Combinatorics
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Lecture - 12
Combinations of Multisets – Part (2)

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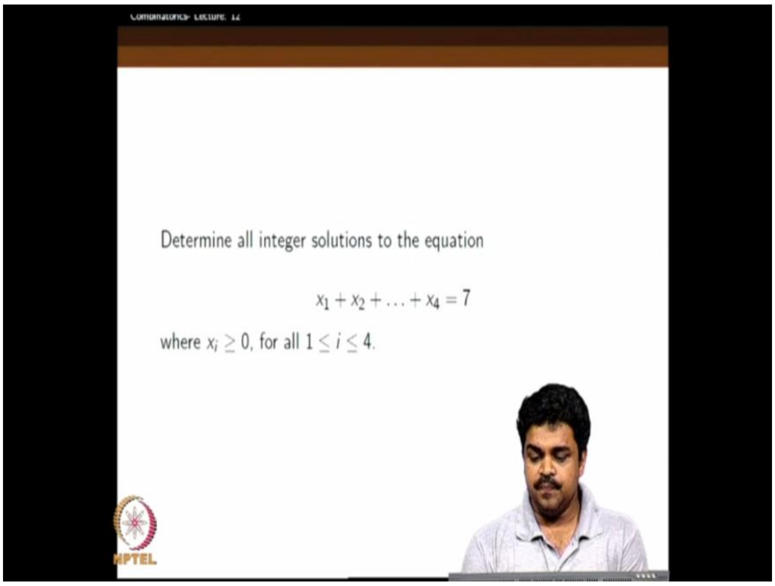

Welcome to the 12 th lecture of the combinatorics. So, we were discussing this problem in the last class. Determine all integer solutions to the equation x 1 plus x 2 plus up to x 4 equal to 7, and the x sides are all greater than equal to 0, it is non-negative integers and now how many possible ways we can get solutions to this or how many solutions are therefore these thing? You know I am claiming that this is the same as the problem problems we were discussing in the last class, what was the problem?

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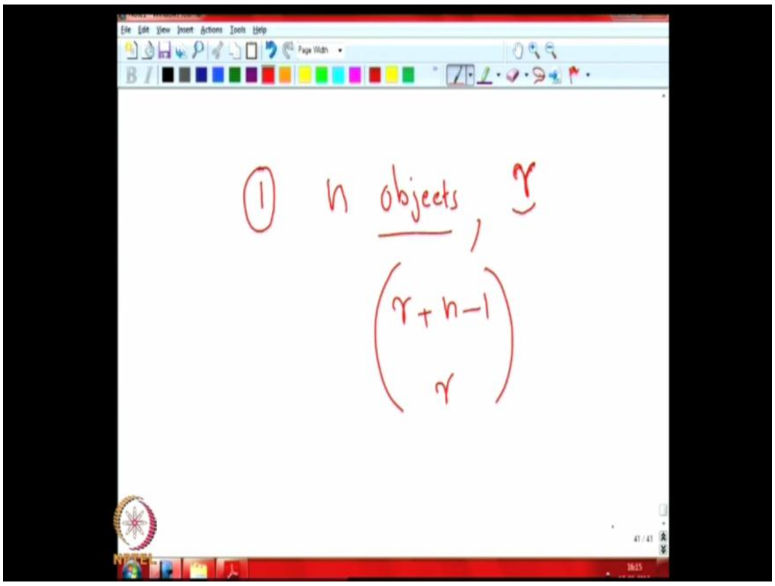

The first problem was there are n objects we called them the n type of objects, sometimes or may be n sometimes you say that n type of objects, and each objects has lots of them, you can take any number of them from them that type or we just say that n objects we can take the same objects several times, repetition number of each object is infinite you can take any number of times if you. And we want to make a selection of r things, r things have to be selected using this from these n things only just that some of the things can repeat. And this right the things are allowed to repeat that is what this number was the possible number of ways make selections of r is r into n minus 1 choose r.

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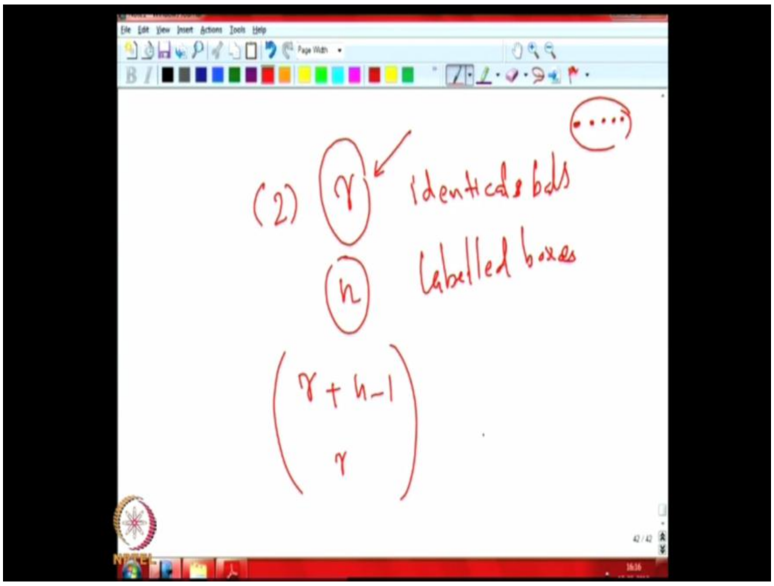

Then the second way which is essentially the same problem was as a balls and boxes problem, balls and bins, balls and containers problem. We have r identical balls here identical balls in the sense that they we cannot distinguish one from another identical balls. And we have n distinct boxes and labeled or distinct boxes you can; you can differentiate one box from another one. Now how many ways you can place this r things into this n boxes, some boxes can be empty here.

So that the number is r, plus n minus 1 choose r the same as above then you remember this r the number of identical balls corresponds to the, the size of the selection we are making in the earlier problem. The n label boxes correspond to the types or the objects we can the types of objects we can take. So the repetition is captured that in many balls can go into the same box, and then we considered a variation or a problem for insistence like when all the boxes have to be non-empty.

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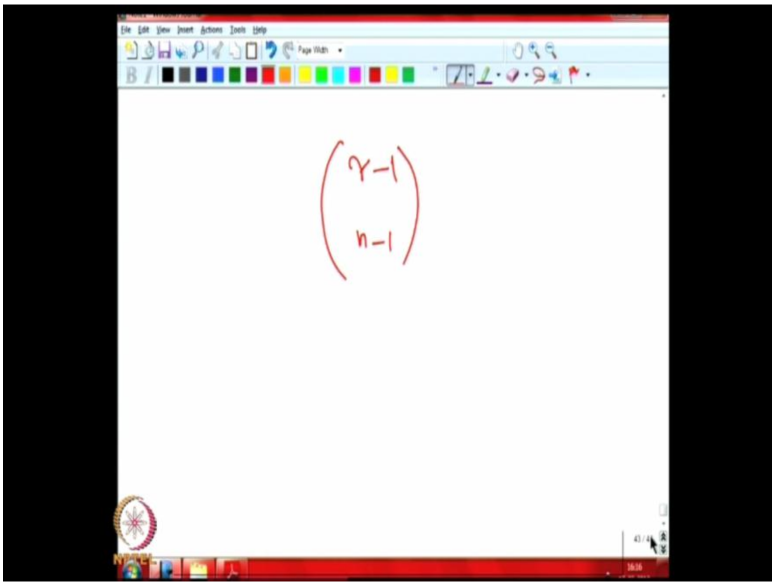

When it is r minus 1 choose n minus 1 just by a simple application of the earlier result. First give n each box 1 object then, apply the same formula we got it right so r minus 1.

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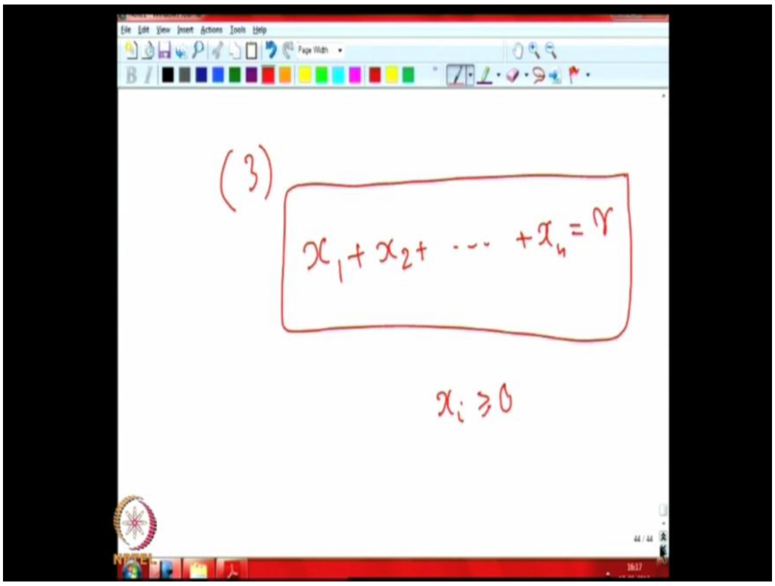

Now, we are seeing this, this third problem namely, this problem of finding integer solutions x 1 plus x 2 plus c x n equal to r is the same here, which each x i has to be greater than and equal to 0.

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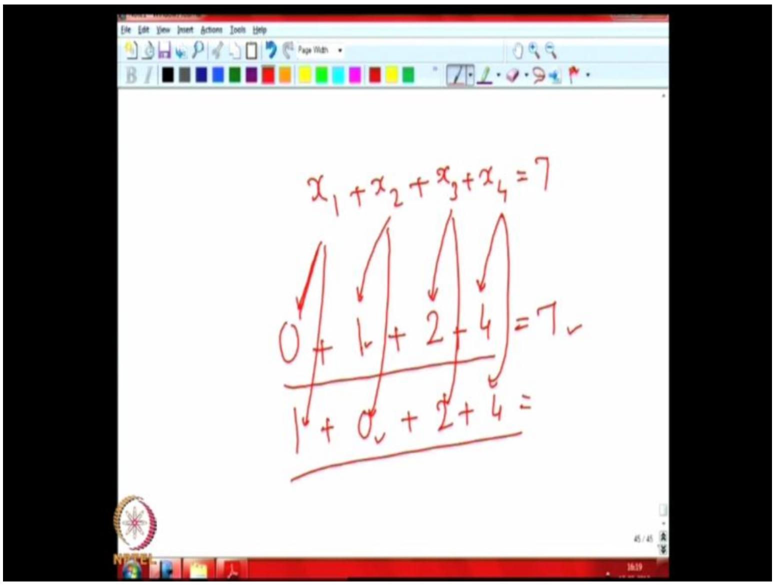

So, to illustrate the problem, We just took a special case nearly x 1 plus x 2 plus x 3 plus x 4 equal to 7 see because what we illustrate here it is just that we want to show the nature of the things for insistence 0 plus, 1 plus, 2 plus, 4 equal to 7. So that here x 1 is equal to 0, x 2 equal to 1, x 3 equal to 2 and x 4 is equal to 4 this is 1 solution. But not that for insistence for if i had taken 1 plus, 0 plus, 2 plus 4 this is a different solution. Because you know it is x 2 x 3 x 4 and x 3 are getting the same values, x 2 is getting different value now earlier it was 1, now it is 0 x 1 is getting 0 so x 1 is getting 1 here. So the order in which we write this is important, we are allowing zeros and we need 4 terms including 0 s in of case, but the order in which we write it is important.

Because the eighth one, the eighth number we write here will go to x i, x i is a scientific number. Just that we are using the same numbers is not enough, that is a different problem how many for instance if we do not give importance to the order we are only interested in which numbers we are using to form 7. And that is a different visor partition, partition problem that we will discuss much later in the course. But as of now you should understand that this is different, for insistence you want to write down all these things. In case you can try to do that, but rather than trying to do such an enumeration in these time we would try to establish the correspondence between this problem, and the earlier problem so earlier 2 problems.

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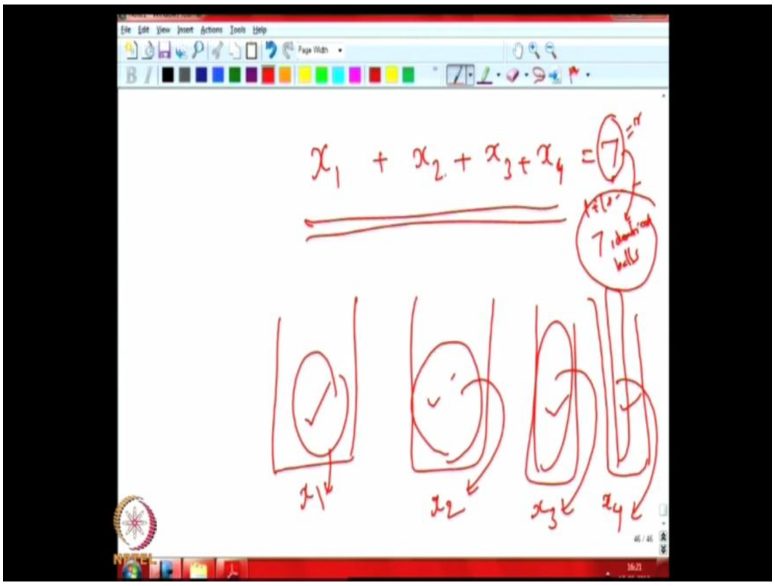

So here we can see that x 1 plus, x 2 plus, x 3 plus, x 4 equal to 7 is the equations. Now let us have 4 boxes this box is labeled x 1, this box is labeled x 2, this box is labeled x 3, and this box is labeled x 4. Now we have 7, these 7 things that mean it is like in some problems we consider 10 rupees is being given to people. Sometimes we considered 7 bananas are being distributed, so here just 7 that this number 7 is essentially 7 units 7 1's 1 plus, 1 plus, 1 plus, 1 7. So we can think that those 1's are given to each of these things, so we can or in other words we can think that they are 7 identical balls here to the correspondence between the earlier problems.

Now the question is the 7 identical balls some of them should go to this box, some of them should go to this box, and it should distribute the 7 identically. The number of boxes which went into this box will become the value of x 1, the number of box balls which went into this box will become the value of x 2 the number of balls which went into this box will become the value of x 3. And the number of balls which went into this box will become the value of x 4. So, therefore, this is the formulation, this is same as the earlier problem. So the how many ways we can distribute this 7 identical balls into this distinct 4 boxes, we can also put 0 into some of them that there is no ball going into this. You can see that there is exactly the same problem.

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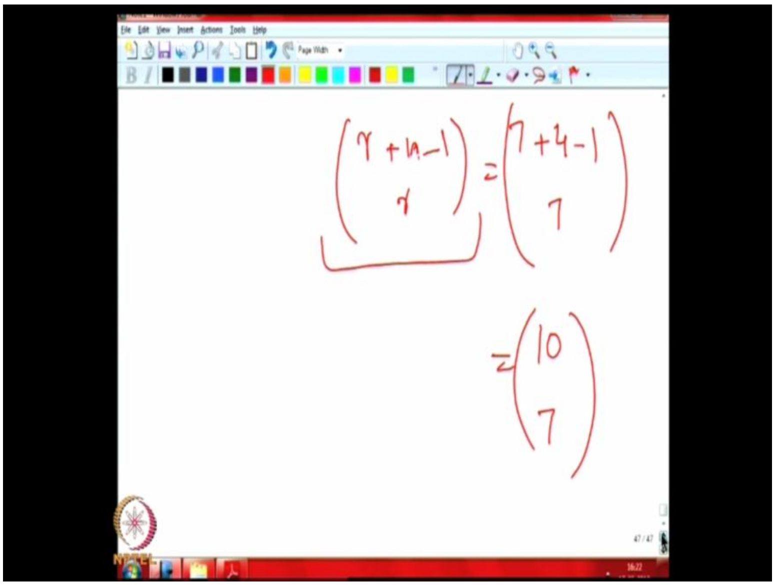

So for instance if you want to therefore, the answer will be, the answer will be, the answer to the earlier problem namely 7 will be equal to r, and then this n will be equal to this. So the r plus n minus 1 choose r that means 7 plus 4 minus 1 choose 7 that is 10 choose 7 this will be the problem. General answer will be this, now if you want to see the correspondence between the other selection problem.

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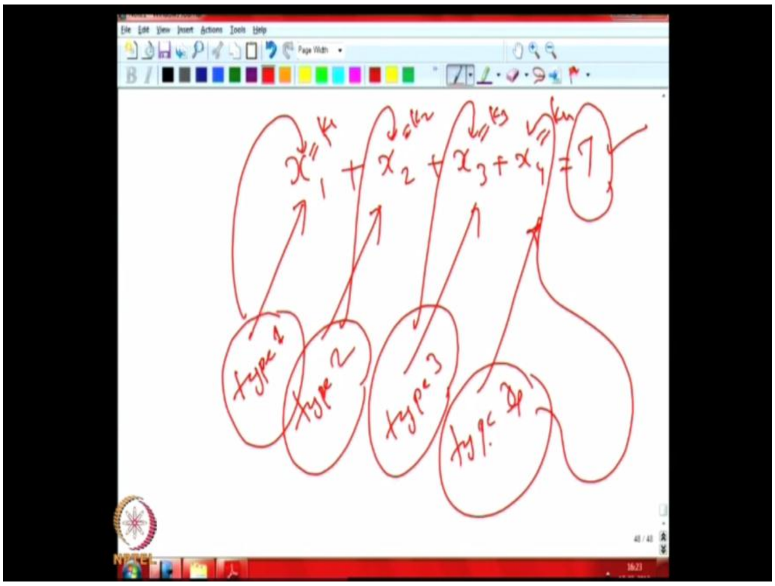

So x 1 plus, x 2 plus, x 3 plus, x 4 equal to 7 we are as into selection problem we just say we want to select 7 things. And this is 1 type this is 1 type, type 1 this is another type, type 2 this corresponds to another type, this corresponds to another type, type 3. And this corresponds to another type or may be another object you can say like that is the first object, second object, third object, fourth object we are taking several times. The same object or we are taking so many things of the same type.

The question is we want to take 7 things of this type, this type, this type or this type, and then how many we take of this type will become the value of x 1? How many or we take of this type will become the value of x 2? How many of this type we take will become the value of this one, and how many of this type we take will become the value of this one, this x 3.

So before and similar the backwards suppose if we get a solution for this thing, it is as good as saying that we have sorry for instance x 1 is equal to some k 1, x 2 is equal to k 2, x 3 is equal to k 3 and x 4 is equal to k 4. That means we will be equivalently saying that we have selected k 1 things of type 1, k 2 things of type 2, and k 3 things of type 3 and k 4 things of type 4. And of course, all of these add up to 7 that mean we have selected 7 things of this types that is what in k 1 plus k 2 plus k 3 plus k 4 equal to 7.

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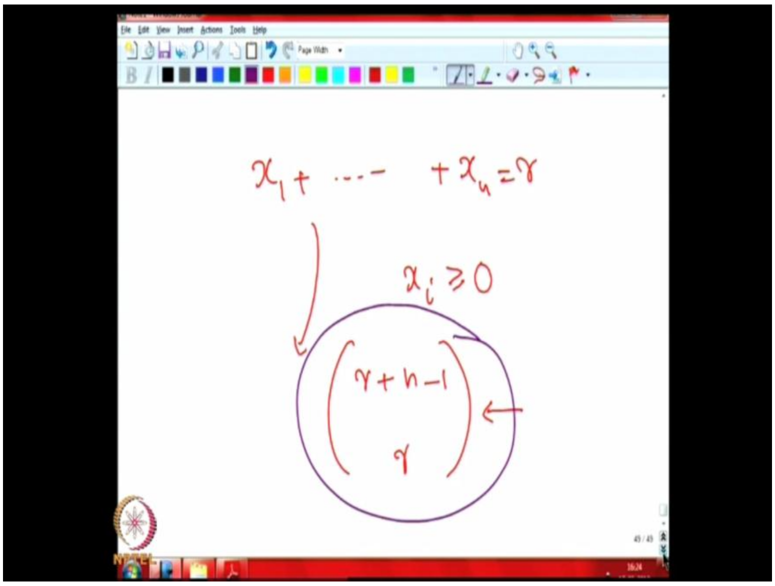

So, we see the correspondence. So the answer to the general problem x 1 plus x n equal to r, where each x i has to be an non negative integer the number of such solutions is equal to r plus n minus 1 choose r this is total number of possibilities. This is the first one, and we just remember that thing namely.

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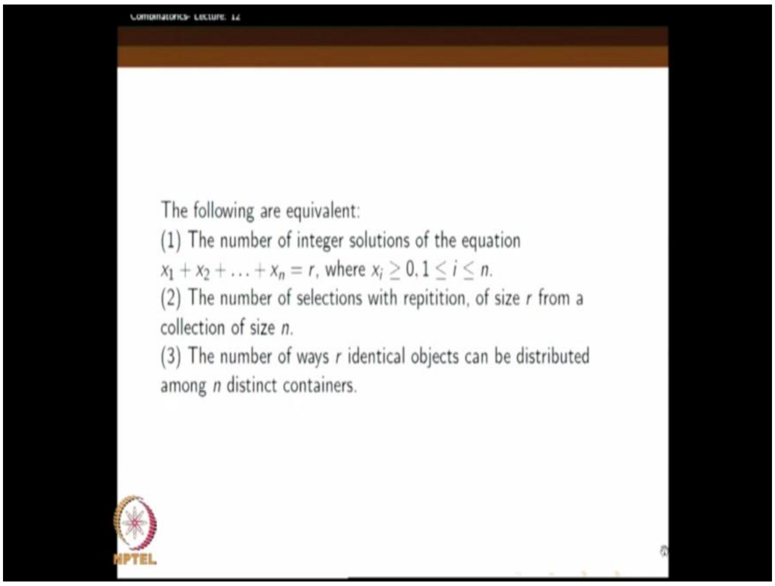

These 3 problems are equivalent, we were discussing these 3 problems all the time the last class and this class also we are continuing with it the number of integer solutions of the equation non-negative integer solutions we mean x 1 plus x 2 plus x 3 plus x n is equal to where x i greater than and equal to 0 for each i. And the number of selections with repetitions of the size r from a collection of the size n, and then the number of ways of r identical objects ways in which number of recent which r identical objects can be distributed among n distinct containers; these are all the same things.

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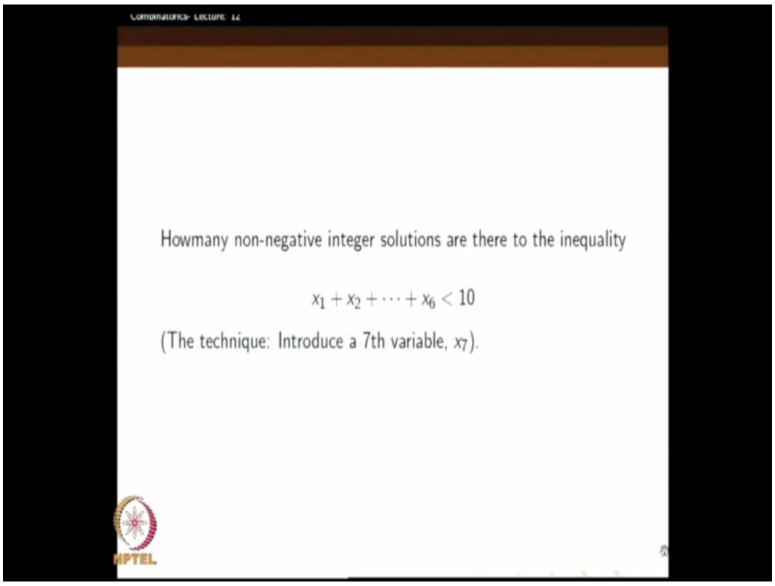

Next problem we consider is this is some problem were we can use the earlier one. How many non-negative integer solutions are there to the inequality x 1 plus, x 2 plus, x 6 is less than 10.

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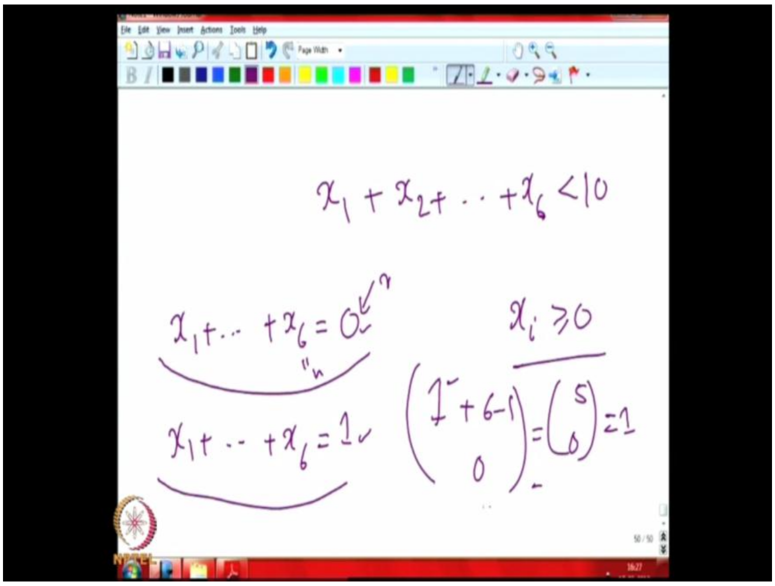

So, the previous answer when you say x 1 plus, x 2 plus, x 6 is less than 10 as what I solved. Now, that means some can only go up to 9 and each x i has to be greater than equal to 0 as usual now non-negative we are getting. Now the 1 way of looking at this, so that we can enumerate all possibilities. For instance 1 solution may be like x 1 to up to x 2 may sum up to 0 it may sum up to 0. So here we know how to find out the solutions, because everything has to be 0. So there is only one solution so our formula will also be that, because r is equal to 0 here, r equal to 0 here and n equal to 6 here. So 6 minus 1 choose 0 that is 1, because 5 choose 0 is equal to 1. So, the other thing you can have x 1 plus, x 6 is equal to 1. Here, how many solutions you can get so instead of 6, we will put 1 here, so instead of because r instead of 0 we will put 1 here.

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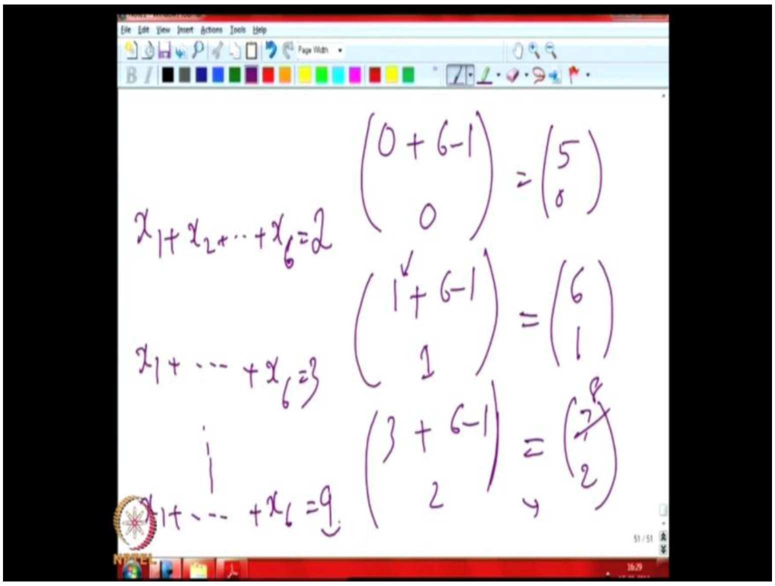

So, this will give us how much this will give us, 1 plus so initially we got 0 plus 6 minus 1 choose 0 that is 5 choose 0. Now, we are getting 1 plus 6 minus 1 0 choose 0 is equal to, so 1 plus r is equal to 1 here, the type of things is again 6, 6 minus 1 and we have 1 here that is 6 choose 1. And then we have the next 1 then the next thing will be x 1 plus x 2 plus so up to x 6 they have to add up to 2 add up to 2. So, r is equal to 2, plus 6 minus 1 choose 2 that is 6 chose 2, and then x 1 plus x 6 plus is equal to 3 so this will become 3 here.

Now, instead of this 6 sorry here it was 7 choose, 7 choose so 7 are there now it will become again 8 and so on. So, we can add up all the things together and then get the answer, but of course, this has a drawback so up to where I should do, it we should do it all the way to 9 x 1 plus x 6 is equal to 9. Because we know that this segment this sum has to be strictly less than 10 integers, so it is 9 the sum is either 9 or 8 or 7 or 6 or 5 or 4 or like that till 0. But then suppose n was this number this instead of 10 we had a very large number, we cannot do this thing because this is we will have to consider too many cases. Therefore, that is not a good idea we would rather try to a get more general solution.

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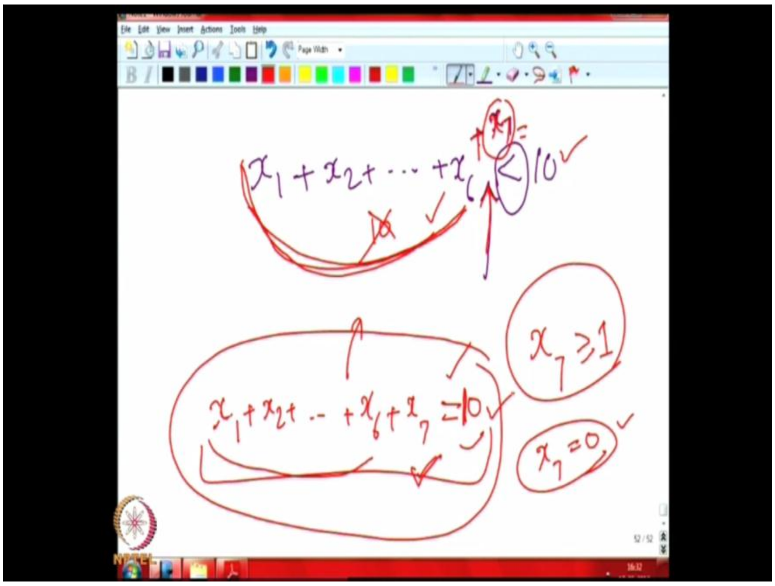

So, what we will do x 1 plus, x 2 plus x 6 is less than 10 is what we told. Now, what about introducing a new variable here say plus x 7, and then make in a instead of this, this let us put a equality. We add this 1 x 7 and then make it equal to 10 means whatever wherever we reach here and then plus x 7 is equal to 10 that has to be that has to be the value of x 7, because sum is automatically determined by what the sum takes. So, of course one condition we have x 7 has to be always greater than and equal to 1 why because we if allow x 7 is equal to 0.

Then what happens is this entire sum we will have to take a value of 10 which is not allowed right it can take only up to 9 if it is only taking up to 9 this last value has to take at least 1. Therefore, x 7 has to be greater than and equal to 1 is the only condition you can see that a solution for this new this thing x 1 plus x 2 plus x 6 plus x 7 equal to 10 this time, any solution to this thing will corresponding give a solution to this 1.

Because you look at this thing, and take a solution take the solution for that and now you look at the value of x 7 it is at least one you remove it from. Then, now if you sum these things together, you will get something less than 10, so that will be a solution for this thing also this earlier equation. On the other hand any solution to the earlier equation will give a corresponding solution to this, because if you have this thing less than 10 you can always find a value for x and whatever is left to each time that will at least one. So that it will be a solution for this new system of equations we have written, a system new equation we have written. So therefore, this solution to there is a bijection between the solutions of this equation and the solutions of the semi equality. Therefore, we just have to solve this new problem instead of this problem therefore; we concentrate on this new problem.

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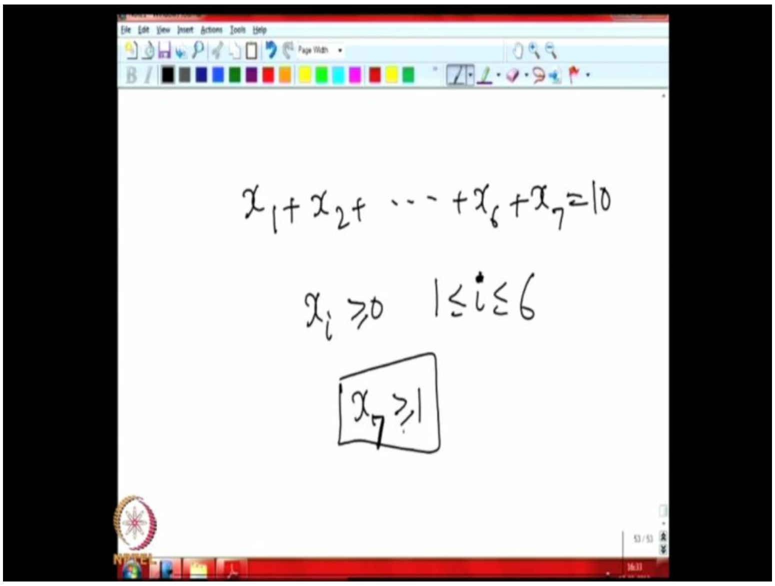

This a nice trick, we converted into a equality so mainly x 1 plus, x 2 plus, x 6 plus, x 7 is equal to 10. Now, the conditions are x i is greater than and equal to 0 for 1 less than equal to i and less than equal to 6 and x 7 has to be at least 1, x 7 has to be at least 1, x 7 has to be at least 1 is the problem. Now, we know how to solve it because you know this corresponds to this we cannot directly apply the previous solution, because you know that the conditions is not all x i is greater than 0. Here x 7 is at least than greater than 1 more, so we will again convert the problem to such that it comes to the form we, we are more to it.

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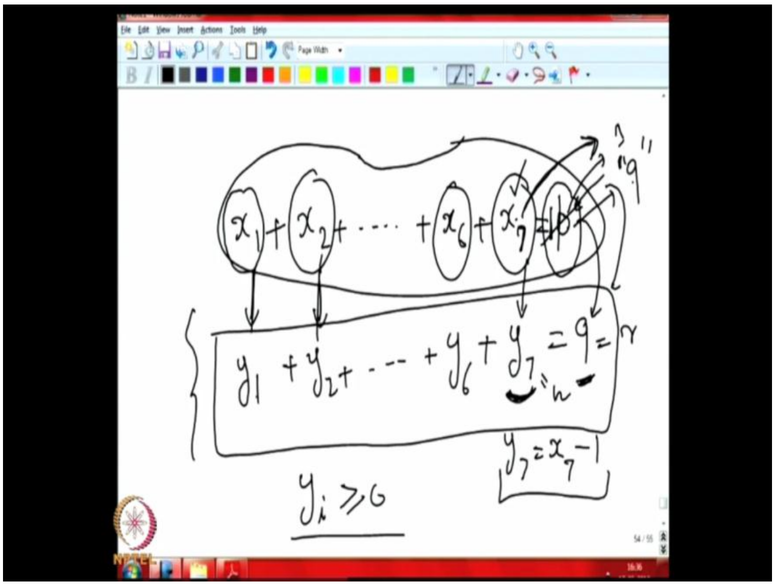

So, this, this condition we want to get rid of so what we do is we write. We write a new system of equations for this x 1 plus x 2 plus sorry x 6 plus x 7 is equal to 10 as for the system. We will write for x 1 we will write y 1 it is equal only, so this is y 2 I am just converting it to another 1 y 6 plus. Instead of here we will say that y, y 7, but y 7 x 7 minus 1, but we know this is equal to 10 right therefore, this has to be 9. We will define y 7 to be x 7 minus 1, see the again the point is that if you have a solution to this thing we also get a solution to this 1.

Because whatever value we have for y 7 we can add 1 more to it and make x 7, so here in this equation we just say y i is greater than and equal to 0 for all i, 1 to 7. Here even if it is 0 initially it will become 1 by adding 1. Similarly, if we have a solution for this one, we will get a solution for this, because this will copy for all of these things for the last 1 we will minus just 1 and just make this. And because this was at least 1 by (( )) 1 it will in the worst case it can become 0 that is all it cannot violate any condition. This 10 will become 9, 10 will become 9 because you know here we are using x 7 we are just minusing 1 therefore, it has to minus. Now, we can ask for the number of solutions for this thing, but we know because there are n variables here so 7 variables here, r is equal to 9 this is r this n is equal to 7.

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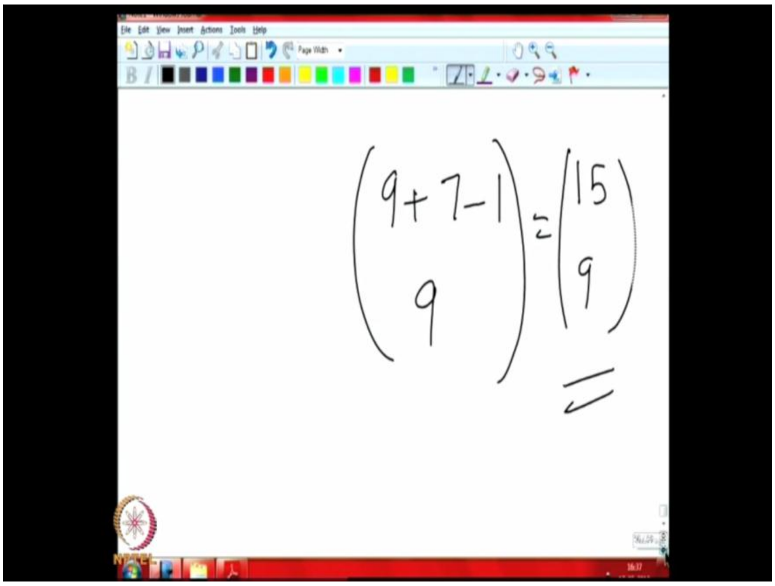

So therefore, we can say that it is 9 plus 7 minus 1 choose 9 this will be the answer, that is 15 choose 9 this will be the answer. The see if this is a this may look like this last 1 so little, little change of variable here, but you could have argued using the balls and bins in a more concrete or a combinatorial way. For instance you remember this each of x i corresponded to a box there, x 1 corresponds to the first box, x 2 corresponds to the second box, x 6 corresponds to the sixth box, x 7 corresponds to the 7 box and so on.

And now, if this was the situation x 1 x 2 x 6 and x 7 was corresponding to boxes when we say that x 7 should have value at least 1, it means it is equivalent to saying that and this 10 identical balls here remember 10 identical balls. So, it is equivalent to saying that the seventh box should get at least 1 ball so you rather give 1 ball to that box that means identical boxes are only 9 left. Because only 1 ball has already gone from it now the condition is that 1 ball is already given to them as soon as you think that the 7 boxes have to get these 9 balls that mean we have to distribute these 9 balls in, in these 7 boxes no more conditions. If you want some boxes can be calculated so that we could have also got these 9, 9 plus 7 minus 1 choose 9. So, this but that initial trick was worth noting, because we converted first into equality by adding a new variable to it.

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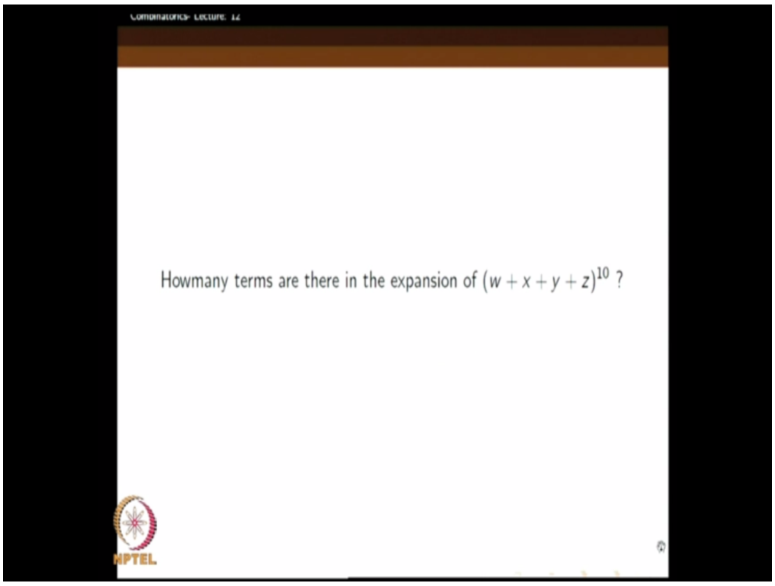

Now the next question we consider is, how many terms are there in the expansion of w plus, x plus, y plus z rise to 10?

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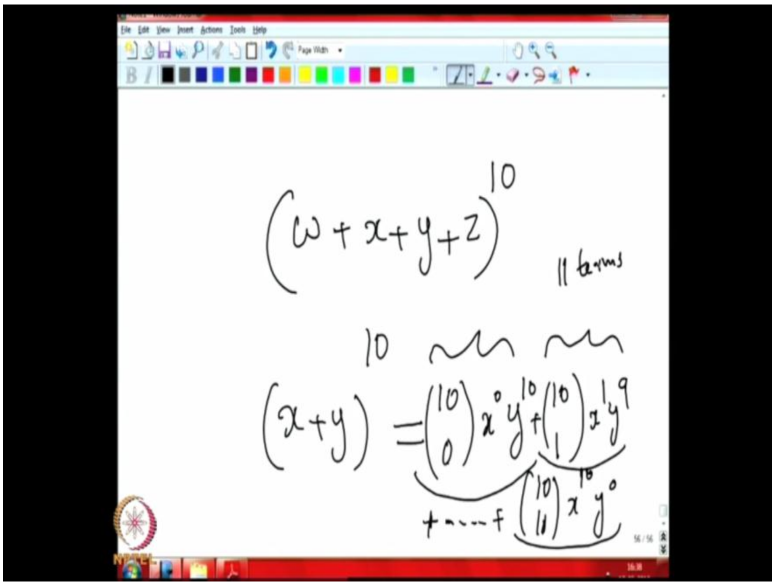

What do we mean by this? So for instance if you had just considered x plus y raise to 10, we have 11 terms namely these are the terms. So, the terms I think terms are rarely defined I will just what I want to try to make it very formal, but from this example if you understand what it will be. So, because the expansion is 10 choose 0 into x raise to 0 y raise to 10 plus 10 choose 1 into x raise to 1 y raise to 9 and so on so on 10 choose 10 and x raise to 10 into y raise to 0. So each 1 of these things are considered terms this is one term, this is second term, another term. And if this is term and there are so we have starting from 0 we are reaching 10 so there are 11 terms in that sense.

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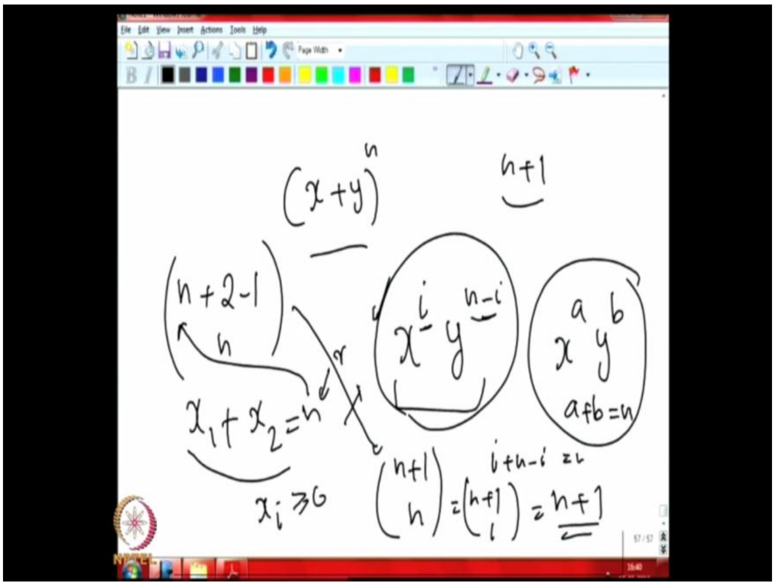

So, the way to attack this form for instance, this is trivial to say that x plus y raise to n has n plus 1 terms. So and so essentially we are asking how many patterns of the forms x raise to i y raise to n minus i are there that is what we are asking. Because that the first coefficient is unimportant so these kinds of patterns how many patterns are there. Clearly we know the condition is that i you know this i and n minus i they are adding up to n that is the way we are splitting this exponents.

So the x rise to a, and y raise to b is a pattern if and only if a plus b is equal to n, this is the pattern which will appear in the expansion of x plus y raise to n. Now this is a question of you know this is same question which are considering x 1 plus x 2 plus is equal to n, and each x i has to be at least 0. And then how many ways can do these things is what and how many solutions are there for this thing that is what we are asking. And we know because there are 2 terms and here n equal to r so r plus sorry the number to types is 2, 2 minus 1 so you can write this r is n. So, this is becoming this, and then choose n this is how much n plus 1 choose n that is n plus choose n plus 1 choose is equal to n plus 1 that is what. So, we are trivial enough we see the pattern question we understand the question, what we are asking and we see that we how we can map it to the pervious problem in this thing.

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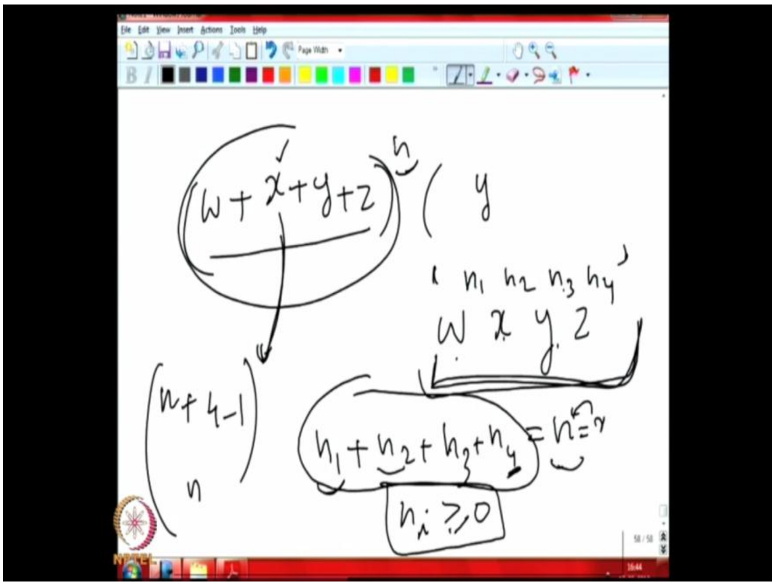

So, when we come back to this x plus y plus sorry w plus x plus y plus z raise to n. And then we are asking for the pattern of x raise to n 1 plus y raise so 1 thing raise to n 2 said raised to sorry w raise to n 1, x raise to n 2, y raise to these kind of patterns will come said raise to n 4. The only condition is that so for this to be a pattern in the a expansion of this thing is that n 1 plus n 2 plus up to n 4 the n 3 plus n 4 has to equal to n. Because they should add up to the total exponent so why is it so because you know we are writing you remember the way we manipulated we worked with the multinomial we proved the multinomial theorem.

We had written this as n products this 1 once again like again 1's again 1's like that n times we wrote. And then each variable is taken so each term here is contributing 1 variable and it is forming a finally, it is forming a term. And we are collecting the terms of similar type so it is like you know an x may be contributed first, the second term may contribute a y, the third term may contribute another y, and fourth term may contribute as said and finally, we will see how many excesses are there that may power that into y power into said power that and so on. This is, this is what a pattern of this kind of a pattern will arise out of them.

And you can see because the total number of things which is contributed, the total number is the number of terms because each term is contributing 1 to the to a final product term. So therefore, this n 1 plus n 2 plus n 3 plus n 4 has to be equal to n that is the 1 and of course, each n i has to be greater than and equal to 0. Because we cannot put negative thing because there is nothing like negative thing, 0 thing is because some x may not be contributed by any term their product their term like this. Everything every time we may get y that is y into, y into y that means y into raise to n will come that means it is w raise to 0 into x raise to 0 into y raise to 0 y raise to n and z raise to 0. Or it is possible that there is 2 y is and n minus 2, z is sets all w and n minus z is this kind of things are possible.

But it is not possible to have negative things, so it is all non-negative integer so that exponents of the x y z and w. So they should add up to n that is the only thing we can understand. So, now how many ways we can get these things that is what how many solutions are there for this equation? Now, instead of x 1 x 2 we should have written x 1 x 2 we just have n 1 n 2 n 3 n 4 for the variables. Now, that number of solutions clearly there are 4 of them here so r is equal to n here. So, r plus 4 minus 1 choose r will be the answer, but r is n here, r is n here so so in this case we use n for that r so n plus 3 choose n will come.

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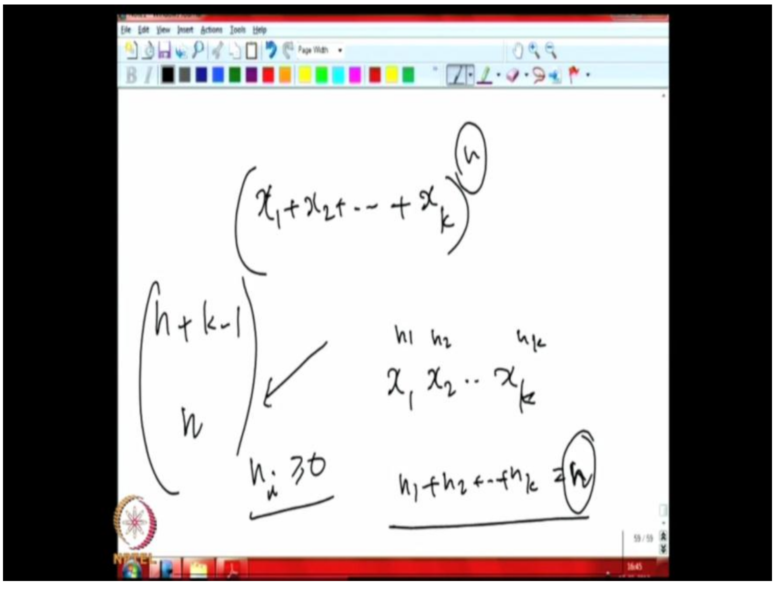

So, in general what is happening is for instance you, you may be talking about something like x 1 plus x 2 plus x r sorry x k raise to n. In this thing how many terms are there how many patterns will be available? So, we are asking how many things of this sort write x to x k and then this is n 1 n to n k the condition is n 1 plus n 2 plus n k has to equal to n and also each, each n i has to be greater than equal to 0. So is the condition so how many solutions are there? Solutions are definitely here instead of r will use n, because there are n this n, this is the n equal to r usually n write r here. And the number of terms is equal to k, so n plus k minus 1 choose n this will be the answer.

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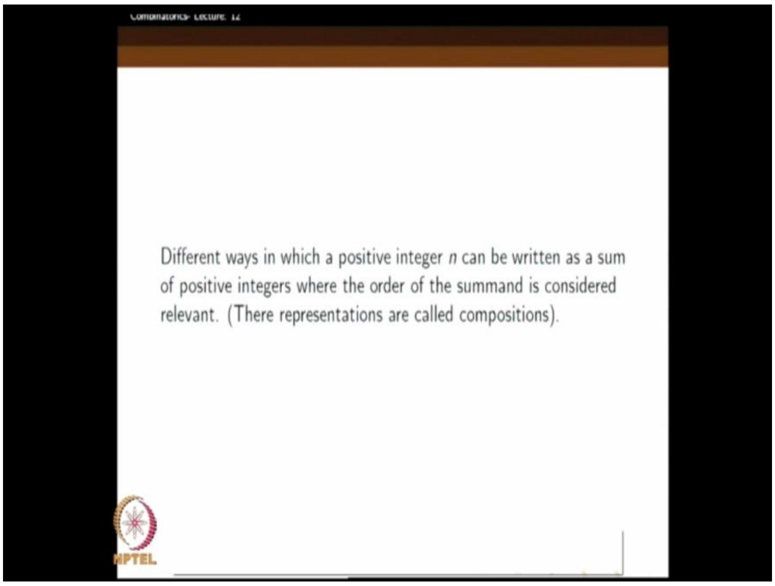

So, this is what we should understand; different ways in which a positive integer can be written and as a sum of positive integers, where the order of the summand is considered that relevant, this representation is called compositions. So, what I will again come back to this, so what we want to do is suppose some positive integer is given we want to represent that represent it as a sum of positive integers itself.

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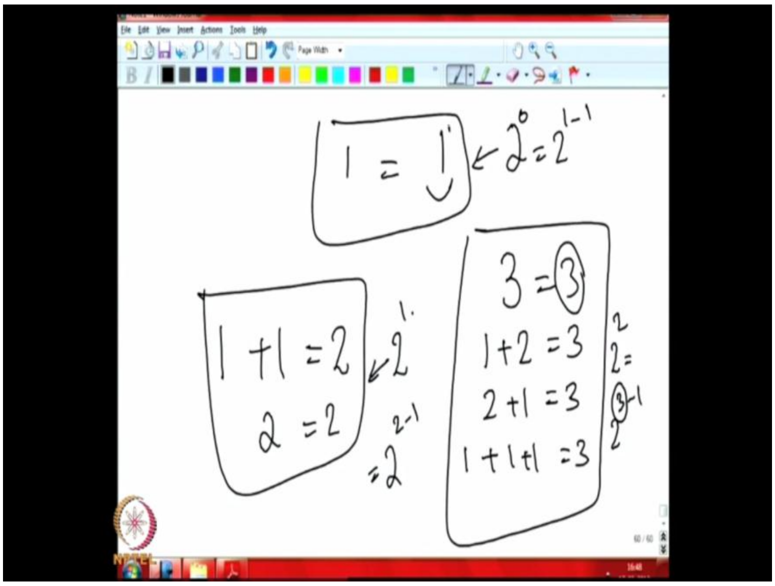

For reference 1, 1 can be written just 1, 1 is to 1 nothing can be done about it. Because if you take 2 you cannot you have to use 2 negative number 2 that is not allowed, so 1 is only 1 we can write only 1 and only 1 way. Now 2, 2 can be written say positive integers you are not allowed to take 0, so you should start with 1 one plus 1 is equal to 2 that is the only way you can do it. Now, what about 3 is on the way we can do it what about 3, 3 you can start 1 plus, you have 1 more way you can also write this as 2 equal to 2 there are 2 ways to do this thing.

And for 3, we can start with 1 single term compositions that means 3 equal to 3 kinds of things and here we can also write it as 1 plus 2 equal to 3. And then we can also write 2 plus 1 equal to 3 the order is relevant here, the order is relevant 2 plus 1. Now we cannot do anything for, but there are 3, 3 lengths splits 1 plus 1 is to 3 then we have 1 plus because 2 should be split 1 plus 2. So, 1 plus 1 plus 1 plus 1 now, once you start with 1 then you have only this 1.

So there are only 4, 4 ways to do these things and there's no more thing.So we can see a pattern here this when I used on this 2 to the power 0 ways to these thing, here 2 to the power 1 is this 1 happens to be 2 minus 1, 2 minus 1 so that 2 raise to 1. So that 2 to the power 2 minus 1 that is 2 is 2, 2 is 2 to the power 2 minus 1 this 2 raise to 1 minus 1 2 raise to 1 minus 1 and here it is 2 to the power 3 minus 1 this 3 corresponds to these 3. And this 1 here corresponds to 1 and and looks like that pattern.

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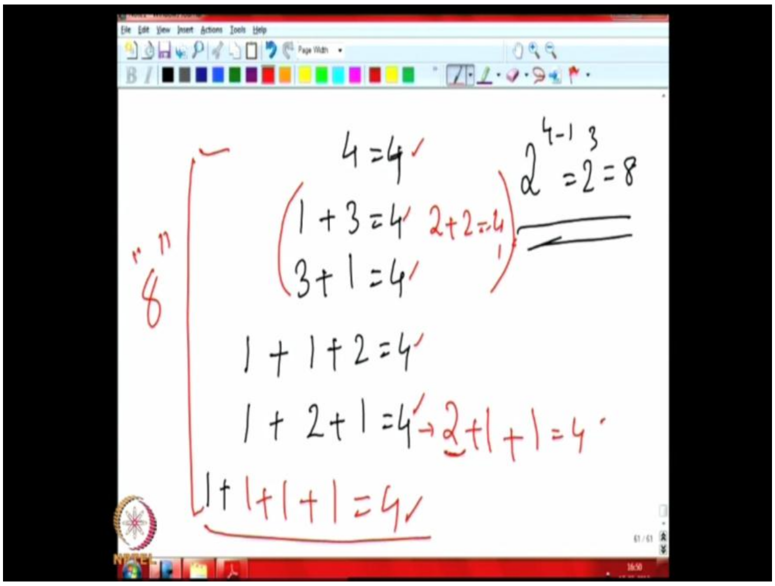

So, we can try with 4, 4 equal to 4. So, what we expect is we want to get the number 2 to the power 4 minus 1, 2 to the power 3 is equal to 8 now, this is what we want now 4 equal to 4. Now, if 2 splits 1 plus 3 equal to 4 we can also write 4 3 plus 1 equal to 4 this 1, 3 splits 3 plus 3 should be split so 1 plus 2 is equal to 4. So, 2 plus 1 is equal to 4 and then 4 splits you could write 1 plus here there are more in fact you can also write 2 plus 2 equal to 4, this is 1 2 plus 2 is 4. And we have 1 2 3 4 5 6, so we have 1 plus, 1 plus, 1 equal to 4. Still only 7 which should be 1 more, here we could also rearrange these.

So here 1 plus 1 plus 2 you can also write this as 2 plus 1 plus 1, that is different this is a different order. But the same numbers, but the orders is different 2 plus 1 plus 1 that 1 2 3 4 5 6 7 8, 8 ways of doing it. Now, we cannot do anything because here we have considered all the possibilities here, we have considered all the possibilities here. Because we started with 1 and starting with 2 here, and there is only 1 way of doing it here this only 1, so we have exhausted. So, it is true that 8 ways of 8 compositions are there remember the definition of compositions I have elaborately done a few examples. So, you should know what it is now.

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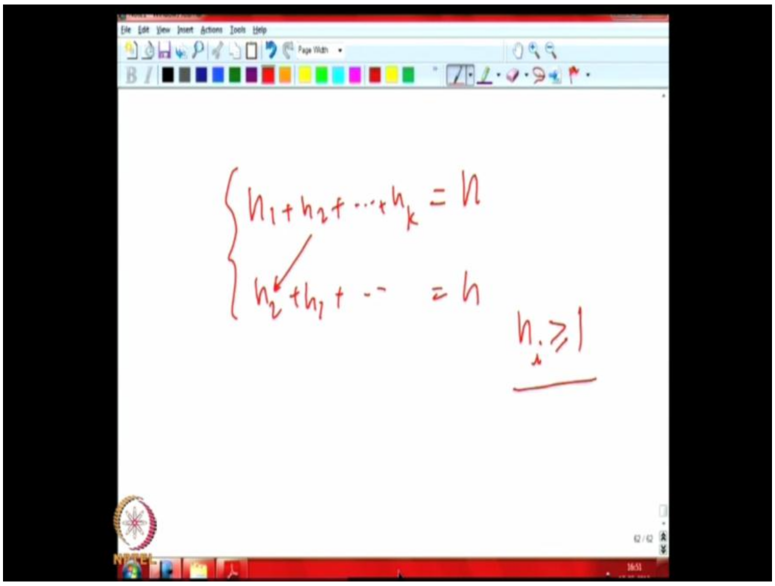

So, it is just that, your positive composition of a positive integer n is something like n 1 plus, n 2 plus n k equal to n we do not mind how many things are there. But each n i has to be at least 1, all are positive integers. Now, the order is important in the sense that same numbers we could have this, you put here and then plus and then keep all the things together still it should be counted separately. So, how many possible compositions are there that is what we are asking. So, definitely one way we see we want to use it use the earlier prob earlier whatever we have done before to solve this thing.

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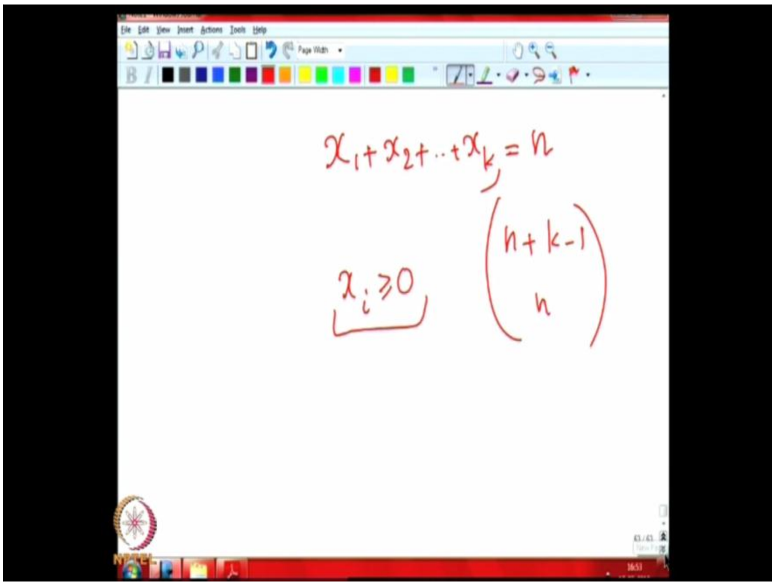

So, for instance we know that, when we write x 1 plus x 2 plus x k is equal to n then the number of ways to do this thing is n plus k minus 1 choose n. But there are some differences here we assumed each x i is greater or equal to 0, there we are not allowing 0 s. So, is it possible that I because see if 0 was if 0 was alone then we could have then we still have a trouble how many things should be there how many k should be there that also we have to figure out. So, before we cannot direct quickly apply the previous this so what we will do is to we will see all possible case.

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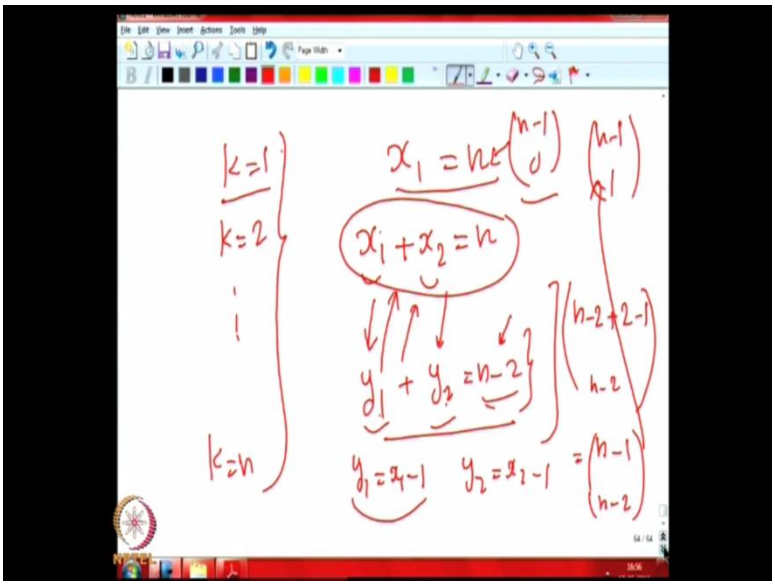

See for instance k can be equal to 1, k can be equal to 2, k can be equal to n these are the total number of possibilities. Now so when we say k equal to 1 case then it will be like an equation x 1 equal to n. Here you know that, there is only 1 possibility to this x 1 has to be n we can say what was to so for k is equal to 2 case it will be like x 1 plus x 2 is equal to n the only thing is we cannot put x 1 or x 2, 0.

So, each of them has to get at least 1. So, see what will happen is 1 is anyway given to x 1 now we can in other words our technique was to convert x 1 into a variable y 1, and x 2 into a variable y 2 where x 1 y 1 is defined as x 1 minus 1. And then y 2 will be defined as x 2 minus 1 that means we are allowing y 1 and y 2 to take 0 s, but here we have to make it n minus 2. Because what we do is we get a solution for this, we will get a solution for this also by minusing a whatever value we have assigned for x 1 will minus 1 from that we will minus 1 from this. And will assign for this thing and satisfy the sequence.

And then anything that satisfies this equation will satisfy this equation also because we are adding 1 to it. This is what it is therefore; now with this n minus 2, we can know how to solve it how much is this here it is like the solution of this equation. Here that its summing up to n minus 2, the answer is n minus 2 plus the number of types 1 to k x so that is 2 minus 1, choose n minus 2. This will be n minus 1 choose n minus 2 this is, this is what we will get. So, we can probably write it as n minus 1 choose 1 and here this is only 1, we can write it as 1 minus 1 choose 0 if you want. See you can you can this is just 1; we can write it as n choose 0 or n minus 0 or whatever in the end we see the pattern decide. This n minus 1 choose n minus 2 by symmetry identity it is n minus 1 choose 1 also.

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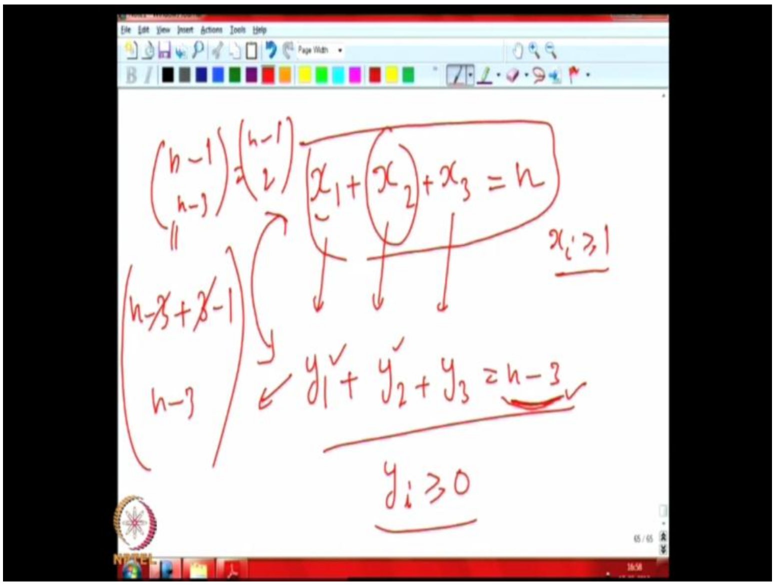

Now, let us see what is happening with 3, 2, 3 terms that is more the sum of 2 x plus 3 equal to n as of as what we want, but each of this exercise here has to be greater than and equal to 1. Therefore, we convert because to use our pervious theorem, we will convert in to y 1 plus y 2 plus y 3 equal to n minus 3. Why because now, we are allowing y i is to be greater than equal to 0 now any solution of this thing will give a solution of this thing. Because to each y i we can add 1 and get the corresponding value of x 1 that will because that is at least 0, this will be at least 1 and the reverse also.

If I get a solution for this thing I can minus a minus 1 from each of them and get the value for 1 this that they add up 10 minus 3 now that is it that is what we want. So, the solution for this corresponds to solution for this, and solution for this corresponds to the solution for this, that there is an equivalent between the solutions and the set of equations. So, there is a bijection between the solution set of this and the solution set of this therefore, we can see how many solutions are there for the second one. And this we know, this we know because this r is equal to n minus 3 is here. So, n minus 3 plus number of variables 1 2 3 it is 3 minus 1 choose n minus 3 which is we can write it as so here as n minus 1 again and this is getting cancelled. So, n minus 1 choose n minus 3 which is n minus 1 choose 2. Earlier it was n minus 1 choose 1 we initially started with n minus 1 0 n minus 1 choose 0 next, we wrote n minus 1 choose 2.

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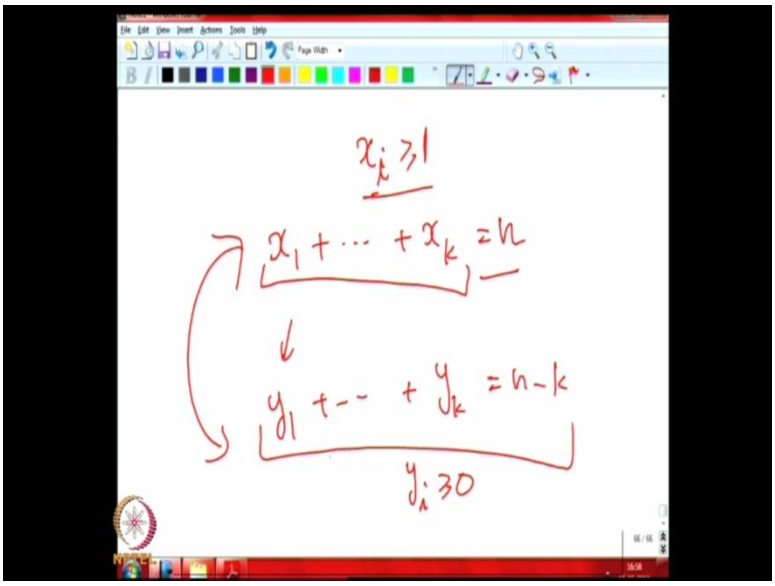

Now, the next one, we can see for instance in the k th arbitrary curve the k k-th term thing, so this x k has to be equal to n. Now this is you know again I will do the same trick y 1 plus y k I will convert this is equal to n minus k. And of course, each y i has to be greater to equal to 0, here it was x i has to be greater than equal to 1. So, now yes, yes we argued earlier we can say that there is a bijection between the solution set of this equation and the solution set of this equation. So, we just have to count the number of solutions of this.

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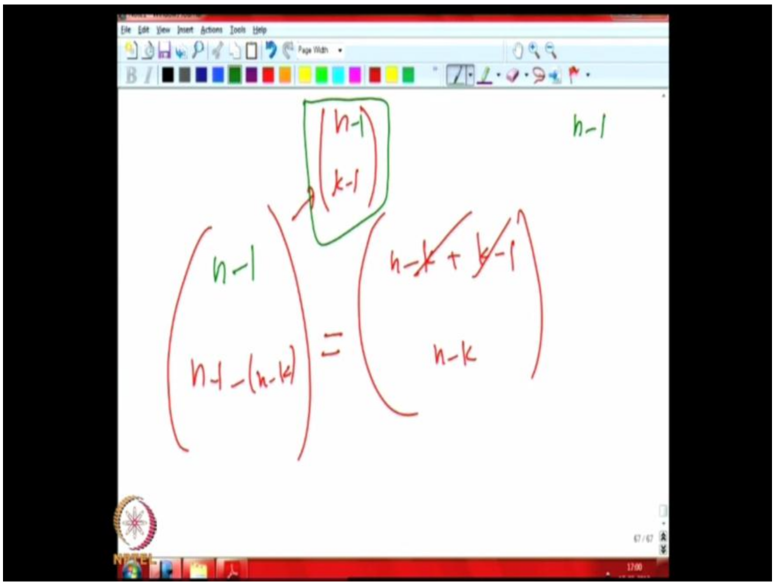

And this is as we mentioned this is n minus k for r, plus k minus 1 choose n minus k. Cancel k n minus k that will be n choose k plus 1,k plus 1 or k minus 1 n choose n minus 1 n minus k. That is n choose n choose k minus 1, so this is what So for k-th k terms we get n choose, this is not n, n minus 1; this was n minus 1, n minus 1 choose n minus k; this is n minus 1 k and k minus 1. So, when there was 0 term we could get minus 1 choose, n minus 1 choose so for k-th term what should we get n 1 2 written the pattern correctly. So, we are getting for 0 term n minus 1 choose 0, one term then there is only 1 term we are getting n minus choose 1 and 2 terms we are getting n minus 1 choose 2. There are 3 term, 3 terms we are getting n minus 1 choose 2 for when the k terms, when there are k terms we are getting n minus 1 choose k minus 1.

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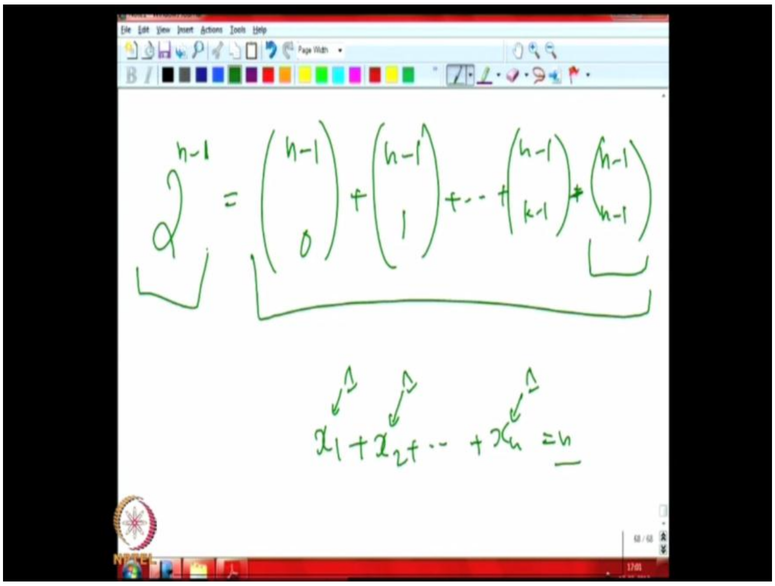

So finally, we can go up to how many terms, this will be n minus 1 so if 1 term n minus choose 0 2 terms n minus choose 1 and finally, k terms n minus 1 choose k minus 1, when we have all n terms it will be n minus 1 choose n minus 1. That is because as you see and there we have an equation like this x 1 plus x 2 plus x n equal to n only solution is 1, 1, 1, 1 for all of them add up to. There is only one solution that is what n minus 1 present, and we know what this is we have already summed up this kind of identities. In the earlier classes this is 2 to the power of n minus 1 certainly we want to get this is the general formula. So, whatever the pattern we observed was valid. So, its correct now so this was we just used our theorem of case this can be solved in much easier ways also. But you should try to solve it in different ways, but we just use this to illustrate our, the formula we have got earlier.

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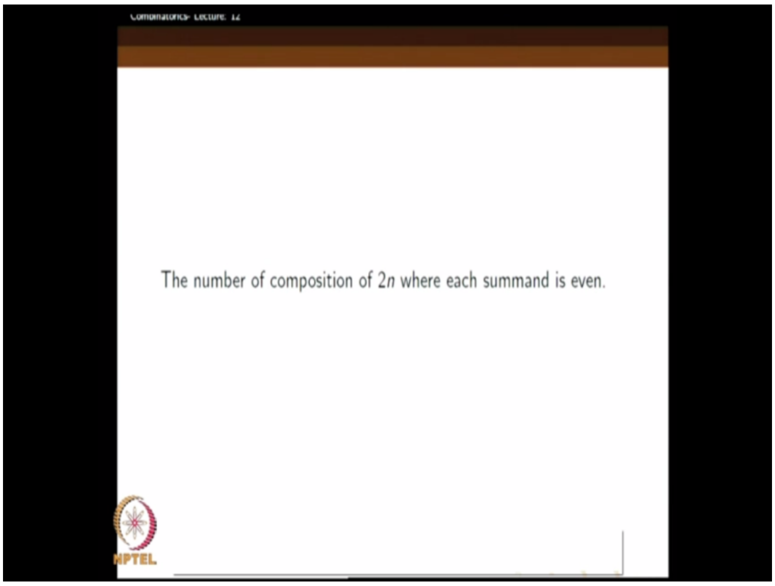

So, this is the number of representations here. And now we can yeah some slight variant, variant of this problem the number of compositions of 2 n, where each summand is even, even.

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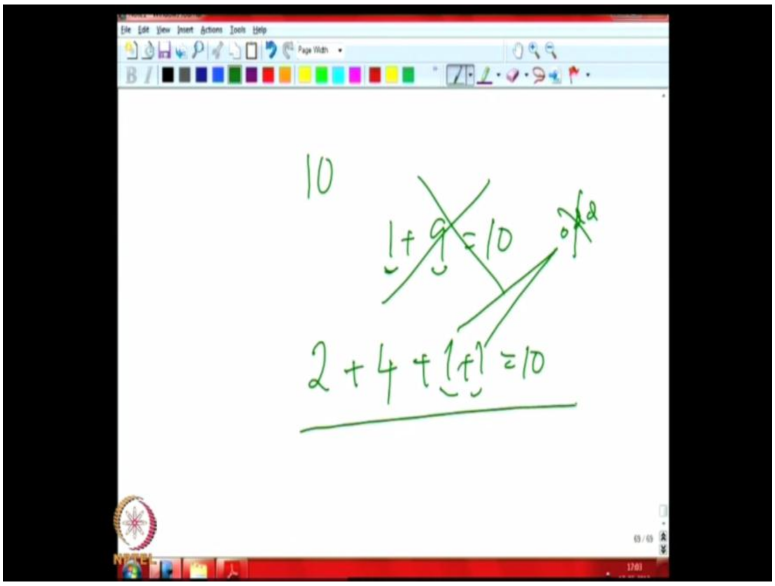

So that means maybe we can take some examples, suppose say 10 let us say now we are not allowing 1 plus 9 into 10 kind of things. Because now here we have 1 and 9 so we are not even allowing 2 plus 4 plus 1 plus 1 equal to 10, because these numbers are odd numbers these are odd numbers these numbers are not allowed.

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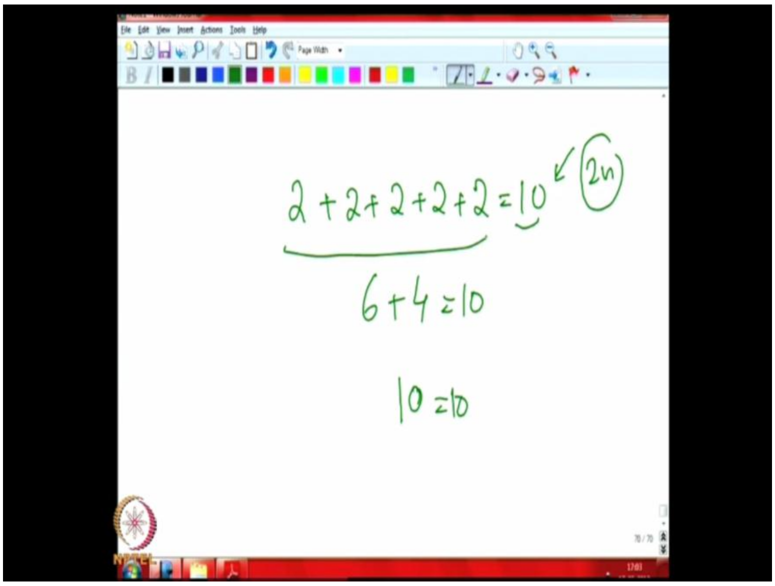

We want all of the summands to be even, so that means 2 plus, 2 plus, 2 plus, 2 plus, 2 is equal to 10, 6 plus 4 equal to 10, 10 plus 10 equal to 10. And that is of course, you can see that we should we can only talk about even numbers, yes we are getting taking compositions of even numbers.

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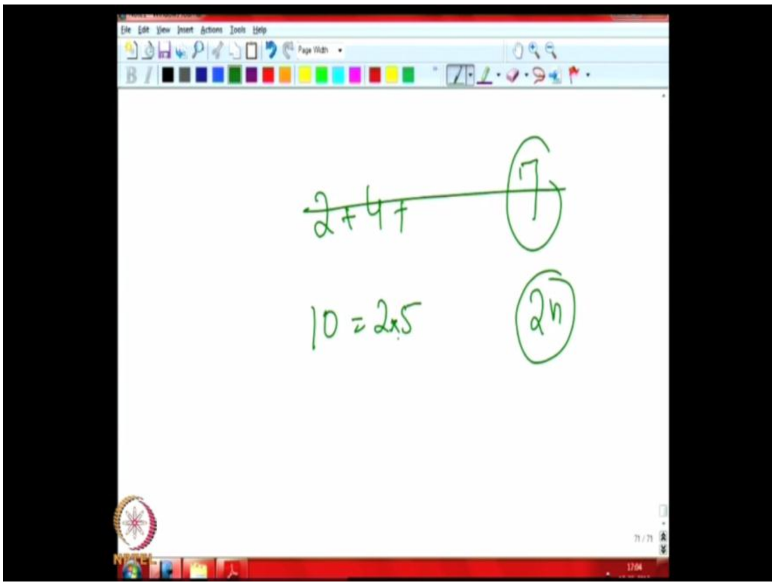

If you are summing up all the even numbers we will only get even numbers, there is no point asking for even compositions of 7. Because how much ever even number you had see without adding 1 number at least we cannot get any odd number. Therefore, we are taking about compositions of even numbers namely 2 ends. So, from the case of 10 is equal to 2 into 5.

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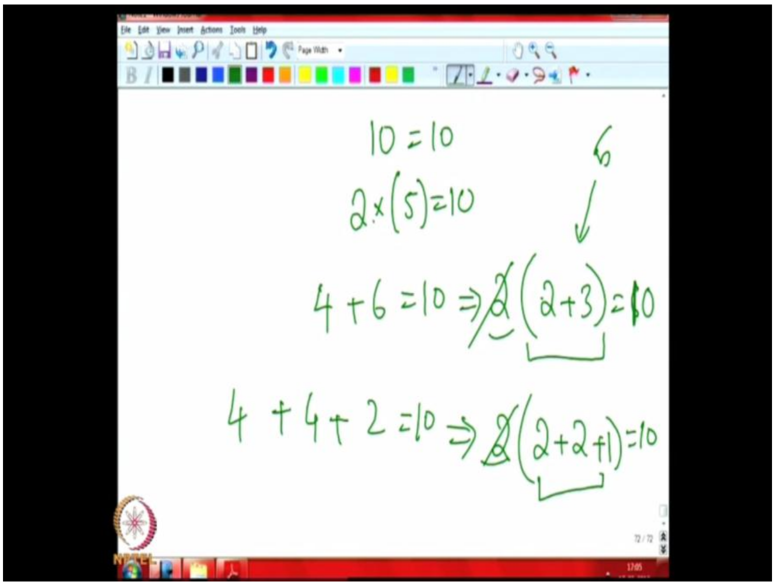

Now, how will you solve this thing easy just that you can see that, because it is an even number in case to 10 let us say 10. We can 1 is 10 is to 10, this we can write as 2 into 5 equal to 10. Similarly, in the for instance when we write 4 plus 6 equal to 10, you could have equivalently written as 2 into 2 plus 3 equal to 10. So, 2 into 2 4, 2 into 3 6, or when we write 4 plus, 4 plus 2 equal to 10 we could have written 2 into 2 plus 1 equal to 10. So, you see every time when you remove this 2, the remaining thing will add up to 5 only, 2 into 5 is equal to 5 is the result.

So, actually we are once you discard these 2's we are looking for compositions of 5 without any conditions, because here we have the condition that all summands we have, have to be even. Now, here we have just the condition that the summands have to be non negative sorry, sorry summands have to be positive and we are not allowing 0's off case. So, and then how many we can do that is we are asking that is what we are asking.

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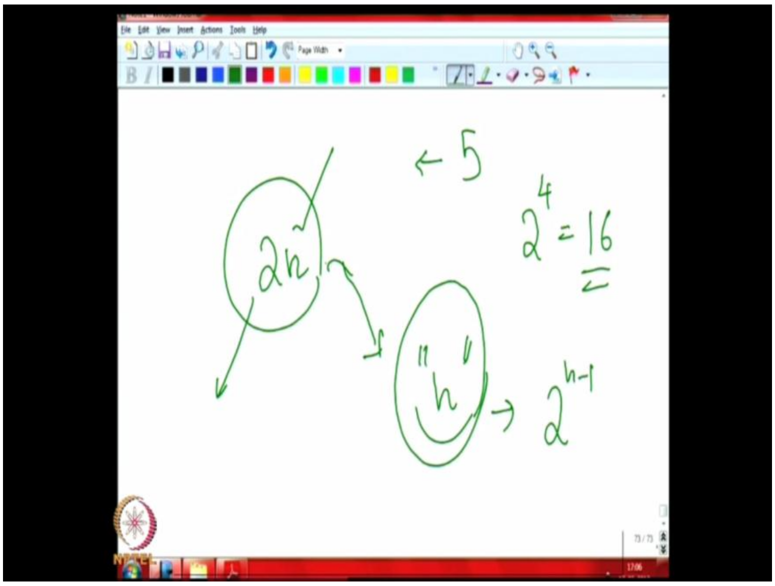

So, the answer will correspond to the number of compositions of 5, number of compositions is 5 which is actually 2 to the power 4 as we have seen 16. So, in general when we are taking about 2 n we will be considering the number of compositions of n number of compositions of n, where there is no restrictions here was a restriction that every summand has to be even.

So, we as, as we were always doing the bijection between a compositions of 2 the compositions of 2 n there is summand has to be even, even compositions you can say, say to the compositions to the normal compositions of n. There is a one to one corresponds to the bijection between so it enough to count this, this is 2 to the power of n minus 1 as we have got.

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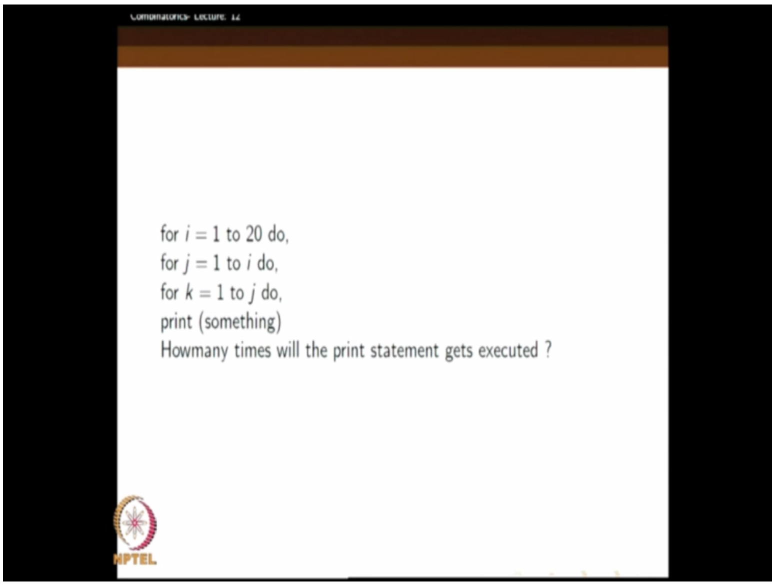

So, that is some next question we better discuss. Now, here is an interesting question for j equals to 1 to 20 there is a program segment 1 to 20, 2 but j equal to 1 to i 2 for j equal to 1 to j 2. Because there is a loop inside a loop in the sense that so there is a nested loops 3 of them nested. So, first outer most variable is i so then next comes our j next j goes from 1 to i only whichever was the value of i at that time k goes from only 1 to j. So, the question is how many times will the statement gets executed?

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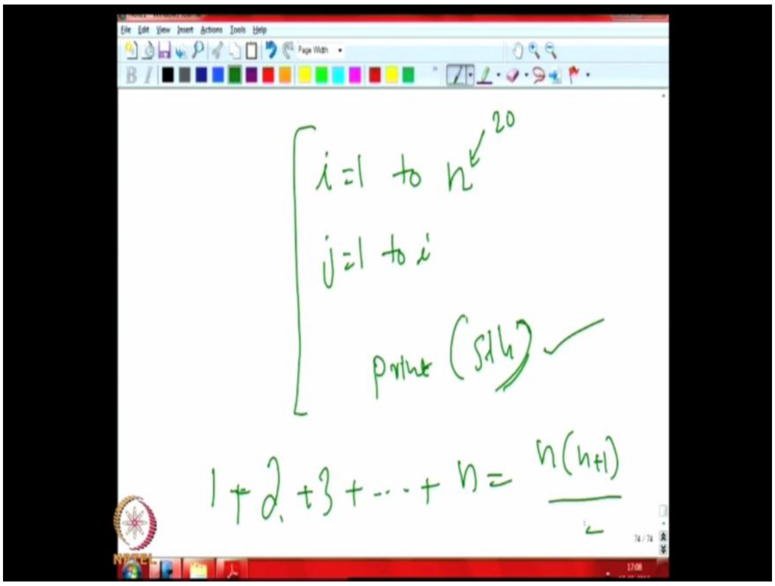

So, you know, so you see for instance there is a simpler question i equal to 1 to n, I put 20 here, but also n j equal to 1 to i. How and then print something say something how many times the, the print statement will be executed? This is a usual question which is comes in exams and all so what is that, because first time when i is equal to 1 j will range from 1 to 1 means j will take one value. So, when i equal to 2 j will range from 1 to 2 so that means 2 times this print statement will get executed. So, then i equal to 3 this j will go from 1 to 3 and then the print will be done 3 times and then like that up to n. And we have to sum this up this will be n to n plus 1 by 2.

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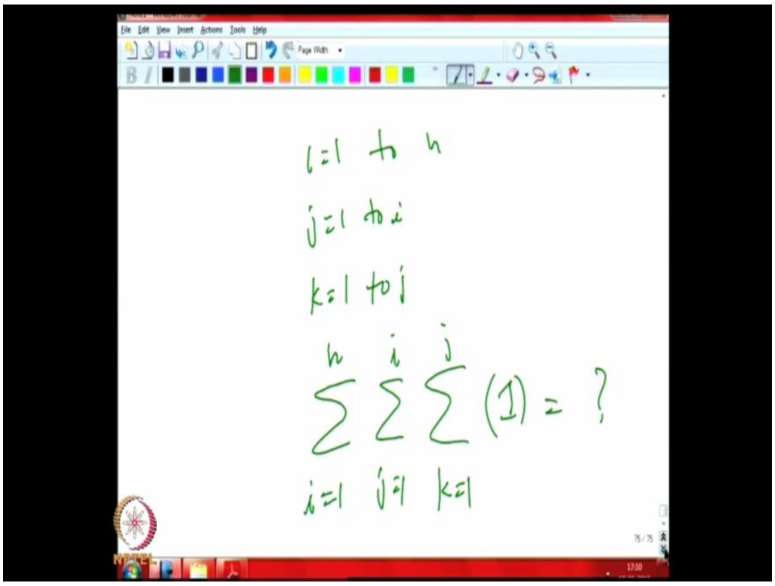

So, but now what will you do suppose this i equal to 1 to n, and now j is equal to 1, i and then k equal to 1 to j. So, we will say that this we have to sum equal to 1 to n then j equal to 1 to i, and then k equal to 1 to j and his 1 is to be summand every time so this only 1 has to be summand for this, this is what we are asking. We are printing once each one of this thing, but what is it counting so this can happen see now somehow with lot of effort you may do it. But the issue is that you may have several such summations not just 3 you can have 3 4 5 6 r summations for them, how will you do it?

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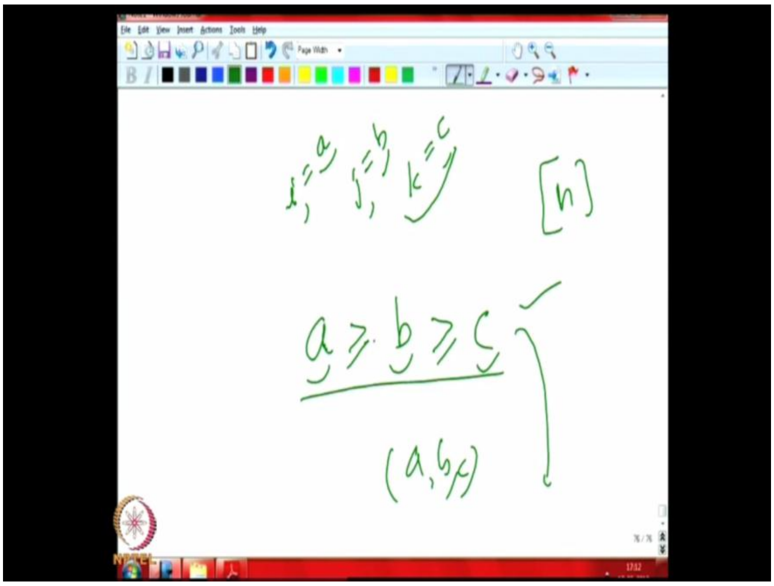

The point here is that you know at whenever your print statement is executed there is a value of i, there is a value of j, there is a value of k. Say value of i is equal to a, value of j is equal to b, that is the value of k is equal to c. You know that a has to be greater than or equal to, greater than equal to b because j is going only up to i right so the value j takes will always be always less than equal to the value taken by i. And similarly, the value that k takes will be the almost the value that j takes this will be true, suppose on how many so the question is how many a b c is are possible.

For instance, if say i is going from 1 to n only the possible values are up to from 1 to n only. So from 1 to n many such a b c is can be created such that a is greater than equal to b is greater than and equal to c. You can see that this solution is will give you the answer the number of this things will give you the number of times the print is executed. Because you know every time you print statement is executed you have a value of i, value of j, value of b and this is such as a b c that is a l is greater than b.

And on the other hand you have an a b c, such that a greater than b and b greater than c and all of them are coming from 1 to n. Then of case corresponding to that, there is a point that which prints the statement i will take that value of a, j will take that value of b, and k will take that value of c that particular value c. And then this is all valid a is equal to that a, and j can reach that particular value of b because b is less than and equal to a. And also k can take that particular value of c and so on so it is just enough to count this thing. So, we will stop here and continue in the next class.