Combinatorics
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Lecture - 15
Generalization of Binomial Coefficients-Part (2)

Welcome to the 15th lecture of combinatorics. So, we were discussing the absorption identity and its companioning in the last class, where we stopped.

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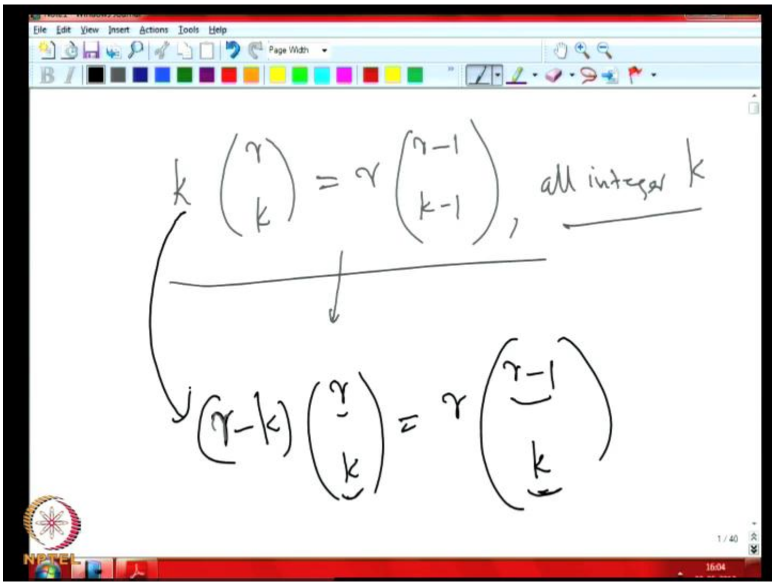

So, the absorption, I have to remind you, the absorption identity was r choose k equal to r into r minus 1 close k plus 1. This is valid for all integer k; all integer k, right. There is no restriction. So, any r in complex number, it can be any complex number. K is an integer. That is all. So, k is an integer because this combinatorial coefficient is defined only for lower index integers.

Now, the companioning in is r minus k. So, we can r minus k into r close k equal to. Instead of k, we use r minus k is equal to r into r minus 1 close k. The point here is that we are not changing the lower index. The upper index is reduced by 1. In the earlier one, both upper and lower index was reduced, but then instead of k, we are using r minus k here, right.

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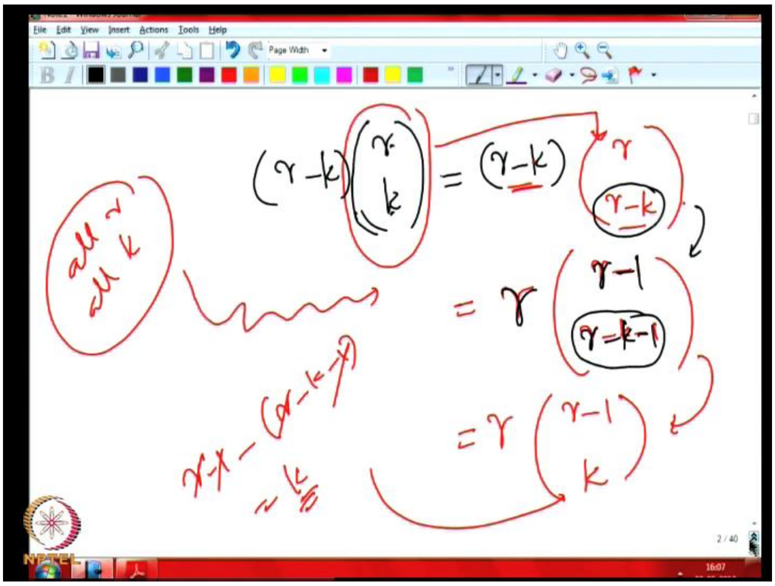

So, we want r minus k into r close k, right. So, first, what do we do? We apply this r close r minus k into r close k is equal to r minus k into. So, this r close k were replaced by r close r minus k. This is by applying symmetry identity. So, that is a why I wrote again red because anyways symmetry identity cannot be used for all r and all k. We have some restriction there, remember, but for the time being, we just assume that we can do that, right. Suppose, we can do that, then you know this and this has become same. Now, this and this, this lower index and this value. So, it is like our previous identity k r close k is equal to r r minus k close. So, instead of here, instead of k, we have r minus k k r minus r r minus k, right.

So, therefore, now this and this are same. So, we can write it as r into using the previous absorption identity. So, both indexes will reduce r minus 1, r minus k minus 1. This is that now we can again apply the symmetry identity. This is r into r minus 1, r minus 1 minus r minus k minus 1. That is this k, right. So, we put r minus 1 minus r minus k minus 1, we get k. So, r r is cancelled minus 1 minus 1 is cancelled, minus 1 minus k. k is k. So, that is what we are writing here.

So, here again symmetry identity is used, but the only issue that this was the second set because this was an absorption identity. We used absorption identity. We used from here to here, but here r minus k was converted to r. So, r close r close k was converted to r close r minus k. Similarly, here r minus 1 close r minus k minus 1 was converted to r minus close k. Both use the symmetry identity, but the symmetry identity works only for non-negative integer r, not even integer r, non-negative integer r as we have seen. So, that is what this is. This is any integer.

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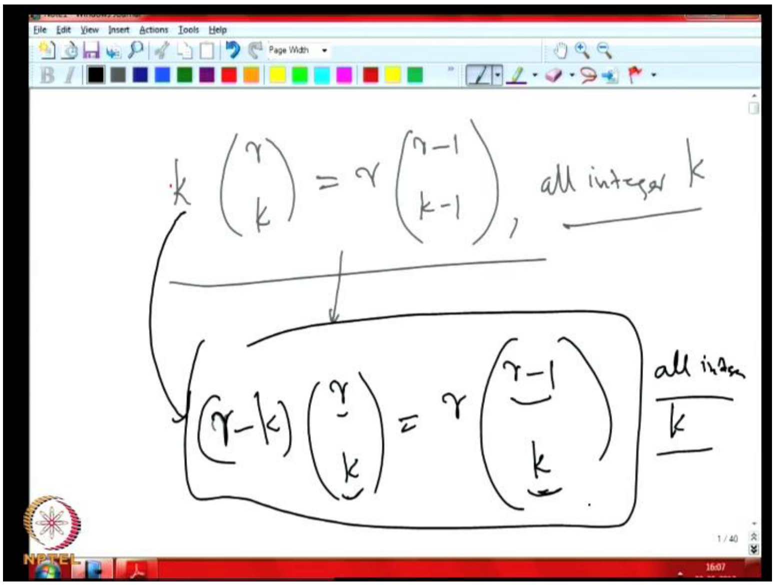

Then, how do we say that this companioning to the absorption identity, namely this one works for all integers, all k, all integers k. That means, there is no restriction on r. So, that is what we claimed. Initially, the proof does not seem to say that right here, but here we learn a new technique.

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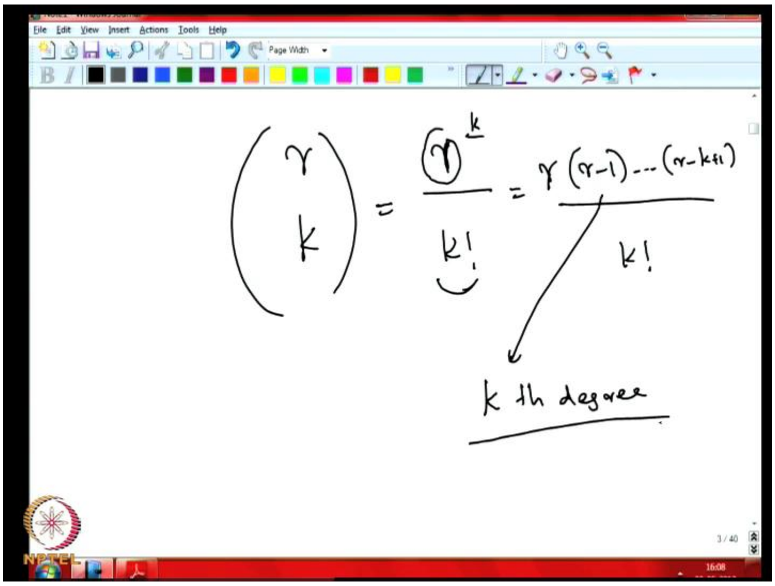

The technique is that see, these are close k, while we have written this r k falling factorial divided by k factorial. So, we could even have considered r as a variable here. So, that will look like r into r minus 1 into r minus k plus 1 divided by k factorial. So, as long as k is a positive integer, this r is a variable and if you expand this, we will get a polynomial in k-th. K-th degree polynomial will become in the numerator divided by k factorial, k being some positive constant here, right.

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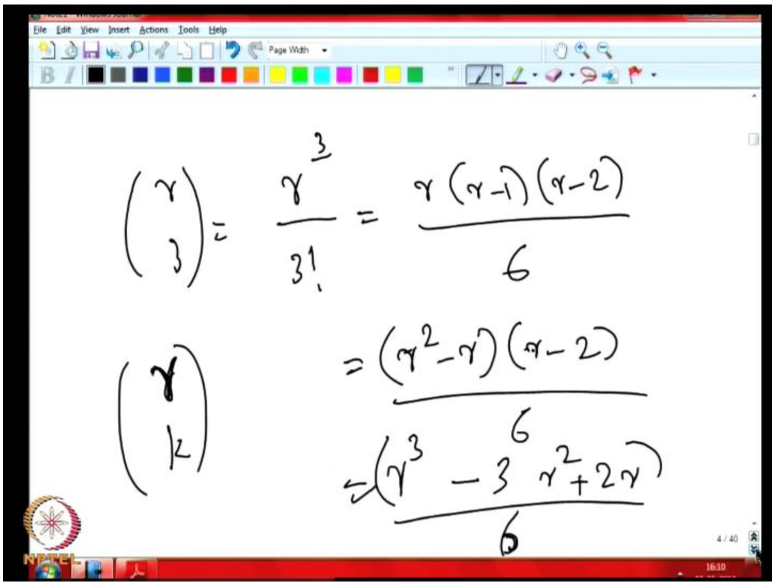

So, for instance, r 3 falling factorial divided by 3 factorial, which is r close 3. This will be r into r minus 1 into r minus 2 divided by 6 which is r square minus r into r minus 2. If I multiply it here first and divided by 6 which is again multiplying r cube minus r square and here, minus 2 r square, right and then, plus 2 r divided by 6 r cube, r cube to minus 2 r square minus r square. That is total minus. We can write minus 3 r square instead of minus 3 r square and plus 2 r divided by 6. So, up in the numerator, we do have a polynomial of degree 3 and that is true for r close k also, right. So, this way, we can interpret this r cube k is a polynomial of degree 3 k in r now. Good. So, that will help us here.

So if it is a polynomial of degree k in r degree k in r, then this identity we wanted to prove look r r minus k r close. K is equal to r into. This is a polynomial of degree, this is a polynomial of degree k in r, this is a polynomial of degree k in r. So, this side, we are multiplying into r minus k, this side we are multiplying by r. So, here, it is a polynomial of degree k plus 1, here a polynomial of degree k plus 1, right. So, we are just comparing them.

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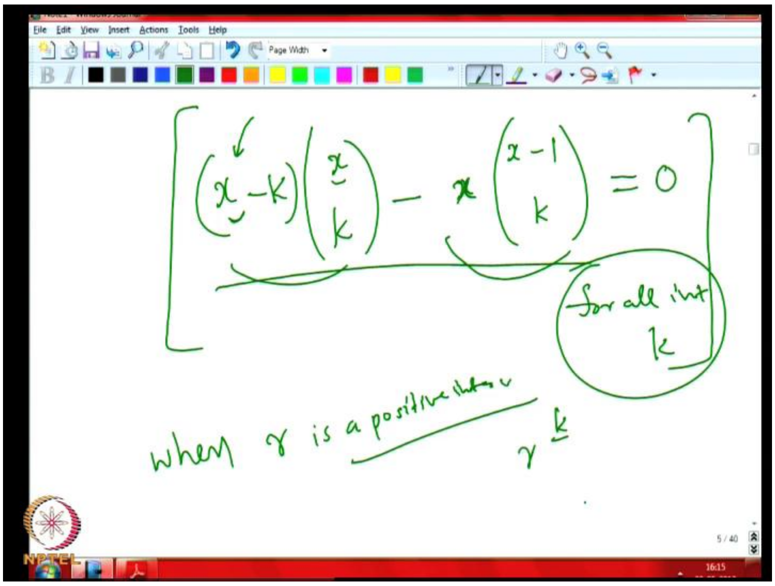

So, for instance we could have rearranged it and told we are trying to prove r minus k into r close k minus r into r minus 1 close k equal to 0. This is what we want to put for all integer k. So, this is a polynomial identity. That means, this is a polynomial of k plus 1 degree here, this is a polynomial of k plus 1 degree here. When you minus the degree is at most k plus 1, that is equal to 0 and we saw that when r for some fixed k, where k is a positive integer, right.

So, assume that k is a positive integer. Why? Because k is a negative integer. We do not have much problem. Y is a no problem because if k was a negative integer, r close k, right. This side will be negative anyway. So, we can say, I can go back to the (()). So, this side will be r close a negative number that will become 0. This side will be r minus 1 close a negative number that will again be 0. Both 0 is equal to 0. What it says? So, it does not matter. It is anyway true. Therefore, we assume that its case 0 or even if when it is 0, that is not a problem because if it was 0 or close 0 is 1 and this r minus close 0 is also 1 and k equal to 0. That is saying that r equal to r. That is correct.

So therefore, we can assume that k is a positive integer. We could have this. We only have assumption that k is a non-negative integer, but now why non-negative integer? Because for non-negative integer k, this definition works and then, interpret it as a polynomial in r.

So, if yeah, if it is a negative integer that was not correct, r k falling factorial by 0 factorial. We won't be interpreting that way, right. So, it is just 0, but yeah so that interpretation would not make much sense.

So, therefore, we will assume that this is a polynomial of degree at most k plus 1, right. Sorry k plus 1, at most k plus 1. Now, the polynomial in r. So, you know that when r is a positive integer, r is a positive integer. This anyway works or r r is a non-negative number, right. This anyway works because symmetry identity which is valid, our previous derivation was valid. When? This derivation was valid when r is a non-negative integer, right because the only wrong thing we did was the symmetry identity because symmetry identity was only valid for non-negative r. So, now we know that there are the several values of r, an infinite number of values of r for which this is true which means when I substitute r equal to those values, any positive integer, this will work out. This will evaluate to 0. That means all those positive integers are roots of this polynomial.

So, remember the variable of this polynomial is here. So, we can instead of r, we could have, we can write x and c, right. That will look better, may be. So, it is more familiar. So, here this would also become x. So, put x equal to any positive integer, it will evaluate as 0. That means how many root this r of this? This polynomial has infinite number of roots, but then you know that a k plus 1 degree polynomial and polynomial of degree at most k plus 1 can have only maximum k plus 1 distinct roots, right. I do not prove it. This is a well known fact.

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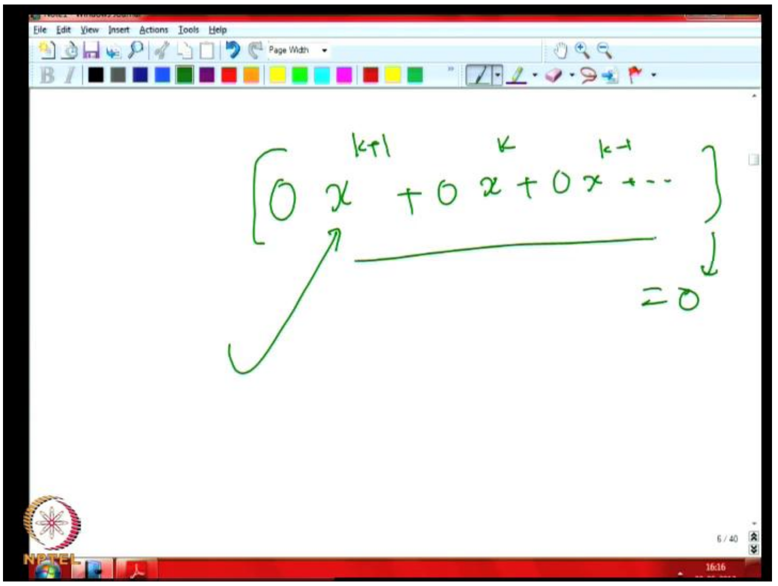

So, if that is the case, what do we infer that say, polynomial if it is not identically zero. For instance, a polynomial of this sort, 0 into x raise to k plus 1 plus 0 into x raise to k plus 0 into yeah, x raise to k minus 1 and so on. This polynomial whichever value you substitute for x will evaluate to 0, right. So, this is 0 polynomial, right. So, this is the only way you can get infinite number of solutions. Number of roots for a polynomial, right. Otherwise, if it is not identically 0 like this, then definitely it has only a maximum of k plus 1 roots and here, we have seen more than k plus 1 roots. So, that means we can infer that it is identically is 0. That means, we have written equation here is always true. For any value of x, it is correct. That is what it means, right. For any value of it, it is correct.

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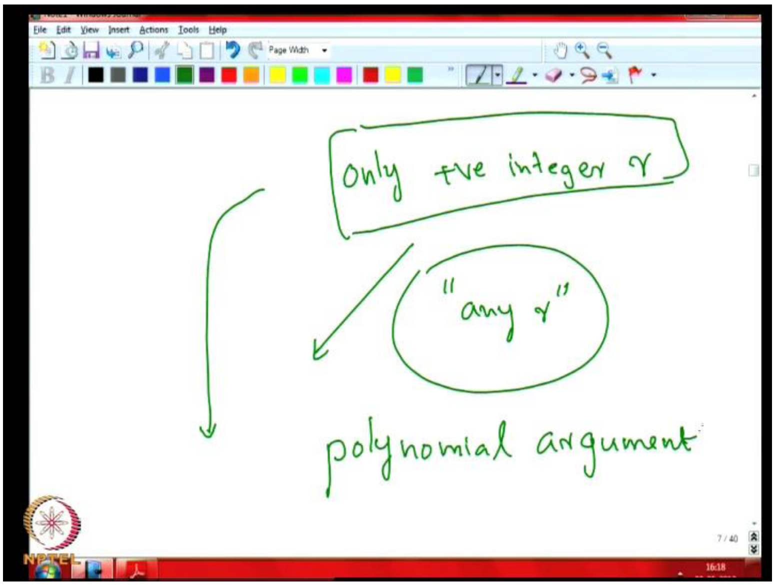

So, in particular, you can put any complex value for x and it will work, right. Fixing a k, you can give any complex value. So, this kind of this argument we proved it for only positive integers r. We only have to prove for positive integer r for symmetry identity we can use and then, we enfold by this argument, this round about argument that therefore, it will work for any r. So, there it was not obvious.

The main idea we used was that it is a polynomial and for infinitely many values of r, it is evaluating to 0 and if it is not r, the polynomial was not identically 0, everywhere not 0. For every substitution of r, any possible value of r if the polynomial was not evaluating to zero, then only for k plus 1 values, maximum k plus 1 values, it could have evaluated to 0 because it is a k polynomial degree, at most k plus 1. So, that is not true. Therefore, it should be identically 0. So, for any value of r, it should work. So, this is called polynomial argument. You see this a very useful argument to generalize the combinational identities.

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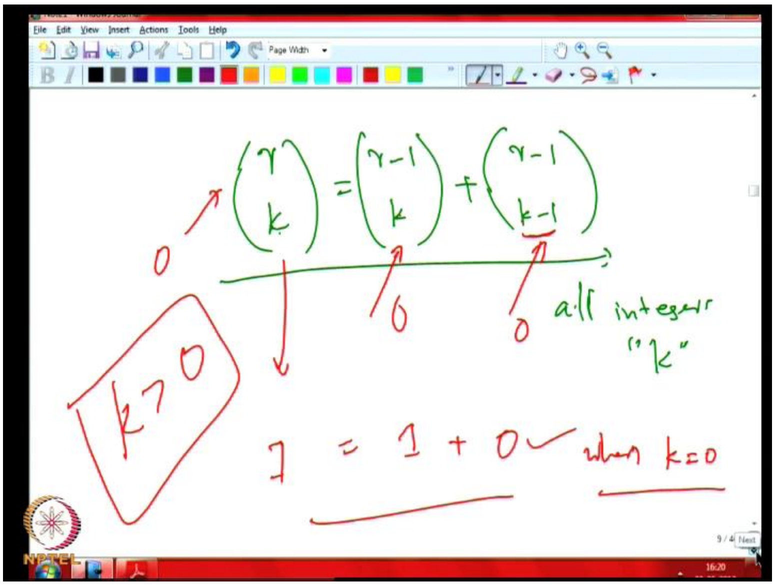

So, you can say that always, it would not work because see if you take n chose n, so this identity, the symmetry identity r chose k equal to r chose r minus. We told that this would not work for any r. See, first of all the identity itself was not making sense, but even then you can see that this is not a polynomial equation. We are having here because this is a polynomial, but here, what this r minus k, right. So, that does not make sense. So, that is the problem because you do not have a polynomial here, right.

So, that is why here we cannot use that method to prove the symmetry identity, but there are several cases we can use it, right. We will see another case here. So, this is the next case. This is the most important formula we proved for the binomial coefficients, namely the addiction formula r chose k is equal to r minus 1 chose k plus, yeah r chose k equal to r minus 1 chose k plus r minus 1 chose k minus 1 and we claim that. So, this can be completely generalized. This works for all integers k. So, we can see whether for negative k, it works or not. Yeah, for negative k what happens is this r chose k, this will become 0 and this will also become 0 because anyway k is negative. This will also become 0.

Everything will be 0. So, it is trivially true for negative k for 0 k. Also, we can check probably. So, r chose 0 if from when k equal to 0. Also, we can check r chose 0, this is going to be 1, right. Then, this is also going to be 1 plus, this is going to be negative 0 minus 1 minus 1. So, this is also correct, right. Then, k equal to 0. Now, we can probably assume that we can come here and we can assume that k is greater than equal to, sorry greater than 0, may be even equal to is not necessary for us.

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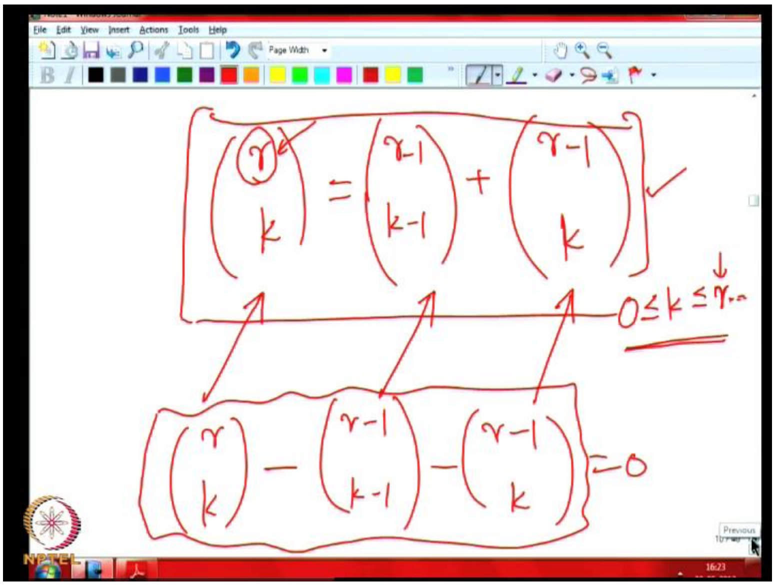

Now, the issue is r. We have generalized to any complex number. So, how will you deal with that r chose k equal to r chose k minus 1 plus r chose, sorry r minus 1 chose k minus 1 plus r minus 1 chose k. This is what we want to prove, but here, we can see that the polynomial method will work because this is a polynomial k-th degree, polynomial in r, this is a k minus 1 degree polynomial in r and this is also a k-th degree polynomial. Overall we can, when we write r chose k minus r minus 1 chose k minus 1 minus r minus 1 chose k equal to 0, this is a polynomial.

This one is a polynomial of degree at most k and that is equally 0. It is what we have written. If we can show that there are infinitely many roots for this already, then it is identically 0. Any value of r will evaluate as and substituted will evaluate to this polynomial. This polynomial will get evaluated to 0, right. So, we just have to show that there exists infinitely many values of r. This is true. In fact, more than k values of r which will evaluate to 0 for which, this polynomial will evaluate to 0.

So, we already know that why? Because for every positive integer r, now this is correct and you have verified it. For instance, whenever 0 less than equal to k less than equal to r, right, we have verified it and it was making sense and otherwise also because when k was greater than r also, this is correct because otherwise, it is 0. So, if r is greater than equal to

0, then this will be 0 r minus. This will be 0, this will be 0 also, right 0 plus 0. So, for all these values, it is 0. There are lot several values of at least we can take any positive integer r which is greater than equal to k and it is going to be true, right as we have seen before. So, by the counting argument or we have seen several proofs of this thing before. Therefore, combining it with the polynomial argument, we conclude that the generalization is correct. That means, we can generalize this r to complex number and since, this is valid, this is about the generalization of addition formula.

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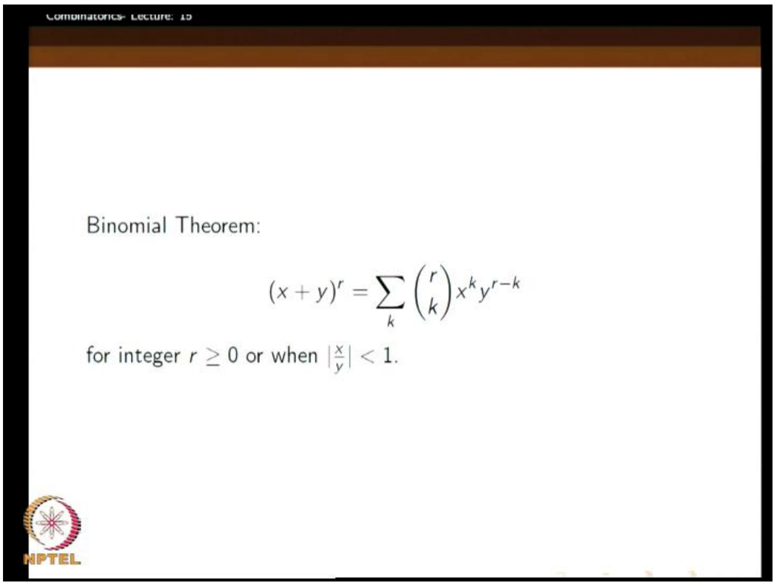

Now, the next one may be it is time to talk about the binomial theorem. This is the general form of binomial theorem, x plus y raise to r equal to sum over all integers k r chose k into x raise to k into y raise to r minus k. This is same as earlier. Just that when r 0 is an integer. See for r is an integer, this is true. Generally true, but when r is not, sorry when r is a positive integer or may be r is greater than equal to 0 non-negative integer, this was true for any x and y, but when r is not a non-negative integer, then we need a restriction that namely, x plus y x by y ratio of x by y. The modulus is less than 0. This is for the sake of convergence. We will see, why this is.

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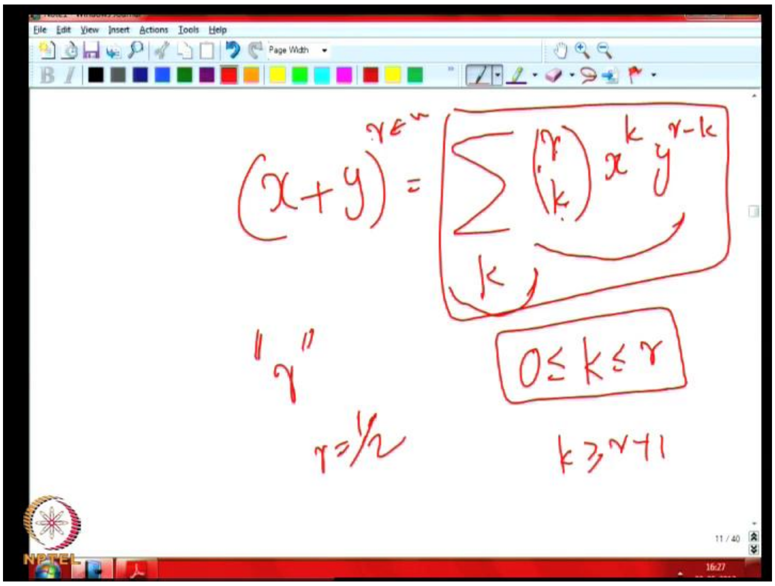

Because we first recalled that x plus y raise to r, if it was let us say this is the way it is written, k for all k r chose k. So, x raise to k into y raise r minus k. So, if k was n, k was a positive integer n, then this will become n chose k, right. X raise to n, y raise to n minus k. The only thing is, we are now writing the sum for all integers k. This negative integers does not make much of a problem because anyway r chose k r chose a negative number is going to be 0. That will go away.

So, that won't contribute to sum at all and then, when stating from 0 onwards, it will contribute up to r because k equal to k greater than equal to r plus 1, this coefficient will again disappear. So, only when k is in between 0 and r, this will disappear and this will contribute when r is a non-negative integer. Therefore, this will be the earlier formula only whatever we have learnt before, right.

So, that x plus y raise to n equal to n chose 0 plus n chose n chose 0 x raise to 0, y raise to n plus n chose 1 x raise to 1 into y raise to n minus 1 plus n chose 2 into x raise to 2 into y raise to n minus 2 and so on. That formula will come from these things, but if r is not, this is valid, we are claiming that this is valid for any r a, even complex numbers for negative integers. Complex numbers like r equal to half. Some fractions, all for any number, it is valid as what we are claiming, right. Just that in that case, this will be an infinite sum. Just we cannot say that only for k in between 0 and r, the terms will remain a such non zero because you know if r is not an integer, it is not r chose k is not going to be 0.

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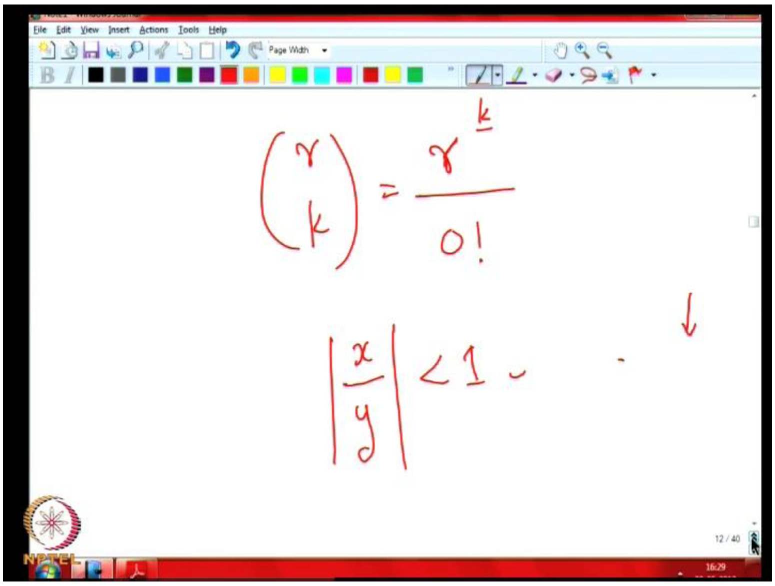

We remember that the formula for r chose k is r k falling factorial divided by 0 falling factorial. Still we can say that when k is negative, it will anyway disappear. So, for positive k, only we have to worry about, right. So, for positive or 0 k, right. So, this is you know when r is not an integer, this falling factorial r into r minus 1 into though it may start with a positive number and go down to a negative number, but will not go through 0 because we did not start with an integer and we are minusing 1, 1, 1, right. Minus 1, minus 2 like that. So, this will not be 0.

So, therefore, they will remain, all of them will remain, the terms will remain. Therefore, we have to sum this. Sum is an infinite, overall all non-negative integer k, right and the next thing if you write an infinite sum like that, we have to worry about the convergence of converge or not. So, for convergence, we need the condition that x by y, these ratios absolute value is less than 1. This is what is written in the second part, right. This is what has been written in the second part, x by y ratio is less than 1. Yeah, whichever is the bigger one, you can say the question about how do we prove it. Again, we are not in combinatorics anymore when we talk about this thing. So, we can say that we are not in really counting now, but we are talking about combinatorial coefficients r chose k, but the proof is also, it is not very difficult. Only thing is we have to use the Taylor series.

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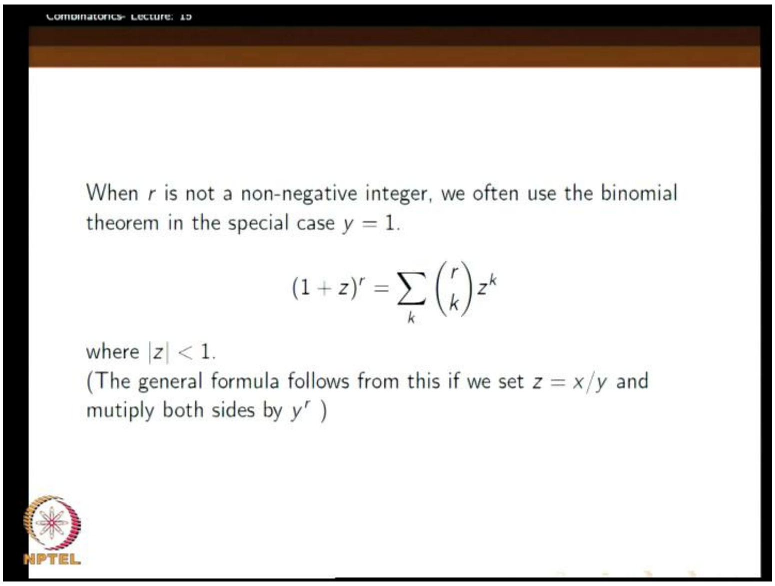

You have to use the Taylor series, but instead of using this, I am not going to give a very thorough proof of that. I will just indicate what it is rather than worrying about. One can learn easily from the calculus books whether than an combinatorial course, but it is important that we get familiar with this form, right. So, usually it is written 1 plus z raise to r. Yes I have written earlier because it can be for something as far as yeah, we will write 1 plus z raise to r is equal to sum over k r chose k z raise to k is what we write, right.

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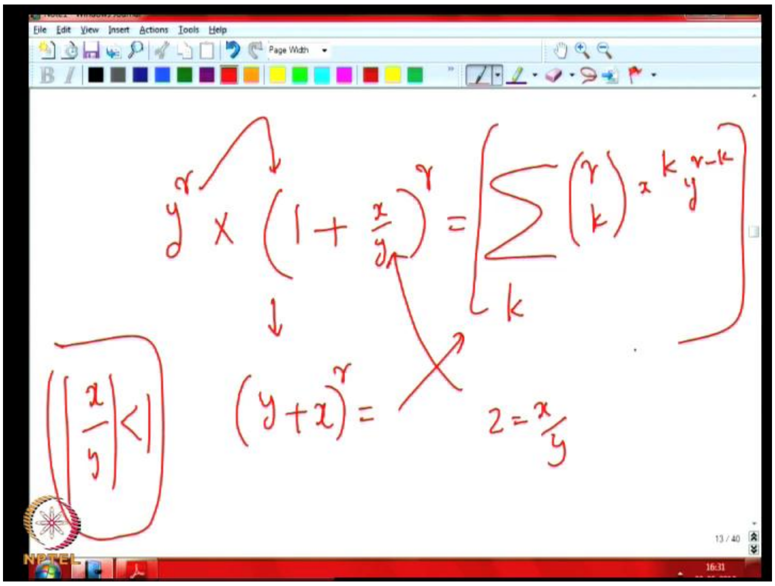

We need modulus of set less than 1 for convergence thing, right. So, not that if you know this thing, the other formula will come very easily by substituting z equal to x by y put x by y here, right. So, what will happen? This also will be x by y x by y raise to k. Now, you can multiply both sides by y raise to r. If here also you can multiply by y raise to r, so what will happen? This will, no because when y goes inside here, if y raise to r into a power, so what happen is this will be written as x by y less than 1 because this will become y plus x to the power r is equal to so into because y raise r is multiplying it. Instead of this, it will become x raise to k into y raise to r minus k, right. This is the way it will happen.

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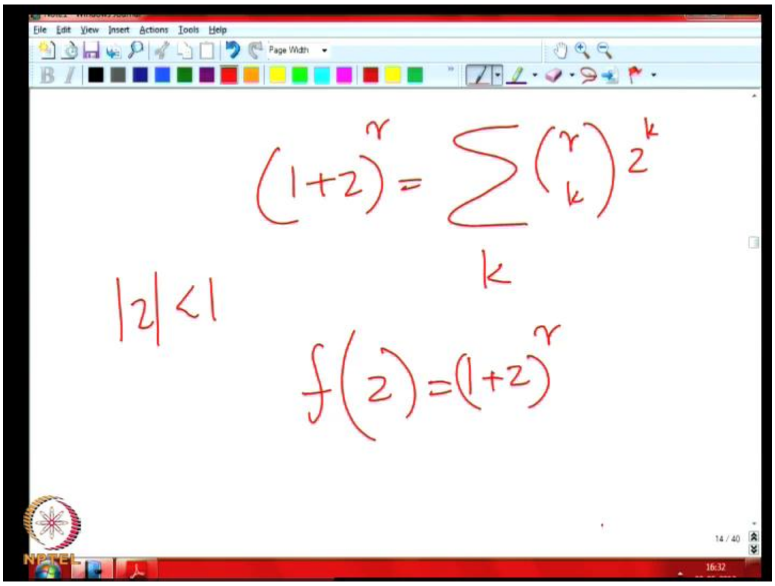

So, we only have to worry about 1 plus z raise to r. So, this is 1 plus z raise to r. What is expansion of 1 plus z raise to r is what they are asking. We want to show that this is sigma k r chose k z raise to k for all integers k. So, we will take a function f of z and let this be the function, right and we will use the Taylor series to expand this thing.

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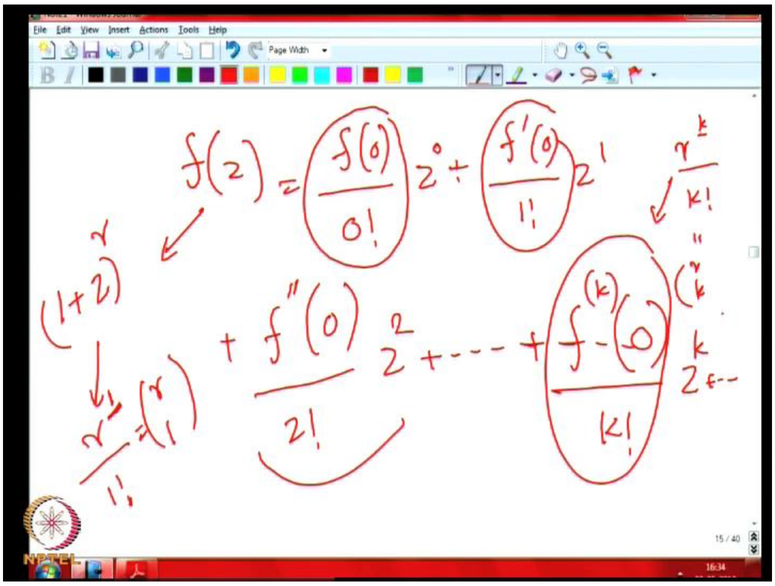

That will be read like this, f of z. The Taylor series f of z equal to f of 0 by 0 factorial into z raise to 0 plus f dash of 0 by 1 factorial into z raise to 1 plus f double dash of 0 by 2 factorial into z square and so on. We can keep on writing like this, right. Anyway, non-negative integers are running proton. We just start from 0 onwards because non-negative integer anyway, it does not contribute, right.

So, the k-th term will be f, k-th derivative of we will evaluate it at 0 and this is k factorial below into z raise k will come, right and so on. This is the Taylor series expansion and then, we just evaluate. So, to get the answer, we only evaluate what is the value of this thing. Therefore, f of 0. It is very easy to see put at z equal to, no what is this function 1 plus z raise to r. This is what we are seeing. So, here we do not have any problem which is put 0. So, that is 1.

So, we can just write it as r chose 0 because that is 1. So, that is this will turn out to be r cube 0 and here, first derivate we can find from this thing and that is not a big problem because r into that is just r 1 falling factorial, right. When you take the first derivative here, you should refer back to your calculus books how you take derivative divided by one factorial. This is what by our definition r chose 1, right. That will come here and here. It is a second derivative. If take two times the derivative of 1 plus z raise to r, that is r into yeah 1 plus z raise to r minus 1 first and again, r into r minus 1, then that divided by 2 factorials or r chose 2. So, like that in the k-th term, we will get exactly r k falling factorial divided by k factorial which is r chose k, right. So, that is where this is coming from.

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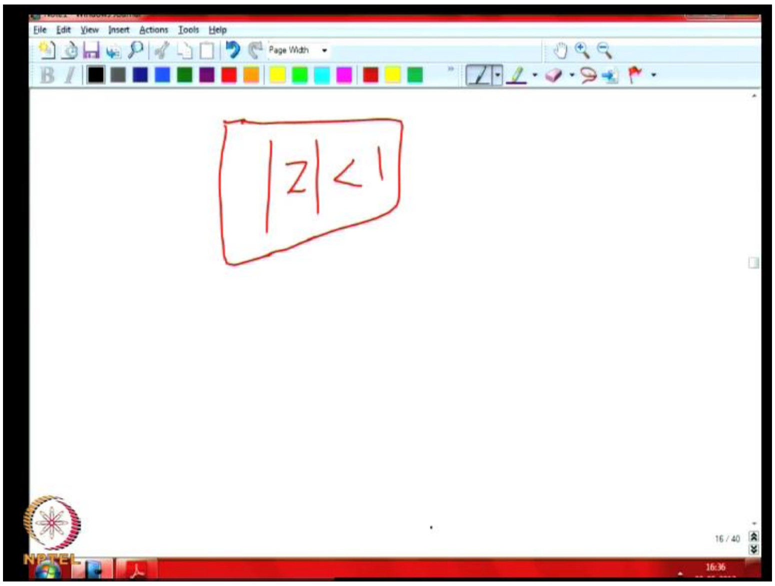

So, the only thing we have to now worry about is the convergence of this thing. The convergence of this thing is ensured if z is less than 1, right. So, that also I just leave out because when it is less than 1, you have to estimate, you have to I mean estimate of how big these things are there. So, then we will see that will be anyways converged, right. So, I will leave it to you to figure out. Anyway, it is this also knowing the proof is not very important here. Knowing the formula is more important, but you can always if you are very curious and want to be clear about it, you can go back to a calculus book and verify it, right.

So, it will be the Newton's binomial formula, binomial theorem. So, that will be available in usual books, right then usual calculus books. So, the Taylor series or the details of all these things you can get from them, one of the usual books dealing with this subjects is colorless. Now, the only thing before leaving, so I just want to, before finishing this discussion, I want to talk some special cases namely, the negative integers because we are doing combinatorics like mostly we will not get into complex numbers, any complex number and all we may get into, but still this negative integer seems to be quite interesting.

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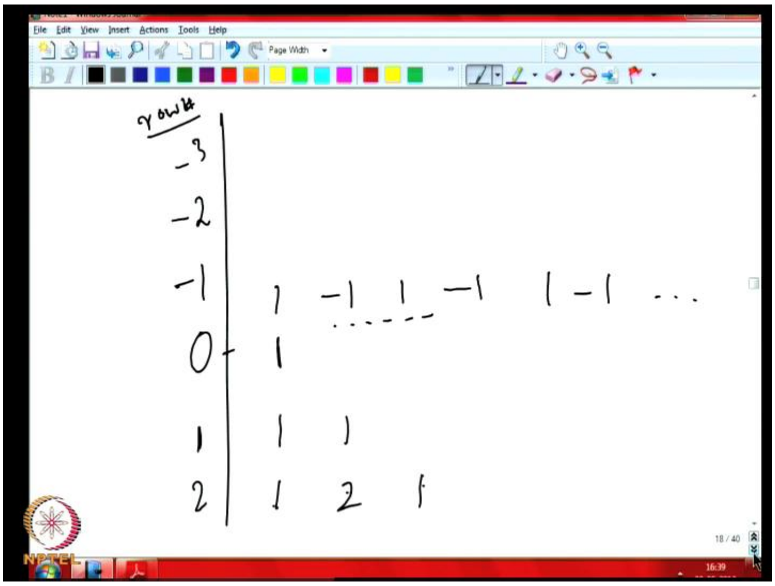

So, we remember we had a Pascal. We wrote something like 0. So, this was a 0, 0, 0 but then we wrote rry. I am just writing it as 0 through row number here. Row number 0 through plus 0. Sorry, 0 through n chose 0 chose 0. That is one and then, we wrote 1 1 because first row is 1 chose 0 and 1 chose 1 and then, second row is 1 2 1 because this is 2 chose 1, 2 chose 2 chose 0, 2 chose 1 and 2 chose 2 and so on. So, can we write about minus 1 through minus minus 2 like this, backward. So, we can complete it this way. So, this row is a 1. So, we can just consider this row one. This is minus 1 and this alternate 1 and minus 1, right 1 minus 1, 1 minus 1, like that right.

So, how will we do this thing? How will I get it? So, one formula which helps is the addition formula n chose r is equal to, n chose r is equal to n minus 1 chose r minus 1 plus n minus 1 chose r, right. So, for instance if I want to find minus 1 chose some k, we are going to get k column. So, what will I write? Minus 1, minus 1. So, maybe we can use it here, right minus 1.

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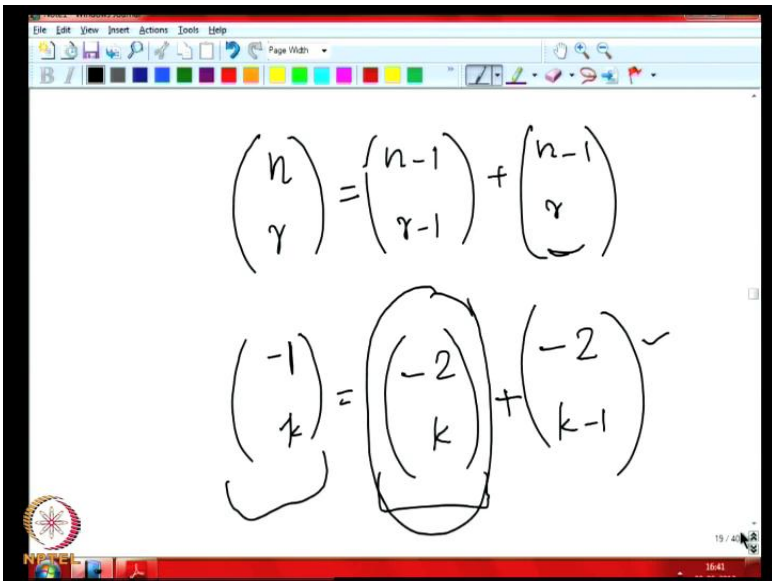

We can use the addition formula in the following way. So, n minus 1 chose k minus 1 minus 2. It is, yeah that is minus 2 chose k plus minus 2 chose k minus 2, right. Suppose, this is the one we want to write and we already know this thing, then we can write this minus 2 chose k as minus 1 chose k minus minus 2 chose k minus 1, right. What we can do or maybe we write it as 0 chose. For instance, if we write this, if the row for minus 1, so let us see we can get this thing using this and this, right. So, that means in the Pascal's triangle, we just go when you want write the minus 2 column. We just go one row before, one column before, just behind and then, add the thing below it. That is what you see.

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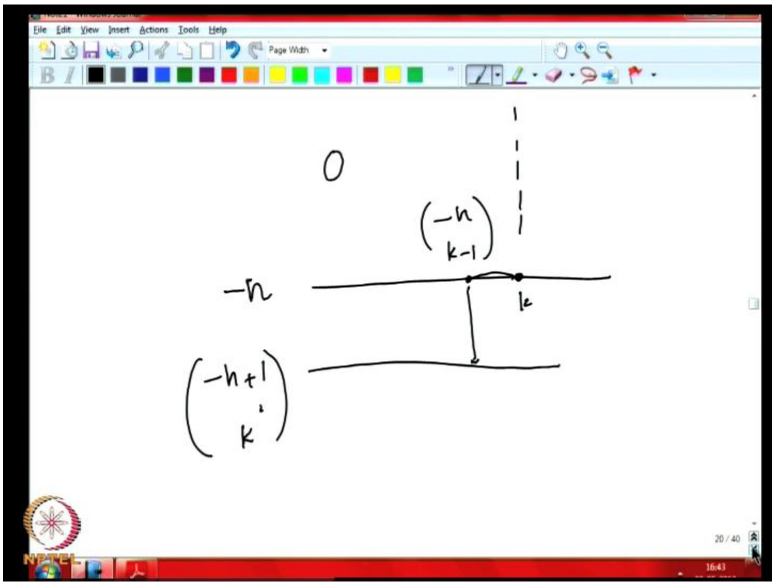

So, for instance if I want to this on minus entry for minus n, so here we want to enter k-th column entry. So, we go before, so that is minus n in k minus 1. It is a previous column. So, we can pick up this entry, so that we should minus this thing from just above 1 minus n plus 1 minus 1 chose k. This minus chose will give, maybe we should check out some of the entries or the initial entries are I will do not do this thing.

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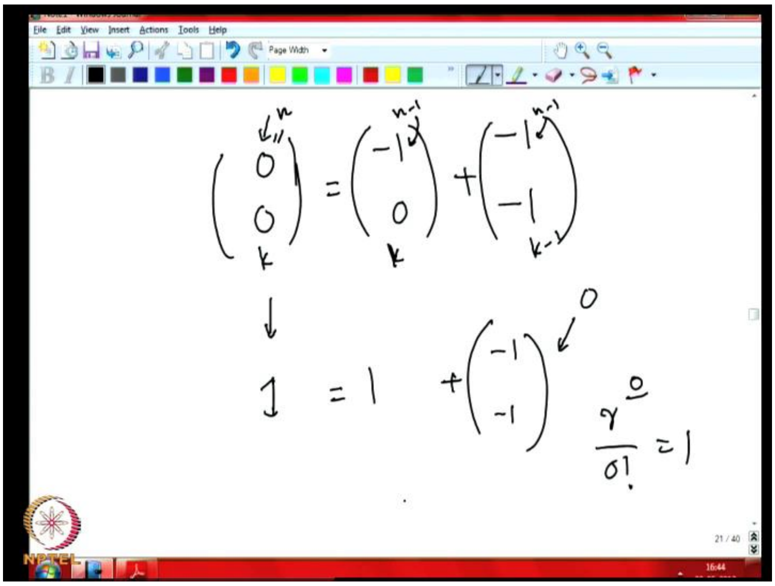

So, this is we will write like this, fine. We have 0 chose 0. We will write like this, 0 chose 0 is equal to minus 1 chose 0 plus minus 1 chose minus 1. Is this correct because just put n equal to 0 here, and this is n minus 1 and this was our k. So, this is our k minus 1. Sorry, this is our k and this is our k minus 1, right. Then, this board, these quantities are one because whenever the lower index is 0, by definition it is 1, right because r to the 0 following is 1 and we have 0 factorial. This is 1. So, this 1 plus 1 plus this minus 1 minus 1. So, this has to be 0. This also will define 0, but that is 0, right and yeah, this is they are doing and we are minusing this from this, right.

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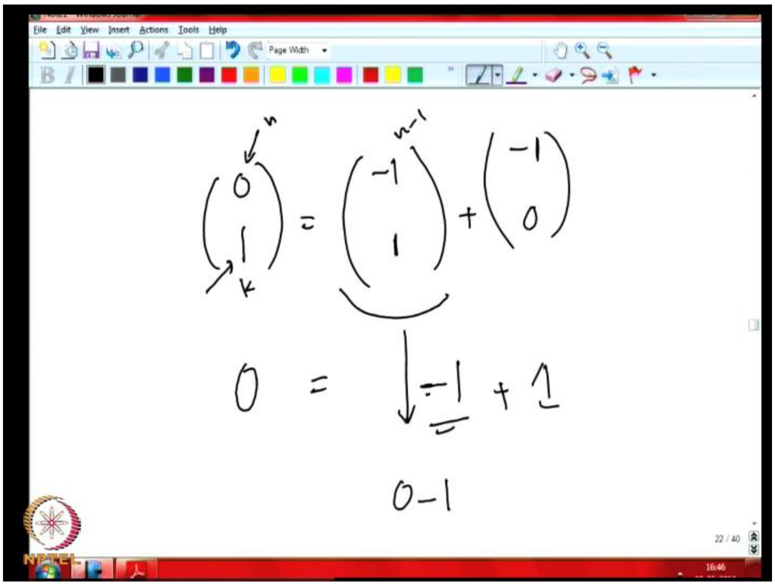

Now, we can right now say that 0 1, what is this 0 1? It is equal to this our n, this is our k, right and now, this is 0, same k n n. Sorry minus 1, that is n minus 1 and here, same k and the minus 1, 0. Here, it is 0 chose 1 is 0, right. This is 0 and here minus 1 chose 1. This if you want to evaluate and this is again 1. So, this is again 1, this quantity is again 1. So, therefore 0 minus 1. So, this quantity will be 0 minus 1. So, that is minus 1. Like that, we can see that this addition formula we can feed back and fill the entire Pascal's triangle backwards. Backward means in the negative direction. Also, we can fill, right. So, what you do is we just starting from 0. So, knowing that for all minus k chose 0, that is all 1, right and then, we could fill the negative one row. We can fill that row minus 1 minus 1, 1 minus 1 and then, we can also fill it backward and similarly, we can go putting the minus 2 things, right. Well, then completing everything is a tedious job.

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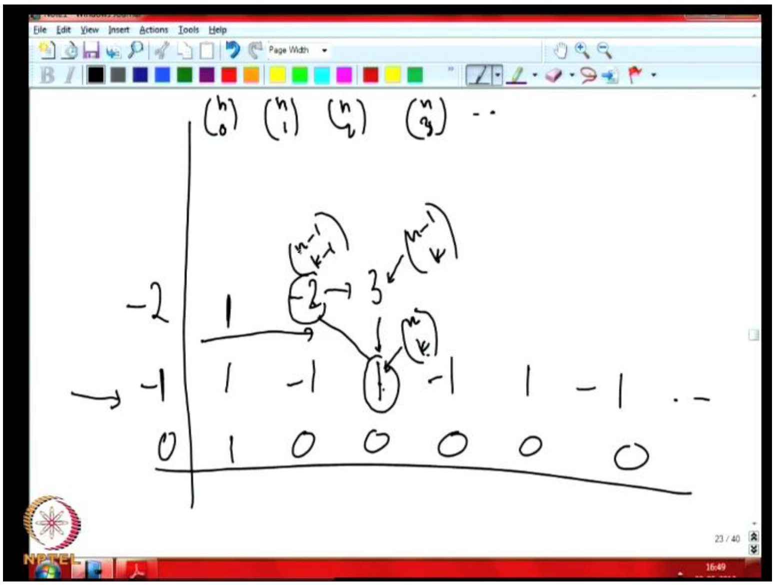

So, what we do is we write down the things and then, check that is easier. So, 0, this is 1, 0, 0, 0, 0 row like this and then, here we have 1 minus 1. This side corresponds to the n chose 0, n chose 1. These are the columns. N chose 1, n chose 2, n chose 3 and so on, 3 and so on. This is 1 minus 1, 1 minus 1, n chose 1 is minus one and then, 1. So, this 1 minus 1, right plus 1 minus 1 and so on and here, this is minus 2. For minus 2, we will get right. So, here it is 1 only, right and then, here it is minus 1 minus 1 minus 2. Here, it is 3, 1 minus minus 2, 3. This is 3. See actually this will correspond to our n chose k. We are evaluating, right.

So, this will correspond to our n minus 1 chose k. We are evaluating, right. So, this will correspond to our n chose k, k and this will be n minus 1 chose k minus 1. So, we know this already because the rate below is already computed. So, this we know and this we know because this is been constructed like this. So, this minus, this is this, right. So, n chose k minus n minus 1 chose k minus 1 is equal to n minus 1 chose k.

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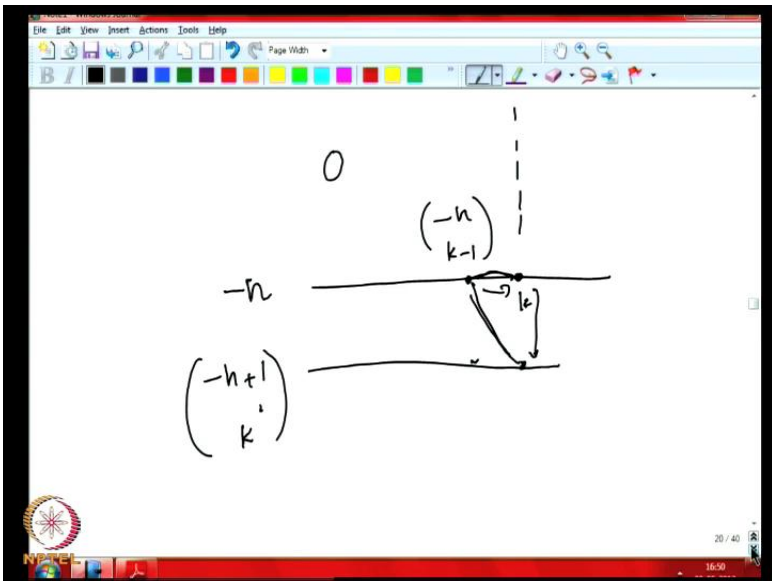

I think I wrote it wrongly somewhere here. So, yeah sorry, I do not know where I wrote. So, we are actually computing this, this one right. Knowing this one and this one, this comes from the row may be the way, I give it. May be this was conducted. This is what we just wrote below it, right the k-th quantity and then, we are minusing it, the thing below. This and this is minus and then, we got this, right. This way we got it, yeah.

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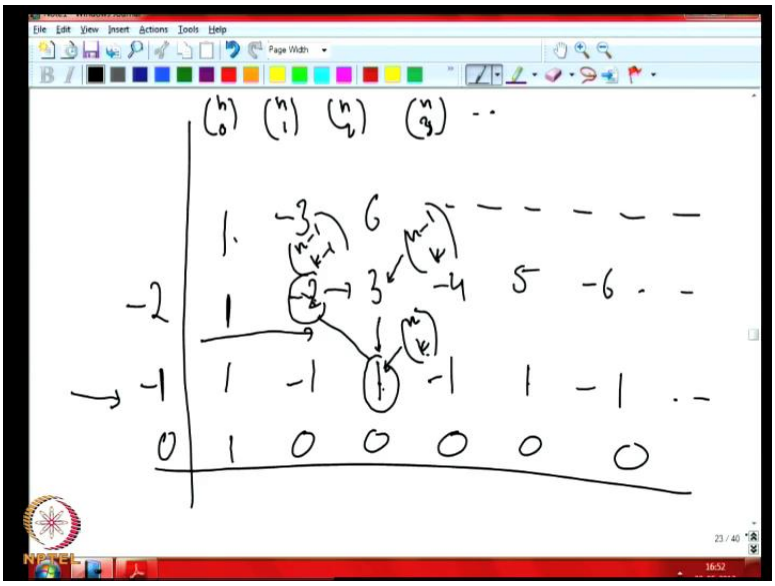

Now, you can compute it, complete it. So, here minus 1 minus 4. This will be minus 4, right and this will be 1, minus minus 4 will be 5 and then, this will be minus 1 and minus 4 will be minus 6 and so on, right. Then, similarly, you can, this is always 1. So, we can do this thing, minus 2 minus 1. This will be minus 3 and this will be 3 minus 3 that is 3 minus minus 3 that is 6, right and so on. We can fill it like this, ok.

So, this is the way we can fill the Pascal triangle, say backward. Backward means upward negative numbers for the negative numbers, right. So, just to use the addition formula properly, if I have confused the thing, just think about it carefully. There is nothing complicated going on, just that we need to recursively fill it, right. We assume that we know 0 through and we also know the n chose 0 column because it is all ones, right and from that we can fill it. That is what now why do you fill it to just get a feel of what it is. After all, we can write on.

When you want to write on the binomial formula, we can also write for negative, n negative integers. We can always pick up the coefficients on this table. Now, some interesting thing we cannot here namely, that the numbers we are seeing here are not at all unfamiliar. We said the same number which we seeing the other pot, for instance I have been seeing this formula. Yeah, so I do not know whether I did not write it anywhere here. It seems our or it disappear.

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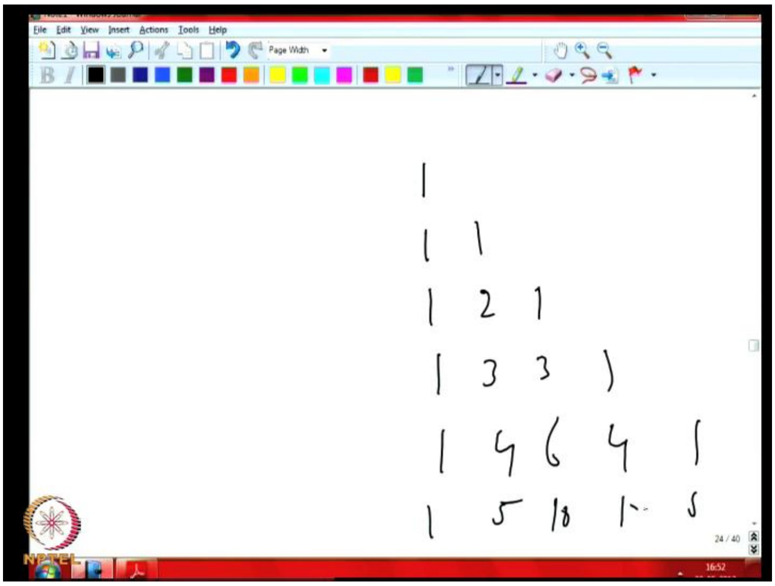

So, here if the earlier, the later columns will be 1 or 1, some points of time it will be 1, then it will be 1, 2, 1 and then, 1, 3, 3, 1 and 1, 4, 6, 4, 1 and 1, 5, 10, 10, 5, 1. All these number which are appearing are actually appearing here. Also, if you look at the patterns, we can see that I have not written lot of column. May be from what I have written, It may not be so easy to see the pattern. If you here self write down lots of them, several column, hence compare and see that these numbers are same and see which number are in the negative side, which number corresponds to number in the positive side. Try to map the things and find out the formula. We will see that.

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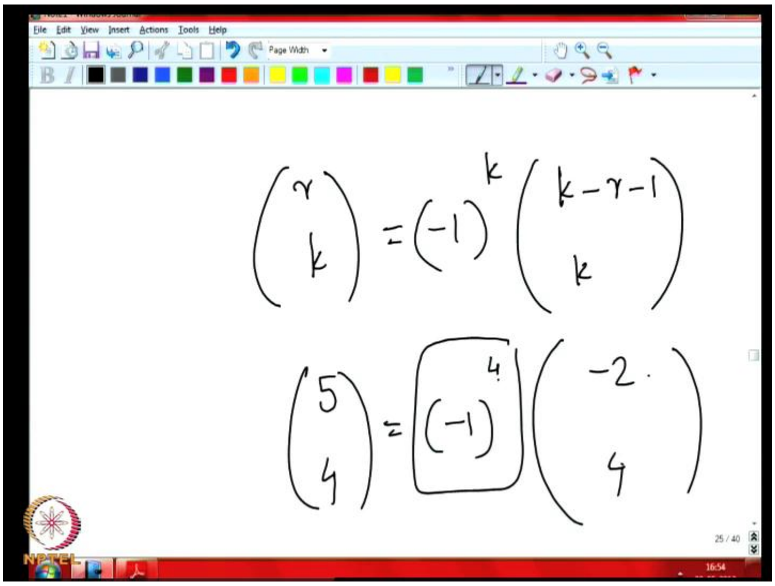

So, this is the formula we can get r chose k is equal to minus 1 raise to k, k minus r minus 1 chose k. That is how we have seen r chose k is equal to this formula. You can guess minus if you stare at the Pascal's triangle for something sometime, right k minus r minus 1, so that it make sense. For instance, let us say 5 chose 4 will be equal to minus 1 raise to 4. That means, there is not suppose negative sign into 4 minus 5 minus 1. This is how much? This is minus 2, right. This is minus 2, minus 2, 4 or if you want to try if the minus number, say minus 3 chose 5.

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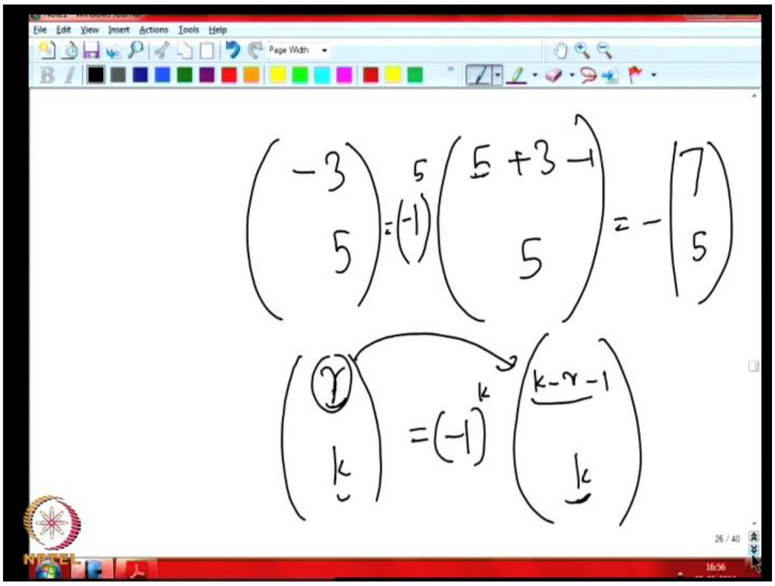

If you want to find, so that will be 5 minus of minus 3 plus 3 minus 1 which is actually 8 minus 1, 7 chose 5, but we have minus 1 raise to 5 here. That means, this will be a negative. This will be negative 7 minus 5, 7 chose 5. So, like that there is a relation between these things. So, column wise if you look at any r chose k, we will see the same number somewhere else in the same column namely, in this position k minus r minus 1. The same column k corresponds to the column lower index corresponds to the column and some number. If it is negative or positive number corresponding, it is a negative number, means this will be a positive number, right and if this is a positive number, this is going to be a negative number; if k was bigger than r, then already 0 and then, yeah that is different issue.

So, otherwise for instance we can see that here, it is suppose r is positive and then, k minus r is k is smaller than r, then k minus r is going to be a k minus, r is going to be negative, minus 1 is going to be negative number and the reverse is happening, right. So, this is minus 1 raise to k. Here, you can see that negative and positive signs are there that you have to discuss. So, I just look at the proof of these things. Proof is also not very difficult.

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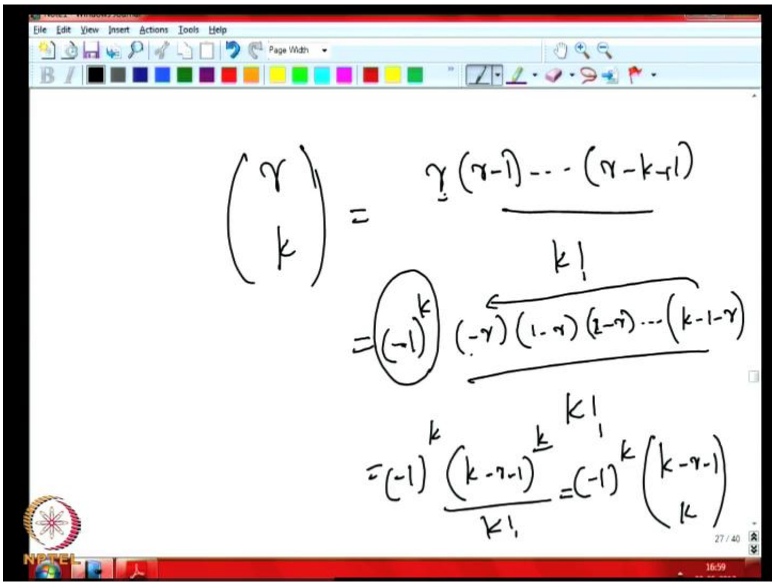

So, for instance, r chose k is equal to we can just say, this is r into r minus 1 into r minus k plus 1 divided by k factorial. Now, just multiply by minus 1 raise to k, right. So, r chose k. So, for a non-multiply, what we do is we just extract minus from 1 from each of them. So, for instance if I extract minus from 1 from r, that will be a minus 1. This will become minus r. So, when I extract minus 1 from 1 r minus 1, it will become 1 minus r and this will become square. One more minus one comes out and then, the next one becomes 1 minus, sorry 2 minus r and here, it will become 3.

So, finally this will become minus 1 raise to k because there is k terms here and then, the last one will become k minus 1 minus r, right, k minus r minus 1. Now, you see that this minus divided by k factorial. So, we can read it downwards, right. So, this is r. Sorry, this is minus 1 raise to k into k minus r minus 1. So, next is k minus r minus 1 minus 1, right and so on. It is coming down, right. Therefore, this is k minus r minus 1, k following factorial divided by k factorial. This is equal to minus 1 raise to k, k minus r minus 1 chose k. This is the definition, right. So, the way to remember the formula is this.

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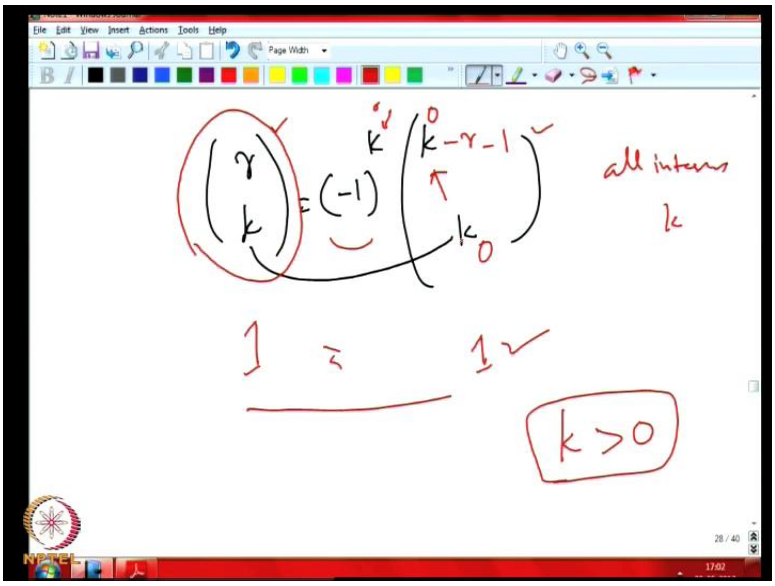

So, when you say r chose k, first you write minus 1 raise to k and remember that lower index is same. We are in the same column and then, we write k. This k here is like that, and then, we minus r negate and then one more minus. This is what is called; this works for all integers k. So of case, the other cases, therefore instance here what we have done. We have first of all we assumed that r chose k can be written like this which assumes that k is greater than equal to 0, otherwise we cannot do that or k is at least, yeah. So, this formula is correct, k is greater than 0.

Otherwise, it is not very difficult to check this thing. For instance, k is equal to negative index, both are 0. Therefore, it is anyway correct. Therefore, we can assume that k is equal to 0. Also, this is r chose 0 and then, this is 1. What is this? This is put 0 here, 0 minus 1 chose r minus 1 chose. This also, yeah this is also 1, right and this one, because k equal to 0 will be 1. So, 1 is equal to 1. That is correct.

So, we can assume that k greater than k is a positive integer, right. We can assume that k is a positive integer and therefore, this is correct. This formula is, you can expand that way and then, we can always extract minus 1 out of this thing that is not wrong, right. So, end of case, there is one thing, if we can also consider that k is greater than r. If k is greater than r, this is going to be 0. If k is greater than r, it is going to be 0 and here, what is happening is k is greater than r. So, we have r chose something bigger than r and then, k. It is time now, so we will discuss in the next class.