Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Lecture - 14
Sterling's Formula,
Generalization of Binomial coefficients-Part (1)

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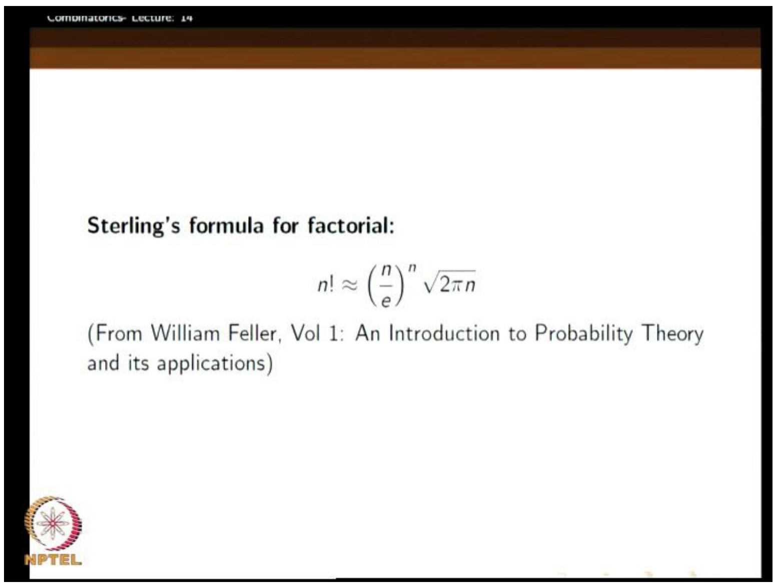

So, welcome to the fourteenth lecture of combinatorics. In the last class we were discussing the Sterling formula, Sterling's formula for factorial. So, this is, when we ask how big is this parameters n, c, r, n, p, r, etcetera, so one tool we can use to kind of approximately know how much these quantities are is the Sterling's formula. So, this is n factorial is equal to n by e to the power n root of 2 pi n. So, I am taking, it is not Feller's formula, so I am taking this, whatever I am discussing is taken from William Feller's introduction to probability theory and its applications, that is why I just wrote it there.

Now, in the last class I was telling, I will kind of give the starting of the proof. So, of case I can just keep the proof, because it has nothing much to do with combinatorics, but still to make sure, that the curious student kind of understands where it is coming from, so we will like to begin the proof and then drop it, because then I will give so because Feller's book has the complete proof, the student can go and check it there. So, this is the way it goes about.

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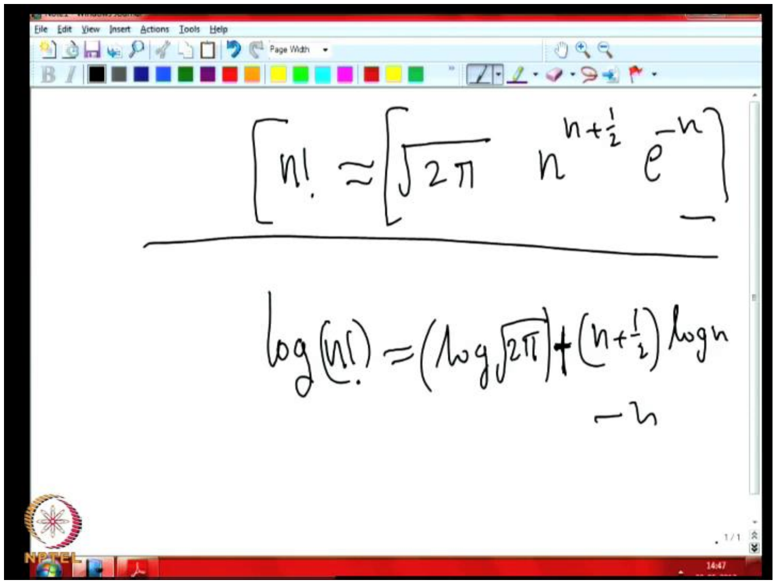

So, we want to show, that n factorial is approximately equal to root of 2 pi into n to the power n plus half e to the power minus n, this is what. So, in other words, we want to show, that as we discussed in the last class, the ratio of these two quantities, LHS by RHS will tend to 1 as n tends to infinity. So, the trick is this, how will you guess this part is the first question, right. So, we work with, so we will rather work with log of n factorial and we want to show, that it is approximately equal to the, so it is standing to the log of this thing so that this formula will come, that is approximately some of the n log of this 2 pi and then into n plus half into log n. Yeah, so when we say log, so this will be plus, ok, plus n minus n, this is what we will show, right, this what we will show. So, in other words we will be working with log rather than directly, rather than working directly with this stuff. We will work rather directly, rather with log, right. So, say, I just took the logarithm of this things, sort of show. So, then whether this is the way now we are going to do.

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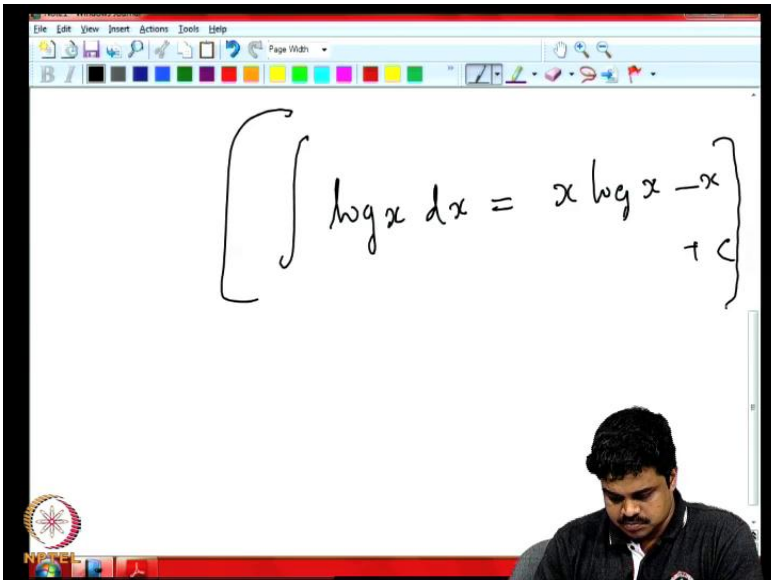

So, then we noticed, that when you integrate log x dx, right, so it is something like x log x minus x plus c. So, it is just, you can try differentiating on this side and we will see that this is coming out. So, therefore, so what we are going to do is, so I would not try to, I would not show the integration as such, but I will just remember these things.

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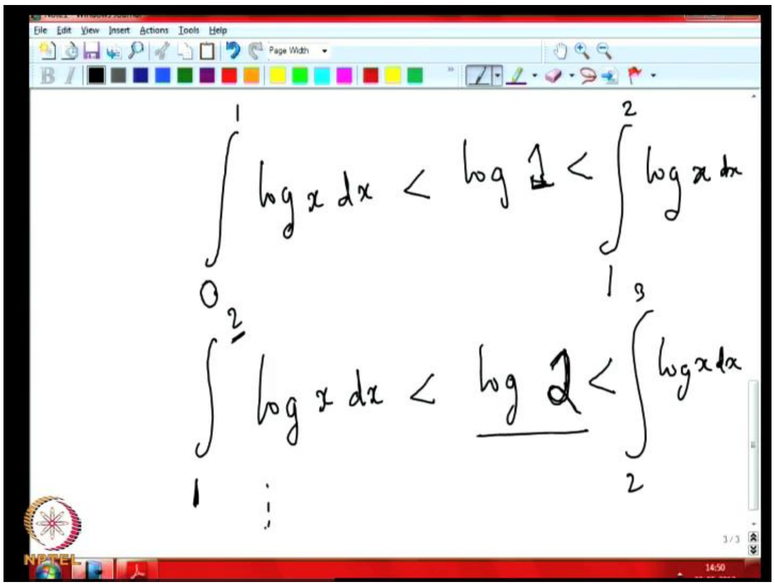

And then we notice, that it is ok, I can try it from 0 to 1 log, see if I write 0 to 1 log x dx, so this is definitely less than log 1 because this is a range of 0 to 1 and this curved log x is integrated and this log 1 is the biggest in this thing. It is a, it is an increasing function, increasing from 0 to 1 and then you see this unit. So, therefore, this portion is going to be strictly less than log 1. Of course, I, we have to go back to your calculus course and then remember all these things and this is less than equal to, definitely if you go from 1 to 2 log x, right, this is what is happening, dx. Now, we can do it from 0 to 1, then so we can do it from 1 to 2. Now, here, so this is log x dx less, less than log 1 log two because this is the higher range. So, log 2, because highest value into 1 is what we are taking here. So, because this is 1, so integration is happening from 1 to 2, that is, the difference is 1. So, the biggest value into 1, we can (( )) for that right less than, so 2 to 3 log x dx.

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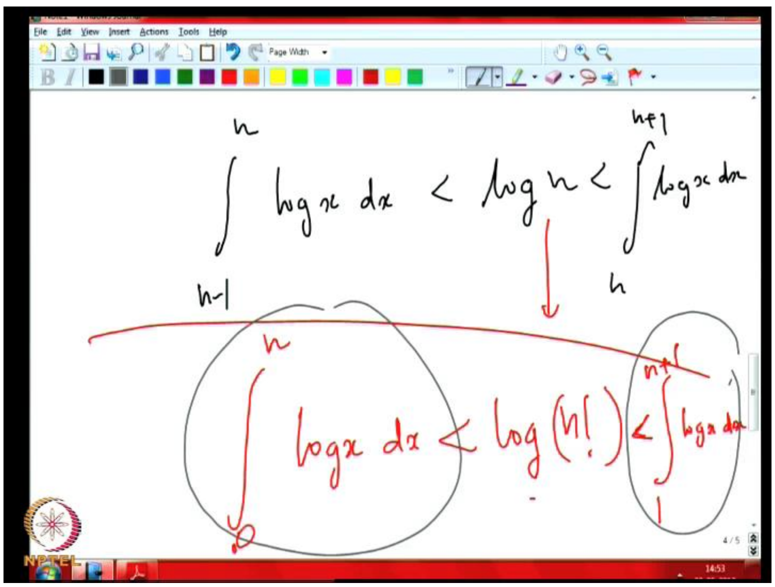

Now, we can keep on writing it till say, n minus 1 to n log x dx. This is less than log n, this is less than n to n plus 1 log x dx, right. Now, we sum up all these things starting this one, this one and this one. So, this, entire this, below, before whatever this less than equal to sign over certain, if we sum up what do we get? When actually integrating from, when actually integrating from 0 to, 0 to n, right. We started from 0 to 1, 1 to 2, 2 to 3, like that, n minus 1 to n minus infinite. So, 0 to n, it is log x, log x dx is what is coming here. Here what is happening is, you started with log 1, so that will be added, log 2 plus like that log n. Let this, this is less than equal to something else, what is (( )).

So, here what was this? so remember here it is, here it is log 1, then it is log 2, next will be log 3 and all the way log n. So, we are adding up all of them, so this quantity will turn out to be say, log 1 and 2, log 1 plus log 2 plus up to log n. So, when you take, when you take all of them inside log, so or when you combine them, that happens log of 1 into 2 into up to n, right. So, this can be rewritten as log of n factorial, right, log of n factorial because when you add up all those logs and if you just want one log what happens is, this we take the product, right. So, this is less than equal to integral of 1 to n plus one, right, log x dx. Now, we know, we have seen, that log x dx when we integrate we will get this stuff. So, we can substitute the limits in that, write 0 to m here.

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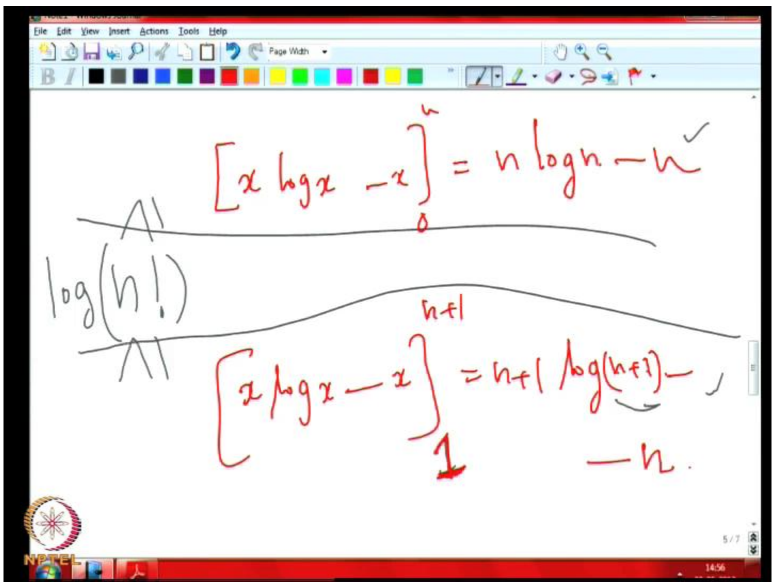

So, we substitute in x log x minus x plus 0 to n what will we get? So, we will get n log n, right, minus n we will get, right. And when you substitute in this x log x minus x n plus 1 is substituted, we get n plus 1 into log of n plus 1 minus n plus 1 will come, write minus n plus 1. So, we will write it as minus n minus n plus 1, right. So, now you know, you are, so this is the upper. See, what was this n factorial was in between this and this is what we have seen, right. So, this was less than equal to, this was less than, this is the way, so this quantity is a lower quantity compared to log n factorial and this quantity, this is a bigger quantity compared to log n factorial. So, so this log n factorial, so log n factorial will be sandwiched between these two quantities and log n minus n and n plus, n plus n, n plus 1 log n plus 1 minus n plus 1. So, we can kind of think that it is in the middle, somewhere in the middle. For instance, if I add up these two things, this one and this thing and try to divide, so this will be n log n. Let us say, make some minor approximations, this log n plus 1.

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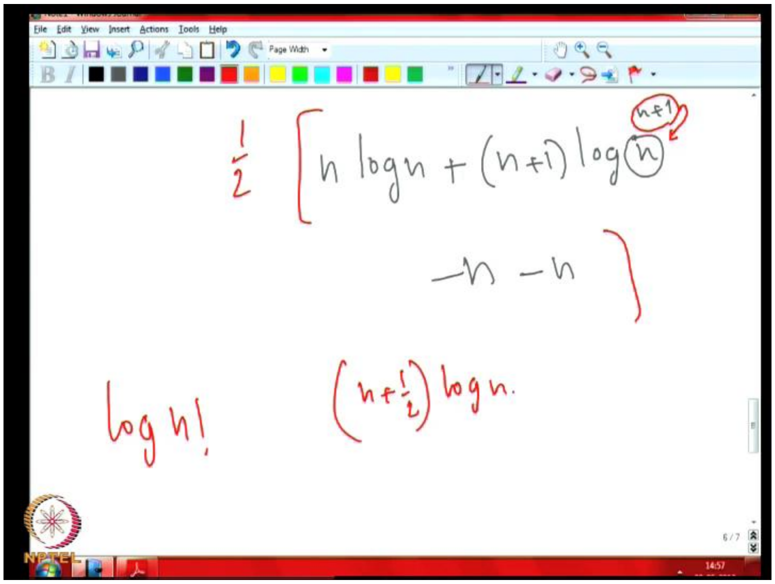

We will just consider as log n only, that is, n plus n, so it will become n log n plus n plus 1 log n. Yeah, if this was log n plus 1, remember, but we are just putting log n here and then here minus n and then here minus n and say, plus 1, right. So, this is what, this is what we got, right. So, sorry, this here I have written all wrong thing because n plus 1, so this is 1 to, sorry, still it was 1 to n plus 1, so you put n plus 1 log n plus 1. So, we have to minus this, this is not from 0 to, right, so minus something here, right, so sorry, minus something here. What is that when you put one here for x? This is 1 into log 1 that will be 0, so this so there will be a minus of, minus 1, right. So, if this is minus n plus 1 and then here n, minus of minus 1, that is plus 1 will come. So, when you n plus 1 log n plus 1, this minus 1 and this 1, this is minus 1 and this 1 will cancel. So, finally what would remain here is n plus 1 log n plus 1 minus n because see I just forgotten.

Earlier when I discussed it before, I just forgot, that there was a 1 here, so there will be contribution from here, that is why, that was why minus of n plus 1 was coming. So, now we corrected it. It is just now this, so therefore that is just minus n, minus n, right. So, now if you take half of this entire quantity, so see, here is the only place where I just changed for convenience. You can, just because this is not part of the proof we are just trying to see approximately where is the middle of this bounds? What is there is a lower bound; there is an upper bound. So, like what is the average of those two bonds, so that we can compare that sandwiched quantity, mainly log of n factorial with that, right. So, instead of, because to make it, so if I add n log n and n plus 1 log n plus 1, you will not be able to simplify, so we will just n plus 1. Instead of n plus 1, I just put log n. So, this is, this will become half of log n, so this is just n log n, right. And here is a half log n, it is half into 1 log n, that is, n plus, I can write n plus half, log n, right. And then here minus 2 minus n by 2 is minus n, right, minus n. 2 minus n is minus 1, 2 minus n will be minus 1.

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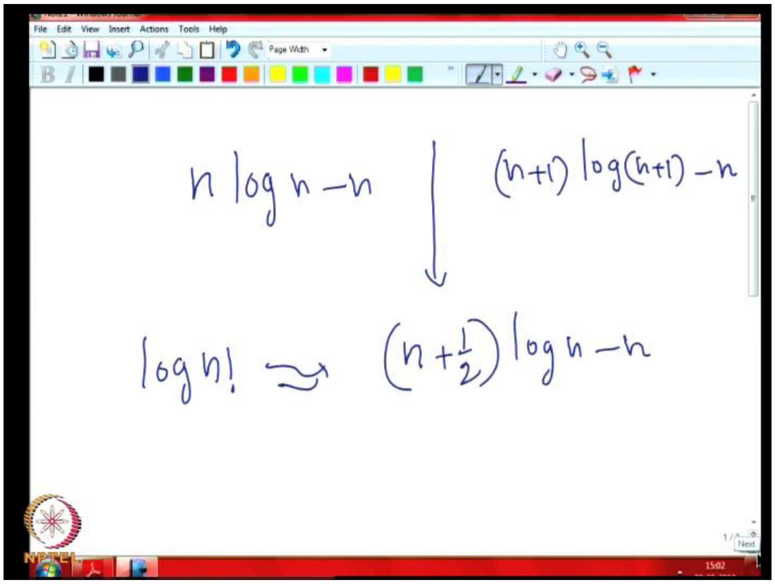

So, what we have seen is because log n factorial is sandwiched between these two quantities, n log n minus n, n under n plus 1 log n plus 1 minus n, right. So, approximately, the, so we have done a calculation, we are just putting instead of n plus 1, I have just put log n, approximately the midpoint of from here to the average of this thing, right. Middle, if you take this plus this by half, if you get something like n plus half log n minus n, so we think, that may be log n factorial will be very near to that, right. It will be very near to, that is, (( )) thing. So, we, with that intuition we tried to evaluate the difference between these two, log n factorial minus n plus half log n minus half.

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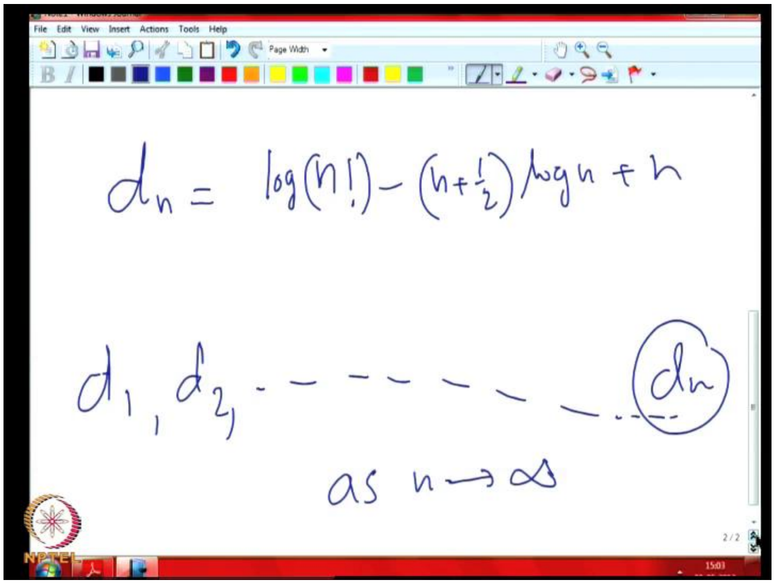

Since we are, we are interested in this quantity, log n factorial minus n plus half into log n plus n, right, plus overall minus. So, let us define this as some sequence, d n, so we will have, for n equal to 1, d 1, then n equal to 2, d 2, and so on, right. So, we can write certain inverse, right, and so the question is, as n tends to infinity, what happens to this sequence d n?

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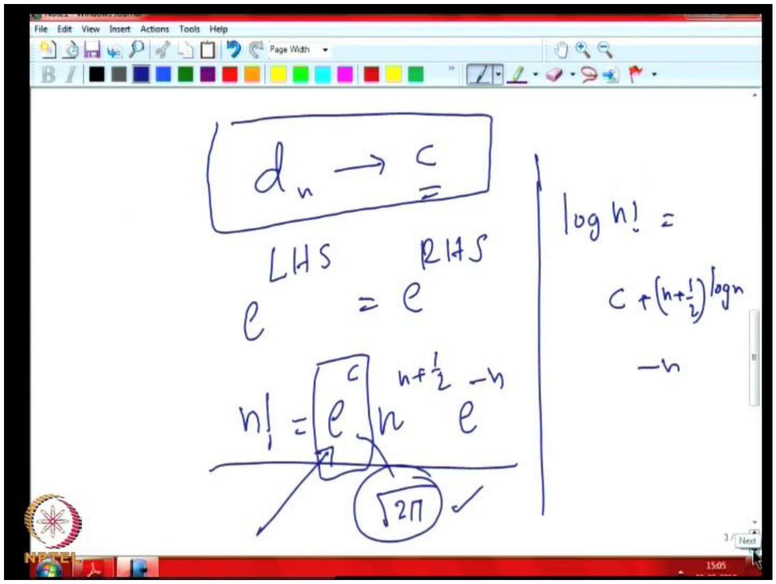

So, the aim is to show what we will, what we have to show is d n tends to some constant c. If d n tends to some constant c, some constant c, then what will happen is, this quantity will tend to the, tends to constant c, right; this will tend to constant c. Now, we will, we can rearrange and that will say, that log n factorial is equal to c plus n plus half into log n minus n as n tends to infinity. Now, you take the e power LHS equal to e power RHS, so that will tell us n factorial. Yeah, if you do that e power LHS equal to e power RHS in this following equation, that means, log n factorial equal to c, c plus, right, c plus n plus half log n minus, right. So, if we take e power LHS, so this will be n factorial, this will be equal to e power c into n raise to n plus half into e to the power minus n. The only thing is, what is this, right, this requires a little more work and then it will be shown, that this will be (( )) 2 pi rho 2 pi, right. So, this I will not do, so this is and again, so yeah, of course, proving that actually d n tends to something requires little bit of effort.

I can do it but to save time I will skip it because (( )) it has nothing much to do with combinatorics. I just gave this much to give the student feeling of where it is coming from, maybe how one can know about it. If he is really curious about the complete proof he can probably read it from Feller’s book, which I have mentioned it before, right. So, yeah it is really important to know the proof. More important is to know the formula well and to be able to apply it when the situation comes.

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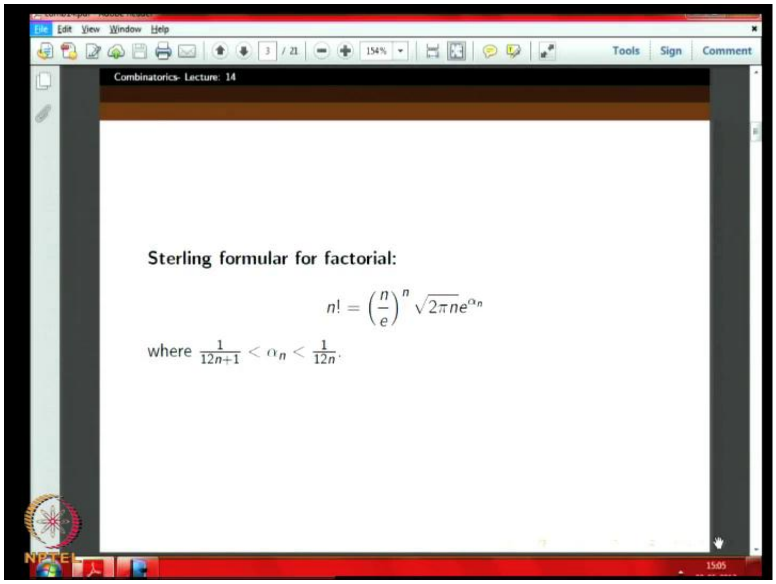

Now, I will just mention, that we wrote at approximately equal to, we can make it little more clearer by saying the n factorial is equal to n by e raise to n. This is your n to the power n, e to the power minus n is already there, that n to the power half we have taken inside this root 2 pi n, right. That root n is coming here, so 2 pi n, but other than that, that approximately that is e to the power alpha n we have captured, that the remaining difference will be captured in the (( )), all are fine, where alpha n is some quantity. Again, we do not know exactly what it is, but we are telling again it is going to be in between 1 by 12n plus 1 and 1 by 12n.

So, it is quite a small quantity as n tends to infinity, e to the, it is going to be e to the power c root is, tend, it will tend to c root, but so therefore, whatever we previously did is correct, but even for large enough n, right. This is, if we take some of, one of this, well, either 1 by 12n, we will get an upper bound. Here, we can put e to the power 1 by 12 n here for this thing. Now, this will be n factorial less than c, it will be bigger, but only slightly bigger because instead of putting 1 by 12n, if we had put 1 by 12n plus 1, you will get it lower bound. Also, you can imagine, that yeah, so you are thinking that some very large number, say may or may be even 100 if you put right, so this quantity into e to the power 1 by 1200, right, will be an upper bound while this quantity, that is, n by e to the power n into 2, root of 2 pi n into e to the power, so where n is equal to 1000 there, ok, of case into the power e to the power 1 by, 200, 1201 will be lower bound. So, it is not very far, right. It is quite, is the, we can see, in most of the cases we can just say, that there exist a constant c 1 sufficiently larger, that this n factorial is in between n by e to the power n root 2 pi n into that constant 1 and the same thing into constant 2, right.

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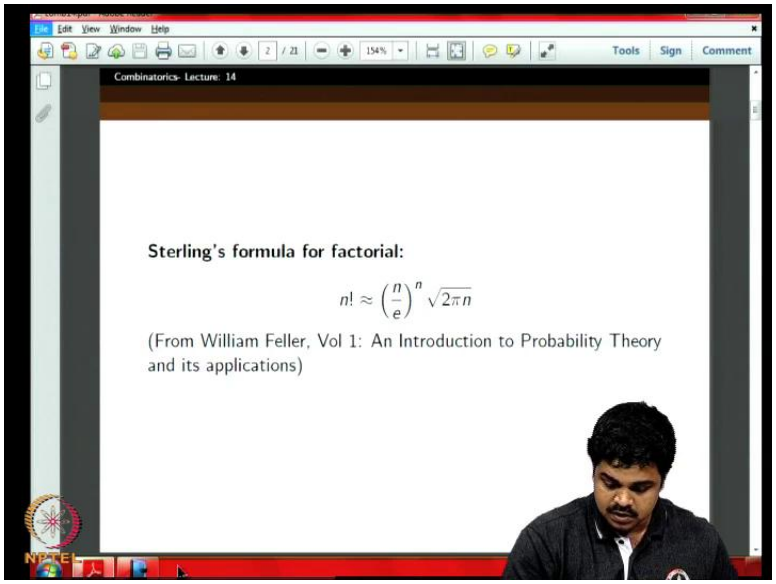

So, in between some constant times, this, this means, ok, in the earlier what time, what we have written approximately in the earlier, like this quantity into some constant 1 and this quantity in some constant 2 or constant 2 is slightly bigger than constant 1 it is very close constants, so that is n factorial will be sandwiched in between that, right. That is what it means. Most of the time we can just chose it that way, right. So, I think this much discussion is enough for this thing.

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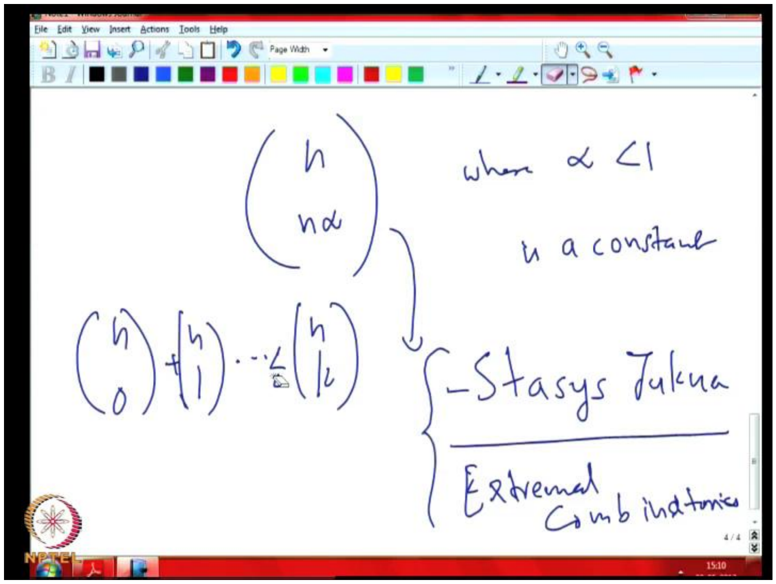

You will for instance, if you want some more, like readymade formulas for say quantities, like n choose n alpha, where alpha is a constant, a constant alpha, say alpha less than all of case is a constant, is a constant, is a real number, is a real constant. Then so you, you will get some good formula. So, by applying the Sterling formula you can derive that, you can apply doing this.So, but in Stasys Jukna's extremal combinatorics, you can see some exercises.

In the first, Jukna, so so I have mentioned this book in the first lecture, it is, this is for extremal combinatorics, extremal combinatorics, combinatorics, so you can get some exercises where you can, anyway you have to, that is only exercises you have to work out, right. And it is not very difficult, you just substitute the formula and then manipulate and also some partial sums, n choose 0 plus n choose 1, say up to certain quantity. If you sum n chose k what can you tell about some upper bounds, right? These kinds of questions are dealt with in some other exercises in Jukna's book. So, so yeah let us, so we can, you can go and read it.

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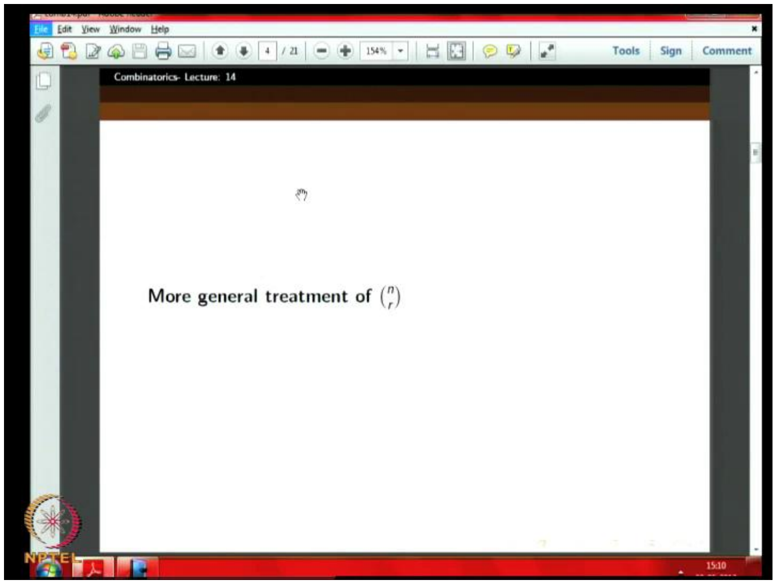

So, in that, now we will move on from this thing to the next topic. We want to consider is the following. This again, some slightly special topic may be most of the combinators courses may not take it, so starting from here, may be couple of classes I will deal with this thing. This is, most of this material is taken from the book Concrete Mathematics, Concrete, so this is taken from the book Concrete Mathematics.

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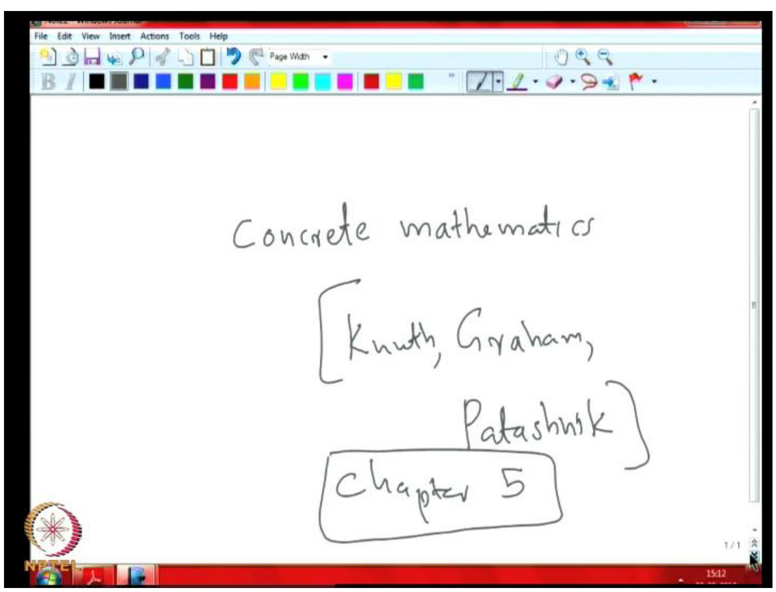

(( )) this is again concrete mathematics, so as you can take it from the book Concrete Mathematics. It is a famous book by Knuth, Graham, Patashnik, Patashnik; Knuth, Graham, Patashnik. So, yeah, so the, of course, there are lot of material in that book. Therefore, are lot of material in that, of this type and the book, the book, chapter number is 5, this is chapter number, chapter number 5, what the kind of material i am talking about. I have just taken some of the initial material to get a, give a feel of that, right. So, the rest of things if you are interested, you can read; the student is interested you can read from that book. This is slightly different kind of treatment, so so this is what, yeah, so this what I got to do.

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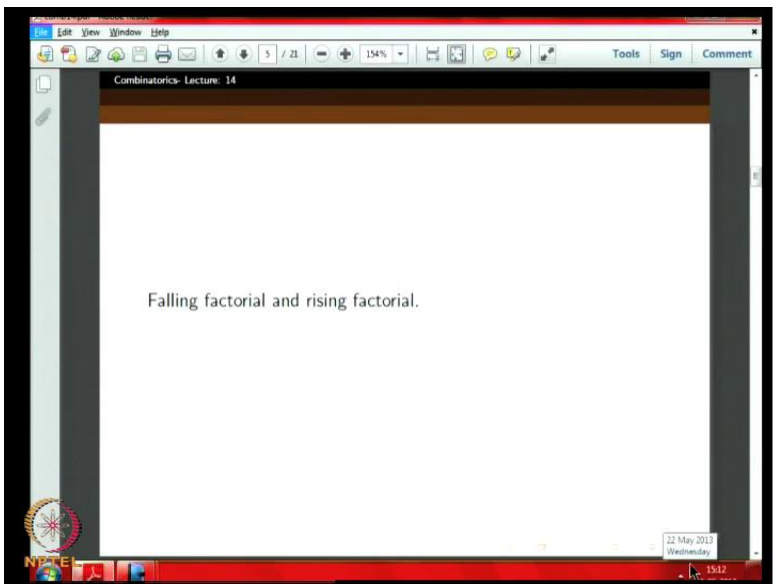

So, first let this mention about falling factorial and rising factorial.

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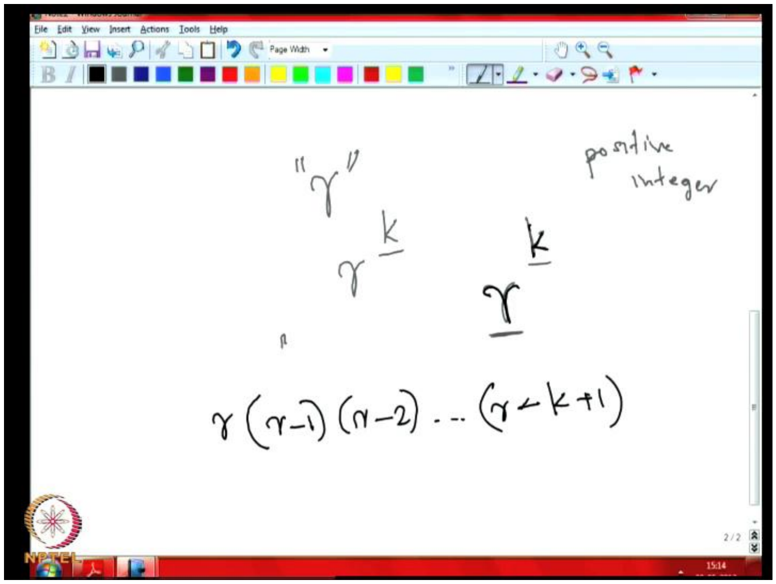

So, we, we have not mentioned this, so yeah, the falling factorial means, suppose you, you have given a number r, so this can be a real number, it can be a complex number. So, the falling factorial is written like this, falling r k falling k being say, suppose let us say we are, it is a positive integer, positive integer, we just consider k to be a positive integer, r can be a complex number, a real number, whatever. And then what is this r falling factorial? It is this sort of r into r minus 1 into r minus 2 into say, r minus r minus k plus 1, this is the falling factorial, right.

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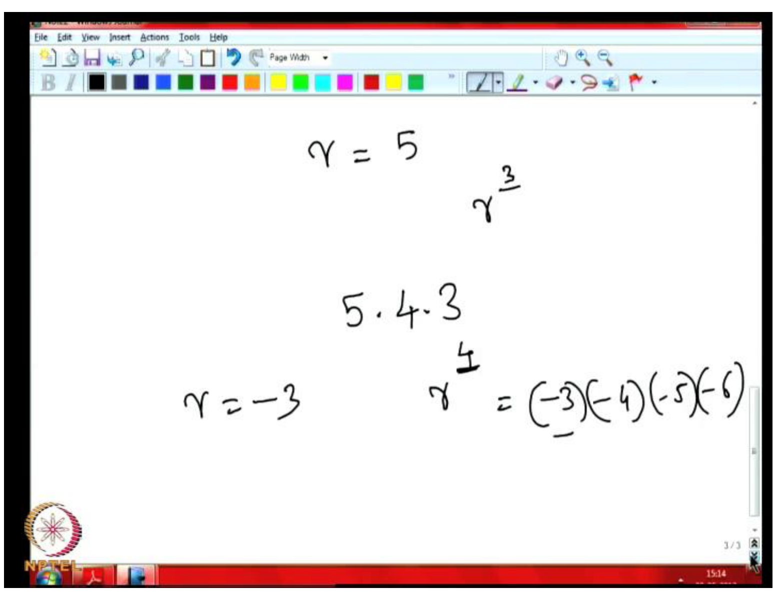

So, for example, let r equal to 5, so what is r 3? So, that is 5 into 4 into 3, which is collect 3 of them downward, right. It is falling. Similarly, suppose r equal to minus 3, then what is r raise to 4 factorial? Far for it will be minus 3 into minus 4 into minus 5 into minus 6. What we have done is starting from minus 3 we go down minus 4 minus 5 minus 6 because we have, we need four of them, right.

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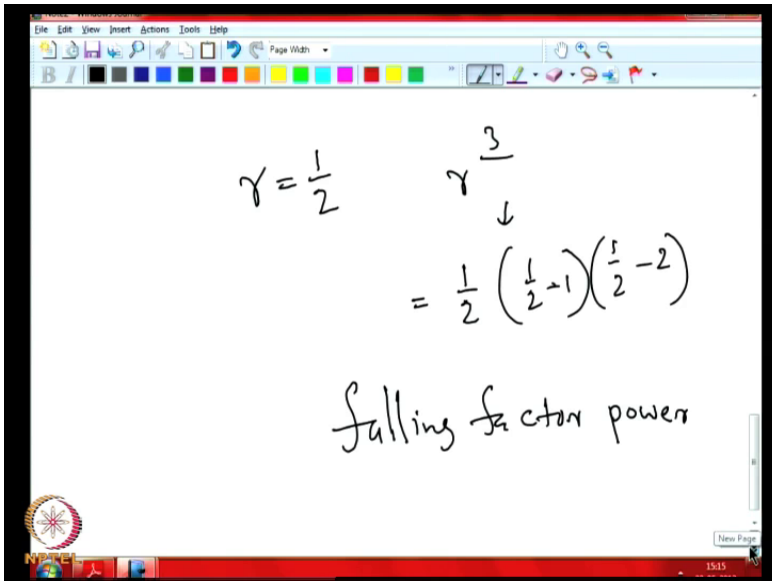

And similarly, you can ask what is, suppose r equal to half what is r falling factorial 3? This will be half into half minus 1 into 2, half minus 2, right. So, 1, 2, 3 terms are written starting from there, so this is the falling factorial, r 3 falling factorial 3, r to the falling power 3, right. So, the, it is a falling factorial power, say falling factorial power, that is the way we can call it, falling factorial powers r to the power 3. So, r to the falling power 3, so so so then we can, so yeah, for instance, we will have certain doubts.

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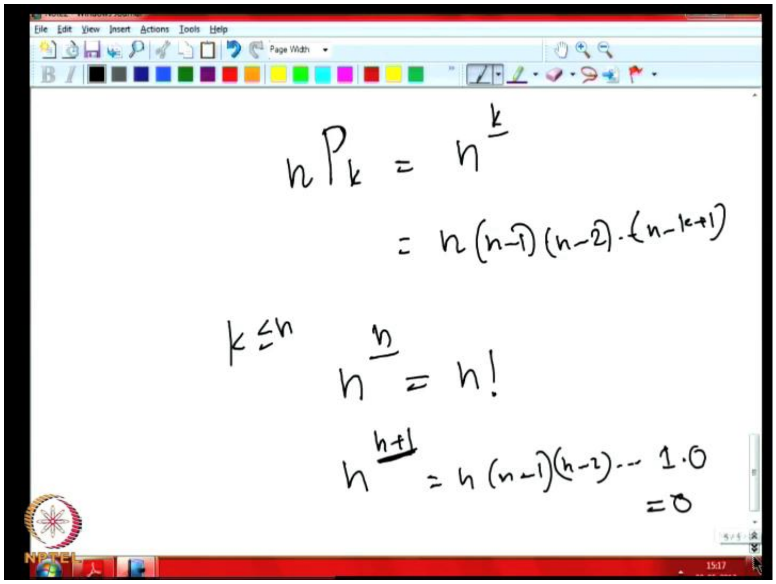

Now, so one thing is, like whatever we have seen up to now for a combinatorial, some, something like n P k kind of thing, so we, this is essentially n k falling, right. So, because we were saying, that this is n into n minus 1 into n minus 2 into, so n minus k plus 1, so k terms are written downward, right. Note, that if k was, say, as long as k is less than equal to n, this will make sense because this will go from n to 1. So, for instance, n, n to some number, which is still bigger than 1, so that is ok. So, for instance, n, n falling factorial will be, n factorial itself, right. So, now what will be n, n plus 1 falling factorial? So, then what happens is, here we will write n n minus 1. So, n minus 1, then n minus 2 and all the way we reach 1. Then we also have to write 0, so this will become 0. So, anything bigger than n here in the following power, so that will make it 0, right.

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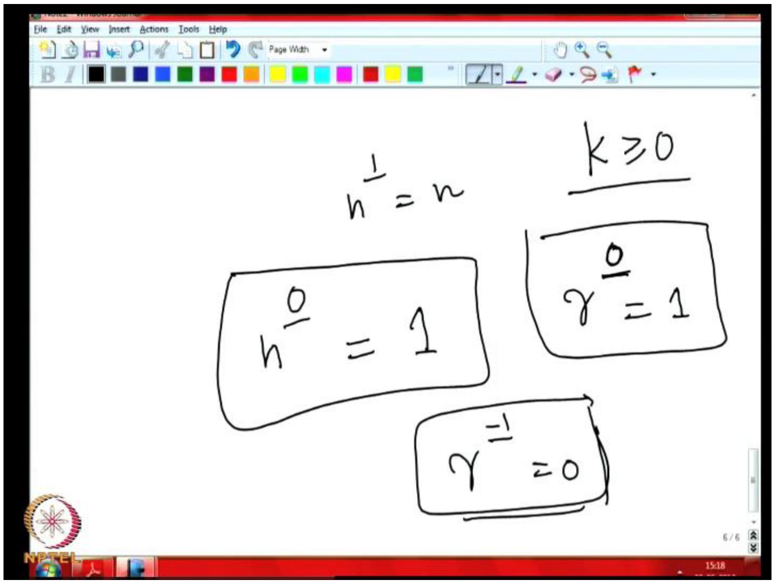

So, similarly, so n 1 falling will be n itself. Now, what is n 0 falling? n 0 falling will be defined as 1, so this is one thing you have to remember. So, r 0 for me is defined as 1. And I guess, we can define for, for negative numbers, if you want, we can, will be defined as 0. So, though we would not, we will not be using it most of the time, so we will be most of the time considering only k greater than equal to 0 and non-negative integers. But if at all, so we can also define for negative numbers, it is 0, but this is important because this is, by definition this r 0 falling is 1.

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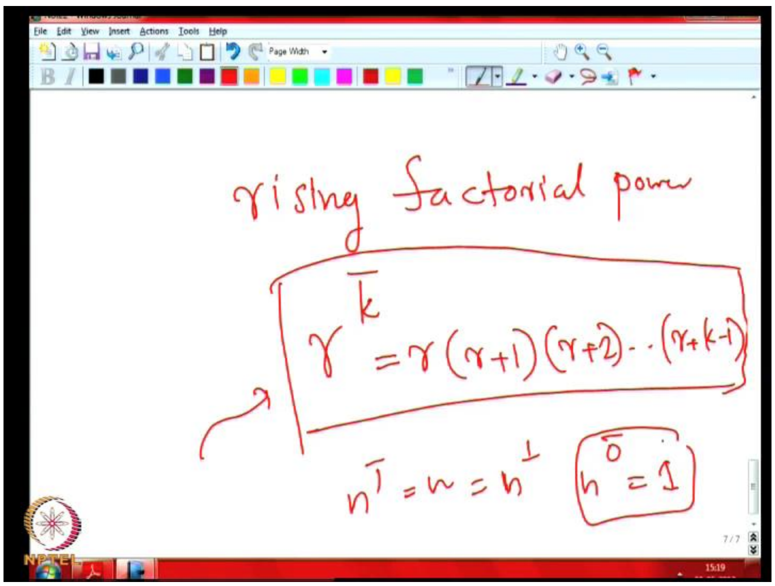

Similarly, we have a rising factorial also, rising factorial power r, so it is written like this. Instead of a bar below we rise above, so r into r plus 1 into, see, instead of r minus 1, we will write r into r plus 1 into r plus 2 into all the way r plus k minus 1. This is what rising factorial power, right. So, this is, this is the definition corresponding definition. For all these things, whatever we, so we, whatever examples we gave before for rising factorial is essentially, so sorry, falling factorial, so we can also try with rising factorial. So, there is not much difference. So, may be one of the special cases we can mention. n, sorry, what is n 1 rising? That is again n, there is no, no difference, n rising, n falling. When, when you just use 1, it is going to be, so for 0 we can again define it as 1.

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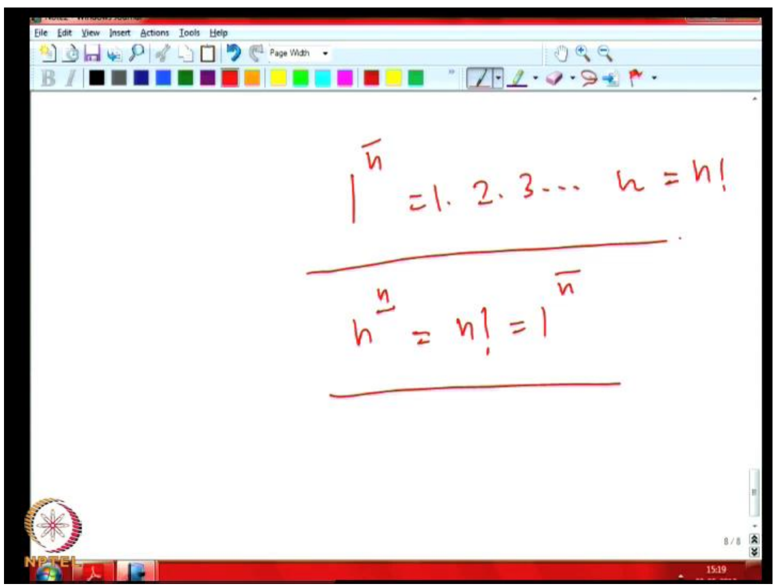

And then the, 1 n rising is 1 into 2 into 3 into up to n. So, that is n factorial. So, remember, the n falling is equal to n factorial that is equal to 1 rising n, right. This kind of things we can remember. These are two concepts, rising factorial and falling factorial.

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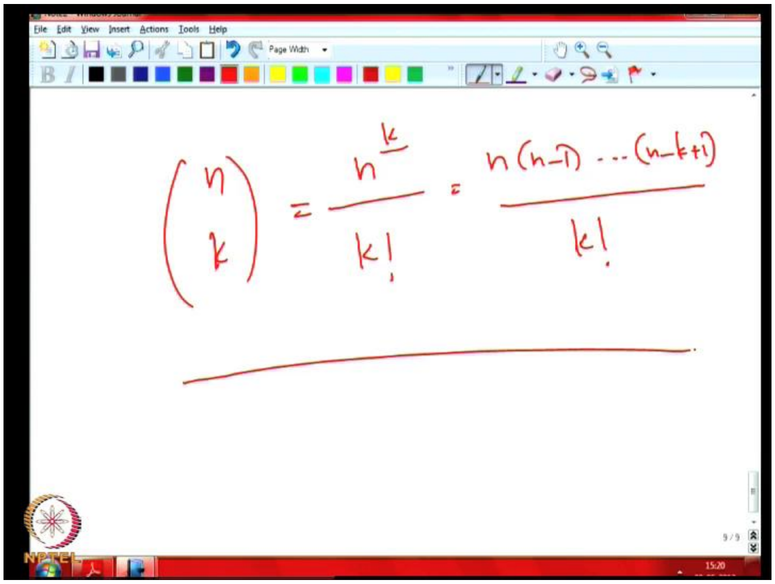

So, and then we can remember, that n choose r, so n choose k, we say n chose k, this was n k falling factorial divided by k factorial, right, using the falling factorial rotation because this is actually n into n minus 1 into up to n minus k plus 1 divided by k factorial.

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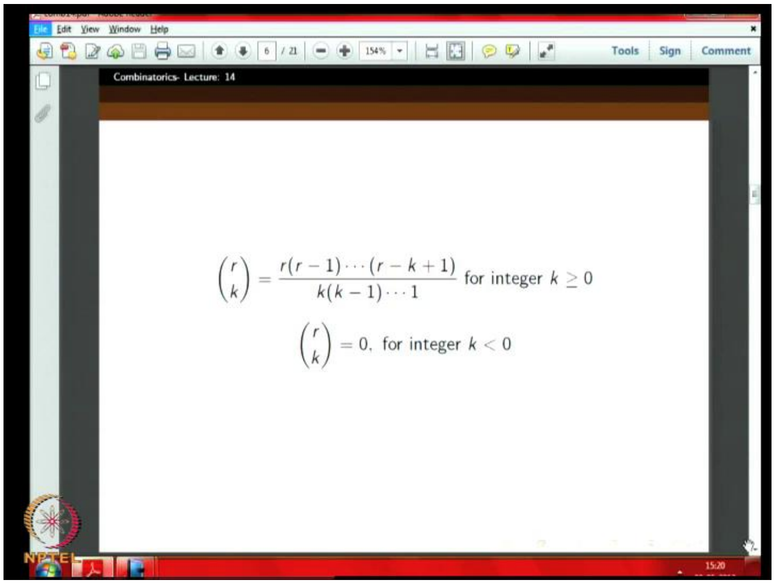

So, this is now, again we go to the next thing. So, yeah, the next thing is to generalize the notion of the combinatorial coefficient, r chose k, right, r chose k.

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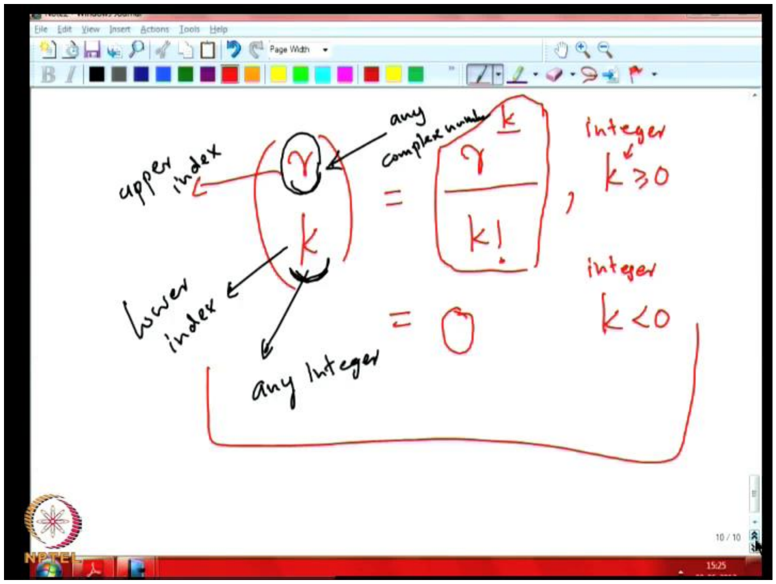

Now, generalized, we just write it as r k falling factorial divided by k factorial. See, the difference is up to, now we were always insisting, that this is called upper index of, yeah, of case, this upper index, upper index, this is upper index and this is called the lower index. So, we are always insisting, that this upper index has to be a non-negative integer, that we were not allowing negative integer, we were not allowing real members, we were not allowing complex number, nothing, only non-negative integers.

Now, we are saying, that we drop all the restrictions on r. You can use any complex number there; that is one important thing in generalization. And then second, here we allow any integer in the lower index. We allow any integer, this is any complex number, here we allow any integer for even negative integers. So, though you will, so the way we define it is, so we just say, that when this is k greater than equal to 0, so integer, integer, then k is integer and then is equal, greater then equal to 0, we will define like this. For instance, here the trouble is, that if you take, if you define like this for k negative, then you have to say what is minus 3 factorial, minus 4 factorial. So, negative numbers factorial, what is that that we do not know, right.

So, we insisted k is an integer and k is greater than equal to 0. When we define this thing, r k falling factorial and then divided by k factorial, otherwise we will just define it as 0, for k less than 0 if, if this is an integer and then less than 0, we will just define it as 0. This is the way we have, say, defined it, we have generalized it, right. So, the upper index is fully generalized to any complex number. The lower index is generalized from being just non-negative integers, say, any integer, but just that when it is a negative integer we will always define it as 0. And then when it is, yeah, a non-negative integer upper index is r falling factorial, upper is actually, this is ratio where numerator is r falling factorial, k, r to the power k falling factorial, r to the k falling factorial power and then divided by k factorial, right. And then so of case, this is well defined, right.

So, now, of case, we can ask why, why are we restricting k to be integers? So, the argument in Concrete Mathematics is, the book is, that it is application-wise, as far as the applications are concerned, this much generalization is usually sufficient, but they have given further generalization. The interested student can read it from the book Concrete Mathematics, so we just stick with this, this much generalization and go ahead. So, the point here is to say that, so if I generalize it, there are no more combinatorial interpretations for these things, right. So, r is a complex number and so there is nothing like we cannot define r choose k as the number of ways of taking k subsets of n, r, l, m set because there is no r, l, m, n set. Now, so it, it does not make sense to think like that. So, this is just defined that way and just that when r, s are non-negative integers, it coincides with that counting interpretation where it is the same, right, whatever the number of way of selecting k things out of r things, right.

So, so we, if you would not, this or those identities we proved, we cannot just take them because we always proved assuming that there is a combinatorial interpretation, there is a counting argument in as more than the counting argument works only when r is non-negative in nature and right. So, therefore, we will have to redo some of the identities if we have to work with these things is this generalized stuff, right.

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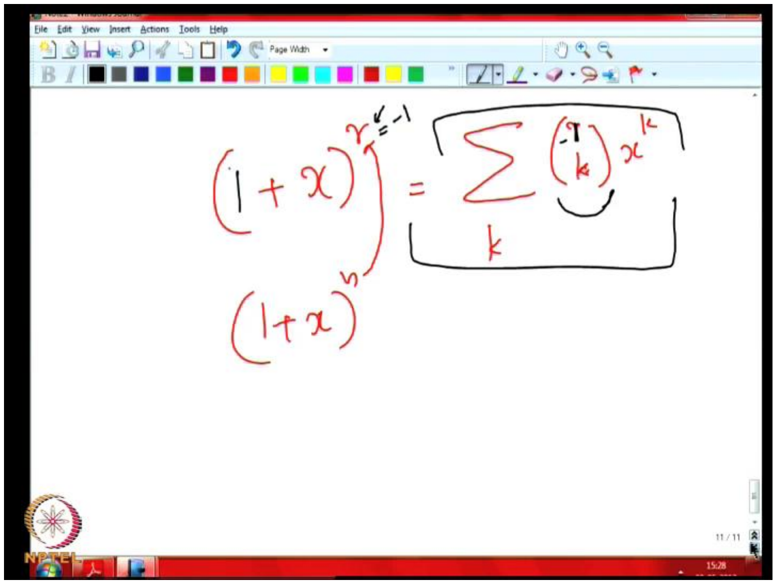

So, we will of case, one can ask why do you want to generalize? The reason is, that there are lot of applications, so there are, of case, because as we have seen soon, even the binomial theorem can be generalized, is to 1 plus x raise to r or we were seeing 1 plus x raise to n, where n is a non-negative integer. But now, here we can we can use instead of n we can put any r where r is a complex number or real number. So, now this will have the same kind of, yeah, expansion just that it will be now like infinite sum. So, r chose k into x raise to k, something like this, right. So, or may be, yeah, we can, we could have always written as x plus x plus y raise to, right, so that will give us y raise to, right.

So, if we see this is the form general format, 1 plus x raise to, we will see how we can, we can get x plus y raise to r from this thing. So, I will, I will just, this, postpone the discussion to the correct time, right. So, this is, so the point is we can generalize it for over, overall k. So, therefore, so in, in the sense this is going to be very useful in some cases. For instance, we could have put instead of r you can put a minus form here and then work with it, right. So, therefore, the minus 1 where you can put r equal to minus 1, so all these things will make minus 1 to the chose k will make sense, right. For every k you can evaluate this stuff and then write an infinite formula for that so that expansion will turn out to be useful, right.

So, so therefore, if we, since it comes, it, it becomes, it becomes important in many other cases so far in counting, probably it is not, it does not come. So, you should, so often so we should be aware of such generalization when we, because we may encounter it some more, right. So, therefore, we will have a quick discussion of it. So, now we will see, what, what all things we have to be careful about when dealing with this stuff, right. This is what the intention of this discussion is.

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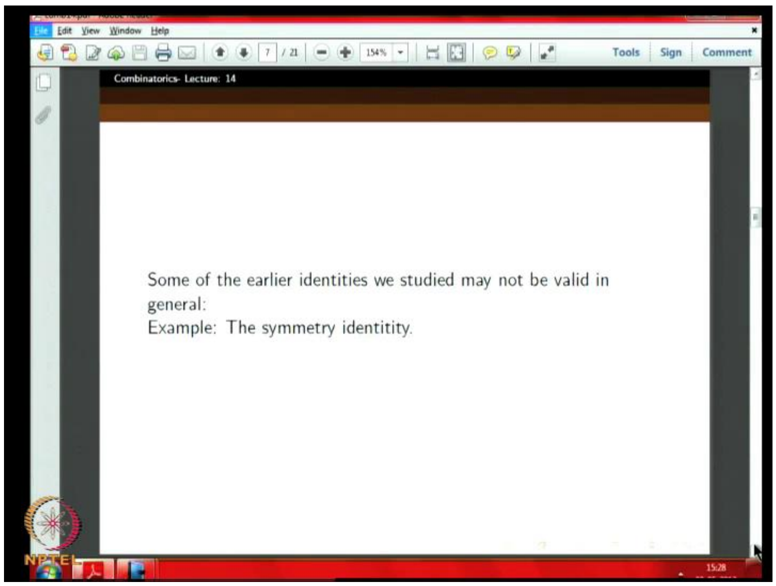

So, nothing serious, but just in passing we will consider some of the most important identities and see what, what difference is there. For instance, a symmetry identity may be the most, one of the most important ones.

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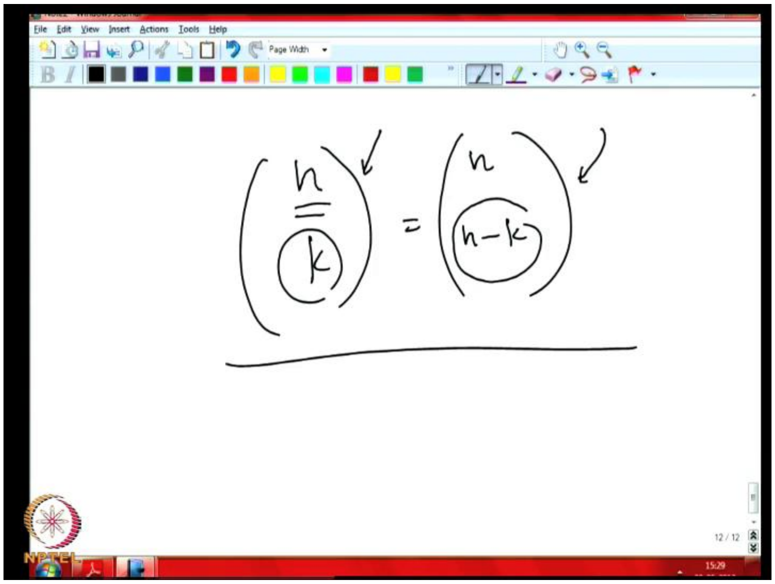

So, that is, n chose k equal to important and simple, n choose n minus k, we have seen, that there is a combinatorial interpretation, for that you are taking k things out of n things. So, that means, we are leaving out n, n minus k things. So, therefore, the number of ways of selecting k things out of n things will be equal to the number of ways of selecting n minus k things out of n things. This is what we, how we prove it, but then we assume, that there are n things. That means, n is a non-negative integer, right, out of that we have to select.

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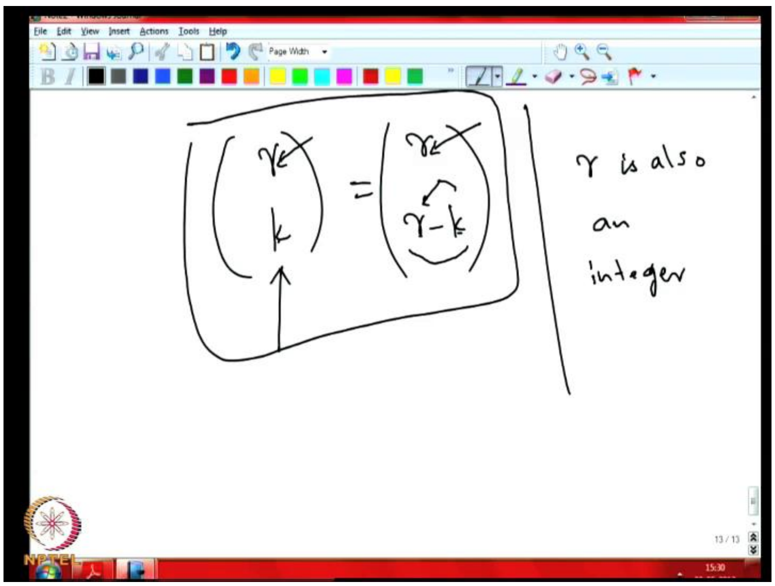

Now, but now we want to show this one, r choose k equal to r choose k minus 1. Now, we see what all restrictions are there, sorry, r choose r minus k. So, here other generalization will make too much sense because though we told r can be any arbitrary complex number, so this is an integer, right. This has to be integer to make sense, r choose k, so but then r minus k, if it, r was not an integer, r minus k would not be an integer. So, this would not make sense at all. The second part, right, the RHS would not make sense of, so automatically this formula, put the, this identity put the constraint, that r is also an integer, an integer, an integer, right, then only we can even talk about this identity. So, though we have generalized, that is not going, this, such formula is the identity, is not going to generalize, ok, fine. Let us still keep r integer.

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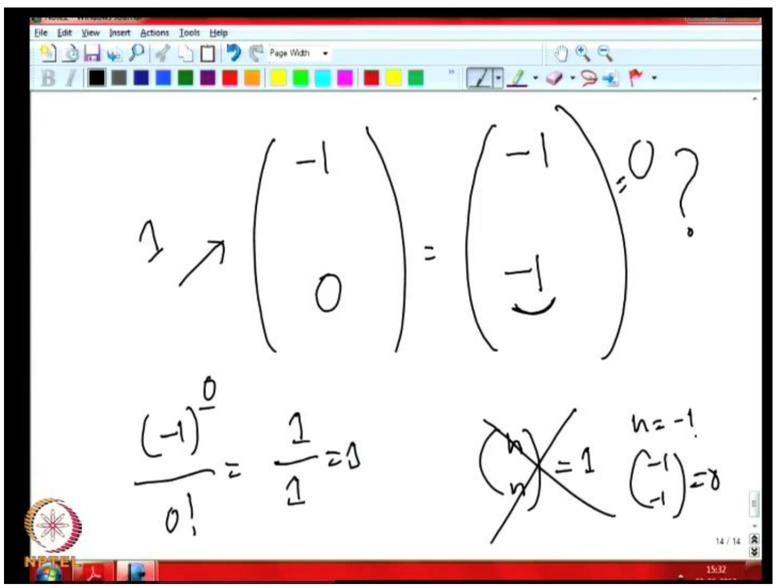

Is it possible, that I can work with all possible integers? The point is, that even that is not true, right, because even that is not true because see, you can see, that yeah, so for instance, if I, if we take minus 1, right, so minus 1 choose 0, we can write, so minus 1 chose 0, so we need minus 1 chose minus 1 minus 0, this is what we want. Is this correct? This was what we are asking, right. But then this is what minus 1 chose minus 1, so is it 1?

So, you, we always see n choose n, yeah, we may tend to say, that is this 1, but then this is not correct. When this lower index is minus, right, that is the way we have defined. You go back to the definition, we see, that if k is less than 0, irrespective of whatever is the value of r, we defined it as 0. So, this is, so this is not true when, for instance, n equal to minus 1. So, minus 1 chose minus 1 is zero, not 1. So, this is going to be 0, right. But this one, this one is 1, right, because what, so this is minus 1 to the power 0 falling factorial, which is 1 divided by 0 factorial, right. This is 1 by 1 is 1. So, here it is 1 and there is a 0, so this is not correct.

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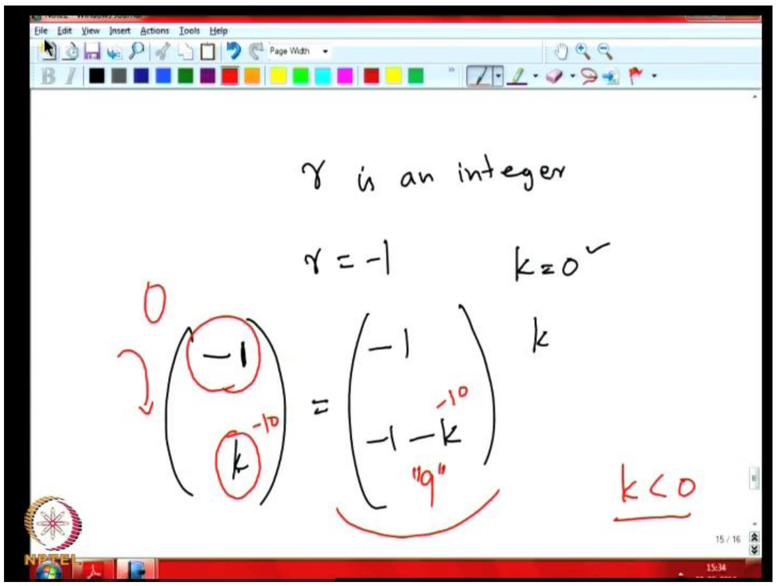

So, so the, our generalization is not working even with the assumption, that r is an integer, r is an integer, right. When, when I took r equal to negative 1, it is not working. So, with k equal to 0, of case, so we can try with k some other negative number, so 0 and some negative number, we can say, r choose so that r, r is some negative integer, something like minus 10, so minus, ok, minus 1, we can say minus 1, say minus some negative integer, let us say k, k is a negative integer, right. So, because if k is a negative integer, so we want to get minus 1 choose minus 1 and minus k, right. This being a negative integer, maybe we can put a value there for just, let us say, let us say, this minus 10, so minus 1 chose minus 10.

But we know our rules is, when k is less than 0 irrespective of the value of r, this, this value is going to be 0 here. What is happening here, it is minus of minus 10 that means 10 minus 1, it will become 9. Then right minus 1 choose 9, but there it is a, there is a value because what will, what will you do, so that it will just calculate minus 1 into minus 2 into minus 3 into, like that it will go downward, 9 times, right.

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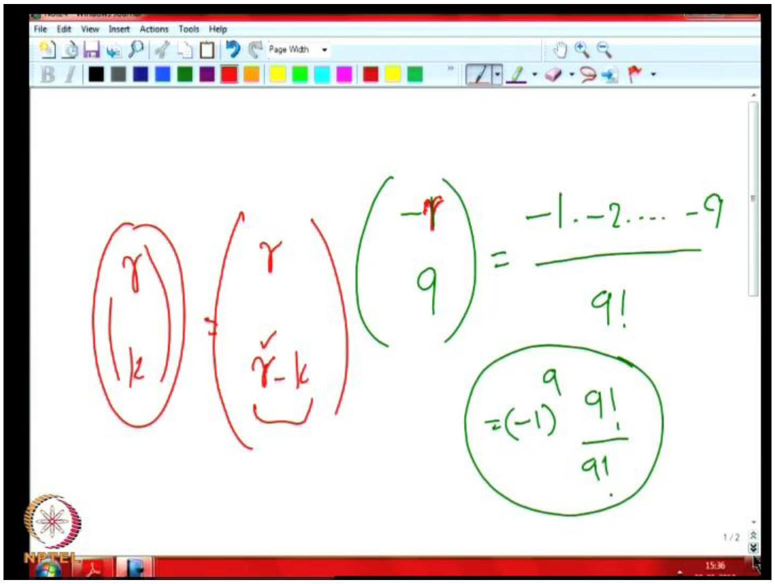

When minus 1, minus, minus, minus 1, minus, yeah, minus 1, yeah, so what, what I am saying is 0, so what I am saying is, yeah, this minus 1 choose 9, right. Minus, minus 1, yeah, minus 1 chose 9 will be equal to minus 1 into minus 3 into, so up to minus 9 plus 1, right. So, 9, sorry, minus 9, 9 of them, right, so divided by 9 factorial. So, this will be some minus 1 raise to 9 into the 9 factorial by 9. This is the way to look, so therefore this is going to be a non-zero quantity, but on the other side we always get a negative quantity.

So, this is, you can, you can easily, there is about putting minus 1 above is not very important. If you had put something and say, minus r, some negative quantity here, negative number here and then you have tried it with some, that r chose k, if k is negative. Anyway, this side is 0 and then the comparison is, whether it is equal to r choose k, this one. So, this, this, already negative quantity, negative minus, negative quantity, there is a positive quantity and whatever it is, this r falling factorial is defined and because it is a negative number. It will, it is anyway, it would not become 0 and then because a negative number, it will, minus, minus, minus, it is going downward, right, divided by r minus k factorial. See, r is a negative number; this is also a negative number, right. So, yeah, k minus, right, so that is, so therefore, in many cases, so you will see, that it is not going to work.

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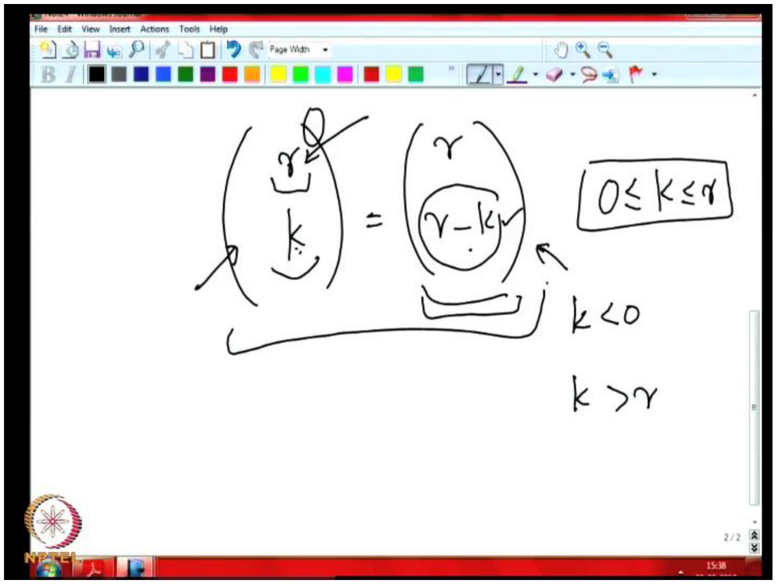

But on the other hand, when we take r is, so once we take r is positive, this issue is not there because this will work because we know, r is positive means, positive integer, positive or 0, non-negative integers. So, we know, that it is already working, so this is because we know that r choose r minus k was 2, just for all positive integers because by counting r we have already shown that. But in k is, that k is in between 0 and r, this we already know. But on the other hand, if k was less than 0, definitely this will be negative and this will be r minus a negative number, this will be r choose something bigger than r, so that will be again 0, right, because the falling factorial becomes 0, numerator things will become 0.

Now, if k was greater than r, so this quantity is anyway 0 because here r chose something bigger than r and r falling factorial k will become 0, then definitely it will become 0. Here, r minus k is going to be negative because it is negative number by definition, it is going to be 0, so 0 equal to 0, so that is not a problem. Once r is a non-negative integer, whether k is positive or negative, we see that, it, this symmetry identities floating, right. So, this was a discussion to bring out the certainties.

So, that means, we have to be careful, that just because we proved the identity in the combinatorial setting, that means, n was positive, r was non-negative integer, it does not mean, that it works in the generalization, it need not work, right, so that as we have seen in this example the formula need not even make sense, right, so we have to carefully reanalyze it. But of case, whatever we have already done will help us because at least, as we have seen in the proof in the later part, we could use the combinatorial proof. For most of the nontrivial cases other things were trivial, just knotting that both sides are 0, it is, it was easy, in fact, right.

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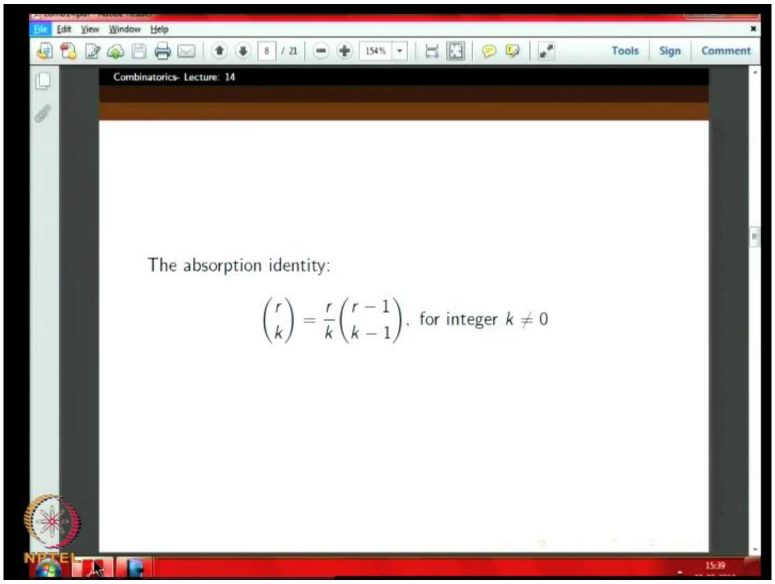

So, we will take another example where it works this, probably we have not seen before, yeah, so this is called absorption identity. This means, r chose k equal to r by k into r minus 1 into k minus y and this works all integers k not equal to 0. There is no restriction on r, so I will repeat once again. Yeah, we want to show, that r chose k here.

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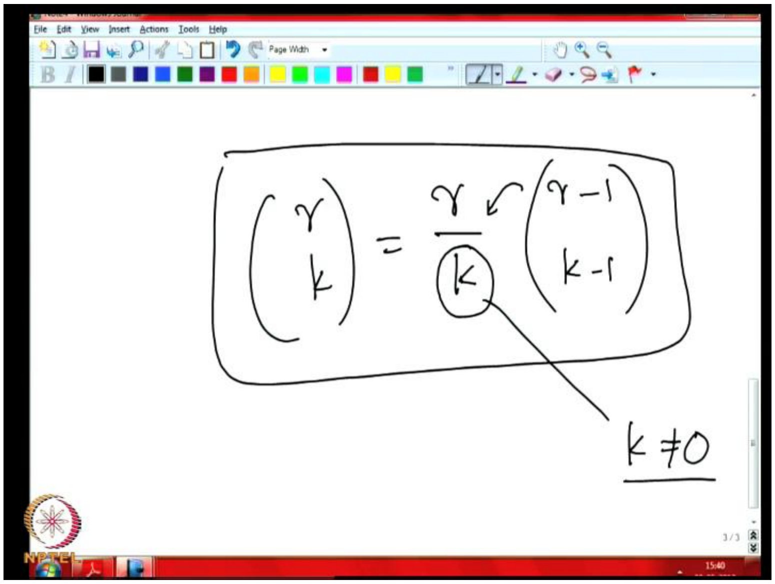

We want to show, that r choose, yeah, r choose k equal to r by k into r minus 1 chose k minus 1. We would not have any restriction on r. This formula is always correct, whichever is the r. k also we do not have any restriction except that we are not allowed to divide by e 0, right. So, k naught equal to 0 is required, see k has to be an integer, of case, otherwise the binomial coefficients are not defined. So, k naught equal to 0, k is an integer, no restriction on r. So, what is this formula doing? r chose k is equal to, if we are converting into r minus 1, chose r minus k, but what is coming out is r by k. This is what is it. This is easily proved, of case, so there is nothing difficult about it.

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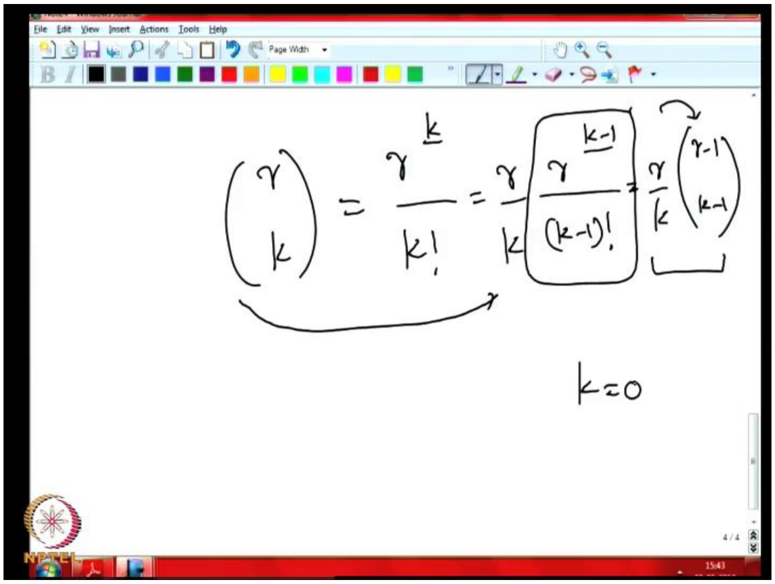

So, this is because r choose k, as we have seen, is r k falling factorial divided by k factorial and this is what r into r k minus 1 falling factorial divided by k into k minus 1 factorial, which is r by k into r minus 1 choose k minus 1 by definition, right, by definition. So, the only thing is, or maybe you have to worry about when k equal to 0, what will happen? So, k, k equal to 0 was the only case where this factorial interpretation, this for example, this was rewritten as this, right. So, this was not correct, when k, k, the, the lower index became negative because so here, upper index was working for all k greater than equal to 0, right. So, if greater than equal to 0, this was correct, right. And then for instance, it suddenly k minus 1 became negative, right. So, then this, the entire thing was not making sense, but you know, k equal to 0, k you are not considering. So, therefore there is no such issue.

And if k was negative, if k was negative, we do not need this proof at all because this is 0. So, whatever we are trying to prove, this is 0 and this is also 0 because k is negative, k minus 1 is also negative, both sides are negative, right, sorry, both sides are 0. r choose a negative number, r minus 1 choose a negative number, both sides are 0, so therefore we do not have, we, we are, it is trivially true, otherwise we just have to use this, right. Write it as r k for n factorial by k factorial, so now separate out r and k and here k is not equal to 0. The k minus 1 factorial is still defined, therefore, right, this is also defined because k is the number, which is bigger than 0, ok, right, therefore k minus 1 is there. So, therefore we can reinterpret it like this, so therefore, that is, that, that is why it is correct. So, therefore it, it works, it works in all cases except when k equal to 0.

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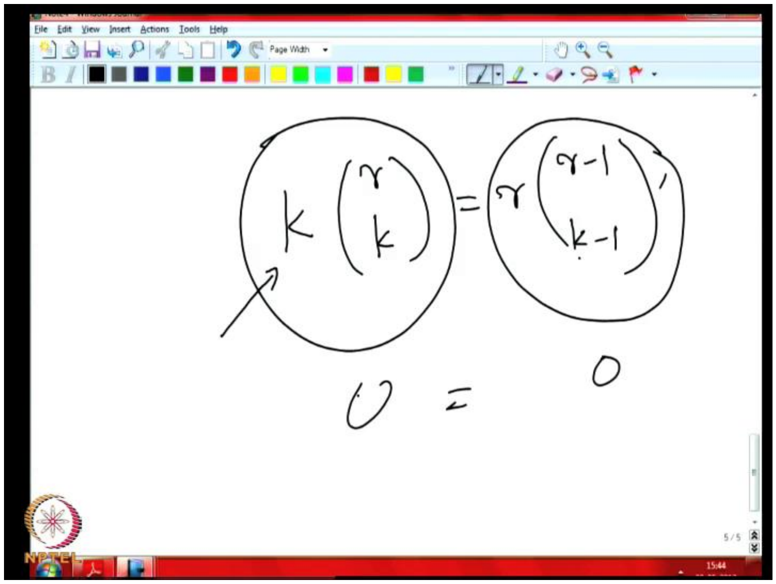

Suppose we want to make it to work for k equal to 0 also. What we can do is we can rearrange it as k into r choose k, this equal to r into r minus 1 choose k minus 1. Now, there is no restriction for every integer k, it works because the only case it was not working was when k equal to 0. Now, k equal to 0, this becomes zero and this k minus 1 will became a negative number and therefore, this will become 0, there both will become 0. So, therefore, that absorption identity works for all k.

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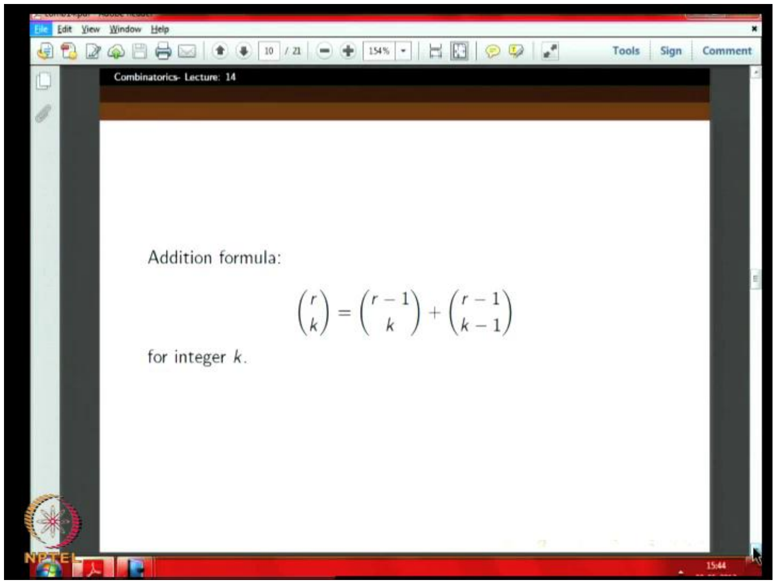

Now, yeah, now what we are going to do is another one, namely, some companion for these things in the sense that We, in the previous what we did is r choose k, like by pulling out something namely r by k, we convert it into r minus 1 chose k minus 1, both upper index and lower index was reduced by 1.

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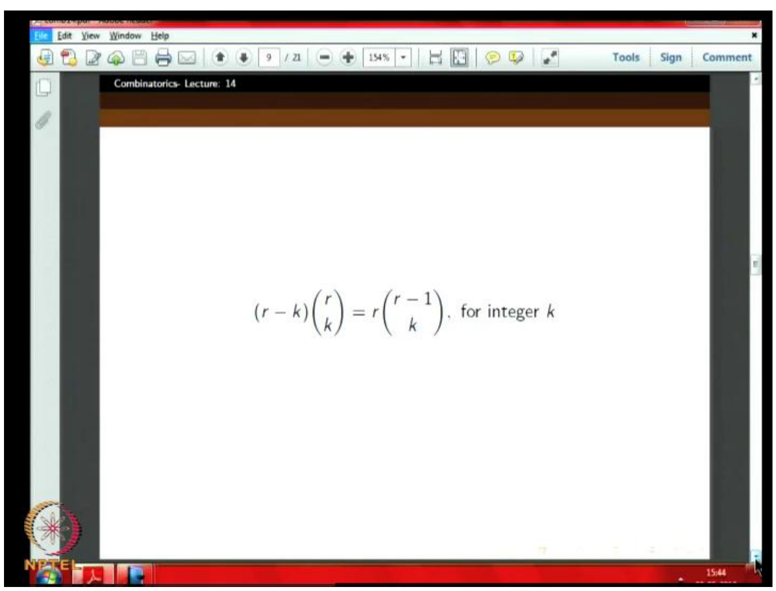

Now, we keep the lower index, intact. r choose k we want to change to r minus 1 chose k, that only the upper index reduces by 1. So, the difference is, that now it was k into r choose k is equal to r into r r minus 1 choose k minus 1 here instead of k is that the remaining, what r minus k? So, r minus k into r choose k is equal to r into r minus 1 choose k, for this also works for all integers k, right. So, there is no restriction on r and there is no restriction on k. The restriction on k in the earlier cases also coming only because we are rotating the ratio of form r by k form, that is why r by k, it, because we cannot divide it by 0, that is why we had to say k naught equal to 0.

Now, in this case we are writing like this, so we do not have a problem for all the, even any case, right. So, ok, so right, so now we will not have any restriction on any case or equal to k or even when r equal to k equal to r, we do not have a problem. See, r equal to r an integer, right, k equal to r and k and r an integer. Even in that case we would not have any problem, so because we may wonder what if it becomes, so we will show this is true, but we will do it in the next class. But how we will do is by using the previous identity and symmetry identity.