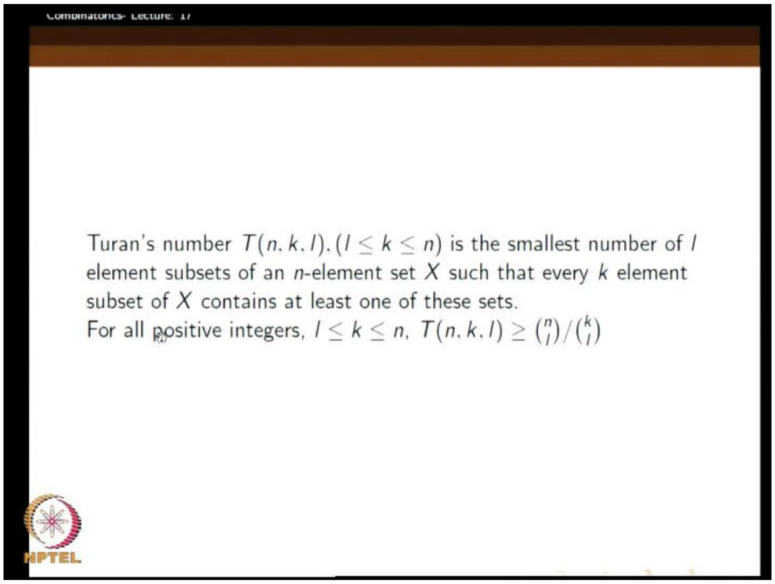
Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

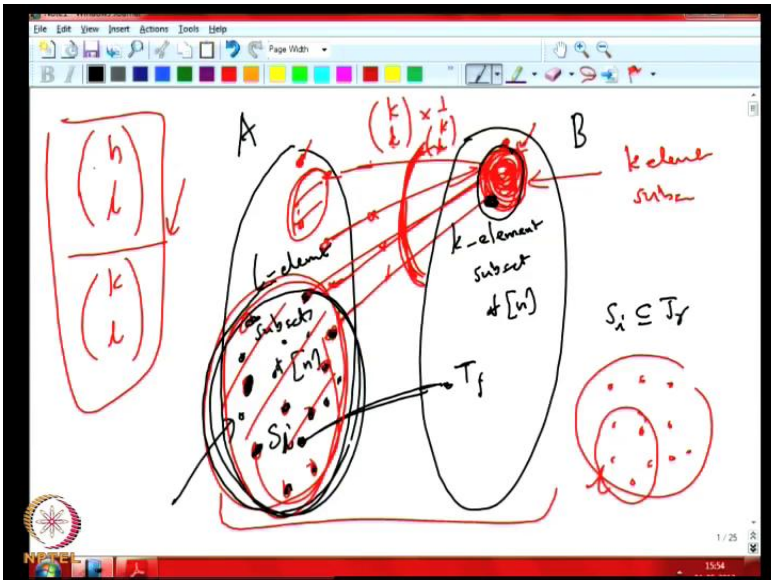
Lecture - 17
Double Counting– Part (2)

(Refer Slide Time: 00:32)



Welcome to the seventeenth lecture of combinatorics. So, in the last class we were discussing about Turan's number giving a lower bound for Turan's number. We will go back to that. So, this is the problem, we defined Turan's number as the smallest number of l elements subsets of an n element set x such that every k element subset of x contains at least one of these sets, right.

(Refer Slide Time: 00:54)



I was trying to give a picture using the bipartite graph. This is many times this is a very nice strategy to draw bipartite graph when we are talking about set systems and we can think about it, or at least, we will get some clarity about what we are asking for, right. So, we told that we are taking, sorry, we decided to take k element subsets of n, on the the, right side. So, let us say this is A side, and this is B side and l element subsets in, in this side. So, when I say n element subset, each n element subset is conceptually considered as a vertex above bipartite graph now, and now it became two vertices. We inserted an edge, we put an edge, if this l element subsets, say we will say A, and so maybe can use S 1 and this is some T, yeah, T 1, say S i and this is T j. If S i is a subset or subset equal to T j, then I will put an edge between them, so I will get lot of edges here depending on, right.

So, what we are interested in is, say, in, is in selecting certain subsets here, so certain vertices here such that any vertex taken here as an edge into this set. It should not happen that there is a vertex here, which has no edges. So, it is all its edges are going to here, means, any subset of this k element subset are not taken; no, subset of this k element subset is not taken, right. So, so that in some other what we say is that at least one of the k element subsets will have to be taken.

So, we can see, that here if you get a, consider a k element subset here how many edges will have a vertex here of this thing? How many edges will go out of it? See, though I have drawn a big this thing, it is, it should be considered as one vertex because this is, this is one k element subset, subset, which is thought of, say, vertex of the bipartite graph. Now, how many edges will come out of these things? Exactly k choose l edges will come out of it, right, because there are l, possible l element subsets. Often this we are planning to select a few from this side, so one of this k choose l things should be taken when I pick up this thing. So, if you are familiar with probability or something, the probability, for instance, if I select, so there are n choose l of this things here, n choose l of members here in the A side because there are l, the number of l subsets of n is n l n choose l.

Now, if I pickup, so if I, if I pickup a subset by tossing a coin, maybe, which is head with probability, say, k choose l, right, so 1 by k choose l. So, when it is, like for each of this vertex here, each of this subset l element subset here, I may pickup, that I will pickup in my collection with probability 1 by k choose l. So, if you are familiar with probability, otherwise you can discard what I am telling, so then it looks like, so for if I do this thing for each or then toss this coin for each of the n choose l members here. So, then I expect the expected number of set vertices I am planning I may take on this side, is this right. So, n choose l by k choose l because 1 by k choose l is the probability of picking one of them. n choose l things are there out of that so that, therefore, so you can expect, that the expected number of vertices taken here, I mean, l element subsets picked up will be n chose l by k chose l, right.

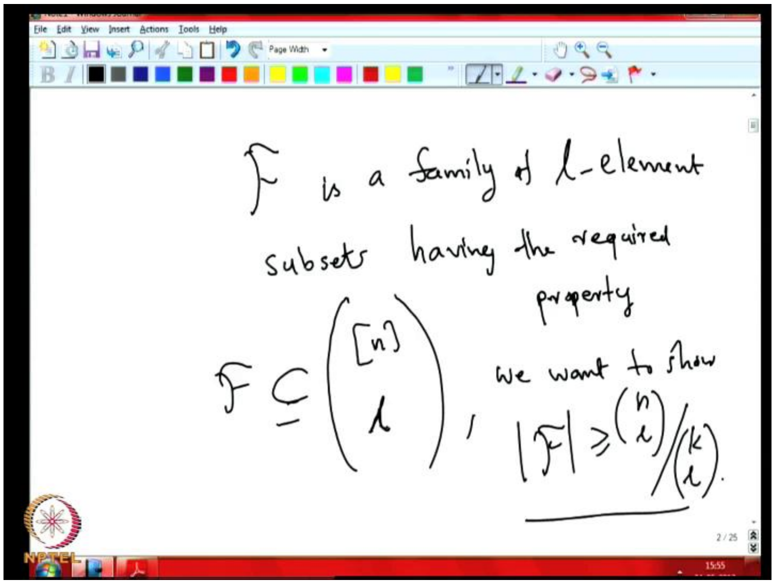
Now, so so from this side for each vertex I want is, here, k element subset here if vertex here means a k element subset. The way, what we want is one of each subset, that k element subsets, l subsets if you consider, that should be picked up. So, there are k choose l of them because each one of those things are picked, the coin is tossed with probability 1 by k choose l, so we expect one of them to be in the same picked up, right. So, therefore, it makes sense in that sense, right. For instance, somehow we may be able to find one subsets, but then of case, so this was an intuitive or suppose, if you are ably aware of what is probability and so approximately, for instance, how to randomly select things and things like that and then that makes, so therefore this statement makes sense.

So, it is like if we had picked up n chose l by k chose l, of case, at least that much you should pickup. This is what you are saying, of case, yeah, so it is just because we expect, it does not mean, that we should be able to hit, but what this theorem says is, suppose we go below this thing, there is no chance, right. You have to take n chose l by k chose l, otherwise we, we, do not have chance of satisfying everybody. That means, one of the neighbors of it will be picked up, right, that is what I said. So, we, we need to take at least that much (( ))

Now, we will do a counting argument to show that. So, this is, of case, this is all intuitive argumentation, which is to, of case, this is not even an argument for this to prove this theorem. It was just to give a feeling what we are trying to do, right, what this quantity on, the, the RHS is and what we really want, right. So, what we want is this will also something like, you know, if, if we consider k element subsets, we list down the k element subsets, right, for each of them. So, we want a representative set such that every k element subset has a representative in our, in our l, l element, in fact, l.

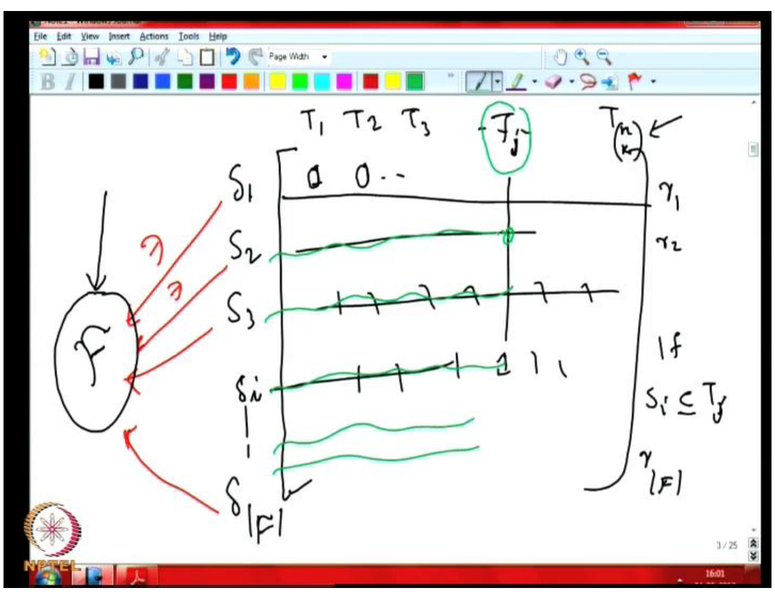
So, for instance, if its committee is, we were talking about the k, k member committee is formed, so we should have at least l members in the committee, right, or again we go back to our previous example, this red people, red subsets are the subsets, right. For instance, I said committee, we, we bribe this thing, so if that, that example does not work exactly here, but suppose you have to bribe each l, l element committees separately. Suppose, you know this committee is fully bribed, this committee is fully bribed, right, or corrupted or something like that. Now, I want, for every k element committees here I want a one of the corrupted l, l set (( )), that is what we are saying, that is what we are trying, ok. So, so approximately, so we would not show, that k choose l by k choose l is a requirement, right; at least that much is required.

(Refer Slide Time: 09:09)



Now, to show this thing what we do is, suppose some f family of l element subsets, say f is a family of l element subsets, is a family of l element subsets having the required property, so not that f is a subset of n choose l. So, this is notation to, notation for all the l elements subsets of, l subsets of n, ok. This is subset of this and now we want to show, that we want to show cardinality of f is at least n choose l divided by k choose l. This is what we want to show, right.

(Refer Slide Time: 10:28)

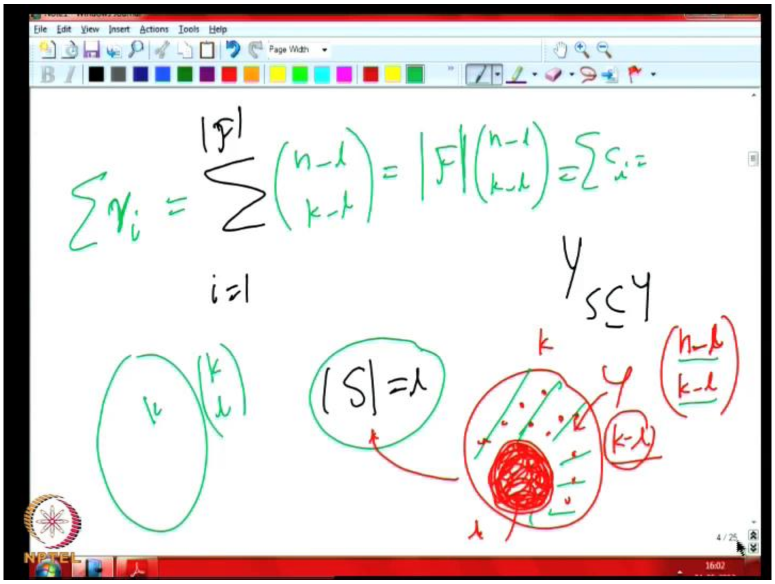


So, now we will again use that double counting technique. So, this time what we do is, we index the rows S, the members of F. So, let us say, S 1, S 2, S 3, these are all l element subsets, multi of F, look this coming from this, right. These are the members of F, right. These are the members of this, but these are all subsets again, l element subsets.

Now, this time we index their ways by the k element subsets, T 1, T 2, T 3, T n, choose k, all the k element here in the columns, we have n chose k columns because each k element subsets has a corresponding column there, that there are not choose, we do not have n chose l rows because we are only interested in the hypothetical family we have constructed. So, we claim, that this family is one satisfying the property, right. And then we have listed the members here and we are interested in the cardinality of that family.

And now, again what will be i j-th entry? So, S 1, T 1, here I will put a 1 if S 1 is a subset of T 1, otherwise I will put a 0. If S 1 is a subset of T 2, then I will put a 1 here, otherwise I will put a 0 like that, right. So, similarly, so if I take a S i and a T j, a 1 will be put if S i is a subset of T j, subset equal to T j. Not that these are all l element subset, l is less than equal to the cardinality of each subsets corresponding to columns because i have k element subsets, right. Now, what are we interested in? We just count it into different ways, namely we count the now 1s, number of 1s in the matrix row-wise, so how many 1s are there here? How many 1s are there here? So, as we told here, r 1 and r 2 is r cardinality of, we will count so that is easy to do data equal to 1 to cardinality F.

(Refer Slide Time: 12:47)



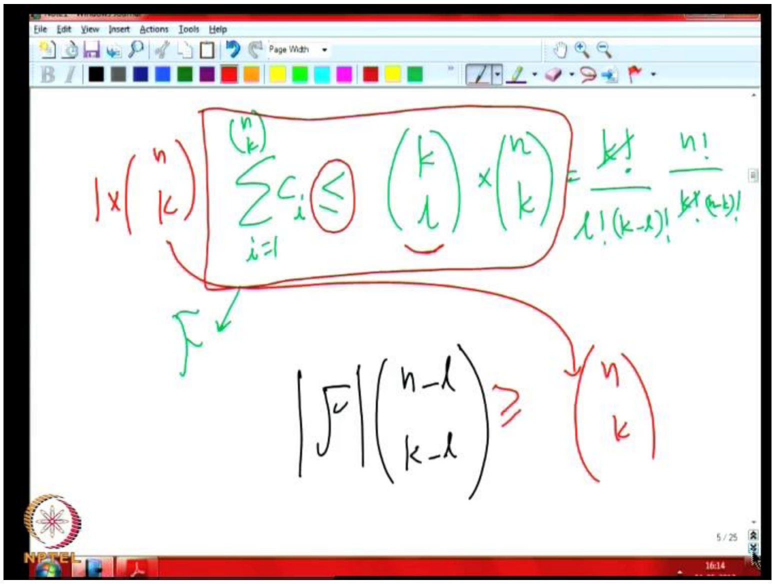
So, we want to count the number of 1s in each row. You can see that because this is an l element set, so a row corresponds to an l element set. They are asking in how many k element sets it is a subset of, I mean, they, how many k element subsets are there? (( )), it is a subset of that will be a number of 1s in it, right. So, for each element you can easily see that for each l element subset the number of 1s we see in the corresponding row will be same though it will be in different places that will be same, why? Because it is a same number, so you give me an l element subset, right.

Now, I want to know how many Ys are there such that y contains S. Now, it was a picture like this. Suppose this is a Y, so now S has to be here, but this is already l, now this total wise k. I am right, so now the question is we want to decide the remaining elements, namely k minus l elements of y. In how many ways I can decide it because this l things are already decided because that is exactly S. If we exit right now we can decide this thing, but there are total n minus l remaining elements because the universe has l elements. We have already fixed the l elements corresponding to this S. Now, the universe has n minus, so l remaining elements out of that we have to take this k minus l elements, so k minus l ways of picking it up, right. So, this portion, so this portion can be decided in n minus l choose k minus l. Is it clear?

So, because so we want the k element subsets of universe n, n element subset such that a given l element subset is there inside it. So, the way to do it is we just pick up the members of that l element subset first and keep it aside and now decide their remaining k minus l elements, which should make up the k set because this element subset, this l elements has to be there anyway, but the remaining k minus l elements can come from n minus l possible elements. Why n minus l? Because total number of elements in the universe were l, but this particular l elements we kept apart. Now, they are already gone, right. We cannot select again those things, so out of the n minus l remaining things we have to select k minus l of them. So, that is saying for every l element subsets.

So, the number of 1s you see in each row will be equal to n minus l chose k minus l, so we can just write n minus l choose k minus l here, right. But because this is a same number, we know this is equal to cardinality of f because this is summation, same thing is summed up. So, this is F into n minus l choose k minus l, this is what our sigma r i is, right, sigma r i is. Now, this has to be equal to sigma c i, the column sum, and what is that the column sum says? So, you pick up a particular k element subset, how many l element subsets are there inside it that is what x is. I mean, how many l element subset does it contain from this F? You know, that is difficult to know. You know that suppose given a k element, subsets, subset given a k element subset we definitely have k chose l, l element subsets inside it k chose l l element subsets inside it. Now, k choose l, l element subsets are there inside it of case. So, there are k choose l l element subsets inside it, but how many of them will come from F, that we are not sure, right.

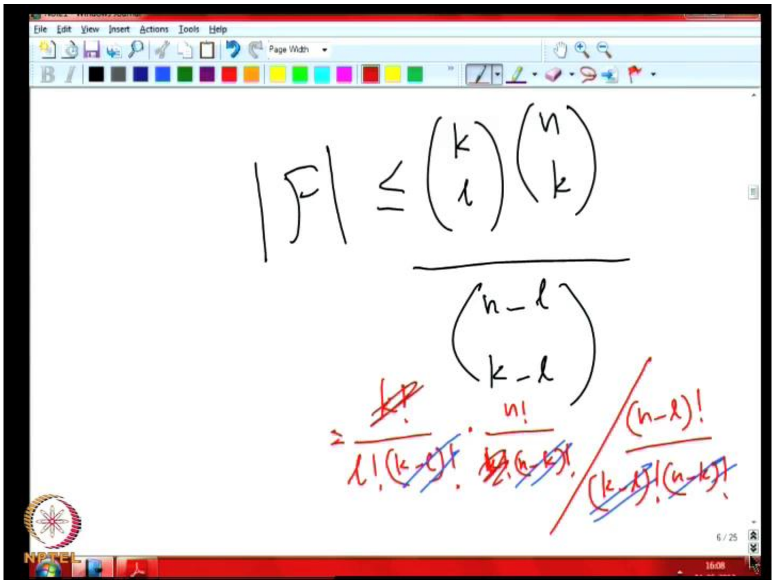
(Refer Slide Time: 17:27)



So, but then let us say, that see the column sum can be at least k choose l, that is clear because you cannot have more than l element subsets, more than k chose l, l, element subsets for a given k element subset, even that much may not be there because we are not considering all l element subsets, S rows only. Some l element subs, namely, that those subs, l element subset, which are inside F are coming S rows, but we have now way of doing because we do not know what, that F is right. We only have a hypothetical F there, so we do not have, so we just put an upper bound, we say that at most k choose l things are there.

So, the sigma c i has to be at most this. So, this is going from i equal to 1 to n chose k, right, because there are k, k, n choose k, possible k element subsets. So, for each of the k chose l can be taken, this is what we get and this is what I can do a little bit of manipulation. Here, k factorial by l factorial into k minus l factorial and then k or we have n factorial divided by k factorial into n minus k factorial. Now, this will go away, right, so k choose l into n choose k is the row sum. So, what do we get from here? So, we get F, cardinality of F into, so in the previous case we have cardinality of F into n minus l choose k minus l, that is equal to, so we compare it cardinality of F into n minus l choose k minus l. This is less than equal to, so this also, right, because k choose l into k choose l into n choose k. This is what we get.

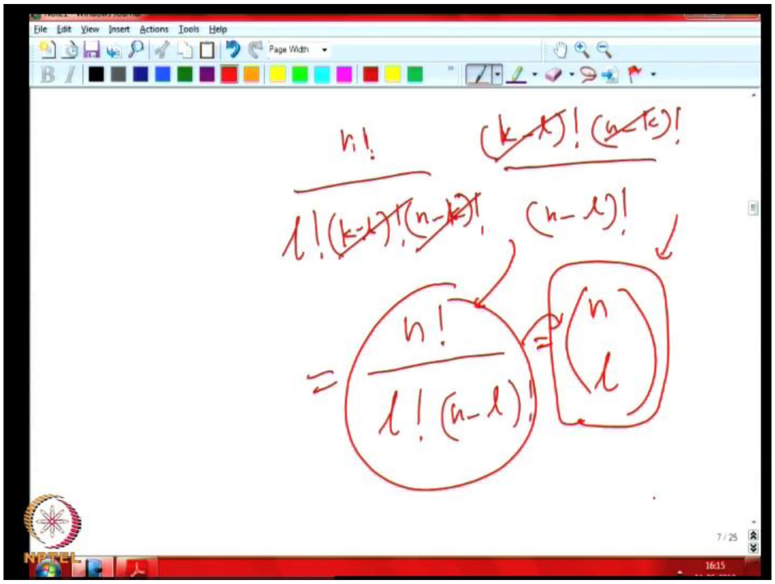
(Refer Slide Time: 19:51)



Now, of case, so the question now is, so we get cardinality of F is less than equal to, I just have to do the manipulation, k choose l into n choose k, k choose l into n chose k divided by n minus l choose k k minus l n minus l choose k minus l. So, we will try to do (( )), so the, so this is, of case, I should not make any (( )), so this is k factorial divided by l factorial into k minus l factorial here. And a numerator, I am expanding n factorial divided by, n, k factorial into n minus k factorial and this total divided by below this is n minus l factorial divided by k minus l factorial n minus l minus k minus is n minus k factorial n minus l minus k minus l. This is just n minus k factorial. So, here the two things will cancel, so this n minus k this n minus k and this n minus k factorial will go away and this k minus l factorial and this k minus l factorial will go away.

And of case, see you have to write it in a proper way. For instance, this into, maybe I can, if you, if you are, if you want to do it, may be I can do it more carefully here at least, so this will go away, at least that much we can, we can be sure. So, instead of doing this thing, let us say this will anyway go away, right. This and this will go away, so therefore, I will write it again.

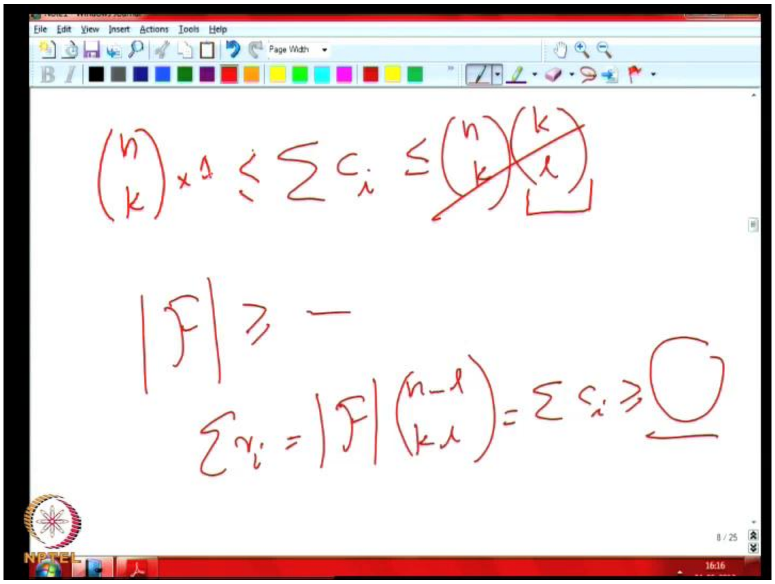
(Refer Slide Time: 21:59)



Once again, n here, n factorial in the numerator, l factorial and then k minus l and n minus k factorial, that I will write n factorial, divided by k l factorial l factorial n minus k minus l and n minus k factorial k minus l factorial and n minus l n minus k factorial in the denominator, right. n minus k factorial here, below, we have, yeah, this, this, the denominator goes up, that is k minus and n minus k k minus l factorial into n minus k factorial divided by n minus l factorial, right. Yeah, now we can cut this off.

So, what do we get? So, finally we get n factorial divided by l factorial into n minus l factorial. This is what we get, that is, n choose l. And finally what do what we want to get? We have to get n choose l by k k choose l, right, n choose l by k choose l n choose l by k choose l. We will get n choose l, right. This manipulation has, once again I will do. So, this is correct, so the k chose l into l chose k here, so we have F into n minus l chose k minus l, right. And we want to show, that F is less than equal to, sorry, F is, oh sorry, so what (( )), I made a mistake because I wanted to show a lower bound, I went for an upper bound. So, therefore, actually, the, the, the thing is I took a lower bound here; in fact, I have to take lower bound here, right. So, but this is not true here. So, because this is only an upper bound, I, I, I took k chose l into n chose l, that is a possible thing. So, therefore let me see, in the worst case we may have just one here, right.

(Refer Slide Time: 25:32)

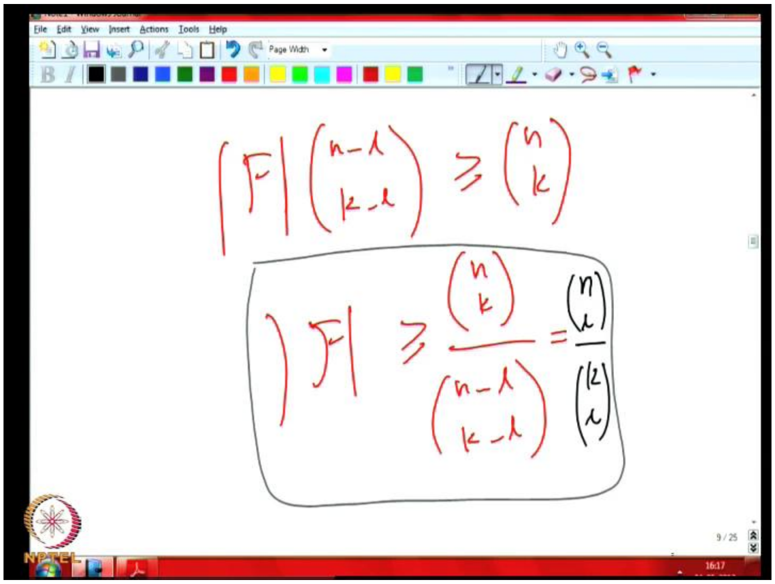


So, we can, we can write, yeah, that is, that is one mistake I made. So, sigma C i, when I say sigma C i, that is definitely less than equal to n chose k into k chose l. But what we want to prove is, that F is greater than equal to something, right. But what we know is F into n minus l chose k minus l is the sum of rows, sum of rows. Now, so this is sum of columns also. So, we want to get lower bound for this, right, so that we will, we will be able to say, that F is greater than equal to whatever lower bound we get divided by this n minus l, this thing. So, that is why I made a mistake. So, therefore, I want to get something here also, right. This, this is not good for me, right. So, what can I do, but because I know that I have, n choose k times, but the, the thing is I cannot say, that for each column I have k choose l entry once in it, that is definitely not true for each column. They may not have k choose l into S because why, because they go back to the matrix, go back to the matrix here.

If you consider the rows, we are on this; rows only correspond to the members of F. So, for instance, if I take a T j, its subsets may be such that many other subsets may be outside this, but one thing is sure, we have at least one subset unit, but that is a property of F, it is not that 0 subsets of T j are listed here, no subset of T j are listed here because if no subset of T j are listed here, then F is not the kind of family we wanted because the kind of family we wanted was a family such that for every k element subset there is one l, one of the l element subset of, it is part of the family. So, definitely I am sure, that at least one 1 can be seen in this row. So, not that every, every member of this row will be 0s, right, so at least one 1 will be seen from this thing, right, that is because F has that property, right. So, therefore, as a lower bound we can take just one, we do not know whether more than one element, l element subset of this particular k element subsets will be listed here in the rows, that is not guaranteed, right. If it is, if this F did not have that property, it can even be 0, but because F has the required property we know there at least one 1 will be same, so we will just put one as a lower bound instead of, yeah, here. So, here, where we have taken k minus l, here we will just put 1 into n chose k as a lower bound. So, now it will, so now here we do not want this, we want greater than equal to, right, 1 into n chose k. This lower bound we will put, so 1 into n chose k, this just n chose k, right.

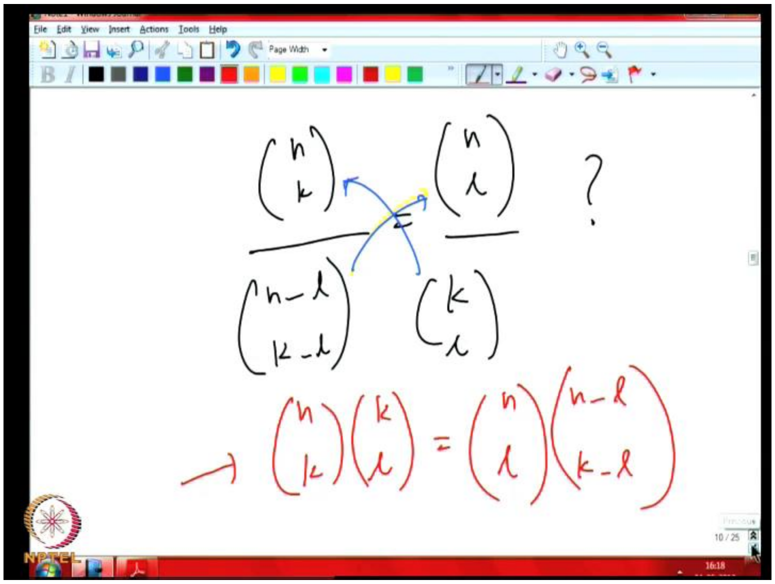
So, now, ok, this manipulations are, were not what we were looking for because this was a, this will only give an upper bound on f. Of case, whatever we are getting as an upper bound for F is valid, but then the problem was, see among the manipulation that we have done is not wrong, but that is not our aim. Our aim, the theorem was asking for a lower bound for F, not for an upper bound for F, right. So, because you know when we take thing what we got is, this is n chose l, right. So, finally upper bound of n chose l is quite valid, but trivial, that is all. So, this, whatever after doing all this manipulations we had got, this quantity, mainly n choose l. But then it is like we have, take the, F is at most all possible l element subsets is what it says, right, n chose l. So, it is a trivial upper bound we got, but it is not wrong because it is giving that information at least by doing these double counting. But we go by the, because the nontrivial thing what we wanted was the lower bound, so we work on that part.

(Refer Slide Time: 29:57)



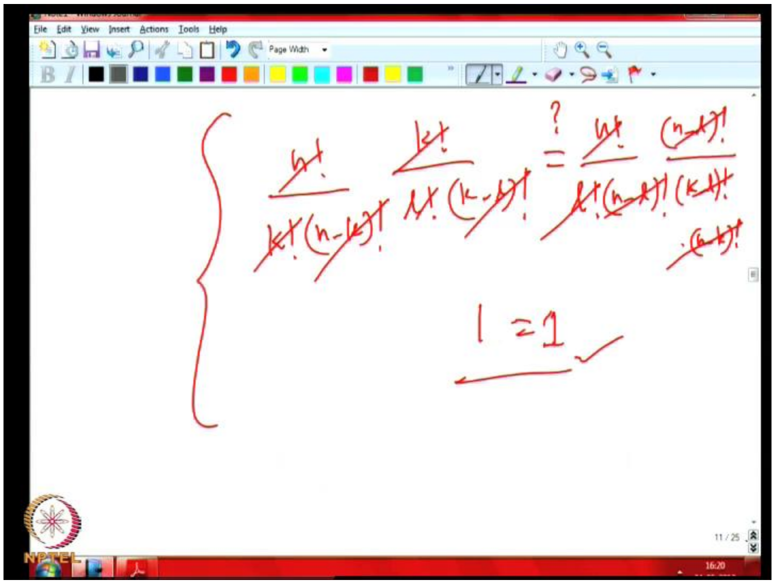
So, we get cardinality of F into n minus l choose k minus l is greater than equal to n chose k, right. So, now we can do this thing. So, cardinality of F is greater than equal to n choose k divided by n minus l choose k minus l. This is vertices, so we want to show that this is equal to, so you know, still we are not through, why? Because you know, this is not, this is definitely true, but this, what we want to show is, that I can go back to the theorem n minus l by k choose k choose l is what we want to show, right. So, so we want, somehow want to show, that this one is equal to, this one is equal to n choose l by k choose l. This is what we want to show, then it will work.

(Refer Slide Time: 31:00)



So, we will work on that. So, n, n chose k divided by, n chose k divided by n minus l chose k minus l. We want to show, that this is equal to n choose l divided by k choose l. Is this true? If this is true, we are done, right, because when we, we got this thing from our double counting argument and if we can show, that this is same, then this is we are done. But this, how do we show this thing, let us, let us try cross-multiplying means, we will multiply this here and this here. So, what, what is that saying? So, that say is, n choose k into k choose l, this equal to n choose l into n minus l choose k minus l.

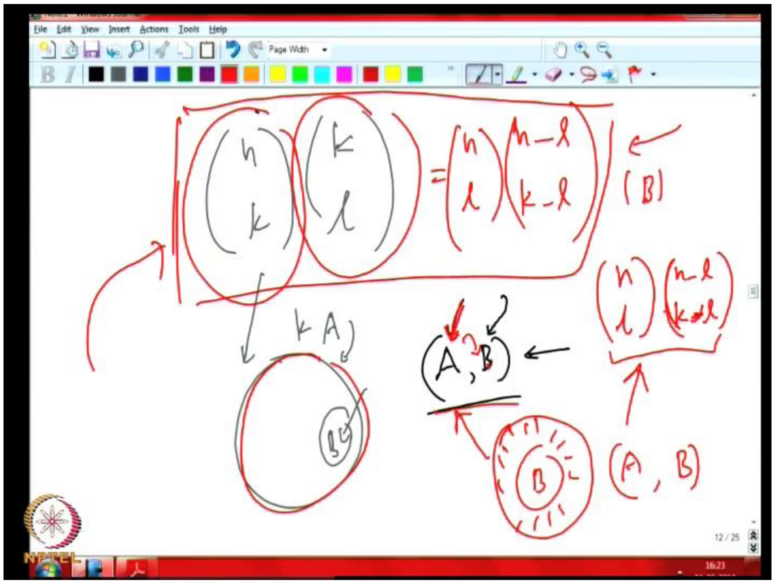
(Refer Slide Time: 32:06)



Now, there are two ways to deal with this thing. One is, just expand it and see, of case, so n factorial divided by the same expanding n chose k into k chose l. n factorial into k factorial into n minus k factorial and this is k chose l k factorial by l factorial k minus l factorial. Yes, in this factorial I want to show, whether I mean to say, whether this equal to n chose l into n minus n choose k choose l. So, there is n factorial into n factorial into n minus l factorial into n minus l k choose l, right. n minus l factorial k minus l factorial into n minus k factorial, so n minus n minus l minus k minus l, you will get n, n minus k factorial.

So, all these things will cancel off: k factorial, k factorial; k minus l, k minus l; l factorial, l factorial; n factorial, n factorial; (( )). So, this is on one is to one thing, just a blind checking by expanding both sides. So, therefore it will, so note, that there is no 0s and all we are cancelling, we are just given k equal to l, k is, this is 0 factorial is 1 only, so all factorials are being cancelled, therefore no problem. So, this is, one is one, so one equal to one. So, it is true or if you want to get a combinatorial proof of that you can try. So, what does it mean?

(Refer Slide Time: 34:03)

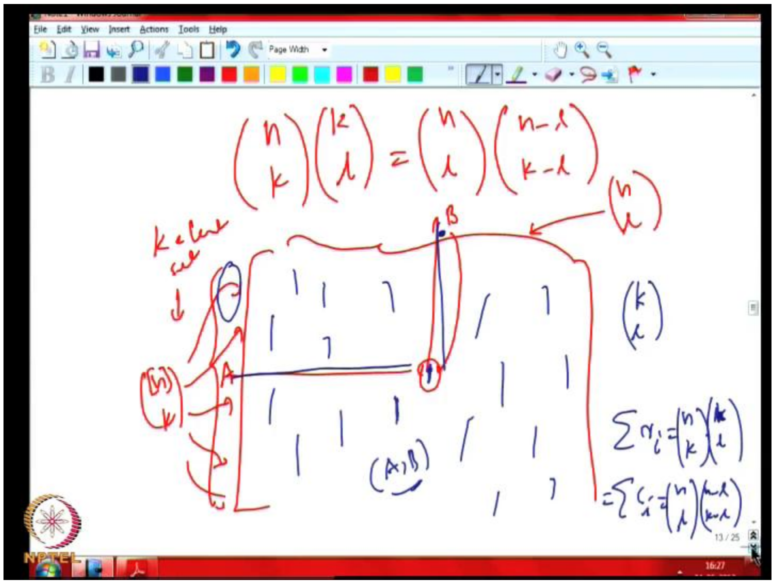


So, this side, n choose k into k chose l, yeah, n choose k, n choose k into k choose l means, yeah, n choose k into k choose l. So, we have first selected a k element subset here, then we have selected an l element subset of it. So, let us say, A is the k element subset, B is the n element subset, then it is like we are counting (A, Bs) pairs, (A, Bs), where A is a k element subset and B is an l element subset, which is contained in A. We are counting this pairs (A, Bs) such that this is, for instance this is the number of ways of selecting, this is the number of ways of selecting the first k element subset and this is the number of ways of selecting an l element subset of that selective k element subset. So, this, the total thing n chose k into k chose l count the number of pairs (A, B) such that A is a k element subset of n and B is an l element subset of A.

But this can be counted in a different way also, how you could have first selected B, namely the l element subset, that can be done in n chose l ways, right. So that can be done in n chose l ways. And after selecting B you can fill in, like we have done before, the remaining because we have already selected l thing from the remaining n element, n minus l elements of the universe, you could have selected any of the k minus l elements, so that you can make a k element subset, right, so that will be B is selected first, then A is selected, right. This will again counting the pairs (A, B) such that B is an l element set and A is say, k element subset of n, which is the superset of B, which is the same as saying that A is a k element subset and B is a subset of A, right. That is, that will give you this thing, so therefore this and this are equal. n chose l into n minus l chose k minus l, both are same, right.

So, here this is also, this also illustrates a usual counting strategy, the counting here, we counted a pair of things, right. We counting pairs, tuples, this kind of things usually happens in combinatorial arguments. And this tuple here, the pair is interpreted in two different ways, first is the first coordinate, the first member as the first selective thing and B is selected after selecting that or then the other way is looking at B and then selecting Bs first and then selecting A. This is also a kind of double counting, right and double two ways we are doing it, right. So, therefore, we can, we can see that. So, both the quantities are equal. We try verified it two, in two different ways, first by just putting in the equations, second by showing a combinatorial argument for that, again a double counting for that matter, right. So, the entire (A, B) is counted in two different ways.

(Refer Slide Time: 37:48)



Yeah, for, for instance if you wanted a, like matrix kind of form for this formula n choose k into n k choose l is equal to n choose l into k minus l, sorry, n minus l choose k minus l, right. So, we could have drawn it like this. We draw a matrix, right, then we list all the l element sets here. So, all l element sets, we can say all k element sets are listed here, namely how many we have. Total there will be n chose k of them and similarly, along the rows, all the rows correspond to the l element set, let us n chose l will be there, so n principal we can write like this. So, that means, the k element subsets are at least, are, are corresponding to the rows. We have k element subsets corresponding to the columns; we have l element subsets, so in columns we have l element subsets.

Now, we can say, that four, the i-jth entry here, this is, may be this is the entry, the row corresponding to A, this is a row, B corresponding. So, if B is contained in A, then we put a one here. Almost the same strategy we did it in the earlier one, right, and then we count like that. We create a matrix, right. Whenever this particular, the set l element subset corresponding to the column is contained in the set corresponding to the row, one row, then we will put a one in it. Then we, counting the total number of 1s here, total number of 1s here, so how many 1s are there?

So, one way is to find the row sums and add up, that is clear because a row sum will be there for a given, how many will be there because given a k element subset, there will be exactly n k choose l subsets, l element subsets of it. So, the row sum sigma arise here will be equal to n choose k into because there are n choose k rows and each row contains a k choose l, k choose l 1s. This is the row sum, and the column sum will be what? Column sum will be, because there are n choose l columns and here, again for each column, that is an l element subset, they are asking how many k element subsets are there such that it is contained. That, as we have argued, it is n minus l choose k minus l, right. So, this and this are equal.

So, that way, ok, we can again see it from the matrix as counting the number of 1s in the matrix, that (A, B) pairs we are counting, we told, that is, that correspond to a 1 here because this A correspond to a row, B correspond to a column, that is this pair, (A, B), right. So, ok, that was a different way of putting it. So, we, but if it is like easier to think in terms of counting the number of 1s in the matrix, we can always do it that way, right. Of case, so what otherwise we can think in terms of counting certain pairs, right.

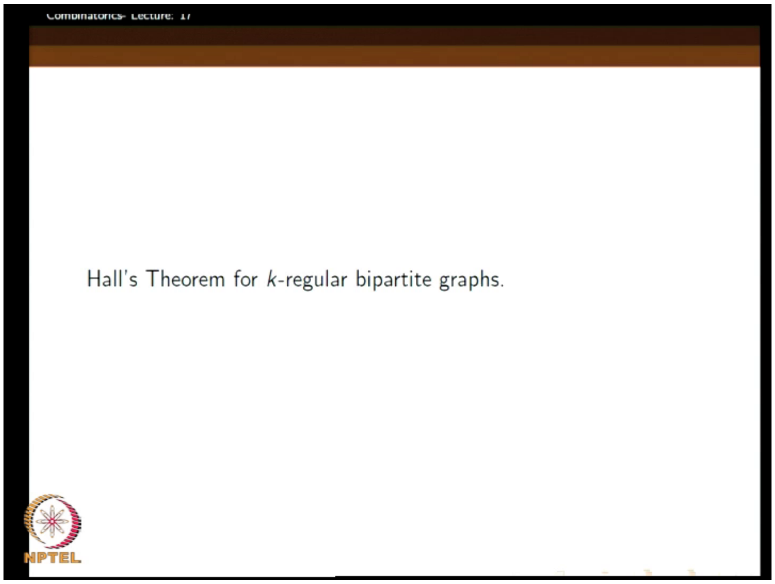
So, now, so I, we have proved it, so I just summarize, that the Turan's number we, we first explained, what is this Turan's number and then see, we wanted to get a lower bound for Turan's number in the process. So, what we did is, we made for the double counting, we made them rows and columns, rows corresponds. So, hypothetically assume, that required family of subsets are known and then we created a matrix for that such that each row corresponds to a subset, l element subset, an l element subset from that family and then each column correspond to any given k element subset there, n chose k of them, there are n chose k columns.

And of case, initially I made a mistake of working for an upper bound by saying, that ok, the row sum, there was no problem, that was F into cardinality of F into n minus l choose k minus l because each row contained, exactly we knew how many 1s will be there because given any k l, k l element subset, it will be contained in n minus l choose k minus l, k element sets. But the other way was not easy because you know we do not have all the n chose l subsets listed as rows, so therefore we do not even know what is this F. We just have a hypothetical property for that. So, for an upper bound we know, that each column can contain only at most k choose l once because it is a k element subsets. How many l element subsets can at most be there, that is what I tried first, so that only led to an upper bound and we show that it led to a trivial upper bond, that was not useful but still it illustrates the double counting argument.

Anyway, whatsoever, but lower bound we need to get a lower bound for the, the number of 1s in the columns. Of case, it can be 0, of case, because a given k element subset, what is just an arbitrary family of subsets you have taken for the rows, then it can be even 0, but in this case this family had a property namely, that you take any k element subsets, at least one of its l element subsets will be in that family, that is why we know, that there is at least one 1 in it. So, therefore, if you add up the 1s column-wise, each column will contribute at least one 1. So, there was total sum, total number of 1s will be at least the number of columns. So, that is at, that is where we got n chose k.

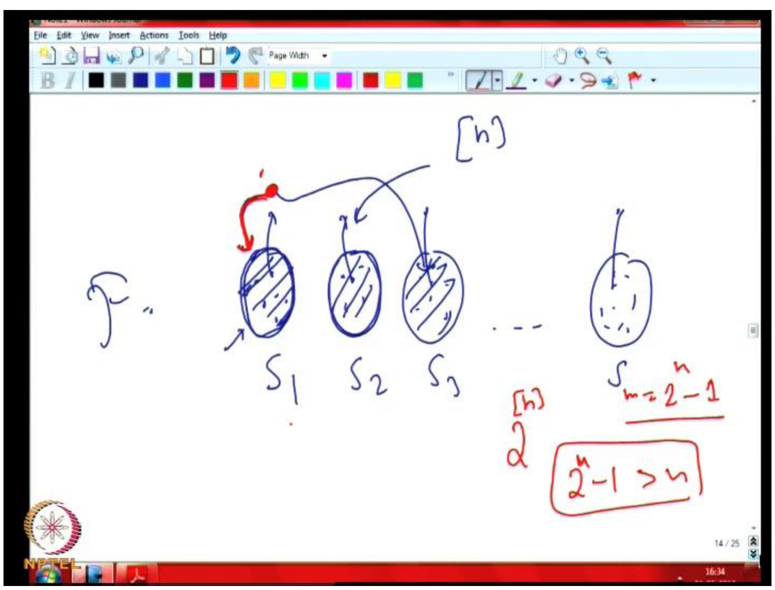
So, we show, that the cardinality of that family into n minus l chose k minus l has to be greater than or equal to k chose l, sorry, n chose k and we divide it and then we got a quantity and the way we gave the lower bound was by another ratio, but we showed that that ratio is the same as this in the end. That again we use the double counting argument when we gave a combinatorial proof. This was the summary of this last proof. So, this is a slight goof up in the presentation, but again, it is not much of a problem.

(Refer Slide Time: 44:41)



So, and then next one we want to consider is, so I quickly give another example of this double counting is again using this bipartite graph things and also, this is a very important topic for not giving you the full proof. So, this is the question we want to consider.

(Refer Slide Time: 45:12)

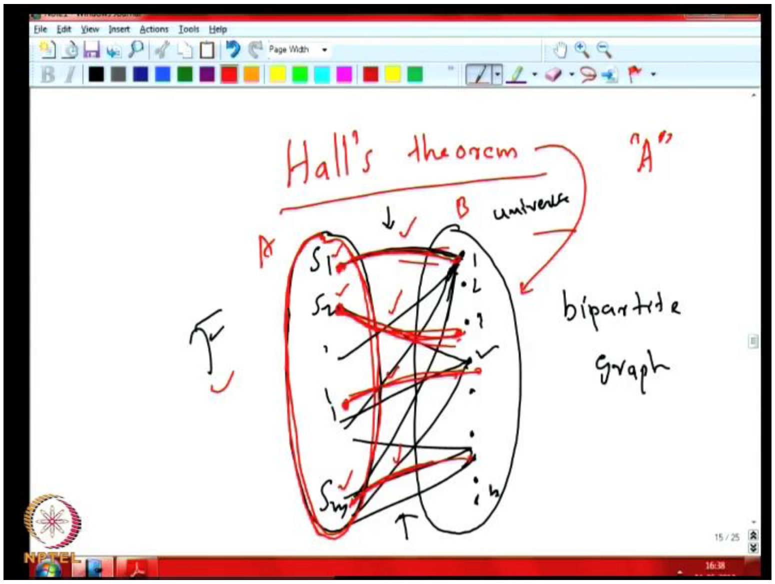


So, we have n, n element, n element universe and then we have some family of subsets. So, let us say these are the subsets, I am just writing it as S 1, S 2, S 3, some S n. So, this is a family of subsets, right. So, now, so that means some members of 1 to n will be here, 1 to n, some members of 1 to n will be here, some, these are subsets of 1 to n.

Now, we want to select a representative from this thing, a representative from this thing, a representative from this thing, a representative from this thing and we should be able to say, that this representative is for this set. He may belong to another thing, but for instance, these are all, some say this is some groups of people, right, say it is possible, that so for instance these are people who, who can speak English, so these are people who can speak Hindi, these are people who can speak another language, so Tamil and so on. Now, it is possible, that one person can speak both English and Tamil, but so for instance, we, he selected as a representative of English speaking people, then you should represent them. You should not help the other, right, other community for, maybe in any discussions. You should represent it entirely this English speaking. So, we need distinct representatives' work, it does not matter whether their representative belongs to another community, also it does not matter, but we want a distinct one for each of them. Sometimes it is not possible for that you can easily see. For instance, if there are too few people and too many sets, for instance if we collect all 2 raised to the n subsets of it, so suppose S 1, S 2, S 3, S n and say, we have S, this is S 2 to the power, so why should I put n, so it can be m, right, so this, this m can be 2 to the power n, means all possible to, to the power n minus 1. If you want to remove the empty set because empty subset, anyway, does not have any, person cannot get a representative for that one.

So, now it is very clear that it is not n is large enough, of case, definitely 2 raise to n minus 1 will be greater than n, right and there are only n people and then one of them represents one, another one represent another one and finally, there will be, still there will be sets, which are not represented.

(Refer Slide Time: 48:08)



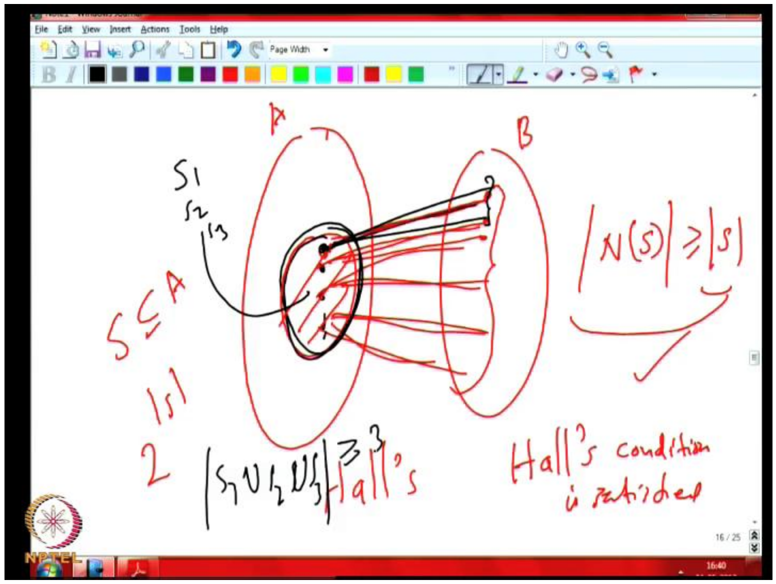
So, the question now is to know when can we have distinct representatives, that this is what the Hall's theorem is concerned with. So, Hall's theorem gives a condition, Hall's theorem gives a condition, which, which gives its equine only condition necessary in sufficient condition for like being able to, so that will, that is necessary in condition for the existence of distinct representatives for all set, all subsets in the family, right. So, though we have put it as a distinct representative and like we have done in the last example, we can also see it as a problem in the bipartite graph.

So, this is, as I saw about set systems, it is nice always to see it as bipartite graph. See, sometimes it can be, we see in the last case, we have n, n, all the n element subsets were, sorry, k element subsets were listed here and l element subsets were, the inclusion was captured in an edge here, what we can do is the universe itself, the members, the people, right, the members of the universe, they can be listed as, say this is 1, 2, 3, up to n, right, this can be listed as the vertices of one side, the sets of the family, this is the family, S 1, S 2, S m can be listed as the other side. Now, how will you, so put the edges. If this member belongs to this set, then try put edges. For instance, inclusion for instance, if this not inclusion, so what is a membership if for each element you can put a (( )) if that, that element is a member of that set. Also, for instance, if this particular set has an edge to this, if it contains it, so that is a matriculation also, so these edges can be put like this.

Now, once you put these edges, so we have a graph, we have a graph, so it is a bipartite graph, right, and both sides of the bipartite graph represent something. So, now what do we want? So, we say, that I want one person to be assigned this thing, one person to be assigned to this set, one person to be assigned to this thing, one person to be assigned to this thing, right. If some persons are not assigned to any, it does not matter. What I want is, for each subset here in F we need a distinct representative, so this person should not be as if, it, if he is assigned to this, this person should not be assigned to anybody else. This is called a perfect matching of this side. So, ok, not perfect matching, sorry, this is called matching of this set, this set. Say, this is A and this is B, a matching of A, this matching, matching of A means, every vertex of A is matched by that matching, right.

So, now the condition we want, Hall's condition is concerned with giving a condition for the, is for the existence of such a matching, right, need not as, as we have explained it, need not happen. For instance, if m is too large compared to (( )), it will not be possible, right, because it will finish off one distinct representative, one representative given for this, one is given, one is given, finally after sometime all the members here will finish off, and then it is not possible to get. So, one of the case that is a trivial condition that we should have, at least as many number of elements in the universe as there are sets, then only we can get distinct representatives, but is that enough? Even that may not be always valid. So, if you, this essentially comes in the graph theory course, it is a very detailed discussion in graph theory books on this thing or also the NPTEL course on graph theory also discusses this problem. So, therefore, we would not get into further details.

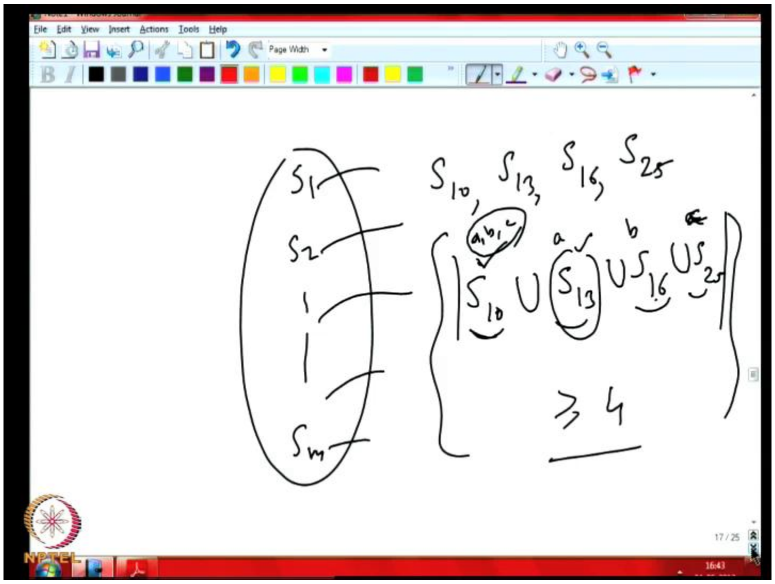
(Refer Slide Time: 52:39)



But what I will just tell, what the Hall's condition tells, so Hall's condition says, so this is A, this is B. Now, if we take any subset of A, so you take a subset S of A. Now, we can see the neighbors of this S here, neighbors of this, this means any member here, which is, which is an edge into this, right. And count the neighbors, so this is called N of S, usually number of neighbors. If number of neighbors is always greater than S for any set you take here, right, non-empty set you take here, if the number of neighbors is greater than equal to even empty set is ok because that will be 0, greater than 0, greater than equal to S, then we say that the Hall's condition is met for this graph. Hall's condition is met for, this is satisfied for this bipartite graph means, take any subset, there are 2 to the power S possible subsets you can take including that empty set.

Therefore, each such subset, the number of neighbors N of S, cardinality of N of S has to be at least the cardinality of the set itself, then we say, that the Hall's condition is satisfied. And Hall's theorem says that if the Hall's condition is satisfied, then it is possible to get a matching of it. We can get a distinct representative in that terminology of set system. For instance, if each vertex on the S, I corresponded to a subset of the universe, then this condition, Hall's condition says if you, for instance, you know, for every subset the neighbors are actually the members of it, right. You take the union of this subsets for instance, this is S 1, S 2, S 3, then you take S 1 union S 2 union S 3, that cardinality should be greater than equal to 3, right.

(Refer Slide Time: 54:46)



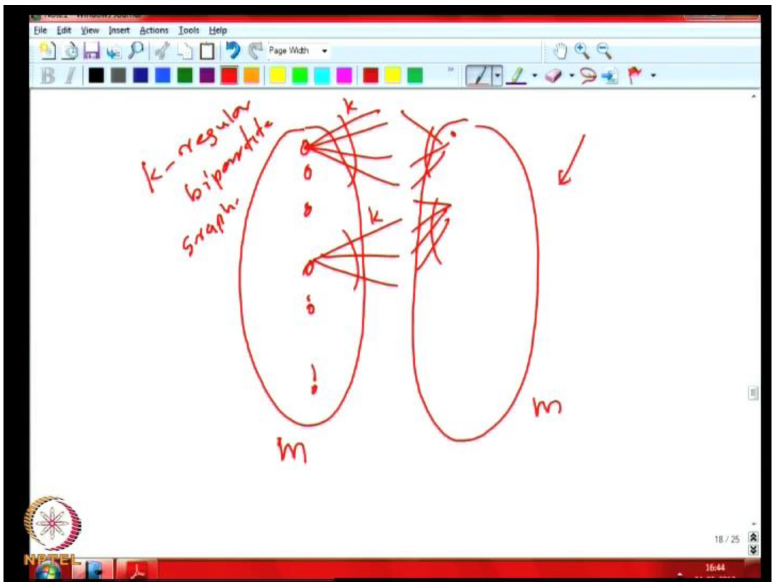
For instance, here, so you can, so this is S 1, S 2, same, right, you can take say S 10, S 13, S 16, S 25, four subsets. Now, consider the union of S 10, S 13, S 16 and S 25 and find how many, how many are there in the union and there should be at least four of them, four correspond to one to this. There are four sets here, so at least four should be there, then only the Hall's condition is met. You know, when the bipartite setting what happens is this S 10 is a vertex in the A side and its neighbors are the members in it, the elements of it, right. And the neighbors of S 13 are the members of the members, the elements of S 13 and similarly, the neighbors of this vertex S 13, S 16 will be the elements of 16 and so on.

Then, you may wonder why so. This, each of them have one element, at least we want it add up to four, not necessarily because it is possible, that what S 10 contains may be contained in S 13 also. In fact, they can be same, right, or maybe there are two. Say, for instance, A and B are here, this can be A, this can be B, this can be another A, right. So, so for A, B, C, so for this can be C, so everything together, it is only A, B, C and C, you, if you take the union, right. So, we do not have four elements here, four sets does not guarantee that the union contains four elements.

So, the Hall's theorem says, but for any collection of subsets you take here and if you take the linear of it and then you, you find the cardinality of a union, if it is at least a number of sets in that union, then the Hall's condition is met. And if the Hall's condition is met, you are guaranteed to have a matching of the A sign, that means, you are guaranteed to have a distinct representative, system of distinct representatives, right. This is what we see.

Now, I can see, of case, one should, you need a proof for that. Of case, we need a proof for that that you should probably get it from some graph theory course or textbook. So, I will not get into the details of the proof.

(Refer Slide Time: 57:17)



But here, I introduce these things, of case, this is an important thing, but I will not get into the proof of that, but rather I would think, it, I will give one special case of it and show, that the Hall's condition is met by doing a double counting argument, namely the case of k regular bipartite graph, k regular bipartite graphs. What is k regular bipartite graphs? So, from the A side the degree of each vertex is exactly k for everything, for every vertex on the A side, the degrees and this side also it is k; this side also it is k. Now, it is clear that, yeah, for instance, we will start using the double counting argument to say, that the, the, the trivial condition is met, namely. So, if there are n sets here, there should be exactly m elements in the universe. Why, because here we are adding up the degrees. I think it is time, I will continue in the next class.