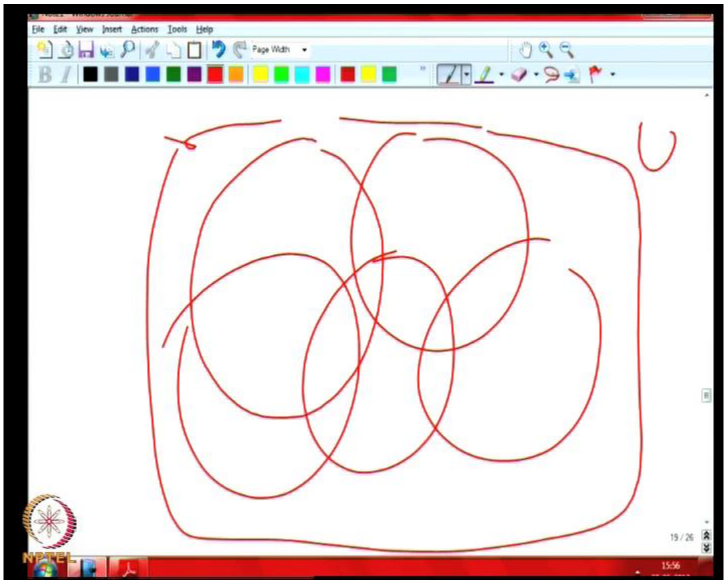

Now, we can use the previous tactic of you can try to use it. We cannot use it, so for writing drawing a Venn diagram, the complicated Venn diagram saying that this we have drawn n sets in the uninverse and then consider the cardinality of U first and then we are interested in the member in U, but those members of U which are not in any of A1, A2, A3, An, right. This is what we usually want to say, but then we will say that ok, first we can try and take all the members of U. Then minus of members of each Ai. That means, first minus A1 cardinality, the cardinality of A1, then minus cardinality of A2 and up to cardinality of An. Then we have to readd the cardinality of Ai intersection Aj, for every A pair A j. Somehow within that, those 3A element, three sets 3 in the set, the members intersection of three sets might have got readded too many times.

So, we subtracted and things like that, but it is a little clumsy to go like that. So, we prove it the other way. We have been abopting the second proof all the time. We were just when two set case, three set case, we had the second proof, but of course we have to write down the formula for that. Looking at the pattern, we can write it down as this.

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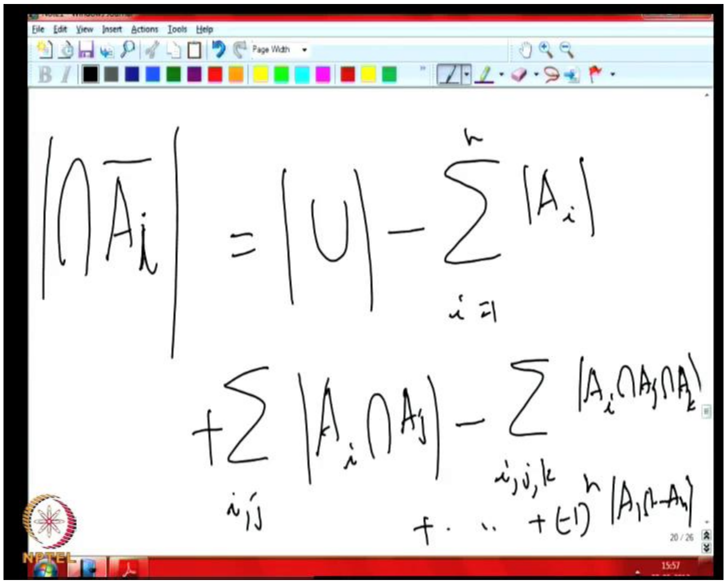

So, A1, sorry Ai compliment intersection, the cardinality will be given by cardinality of U minus sigma Ai. So, for i equal to 1 to n, here all the n's we minus here. Then the plus Ai intersection Aj cardinalities, that is for every i, n, j. Every pair i j will do this. Then minus for every three sets subsets taken together. So, Ai intersection Aj intersection Ak will take that cardinality will be minused and so on. So, alternate plus and minus and in the end, thing we get A minus 1 raise to n, right. We have the cardinality of A1 to An. So, that will be the end. So, these formulas, the question is how do we get these formuals? That is what we have discussed in the last two cases. So, we will discuss in the next class.