Combinatorics
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Lecture - 16
Generalization of Binomial Coefficients Part (3)
Double Counting - Part (1)

Welcome to the 16th lecture of combinatorics, fine.

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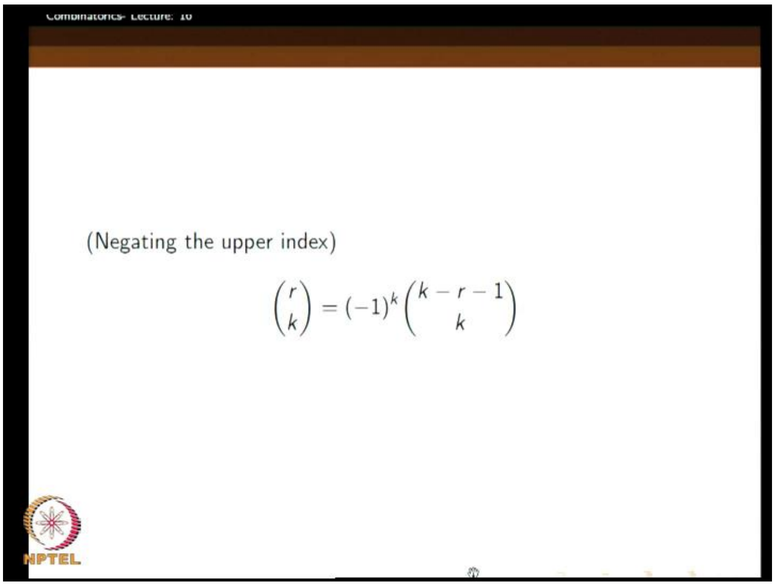

So, in the last class, we were discussing this identity r chose k equal to minus 1 raise to k into k minus r minus 1 chose k.

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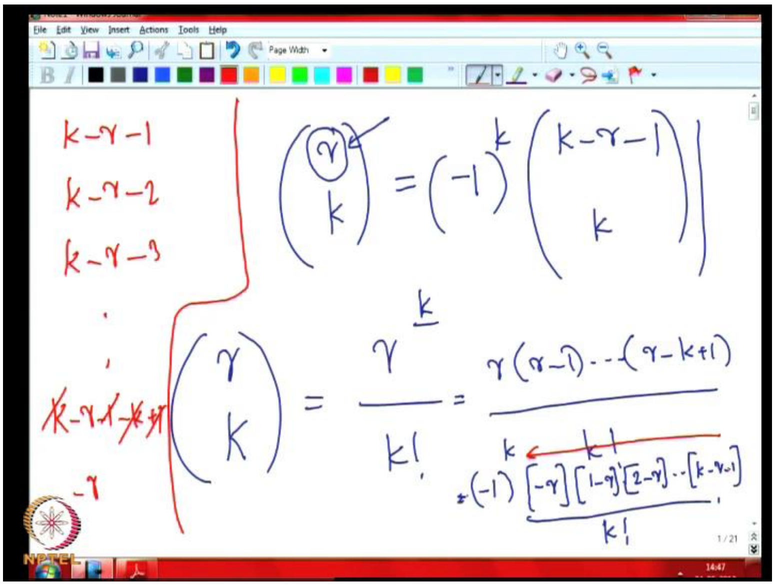

So, just to remind you that we had generalized the binomial coefficients. Now, r can be any complex number and k is integers, any integer. But then when we are talking about this identity, we are talking about r being either positive integer or negative integer. Now, non integer values are not considered. So, that is very clear, because this would not make sense when you say, right.

So, we are only taking about positive and non-negative integer values. So, that is what we show. So, we show from the Pascal's. What we did is we described the Pascal triangle, but again when that is where we did the discussion started, but of case so we can see that the derivation does not need this, fine. So, we will, one minute. So, this is not like that. So, here we have generalized r to all complex numbers and you see the derivation is like this. When k is negative, this is anyway 0 and then, here both sides we get negative values. Therefore, k be negative both sides, we get zeros.

So, there is nothing to worry. We can assume that case of positive or zero quantity and then, we see that even when k equals 0, we say that it is easy because r choose 0 is 1 and this is also 1 by definition and then, this k minus 1 raise to k. 0 will become 1. Also, if we can assume that k is a positive integer and we started from this side, r choose k. This is by definition now because k is positive. We only have to take the upper part. So, k being negative, we do not have to worry. Therefore, this can be written as r k form factorial divided by k factorial. This is the most general thing for we are not putting any assumption on r. So, this is essentially r into r minus 1 into r minus k plus 1. So, k terms we have to let in this k factorial, right.

Then, what we did is we extracted a minus 1 from each of these terms in the numerator. So, that is that k was minus 1 raise to k and that numerator will become minus r into minus of r, minus 1 became 1 minus r into. Then, this became 2 minus r and shown all the v 2. Here, it became k minus r minus 1, right. So, these are as we can see, so we start from the last terms is when you neglect it. Minus k plus, this become plus k, this becomes minus 1, right. If you look this, this is divided by k factorial. If you stay aware of this thing, you can see that this last one.

So, if this starts from k minus r minus 1 and it will be decreasing downward as say, we have minus together this one. Therefore, for instance when you minus r and add plus 1 here, this will, sorry minus k terms. If you come down, we will reach back to here like this. So, for instance if I had written minus k a, sorry here I am sorry. If I consider this was the first term k minus r minus 1 and then, k minus r minus 1 minus 1 that is minus 2 and then, k minus r minus 3 and so on. So, k terms if I consider like this, what will happen is we will get k minus r minus 1 minus k plus 1 will be the last term, right. So, that will cancel off this thing and we will end up with minus r, right.

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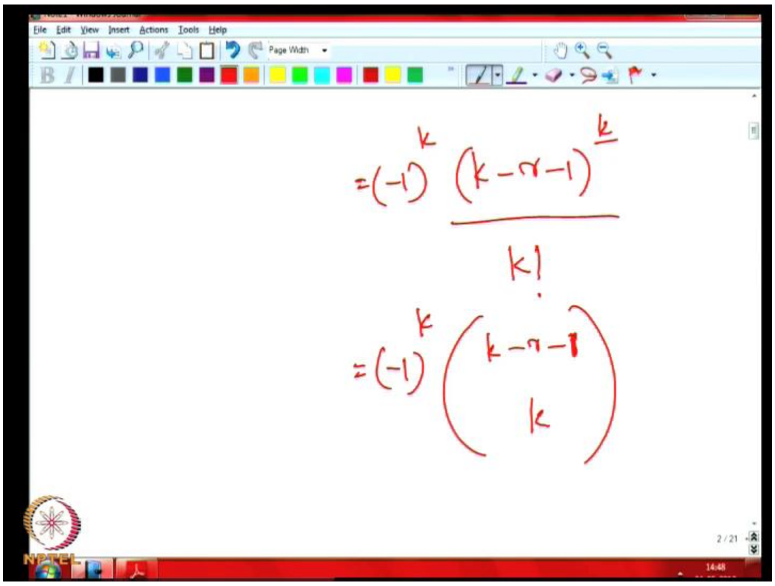

So, this is what we can read backward and therefore, this term which is in the numerator can be replaced by minus 1 raise to k into k, k minus r minus 1. K is following factorial divided by k factorial below, right and this is exactly minus 1 raise to k into k minus r minus 1 chose k because anyway, k is again the positive quantity. Only k we have not changed. So, the only numerator has changed. Now, this is become k minus r minus 1, k following factorial. Therefore, this is how we prove this thing. So, if you start with a positive quantity, this may like come down r r minus 1. This may have a 0, but that is not a problem. So, we are taking minus 1 raise to k into 0 only there because when you neglect it, it will remain a 0. It will read backward.

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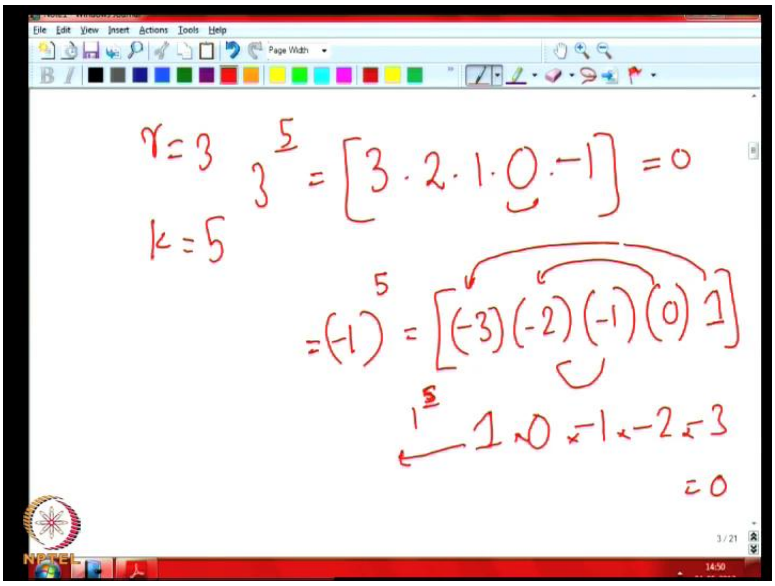

For example, if you start with minus 3 and put r equal to 4, sorry r equal to 3 and k equal to 4, then we have 3, 2, 1. We can put it as 5, 4, 0, minus 1, right. This will be my 3, 5 following, right. Now, when we extract minus 1 out of it, minus 1 raise to 5 will come out. This is taken and then, this will become from each of them, this is a quantity 0 only. So, of course that is not going to change, but I can put whatever, but I am just illustrating that point. So, this will become minus 3 into minus 2 into minus 1 into 0. So, minus 0 is 0 into 1, right.

Now, we start reading from 1, right. So, if try to 1 here, 0 here and inter change the positions that will read as 1, 0, minus 1, minus 2, minus 3. It is decreasing, right. So therefore, this will be 1, 5 following factorial, right. Of course this is also 0. Therefore, this is 0 is trivial that these are equal and 0 is coming here. Both of them will be anyway equal, but I just wanted to bring to your naught is that sometimes 0 may be there.

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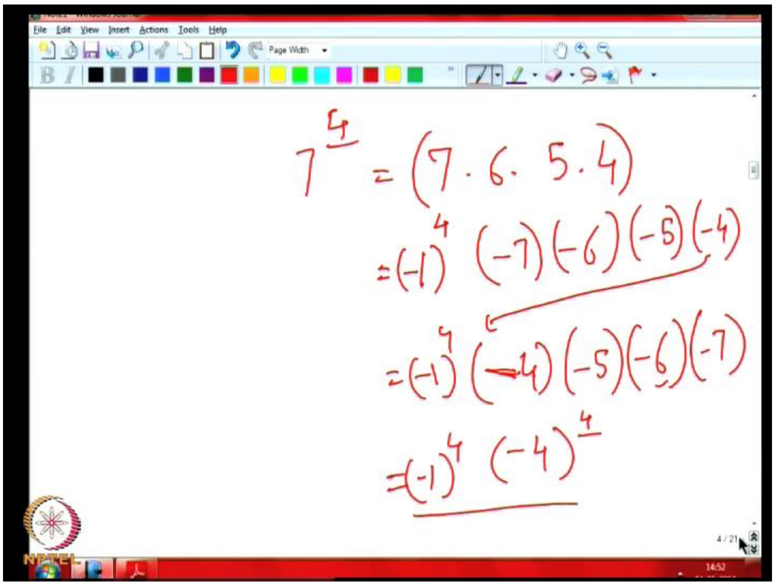

Otherwise, for instance if I just write 7, say four following factorial. This is what 7 into 6 into 5 into 4. Now, if I extract minus 1, this is minus 1 raise to 4 into minus 7 minus 6 minus 5 and minus 4. Now, we can rewrite it as, see the order. We can change. Start from here, right. This will come here, minus that is minus 4 first and then, minus 5, then minus 6, then this is minus 7.

So, that is starting with minus 4. This minus 4 minus 1 is minus 5 and that minus 4 minus 2 is minus 6 and so on. Therefore, this is essentially minus 1 raise to 4 into minus 4 for following factorial and then, divided by of case I had to put four following factorial. So, this is why it is happening, right. So, I just illustrate the calculation just to clear the confusion. So, before that proof works, we do not have to, but I just wanted to make sure that there is a slight clarification I wanted to give about the case.

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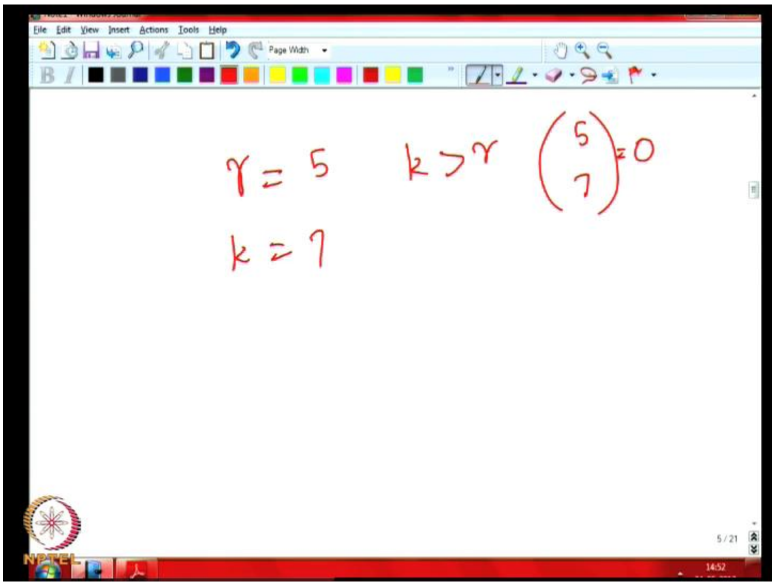

So, for instance r chose k, right. See there is nothing like, see I can also have r equal to half here. It is not going to change the argument. So, let us say the situation of r equal to some 5 and k equal to 7. So, we know k is greater than r here. So, if I chose 7, this is equal to 0, right. As we have seen in the proof, now both cases when you in both sides have 0 and it will become 0 equal to 0, but what is happening here is just to get a good picture of that.

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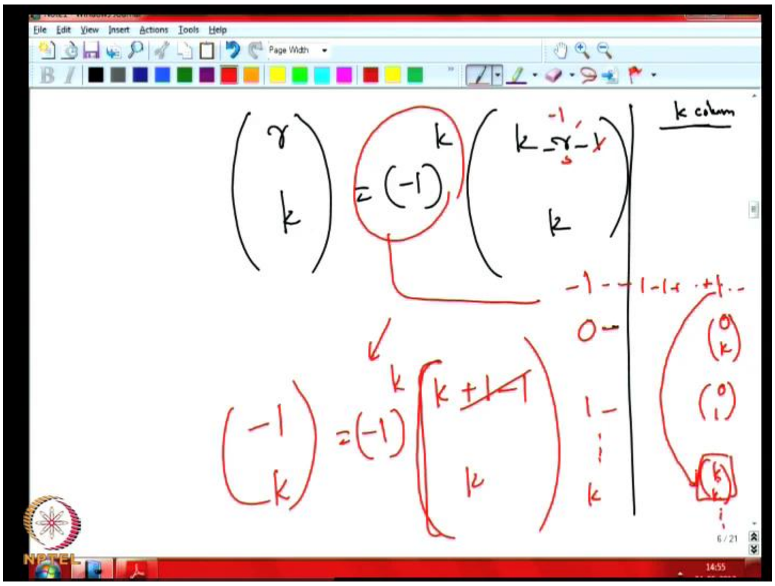

Now, this inequality says, r chose k say equal to minus 1 raise to k into k minus r minus 1 chose k, right. So, if you concentrate on the k-th row, say if we write k-th column in the Pascal triangle. So, what you see is for instance, this is the zero-th row. So, we will see the k-th row 0 chose k, which is 0 of case and then, we will see in the first column 0 chose 1 and like that, but these are all 0's but, till I reach k, right. So, k chose k that is a first real. I mean non-zero entry here. From there on, we will start getting non-zero entries, but in the minus 1 column here itself will have as we have seen in the Pascal's triangle in the last class. Here itself, we will have a minus 1 or plus 1 depending on whether k is a positive or negative, because you know from this thing what it says is.

So, here minus 1 raise to, sorry minus 1 chose k is actually minus 1 raise to k into. So, k minus of minus 1, that is plus 1 minus 1, right. This will cancel of. Sorry, chose k k chose k. That means, it is picking up k chose k. I am neglecting it. So, this quantity here when you put r equal to minus 1, this side will go this, just k chose k. This is just the only difference is that there is a minus 1 raise to k here. So, there can be a, if k is even number that will be plus 1. Otherwise, it is if k is an odd number, this will be minus 1. That is what we had seen that in the negative one-th row. We always had plus minus 1, plus minus 1 like that, right. So, this one actually corresponds to this k chose k, right.

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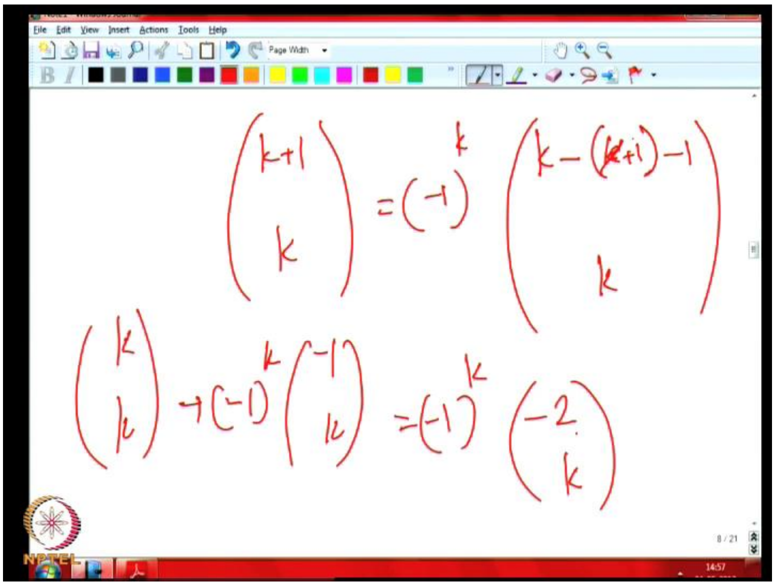

Now, if you want to do it from here, what is k chose k? K chose k, that means put r equal to k and this will be equal to minus 1 raise to k into k minus r, that is this minus k. Then, another minus 1 chose k, right. This is minus 1 raise to k minus 1 k, right. Now, if you want to try k chose k plus 1, what will this become? This will become minus 1 raise to k plus 1, right. The sign changes now because what does minus 1 raise to k? It will be different in minus 1 raise to k plus 1 and this will be k plus 1 minus because r is k minus, sorry k plus 1 chose k. Sorry, I wanted to discuss k plus 1 chose k. So, this will be k. So, k plus 1. So, this will be k k minus k plus 1 minus 1 chose k.

So, 1 k will go. So, we will have a minus 1 minus 1. This become minus 1 raise to k into minus 2 chose k minus 2 chose k. So, this is the way we see that. So, k plus 1. So, k chose k became minus 1 raise to k into minus 1 chose k. So, k chose, sorry k plus 1 chose k will become minus 1 raise to k into minus 2 chose k and so on. So, in that column, there is a correspondence.

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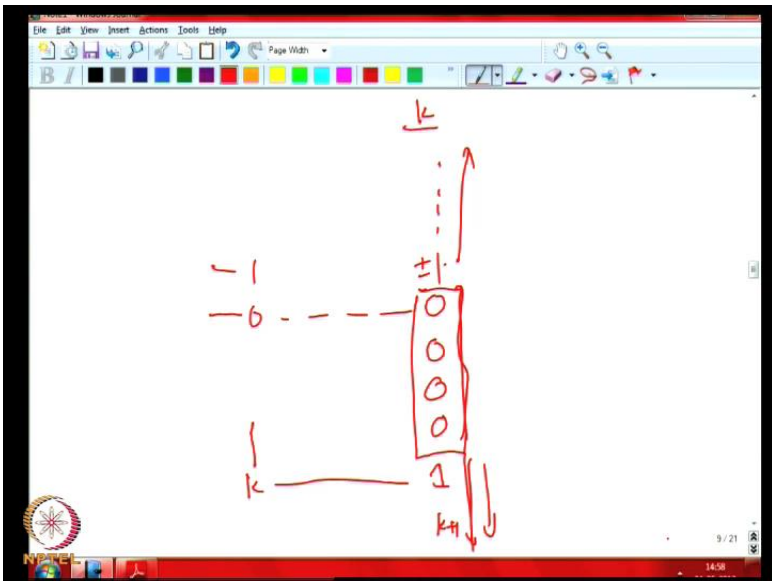

So, in that k-th column, if you have a zero-th column here, you will see 0, 0, 0, 0 for some distance till you get the first entering. The k-th row k chose k is 1 and then, this is k plus 1, chose k is k plus 1 and so on. Like that we will read here, but here from minus 1 onwards, minus one-th row we have minus. So, plus 1 minus 1 depending on whether what is k. Then, here this means sign and then, this will be copied. That is what it says, right. So, there is this few zeros coming in between. Other than that these columns are you can read from here and here, except for the sign like maybe different. So, that is what that inequality says and you can clearly see what is happening. So, why that is true from the derivation we have.

Now, the next one. So, this discussion we stop here. See the discussion about the generalized binomial coefficients; I am not perceiving it further. It is in an interesting topic. One has to carefully do the analysis there. So, we tend to make mistakes when we argue because we can following to several traps like because it is no more counting at. Be very careful when about all the like there were we make sense or not because we just manipulate formulas and this is quite possible that we may make some mistakes, but if you are quite interested in that kind of material, then you should take concrete mathematics, the book concrete mathematics and this chapter 5th contains lot of material on such things.

Now, we move on to the usual kind of combinatorics, namely counting in such like where we are counting the objects and we are talking about how to count and things like that, right. Now, after seeing lot of concepts, now we are back to where we started with one technique, namely the pigeonhole principle. The course started with the simplest technique, namely the pigeonhole principles. Now, we discuss a few more techniques. Another such simple technique is so-called double counting. What is this double counting? It is something like this, for instance you want to count the same thing in two different ways. So, typically we can model this thing using a, say matrix 0, 1 matrix.

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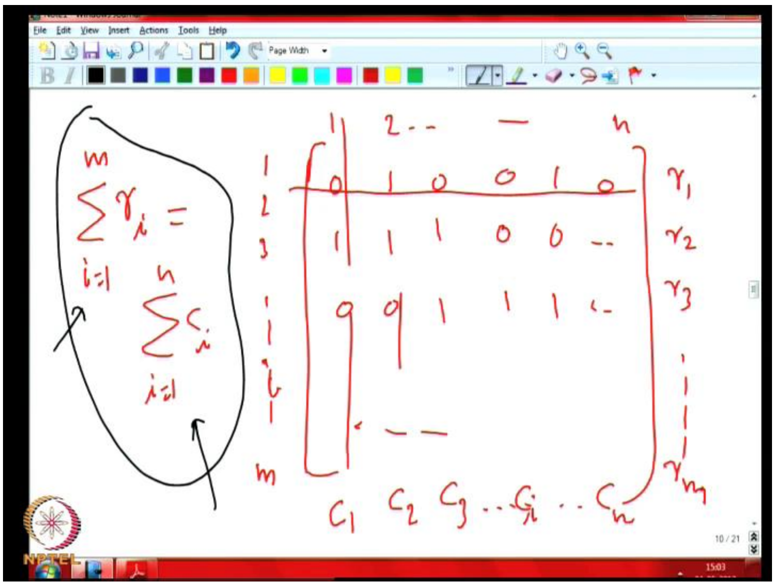

So, we see this matrix. The columns are numbered, say 1, 2, 3 up to n and the rows are number 1, 2, 3 upto m. This is some place we have 0, some places we have 1's, this is the 0, 1 matrix, right 1, 1, 0, 0, 1, 0, 1 like that. Some entries are there, so in i j-th entry, for instance you may get 0 or 1 depending on row. So, that is what it is. Now, the point is that suppose we are interested in discussion how many 1's are there in this matrix. Then, we know that one way to find this out is to count like this. Count like what? So, you just count the number of 1's in the row. So, let us say r 1 is the number of 1's in the first row and we count the number of 1's in the second row. Let r 2 be the number of 1's in the second row and then, let r 3 be the number of 1's in a third row. Then, the r m be the number of 1's in the n-th row and then, definitely we know that the number of 1's in the entire matrix is r i. So, the sum of r i i equal to 1 to m right.

So, we are counting the number of 1's in each row and summing it up, but then we could have done it in a different way also, namely we could have found out how many 1's are there in the first column. Let us call it c 1. Similarly, how many 1's are there in the second column, how many 1's are there in the third column and so on. So, the i-th column, there are c i 1's and then, the n-th column, there are c n 1's. So, we could have add it them together also, right. This is c i i equal to 1 to n, right. Both these quantities are same, right.

So, see this is too simple, but we can model many problems in this way. So, I will give some examples soon, but sometimes we can infer some interesting things by doing this double counting. The double counting refers to counting it in two different ways and then, this comparison may give us some important information, interesting information. This is like without like whether we discuss it or not, we use it even without thinking, but in combinatorics, this is very common, so that we should like we cannot of this thing. Therefore, we discuss a few examples, right.

So, most of this material I am discussing here is from Stasysjukna's extremal combinatorics. The initial examples are taken from there. At a party, the number of guests who shake hands an odd number of times is one. What does it mean? So, in a party, several guests come and what they do is they shake hand. Shake hands with everyone else. They cannot shake hands with themselves, right.

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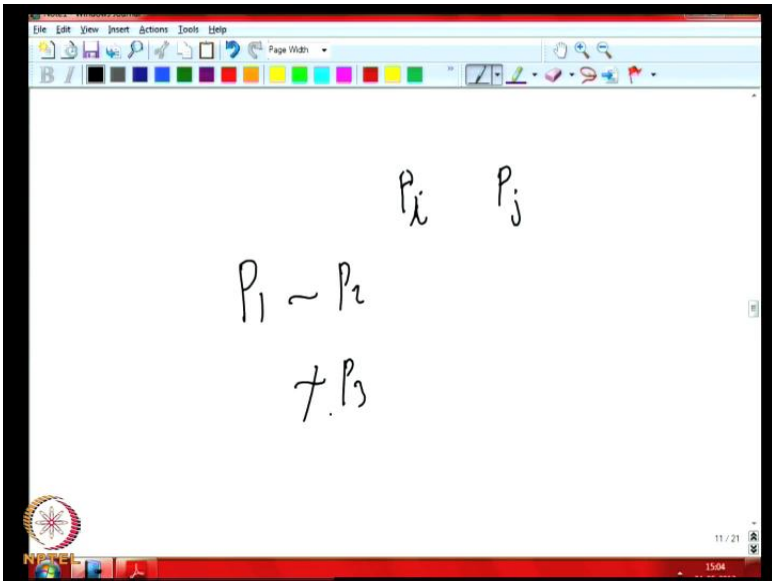

For instance, if a person p 1 will shake his hands with a person, say a person p i will shake hands with p j. If i not equal to j, this is what we are saying that i is not equal to j. Now, the thing we want to prove is that the number or not necessarily, sorry I am sorry. So, this is they do not shake hands with everybody. They would not shake their hands with themselves, but they shake hands with some person who came in the party. It is possible that p 1 shakes his hands with p 2, but not with p 3. This is possible, right. So, one thing we can like if you want to represent it using a graph, right. We had discussed graphs before.

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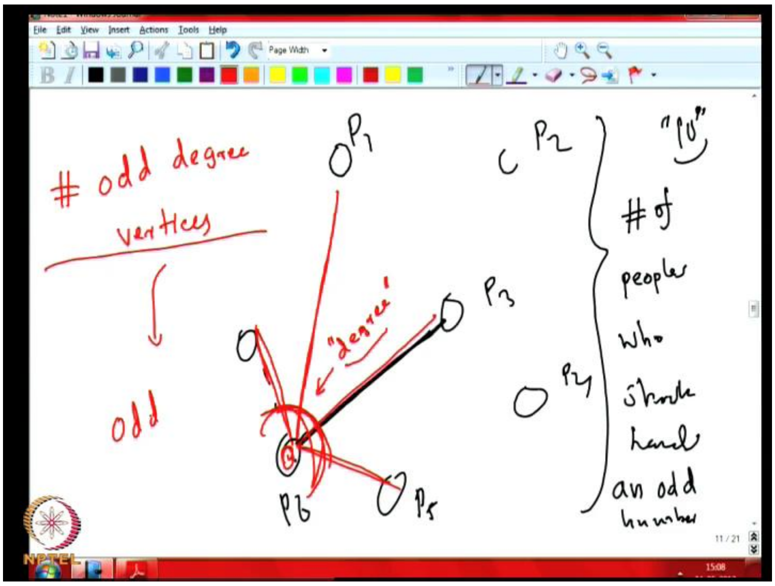

We can have each person as a vortex, say p 1, p 2, p 3, p 4, p 5, p 6 and so on and then, if say p 6 and p 3 shakes hands, we can put an edge between them, but the point we notice that this is a symmetric relation, in the sense that if p six shakes hands with p 3, then p 3 is shaking his hands with p 6 also. It is not possible that p 6 has shaken his hands with p 3, but p 3 has not, right. Therefore, we can always put h between them and what we are interested in is the number of people. So, the graph need not be complete of case. Though I just started telling that it can be, this is a complete gap. Not necessarily we do not need to argue that it is complete graph. We just want to say that here is some graph that is what we can modulate using a graph where persons are the persons who attend the party at the vortices and when two persons shake hands, we put a h between them. We get some graphs, right.

Now, we are interested in the number of people who shook his hands. So, who has shaken his hands an even an odd number of times. Odd number times for instance, once, 3 times, 5 times, we are not interested in. Suppose a person p i has shaken his hands 10 times, we are not interested. So, that is an even number, right. Now, how many such people are there? This is the question, right. So, this question definitely can be post as a graph theory problem. Also, what is the corresponding graph theory we posted? So, what we are asking is when you count go to p 6 and count the number edges going or incident on him, count the number of edges because this is he has shaking hands with him. This is the number of hands shakes that is associated with him and this is actually the degree of this vortex.

We have already discussed what degree is. This is the degree of this vortex, right. A number of times he has shaken hands, right. The number of edges incident on that particular vortex, we are interested in the number of vertices of odd degree. This is what we are number of odd degree vertices. We want to show that the number of odd degree vertices is always odd. This is what we want to show.

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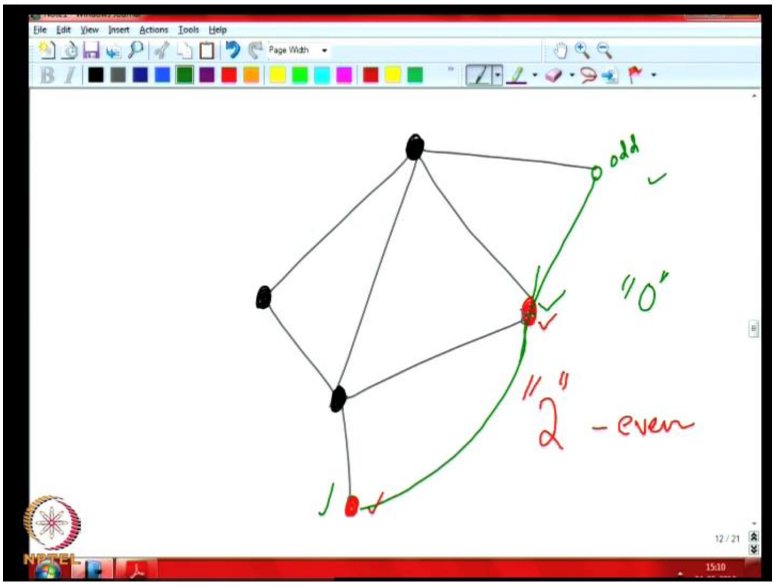

So, we can take a small graph and see whether this is true or not. So, let us say, here is a small graph. I just drew without thinking someone’s graph. So, this is small graph. Now, let us count the number of odd degree vertices. I will just mark it using red color, right. So, this vertex is definitely not an odd degree. This is a degree is 2 here, right. So, this is not now, this is 4 again. This is not an odd degree vertex. So, of course I can use a color and show it this is not an odd degree vertex, this is degree to and this is also not an odd degree vertex. This is a degree 4. This is not an odd degree vertex, this is degree 2 and this one is again a degree 4 vertex. So, this is not an odd degree vertex, but the remaining two are odd degree vertices. This one, this is a red colored one and this red color, this and this is odd degree vertices.

Now, how many odd degree vertices are there? There are two odd degree vertices. That is an even number. Is it possible for me to somehow like make a graph, draw a graph where the numbers of odd degree vertices are even? Sorry, not even that means odd that we are saying that. That is not possible. For instance, I can try to make it odd by putting. So, I connect these two, some other vertex, say I try to connect this person, this vertex. Then, what happens is this becomes even degree vertex, but then here it has become odd, right. So, therefore, gain it is true. So, it is true for any edge you put here or if I had try to connect here, so both of them will become even degree. So, odd degree vertex, there will be only zero odd degree vertices again, any given number, right. So, we can probably try to prove this in these lines, but there is a different proof by using double counting.

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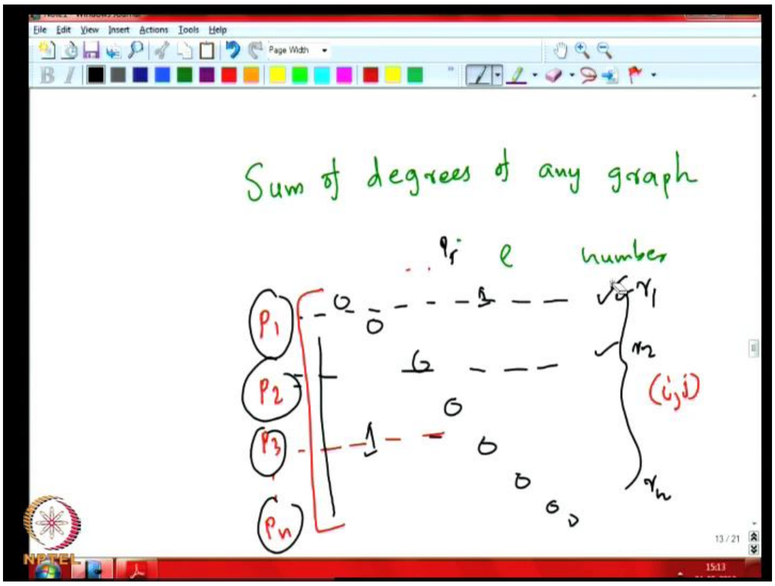

So, this is what we are trying to do. First of all, we note that the sum of degrees of a graph, any graph. So, when I say graph, this is into graph. We are of course not allowing self looks now. So, some of degrees of a graph is always an even number. This is what the key thing here or in other words, you sum the number of handshakes of all the persons, what we get is always an even number. So, I promise that i will work with a matrix. Though it really does not have to, but we can keep this as the model. For instance, here I will keep it as p 1, p 2, p 3, p n as the persons and here along the columns, I will again write p 1, p 2, p n as the person I have persons only.

Now, in the position, this is i j-th position. That means, p i and p j, they have shaken hands. Then, I will put a 1 here; otherwise I will put a 0 here, right. So, for instance here p 3 and say p 5 has shaken hands. Suppose, p 2 and p 2, they have not shaken hands. So, therefore, I should put a 0 there. So, along the diagonals I should have 0. So, of course some of case, this is definitely I am not drawing properly. So, see along the diagonals I will have zeros because p i will not shake hands in other cases. Of course, this symmetric here, if I put in the p i j-th position, there is a 1, then there will be a 1 in the other position also, but now what we are interested in is the total, the count of handshakes, like not the actual number of handshakes.

For instance, I ask this p 1, how many handshakes you have and then, add it to the number of handshakes the p 2 have and the number of handshakes the p 3 have and so on. So, one way is to a row wise count. So, here that is what we are saying that p 1 will say I have say so many numbers and see, p 2 will say I have so many numbers and then, I will add this. So, I had called it r 1, r 2, r 3 because these are the row sums, right r n. Then, that will give me one count, right. This is actually the total number of handshakes, but sorry another way to do so, I am again so this is of course not the way to draw the matrix. So, I have to do it of course once again.

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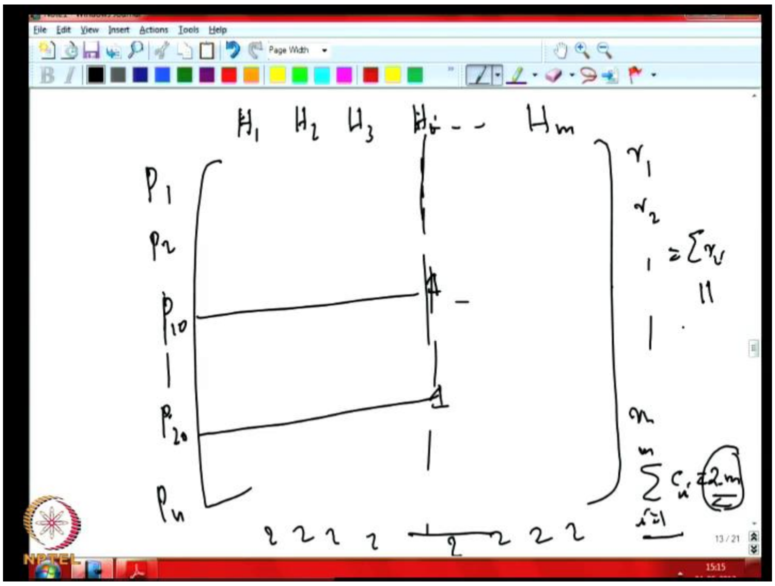

So, I draw a matrix with the number of persons here. So, once again p 1, p 2, p n here and the handshakes themselves. For instance, there are several handshakes which have happened, right. So, earlier what I had done is the adjacency matrix over the graph. Sorry, for that a handshake I will say handshake 1, handshake 2, handshake 3, handshake m. There are m handshake suppose, right.

Now, what I am doing is any handshake involves two people, right. So, suppose this handshake, say I was between say p 10 and p 20, then I will have one, one here and one,

one here, right. Therefore, if some of the columns I always get two-two everywhere, right. Therefore, if I sum the columns, that means sigma c, if I take I equal to 1 to m, so what I get is 2 into m, 2 m, where m is the number of handshakes, right.

Similarly, what we are interested in is the row sums r 1, r 2, r n. So, sigma r i is what I am interested in. So, sigma r i is equal to sigma c i as we have known. So, that is 2 m. Therefore, it is an even number, right. That is what we can do once again. So, it is an easy thing. I just try to put it in the matrix form. So, just not that we are interested in summing the counts of each person in way. The count of a person means the number of handshakes he has made, that is equal to H, the column sums.

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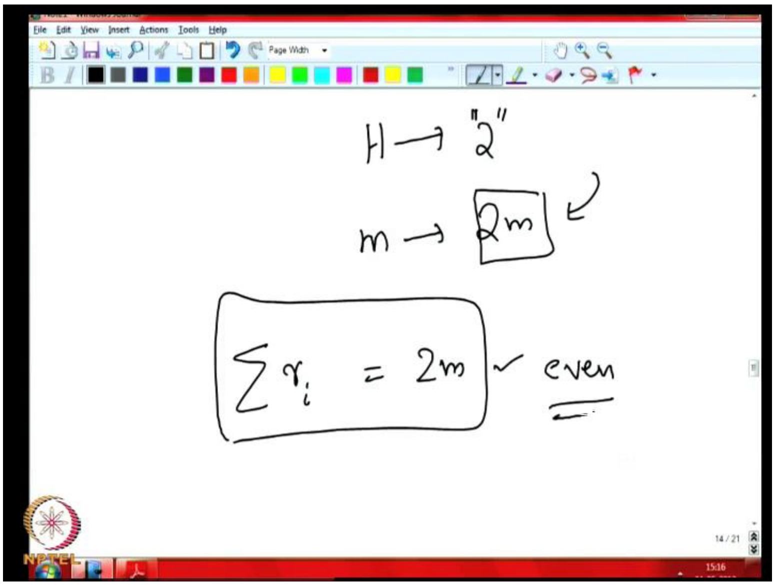

The column sum means a column corresponds to a particular handshake that involves two persons, right. So, there are two one’s in each column. Therefore, if there are m handshakes, whatever m may be 2 into m 1’s will be visible inside in the matrix, the 0 1 matrix, but this is actually the row. Some there you keep for that is this is equal to 2 f. The total number of handshakes is 2 m l and this is an even number, but this is of course an even number. This is not what we are interested in.

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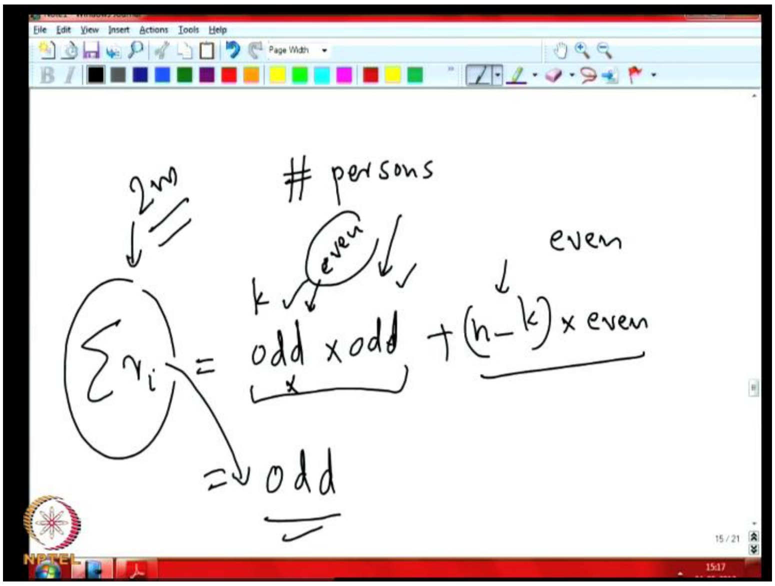

What we were asking is, how many people, the number of persons who has shaken his hands on odd number of times. Suppose, for contradiction, it was odd. So, then what means the total number of handshakes can be count like this odd into odd, because odd number of persons shook hands odd number of times plus and even then the remaining people whichever number of people. Suppose, n minus this number. Suppose, this is k n minus k people, this is whatever I do not know odd or even shook hands in even number of times. The total number of handshakes, that is sigma r I, we were looking will be an odd number into an even number. This is going to be an even number, right.

So, the total it is going to be an odd number because if you add an odd number to an even number, we will get an odd number. So, this is going to be an odd number, but we know that this is 2 m by our earlier double counting argument. Therefore, we infer that there is a contradiction where is the contradiction only place, we assumed is an odd number of people have done odd number of handshakes. That is why, we draw that. So, the only possibility is this is even, so the k has to be even. So, a number of people who shook hands an odd number of times have to be even. This later part has now double counting argument there, but the double counting was captured and what we try to do it in the matrix. Of course, I made a mistake in the initial part, but what we do is we count the number of total sum, right. The row sums, right. That is what we are interested in.

We showed that can be counted by looking from the handshakes point of view and each handshake involves two people. So, it gives count of one-one to each of them. So, two counts will increase to the total, right. Therefore, the total sum will be 2 into number of handshakes. That is all. So, of course we have to, we wanted to illustrate that this counting in a different way will give some information about counting in the other way that was what we wanted, right.

For instance, if actually somebody wanted to count the number of handshake, sorry what one can do is, you can definitely ask each person how many handshakes you have done and then, sum them up and divide by 2. That is what we are saying, right and of course we capture this here saying that in every graph, the sum of degrees of its vertices is two times the number of it is said just hands are even. So, that is what we elaborately proved here and that is it. Now, the next question we want to consider at this, let the f be a family of subsets of z x. Then, sigma x element of x d of x is equal to sigma a element of f cardinality of a. So, here we introduce a slightly different concept.

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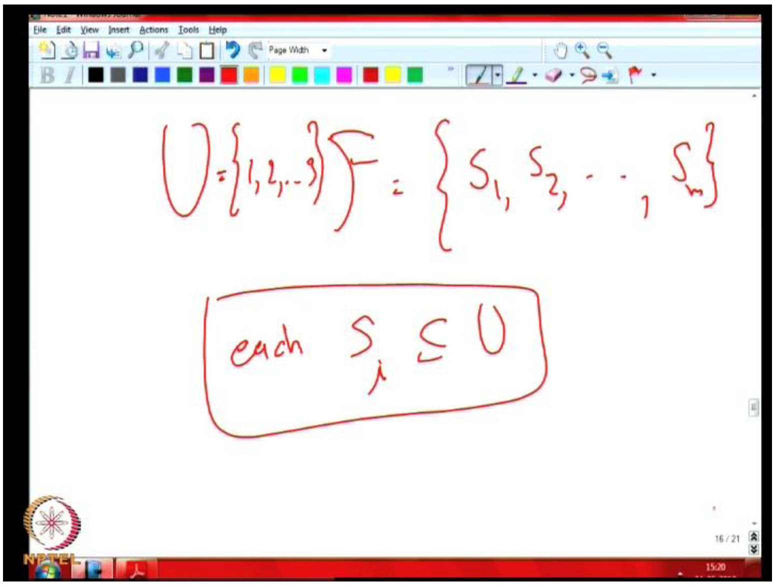

So, here we are interested in family of subsets, just f. So, you be some universe and then, we have a family of subsets, say s 1, s 2 something sum s n where each s i is a subsets of u, right. So, we will assume that you see these are all right sum. So, u is sum. Just see the integers or something 1, 2, 3. So, we can count it in this and these are finite sets, countable finite sets.

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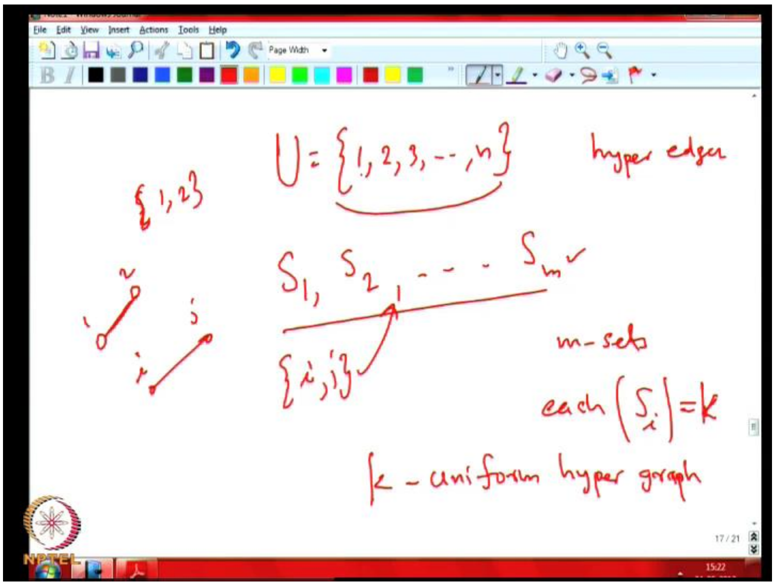

Now, what we are interested in is sometimes we call this as a hyper graph also because you know this is set system. So, when we say it is a hyper graph, so we will say this universe u. So, this 1, 2, 3, 4 up to n. So, if I put like this, these are the vertices of the hyper graph, right. Then, this s 1, s 2 which are subsets of u are the edges or hyper edges, right, edges or hyper edges whatever the edges of the hyper graph see that you can see that a graph is a special case of this thing, where you know in the graph, we are only allowing edges of this sort. The vertices are again 1 to n, the same vertex set, but edges now is always something of this sort, say two element sets.

If I put an edge between 1, 2, if I put an edge between i and j, we will include the set i, j in this collection, right. So, this is what m element sets, so m sets, all right. Each right s i is equal to sum k, say fixed k. Then, we will say that is k uniform hyper graph. When we say that it is a k uniform hyper graph, this was just to introduce second terminology because I may be talking about hyper graph set systems.

When I discuss something that, it is as of now, it is not very much necessary, but you can keep it in mind, this you can think of it as a generalization of the graph, in the sense where the graphs are actually hyper graphs where each edge is a two element set, exactly strictly a two element set and not allowing higher cardinalities for the subsets allowed in the family, but in general, it can be any subset of u, right. So, there it is decimated. This is just a subset of vertices. That is all. Now, here why we told, why we had drawn this parallel between the graphs is because we use certain terminology which is common in graphs, common for graphs. Then, we sit in the case of hyper graphs also.

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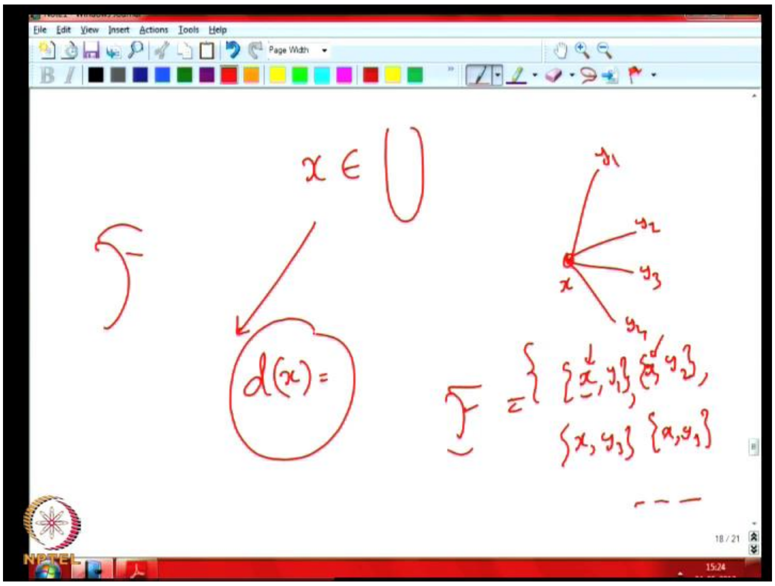

So, for instance let x be a vertex. That means, x belongs to that universe, right. X be a vertex. X belongs to that universe. Then, we can talk of the degree of x. What can be the degree of x in a graph? If f is a vertex, the degree of x is the number of edges in which it is part of, right. So, in the hyper graph case, what we will say is x. So, it will be some y 1, y 2. This is the way we count, right y 4. These are the neighbors of x. We count the neighbors of x. In the hyper graph, we will see these sets xy 1, xy 2, xy 3, xy 4 in the family of sets we are talking about, right. In the f we contain this is essentially the edges, right.

So, it will be lot more, but this will be the edges of the family in which x is part of, right. So, essentially if we are counting the number of sets, some sets will f in which excess n element of right that will be the degree of x. So, we will generalize it this way. Essentially in a hyper graph degree of xx being a vertex, x being a number of the element of the universe that is the number of sets in our family, given family f, such that x is a member of those subsets, number of subsets in f if that x contained in it, right.

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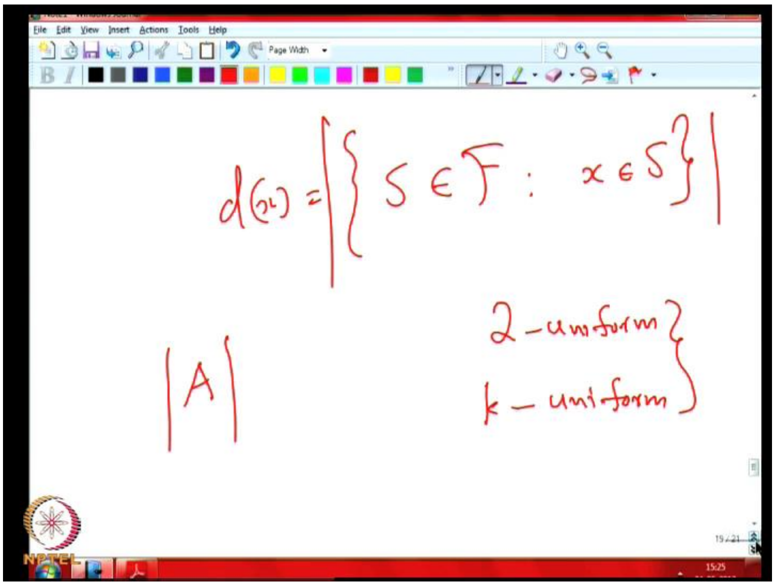

So, that is what once A d of x is equal to all those s elements, such that x is an element of s. So, this is, sorry the cardinality of this how many are there? This is what we are asking. Now, in the graph case, every edge is a two element set. Every edge is a two element set because two end points of those edges, what constitute the edge of the hyper graph. The corresponding hyper graph, but therefore there is two uniform hyper graph as we have already mentioned, two uniform hyper graph and we have mentioned about k uniform hyper graphs, where each edge that means if subsets in the family has exactly k elements in it, but the hyper graph need not be k uniform. Hyper graph is just any family of subsets of the universe, right. Therefore, this is for each set. This will be an interesting parameter what is the cardinality of A; so now we are trying to connect the cardinality of A with the cardinality of, sorry the degrees of x.

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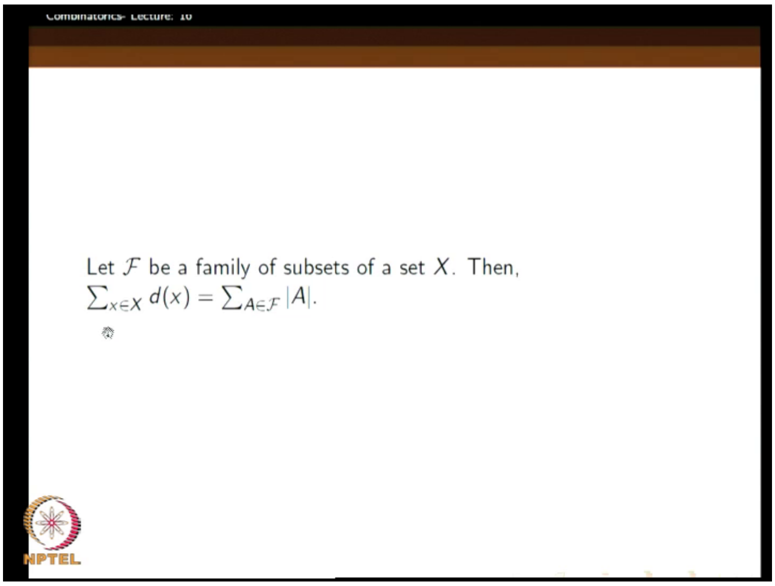

What we did we told you that degrees of each vertex in the universe, we will get exactly the sum of the cardinalities. Is it easy to see? So, we will give a proof using double counting.

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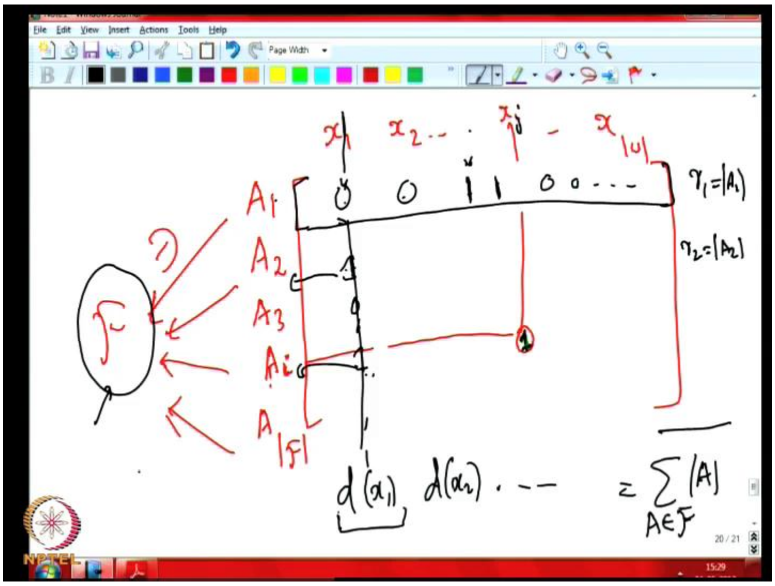

Again, what we going to do is to make a matrix. So, this time we do not make any mistake. These are the index, the rows of the matrix by the members of the family. So, this is a cardinality of F, right. So, here is F and these are all members of F. So, now we will take members of the universe x 2, x 3. So, X cardinality of u, all the numbers of the universe. Now, the i j-th entry will be, what i j-th entry is? So, suppose A i is this, i j-through entry will be if i-th set A i contains the i-th element, then I will put a one there, right. Otherwise, I will put a 0. If i-th set contains x, sorry x j j-th element, then I will put a 1. So, similarly, for each A. So, what will be there in your row? We will see some zeros and then, some zeros, then some ones, then some zero something like that. So, wherever this 0 is there, that means at particular element corresponding to that column is not a part of this set. So, wherever I see A 1 that means this particular element is a member of this set, is a member of this set. That is what this is called the incidence.

If you just read the 0, 0, 0, 1, 1 this as a vector of this row can be read as a vector incidence vector for this set because the positions of the vector correspond to the elements and then, the membership is what is indicated by one’s and non-membership is indicated by 0s, right. So, if you want to get the cardinality of a 1 summing up this row, number of one’s counting the number of one’s in the row that r 1 will be equal to the cardinality of this A 1, right. Similarly, r 2 will be the cardinality of A 2, right and so on. That means, this one side of that formula which we got this, the second the r h s r h s the right side of this formula, this actually the column sum the actually the column sum, right.

So, this is what sigma cardinality of AA in, sorry this is the row sum cardinality. So, for each subset in the family, we are actually counting how many elements are there by counting the number of one’s in the corresponding row and adding the sum. The sum of the row sums will give us one side of the formula and actually, whatever column sums if you look at x 1 and that column if you count, it is essentially the degree of x 1because here whenever I see 1, that means the x 1 is element of that set also. Then, I see another one that means x 1 is element of that set.

So, that way if you count the number of one’s in that column for x 1, I am actually getting in how many subsets of the subsets it is a member of F, right. Subsets from the family F is x 1 part of x 1 a member of, right. So, that is exactly d of x 1 in d. So, if you sum up like this, this is d of x 2 and d of x even. So, the column sums are actually giving the sigma d x’s, right.

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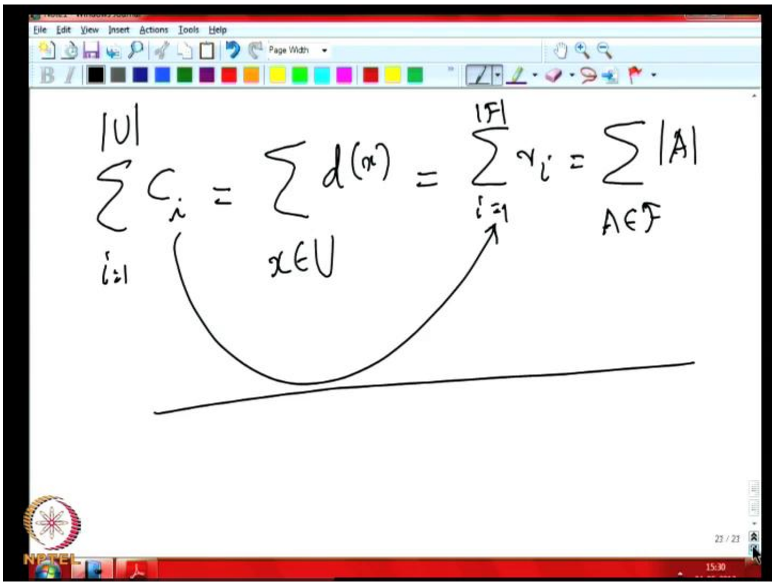

So, the column c i, the sum of columns i equal to 1, 2 cardinality of u, right. These are the number of way each column correspond to n element and this is what will be our d x x element of, right. If this is actually we know that this is sigma r i because this and this are same, right and this is x i i equal to 1, 2 cardinality. This is the number of rows and this corresponds to an element of F cardinality of m. This is the way. So, this is a good example of double counting. So, you should understand that what we have done in the previous exercise is a special case of this.

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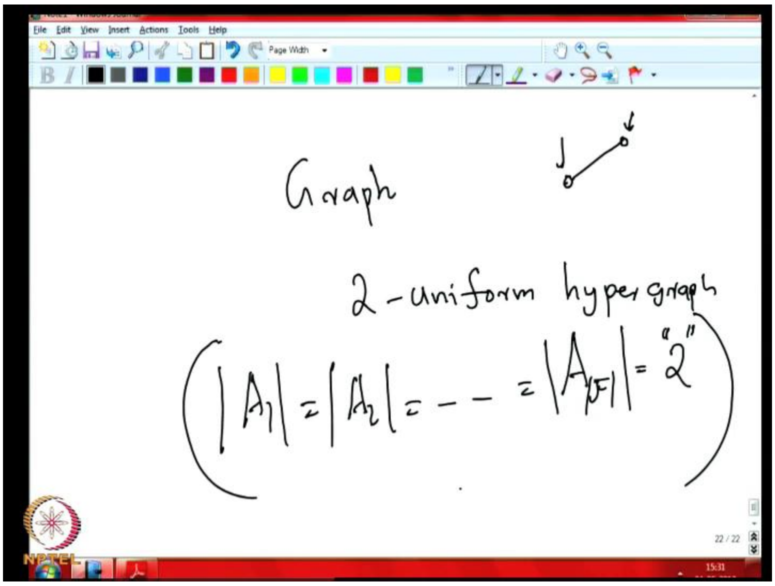

So, for instance as I have mentioned graph is a special case of hyper graph, where it is a two uniform hyper graph, right and then, you know in this formula, if because it is a two uniform hyper graph, each a i that means a 1 equal to a 2 equal to all those a f, right. So, the f will correspond to the number of edges here. So, this is f, this actually is the set of edges. So, this will be all two, right because the every edge is a two element set. It is a two uniform hyper graph. The general in a usual graph is such that two uniform hyper graph because every edge is a set of it is the consisting of its end points, right. The vertex is being the universe, right. Therefore, this is true. So, the sum on the r h s, we show here sigma A element of.

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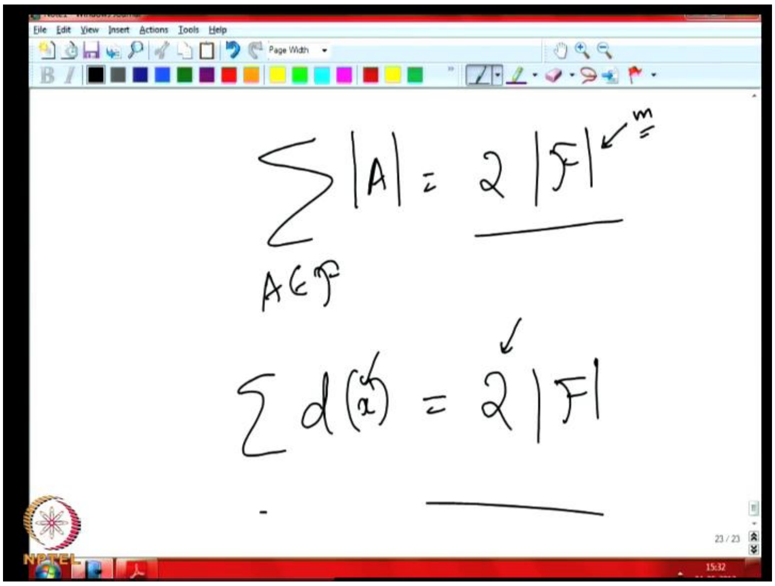

If the sum sigma A element of F cardinality of A, we cannot to be two times cardinality of F, right. This is what we show last time here. We have called n n be the number of edges. There the number of handshakes we called m and then, the other thing was there all sums, namely for each person how many handshakes we made. That means, for each vertex, how many edges are incident on it. That means in how many sets it is part of that is, d of x only right for a person x. That was d of x.

So, sigma d of x shows what we were counting there. So, we in that case, we actually got two f should say if we were talking about the k uniform family instead of two, we will be putting k here and then, this formula would have been much, would have got much simpler form, right that is what we show in the graph case. It is simplest and we could infer that is even and from there, we could even go further and infer that the number of people who had shaken their hands an odd number of time should be odd, right.

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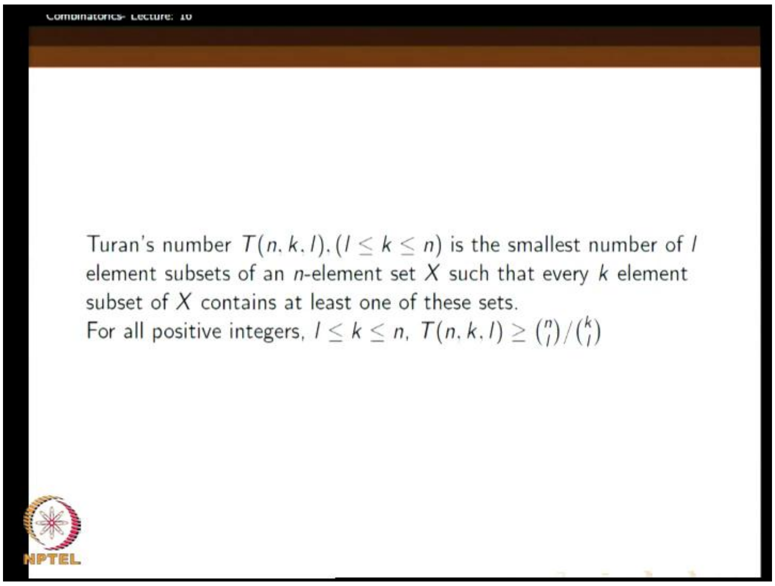

Now, so it is helpful, right. Now, the next one is about the Turan's number. It is another question which is taken from Jukna. So, we define particular parameter for t of n, k, l. It is smallest number of one element subsets of an n element set x, such that every k element subset of x contains at least one of these sets. When we are saying that it has to be at least n chose l by n k chose l, see here what we are interested in is in counting the number of one element subsets having a certain property. So, this is what we want.

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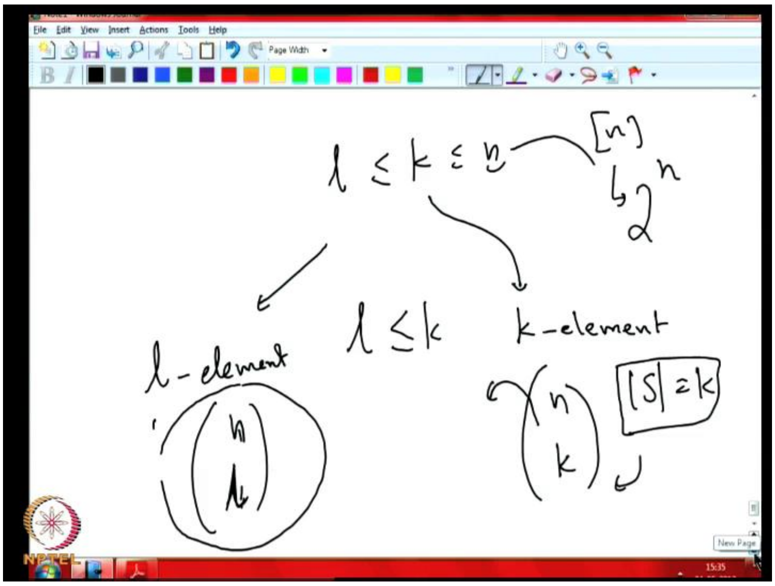

So, you know l is less than equal to k less than equal to n, but n is the size of the universe we have. So, we have total two raise to n subsets of this n. So, n being the universe, but then in this problem, we are mostly considered. We are not really considered about every subset, we are interested in k element subsets. That means, the cardinality of each subset we are taking is equal to k because also we are interested in one element subsets. That means, we are only interested in two types of subsets, namely those n chose k subsets which are such that s s equal to k. That means, those subsets whose cardinality is equal to k, we know that there are n chose k of them, right. Similarly, n chose l of those subsets; I mean whose cardinality is equal to l, right.

Now, you remember l is less than equal to k. Now, our condition is that we have to select minimum possible subsets from this. We want to select minimum possible subsets from this n chose l, such that now if somebody else picks up his favorite k subset whichever he likes, any k subset we should be able to show from our collection. One element subsets in our collection of one subset, we should have one subset which is the subset of this k set, right. That is what we want for instance.

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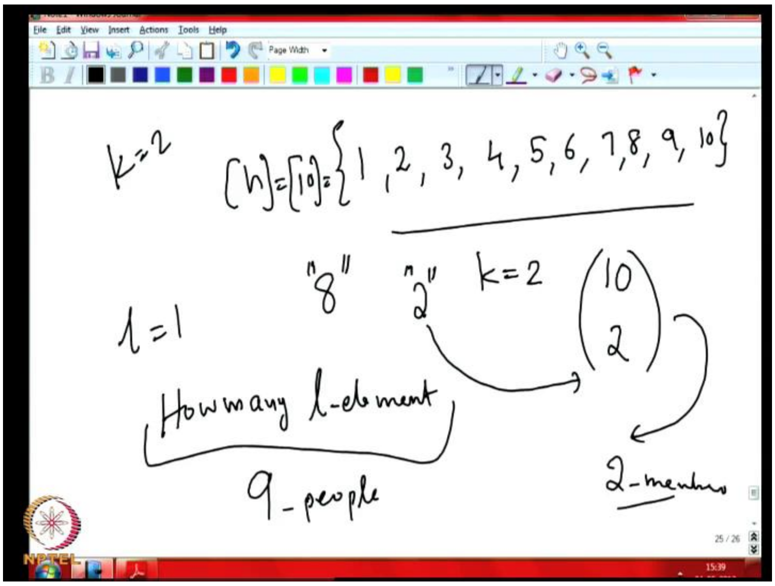

So, we can say that 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is a universe, right. So, illustrating this here n equal to, this is n equal to 10 things. Now, I am interested in, so put k equal to 2. So, there are how many? Two element subset. There are 10 chose 2 two element subset. There are so many of them, right and then, let put l equal to 1. Now, I can ask how many one element subset should I pick up, such that whichever two subset, the other person picks up. Let me call him an enemy. He is coming up to pick up some two element subsets, such that I do not have 1, 1 element from it. So, these are like his forming a two member committee from the 10 people available and then, I have already for instance, I have already low bid several of them, right. I am single person. So, for instance if any of committee forms I want one of the persons to be my person, so that I can get my things done or my k k I can influence the decision.

So, now the question is how many people I should bribe? This is right. Now, you can see that I cannot do it unless I bribe all the 9 people. Why? Because if I do not bribe all the 9 people, then that means I have bribed only 8 people out of it. Still there are 2 people left. He can pick up this. Two people can form a committee and then, I do not have my man in the committee, right.

So, this is the kind of question I am interested in of course. So, it may look like it is a bad thing I am trying to like estimate, but of course this can be a better example that you see that there is a reason to know this, but in our theorem, we are only giving a lower bond. I will have to for suppose, I have plan to bribe people like that, then you know be sure that you bribe at least these many people, so if you want to estimate the kind you had spend on bribing, then at least we will give a lower bond. That is what of course this is not 2 and k equal to k can be much bigger n l and the kind of lower bond, we give here may not even be reasonable. I mean it can be much smaller than the actual number of one element subsets I have to pick up. So, we are only saying that at least we have to pick up this much, right.

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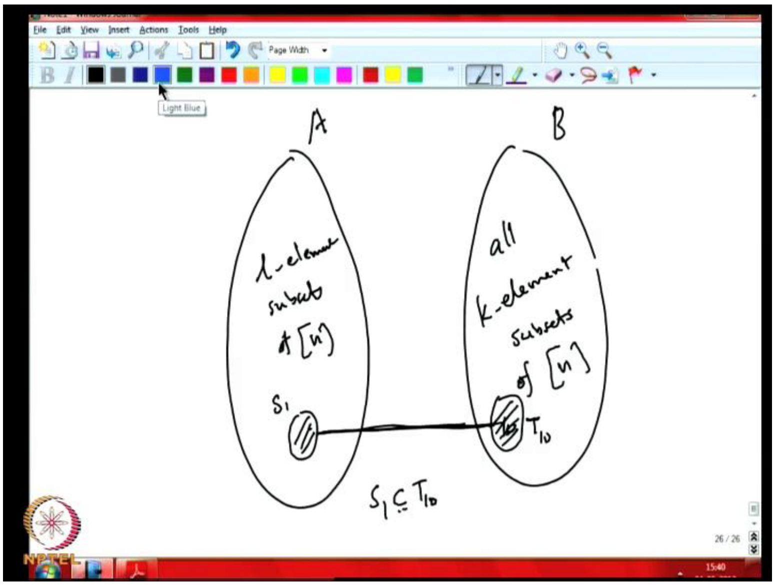

So, see how I will also model it by bipartite setting. This is a bipartite. I am finding two bipartite graphs, right. Here, A i collect all the k element subsets. This is all k element subsets of. See all k element subset of the universe, namely n, right. Here, I will collect all the one element subsets of n. Now, if a particular one element subset and the particular k element subset, here I will put an edge between them, right. This will be seen as a vertex this side of the bipartite and this will be seen as a vertex and that I put an edge between them. If this particular one element subset, say let us call it as 1 and let us call it T 1, T 10 or something. If S 1 is a subset T 10 subset equal to T 10, then I will put an edge between them. This way I can construct a bipartite graph. Now, what I am looking for is I will stop here.