Combinatorics
Prof. Dr. L. Sunil Chandran
Department of Computer Science and Automation
Indian institute of Science, Bangalore

Lecture - 10
Permutation of Multisets - Part (2)

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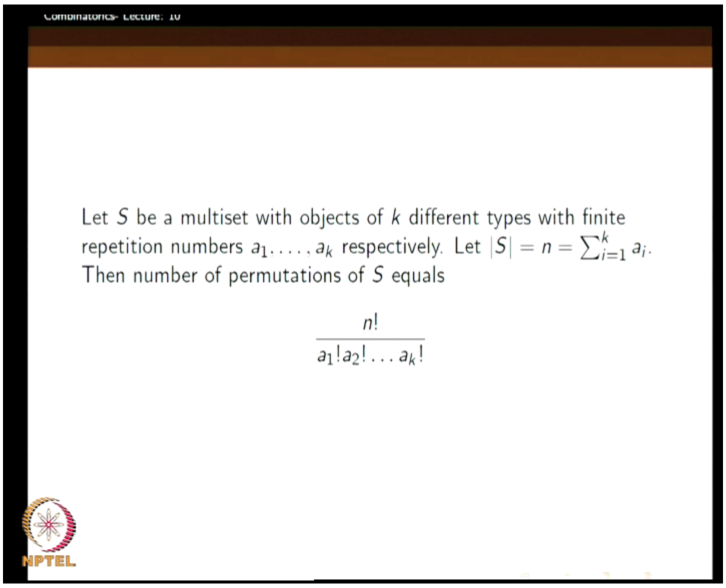

Welcome to the 10th lecture of combinatorics, and in the last class we were looking at this problem, namely the permutations of multi set, right?

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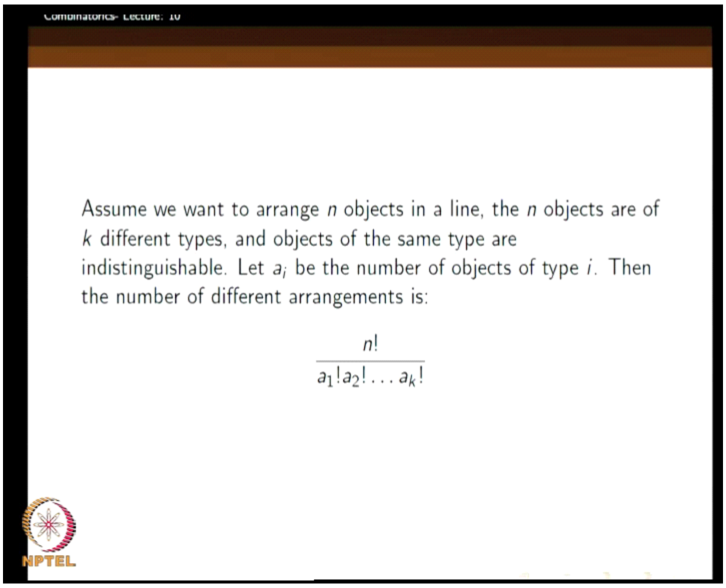

How many ways you can arrange n objects in a line, and this n objects are of k different types, and objects of the same type are indistinguishable. Let a i, a i is the number of objects of type i. Then the number of different arrangements is n factorial by a 1 factorial, a 2 factorial, a k factorial. So, we like this, n objects, it is n objects and seen as a multi set.

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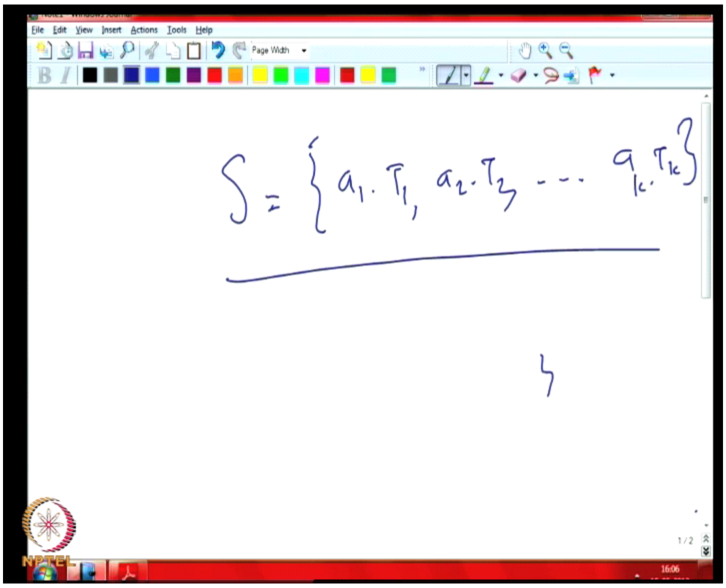

So, how to write the multi sets? That S equal to at a one times type 1, a 2 times type 2 and so on, a k times type k, right? So, the permutations, all the things in the multi sets are made in a, not arranged in an ordered list, right? How many ways you can arrange it? The key point here is that at certain type, there are several of them. Repetition number is what we call for the number of time is set. Certain type comes. They are indistinguishable. For instance, they appear in say position number 10, 13 and 14. So, within these 10, 13 and 14 positions, if we arrange them in a particular type, we won not make any change to the permutation. Earlier, it was not like that because when they were all different, any change would have made a difference. So we see that the number can be counted this way. Now, we will plan in this class is to look at some examples. So, we start with this example.

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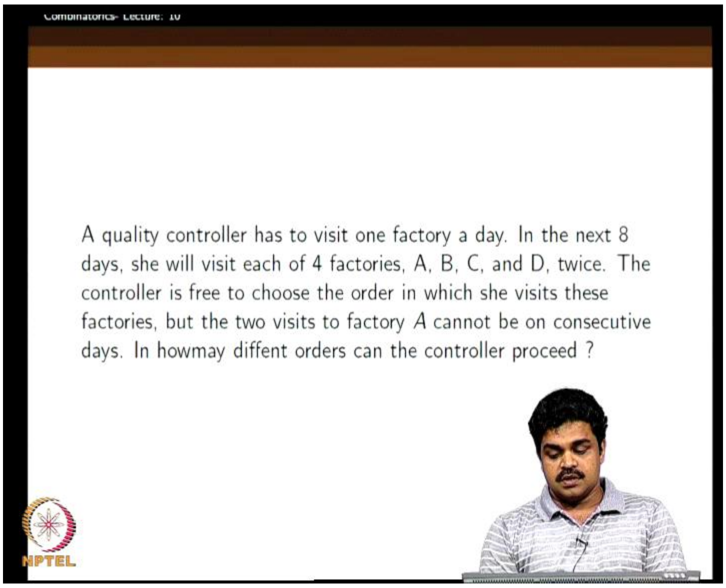

So, a quality controller has to visit one factor a day. In the next 8 days, she will visit each of 4 factories. The factory names are A, B, C and D and each factory will be visited twice. The controller is free to choose the order in which she visits these factories, but the two visits to factory A cannot be on consecutive days. So, you should not visit factory A, today and tomorrow and a consecutive day, right? So, how many different orders can the controller proceed?

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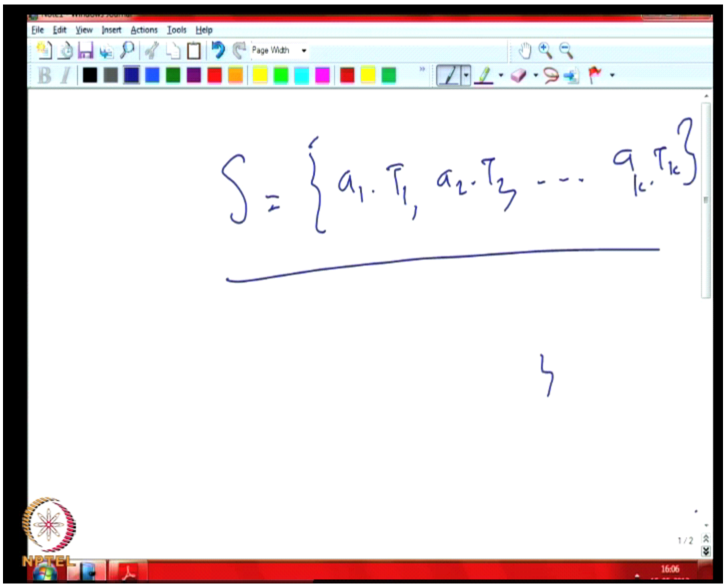

So, we want to count the number of ways she can do this, right order and which she can visit the factory.

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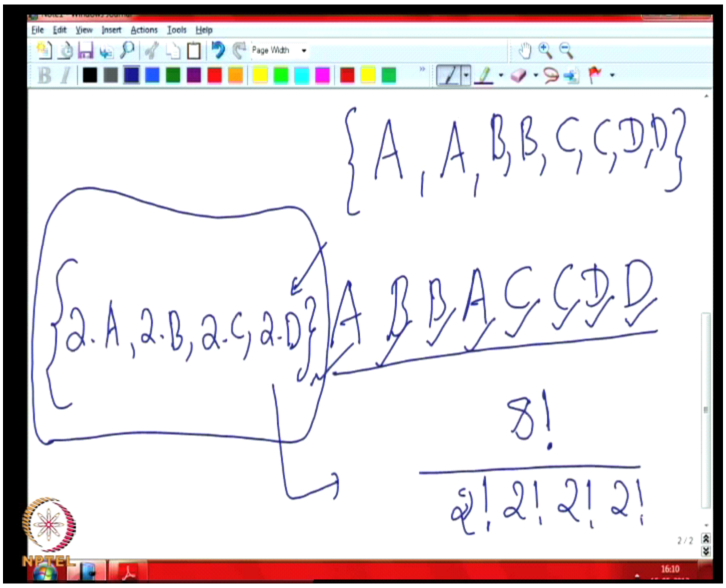

So, what are the things we are permuting? Because see she visits each factory two times, say we can say factory A comes two times, factory B comes two times, factory C comes two times and factory D comes two times. This is the multi set. So, we essentially are interested in some ordering of this sort, right? CCDD, which means first day, she visits factory A, second day she visits factory B, third day she visits factory B, fourth day she again goes to factory A, the fifth day she is in factory C, sixth day she is in factory C, seventh day and eighth day she is in D. So, this kind of a listing is what we are looking for, right? The order in which she visited the factories and therefore, this can be modeled as permuting a multi set here. The multi set is this, AABBCCDD. In our notation, it can be written as 2A, 2B, 2C and 2D.

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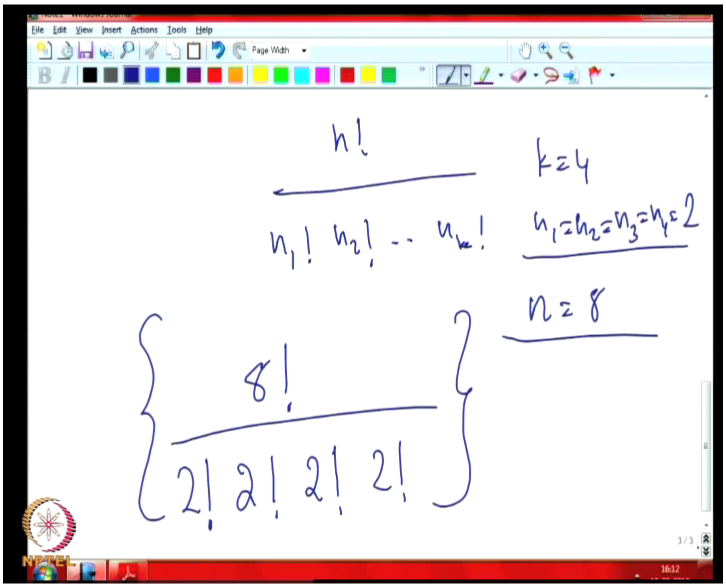

Now, we know the answer. If there was no further restriction because how many permutations are possible for this thing, this multi set that we already know, because a total, the cardinality of the multi set is 2 plus 2 plus 2 plus 2 is 8. So, that is 8 factorial divided by 2 factorial into 2 factorial into 2 factorial into 2 factorial. So, this is by using the formulae n factorial by n 1 factorial into n 2 factorials into n k factorials because k is equal to 4 here, n 1 equal to n 2 equal to n 3 equal to n 4 equal to 2 here and n equal to 8. That is why, it is 8 factorial divided by 2 factorial into 2 factorial into 2 factorial into 2 factorial, but this is not what we are told. We have further restriction. This is not what we want. We have further restriction.

What is the further restriction? The controller, she is not supposed to visit factory A on consecutive days. May be after one visit, the factory has to reset something or there may be a technical reason why she cannot visit the factory on consecutive days. She has to give at least one gap. We will use our subtraction principal here. We would rather find the number of ways she can visit, such that factory A is visited on consecutive days and then minus it from the total being this, right total being. No, total is not this. Yeah, total is this. Total being this total number of ways to visit is this, but consecutive days, if so for instance now we restrict that she has to visit factory A on consecutive days.

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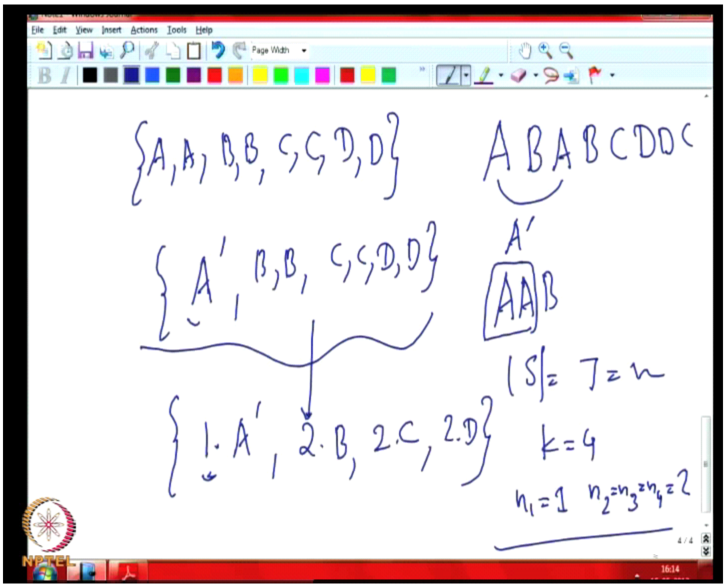

What we want is the other thing, but we will do this way and then minus it off, right? That means, in a permutation of this sort is not allowed, right because she knows A and A are not consecutive here. We want always A A, then B. Whenever A comes, we should see the other A also, right? So, what should we do? The trick is usual. Trick is that we just combine the two A's together into one A. We can say A dash, right and wherever now instead of A multi set of this sort A, A, B, B, C, C, D, D. We would rather have a multi set of this sort, A dash, B, B, C, C, D, D, right?

So, now we will permute this one and whenever a dash appears, we will replace with two A's. That means, A is visited consecutively, right? That is what it is, but this will give you the count, but this multi set is what we in our notation, it will be 1 a dash 2 B and 2 C and 2 D, right? The cardinality of the multi set is 1 plus 2 plus 2 plus 2 that is 7. Cardinality S equal to this is n and now, we have k equal to 4 because there are four types and n 1 equal to 1 because here, it is only one type, n 2 equal to n 3 equal to n 4 equal to 2.

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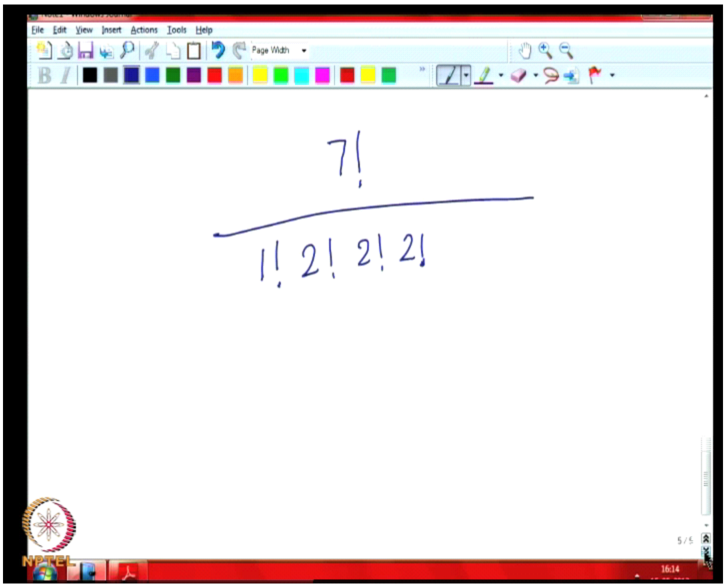

We can easily find the answer for the number of permutations of this sort factories. That 7 factorial divided by 1 factorial into 2 factorial into 2 factorial into 2 factorial. This is the total number of possible ways to visit, such that the factory A is always visited on consecutive days.

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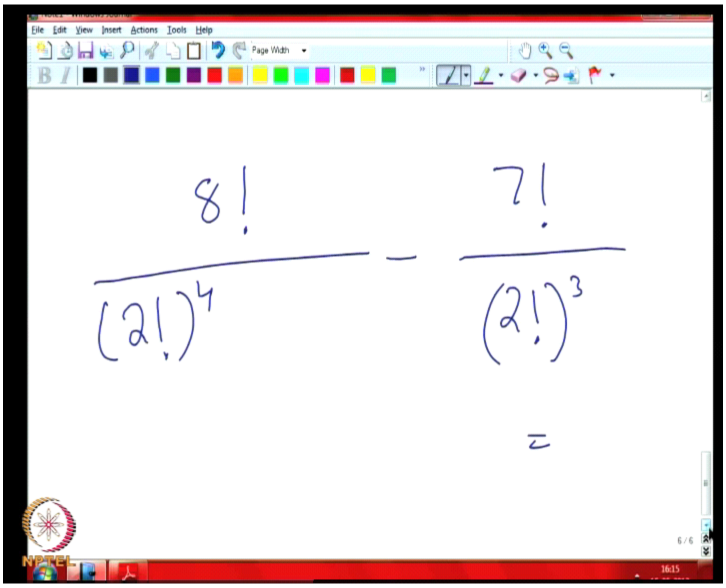

Now, this is not what we want. We want is the other thing, namely the factory A is never visited on consecutive days. Therefore, the total number which was 8 factorial divided by 2 factorial raise to 4 minus 7 factorial divided by 2 factorial raise to 3 will give you the answer. I am not calculating it. So, this will be the answer. So, here we have made use of several notions that they have studied already, namely subtraction principle we have used and this notion of permutation of a multi set we have used, right and that idea of combining 2 S into 1 A something which is very useful. How will you express that idea that to A to come together, right? On this just say that instead of 2 S, we just call A dash and after getting the permutation, say dash is substituted by to A's, right? So, use full trick which you can remember.

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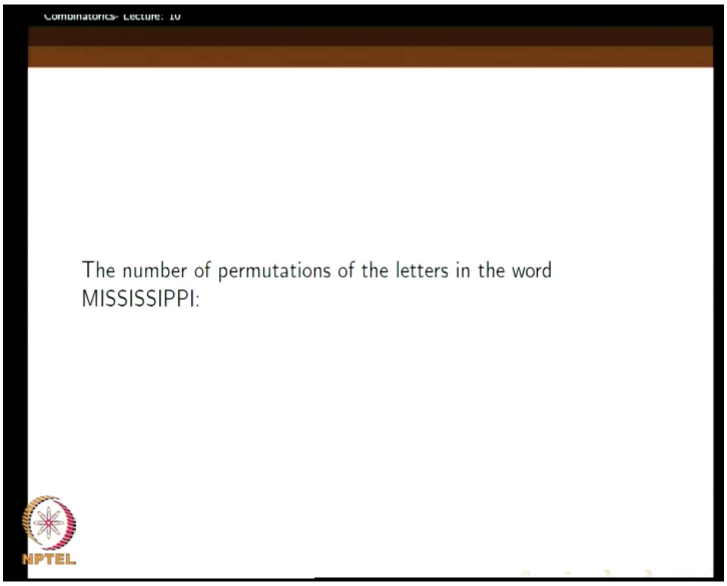

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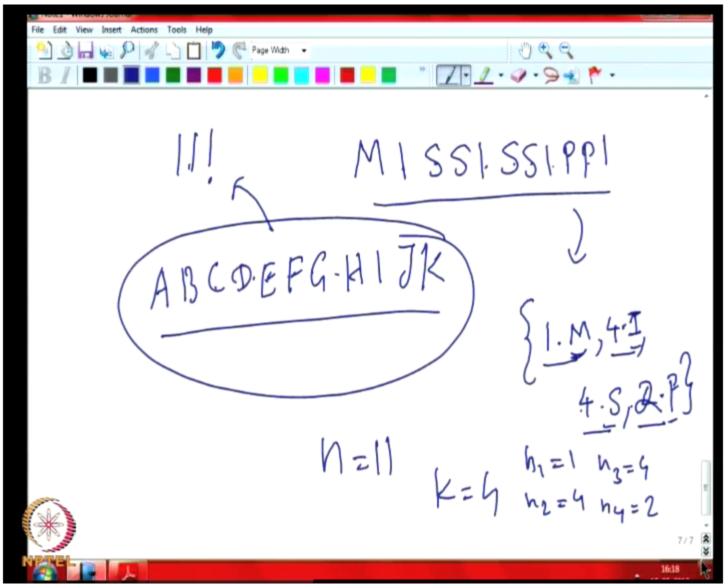

Now, we will look at the next example. Yeah usual question is the number of permutations of the letters in the word MISSISSIPI. How many days you can write the letters in the word MISSISSIPI. The point here is if all the letters are different, you would have got 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 factorial things, 11 factorial possible permutations of this thing. For instance, say a word of this sort A B C D E F G H, right, 1, 2, 3, 4, 5, 6, 7, 8. J, k is a eleven letter word. So, if somebody asks you how many ways you can permute it and how many words you can make out of this thing by reordering them, then it would have been easy, right? We know already that, but here the difficulty is the several letters are repeating. So, this is not just eleven letters. This is not a set of eleven letters; this is some multi set of eleven letters.

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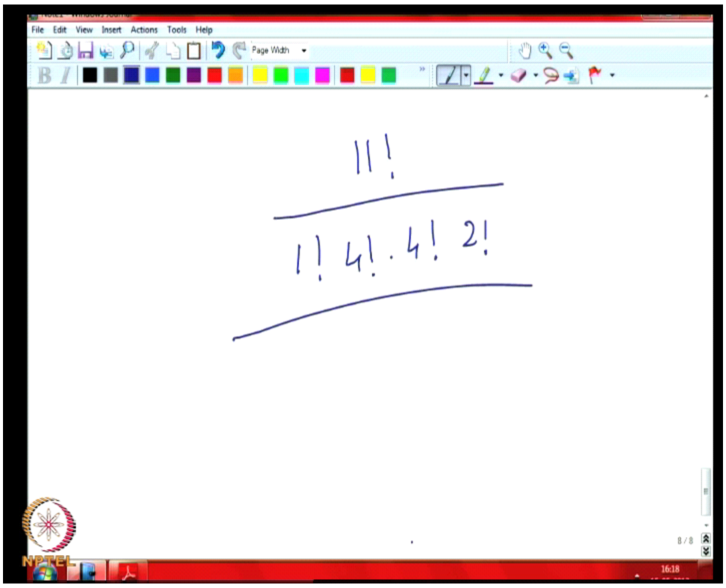

So, the multi set is like this. There is one m and 4 i's and then we have 4 s and then we have 2 p's. This is the multi set. 1 m, 4 i, 4 s, 2 p and that is total 11 a. A plus 2 plus 1 is 11. So, the cardinality of the multi set n is equal to 11. Now, we have four types. K equal to 4. These types are m, i, s, and p. These are the types and then each one of the type that is 1 and 1 equal to 1 n, 2 is equal to 4 n, 3 equal to 4 n, 4 equal to 2. We know the answer. Now, we just have to permute this multi set. That is what he is asking. We do not have to think much that is 11 factorial divided by 1 factorial into 4 factorial into 4 factorial into 2 factorial, right and the next question.

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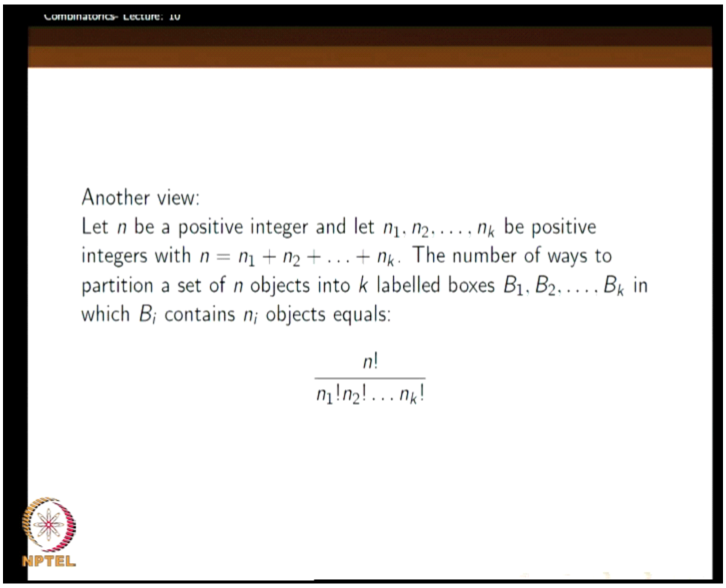

So, we will have a different way of this problem. Now, this permutation of the multi set was one view. We had a multi set containing n things in it. I mean, adding together all the reputations, the cardinality of the multi set is n and there are k type of things in it. Objects are of k types and then each type, the first type is n 1 and second type is n 2. The last k-th type is n k.

How many ways you can order these objects of the multi set? In a list, every tenth multi set. How many ways you can permute this was the question. What in other way? We just told we want n objects to be listed. That is what instead of using the multi set of a analogy. Now, we say we have n objects and we have to place them. These n objects are now different objects, right, different objects, now a distinguishable objects.

We want to place them in k labeled boxes, means boxes are also distinguishable. For instance, the first box you can put second. I mean, you can see the boxes have number on them or maybe the boxes are of different colors. You can immediately distinguish a box from another. The boxes are B 1, B 2, B k, but the restriction is that i-th box should contain n i objects. After you distribute it, i-th box should contain n i objects. So, this looks like a different problem.

So, we in combinatorics classes, we always talk of balls and bins problem. So, here we are talking about indistinguishable balls. That means, these balls, each ball we can identify one from the other, right and the bins are also distinguishable. So, there is this 10 buckets.

So, beans and they can identify one from the other. The key thing here is that the ith distribution. I mean, how many should go to the i-th box is given predefined. Now, how many ways you can put them? The total number is n 1 plus n 2 plus n k, right? How many ways you can do this thing? This term have to be the same number. So, actually the one of the proof was doing this thing. For instance, what we were doing in the first proof, we will look. We kept n positions in the first position, second position of n positions and then this n position we were allotting to each type, right? We are saying that the first type has to get n 1 position.

So, n 1 of them should go to the first type. That means, first types will go and occupy those n 1 positions. How many ways you can select it that can be done in n choose and n 1 ways. Now, that n 1 positions are gone. Now, n minus n 1 positions are left and this out of this n minus n 1 positions, n 2 should be given to type 2 or in other words, n 2 of them should go to bin b 2. That is second one. The type is like now matching with the boxes, label boxes k types, right and this actually, the positions are the distinguishable objectors here and they are going to n 1, n 2, n k something like that, right? The numbers, right? Therefore, this is almost the same proof just that once we understand the correspondence, there is nothing big here. So, it is immediate. So, let us say so I will just describe.

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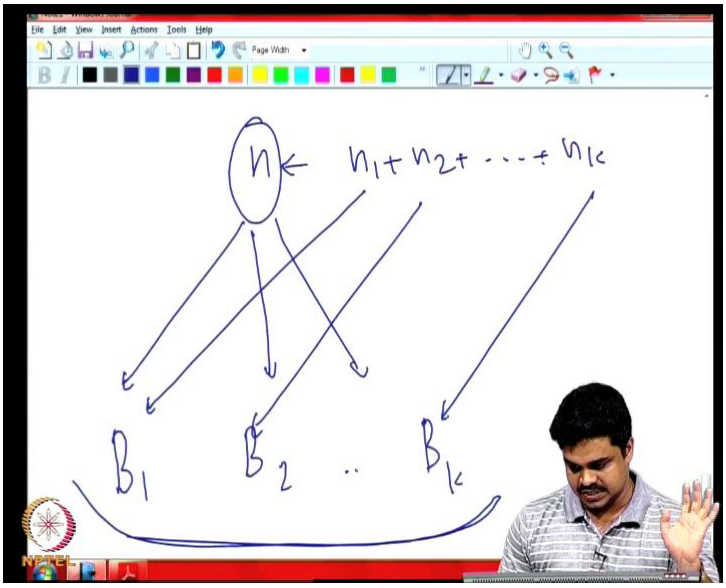

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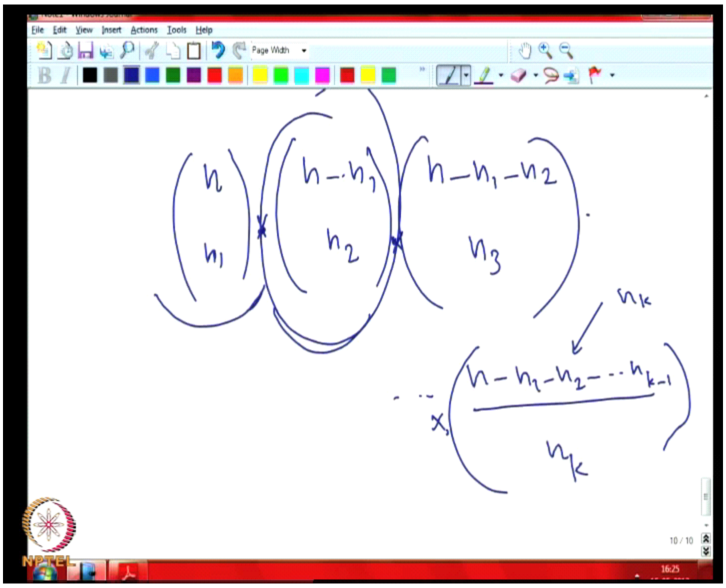

The question here is we have n objects. All can be distinguished, right and this is actually n 1 plus n 2 plus n k. Now, these objects, these n objects can be placed into the b 1, b 2, b k boxes. These many different boxes we have to place, right, but the b 1 should get n 1 of them, b 2 should get n 2 of them and b k should get n k of them. That is the main restriction. How will you do that? So, out of the n objects, you select n 1 thing which should be given to box b 1 that can be done n choose n 1 base, right and the remaining things n minus n 1, from that you choose n 2 things which should be given to box number 2 and from the remaining things. That means, n minus n 1 minus n 2 remaining things are there. Out of that, you can choose n 3 objects which can be given to the remaining objects and like that, we can do n minus n 1 minus n 2 minus.

Finally, with the last box when you take, this will be n k minus 1. We have to select k objects. Actually, this might have become n k. So, we are just selecting all of them because that should be given to the k th box and we have to multiply these things together because you know this is the number of ways we can select that first and one thing to the n 1 things to be given to the first box, and from the remaining things, this is the total possible number of ways to select n 2 things to be given to the second box for any selection of n 1 things to be given to the first box.

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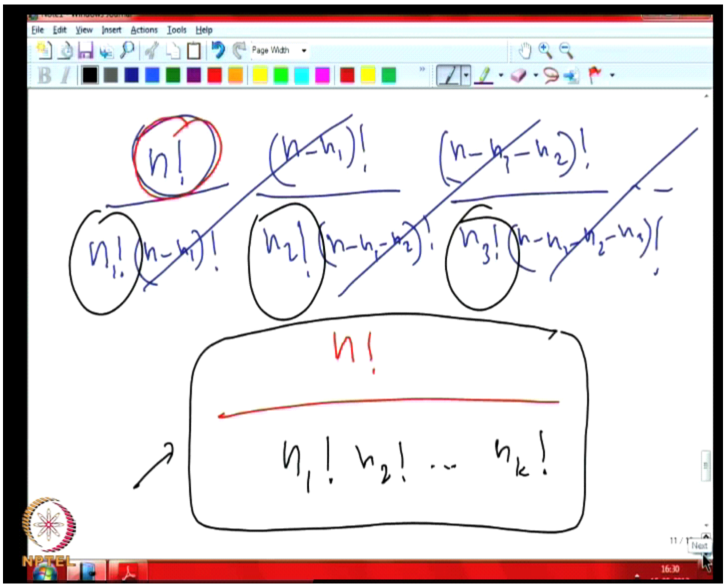

You can combine it with a selection of n 2 things to be given to the second box. Therefore, that is multiplication. So, product role we are applying and therefore, we just can multiply them out. When you multiply them out, we can just expand. This is n factorial by n 1 factorial into n minus n 1 factorial into here n minus n 1 factorial divided by n 2 factorial into n minus n minus n 1 minus n 2 factorial into n minus n 1 minus n 2 factorial divided by n 3 factorial into n minus n 2 minus n 3 minus, sorry n m in, sorry n minus n 1 minus n 2 minus n 3 factorial and so on, right?

So, this will go on like that and you know this will telescope. This will cut like this, right and finally, now what we will remember is this one. Sorry, what will remain is this. This will remain in the numerator, that is n factorial and that the numerator, sorry in the denominator, we have this and this and this and so on, that is n factorial, n 1 factorial, n 2 factorial and n k factorial. You can just work it out yourself. This in matter is just manipulation.

So, this is what exactly the same number is as I told you as you can see from the proof. It is the same thing, same proof that we have done. Actually, just that this n objects in the earlier proof was n possession and these boxes, label boxes actually the types, right? The first type correspond to first box, second type correspond to second box and third type correspond to third box and k-th type correspond to k-th box. There you are telling that the first type occupied n 1 positions. Now, you are saying n 1 object are selected and given to the first box putting the first box like that. That is only difference. Therefore, it is a same number that it is told in a different way, but still you can notice that the problem can appear in. Anyways, you have to interpret in the correct way.

So, this is one model which we can keep in mind. There are boxes, there are balls, right? Sometimes, the balls. So, in some problems, the balls maybe all identical, sometimes the boxes maybe all identical. So, in this problem, balls are so distinct. I mean, all the balls are different. Different I mean, you can distinguish one ball from another. The boxes are also different and you can distinguish one box from another box. So, the speciality here is that the number of things which should go to the i-th box, a specific box, how many things should go is already given to us, right? In some cases, it may not be given. May be we have to work harder in that case. So, it maybe just say, that how many ways you can split it. That may be the question. So, that is it.

So, we will keep discussing about these balls and bins. Problems solve through this course because something which you can keep in mind and then like in the pigeon hole principal, the difficulties not remembering those accept a certain type of balls is mean problems, but rather identifying, a new problem comes identifying that this is this kind of balls in bins problem. Most of the time, you would not know what are the balls, what are the bins and that is the difficulty like in the pigeon hole principal. We were saying that how will I identify, which are the pigeons and which are the holes. That is most of the time that is the difficult task. Once you identify the pigeons and holes, the n application of the pigeon hole principle was almost real, but most of the time to decide which should be the pigeons and which should be the holes was different.

Similarly, to decide which should be the balls and which should be the bins, it will be difficult here. There is this added complication that there are many problems we have to remember. That means, in some cases the balls will not have numbers. That means, they would not be distinguishable. In some cases, the bins will not have numbers, sometimes both of them have numbers, sometimes how many should go to a ball is given, sometimes, it is not given. So, we have to remember lot of things, but at the same time, remembering may not be very difficult, but more difficult is to identify which problem is relevant to which situation when a new situation comes, but anyway, it is when you get more and more familiar. One gets more comfortable with it.

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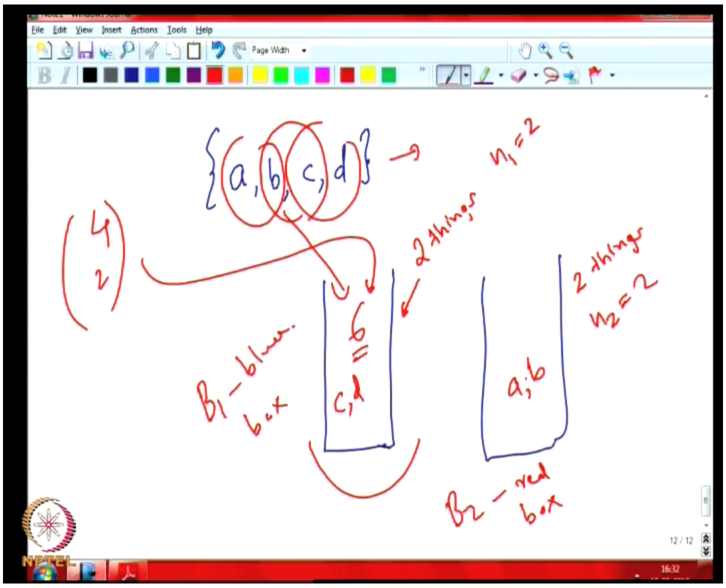

So for instance, one can see the illustrate the point. For instance, you can take this simple example. Let us take a, b, c, d. There are these four objects. Now, these are all different, a, b, c, d, right? One can distinguish from one another. Now, we say I want to put them in two boxes. So, one is two boxes, right? How many ways you can put? Suppose, these boxes are distinguishable, then I can do it in 6 ways. What all ways? Because in this, I can ask in this green bold, right? I can put any of the two things because once you put two things in this, remaining two will go. That means, what I want to do is this. This green box should get two things, right and this red box should get two things, right? So, here our problem says, n 1 is equal to 2. So, our d 1 is the blue box and our b 2 is the red box and this n 2 is also equal to 2. N 1 equal to n 2 equal to 2. So, it is a case of n 1 equal to n 2 equal to 2.

Now, you know you just have to decide what we should put here, right? The other two will go automatically there and we know that this can be done in four choose two ways, right? So, a b can be put here or c d can be put here or b c can be put here or a d can be put here. There are six ways to put it, right? So, 4 choose 2 s. Out of four things, you just have select two and the remaining will go there in the red, but you see suppose the boxes, both the boxes were blue, but red colour was not there, then you know there are now six possible ways are not there. Why? Because you know if I put a b here and c d here and another this thing I could have put c d here and a d here. Both are same because how will you distinguish between these two configurations. I am just drawing it again.

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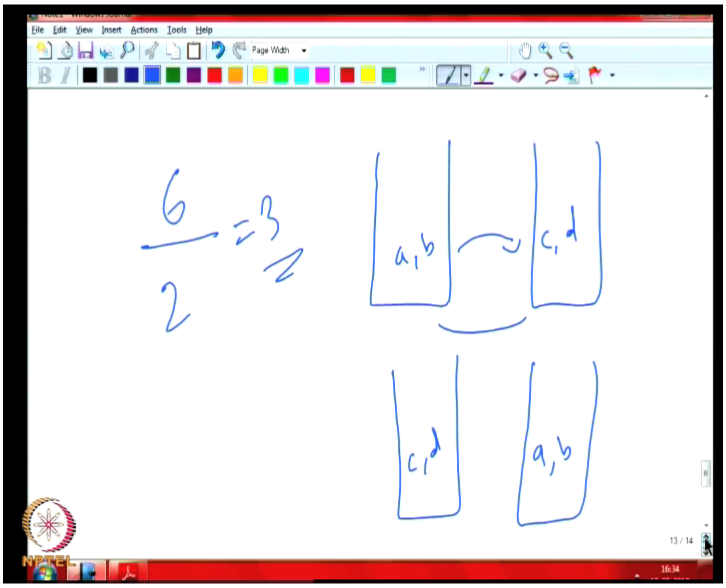

So, this is a b and this is c d and here it is a b, c d, right? Actually, a is just that put this box here. They are not distinguishable. So, we may even think that somebody if just shuffles the boxes, so what earlier we were counting this is two different things. So, we have to divide. Now, by that answer 6 by 2, we have only three possibilities. Now, any configuration, that configuration earlier was counted twice. Now, it is, sorry for n configuration here, it is counted twice. Therefore, we naturally will tend to ask the question. Suppose, I get red labels on the box. That means the boxes are made indistinguishable. They cannot be differentiated.

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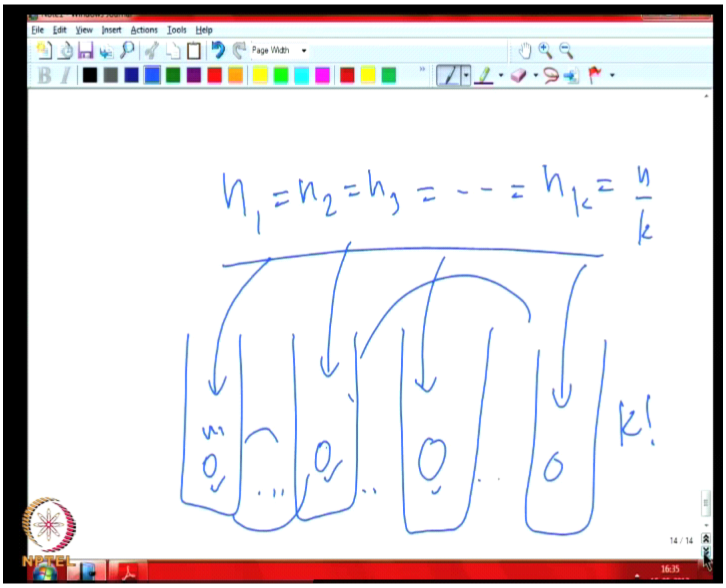

So, all the boxes are same colour or they look same. Then how many ways are there? So, one special case only we consider the case when n 1 equal to n 2 equal to n 3 equal to n k. All k boxes have to get equal number, right? This n by k, right? Here the boxes we are making same k boxes and the numbers which should go into them, they are all equal. Otherwise, again there we will have some difference. Now, how many ways you can do? It is clear that any order if you give because this we have to put something. Here some objects, here some objects, some objects here, some numbers here, some n 1 objects are gone here and n 2 objects are gone here.

Now, permutation of this thing. So, this factorial ways I can rearrange them. This comes here, this comes here. So, let us rearrange the bins themselves that we can rearrange the boxes themselves in k factorial ways, but each of this rearrangement was countered once correspond with the same objects in each box and with one tweeze of line of the boxes. It was countered as a new (( )) the previous counting now, but then for this, we know that this reshuffling of the boxes does not matter. Shuffling of the boxes does not matter; k factorial shuffling will not affect it. Only when we change the internal contents or the boxes, then only we have to worry, right?

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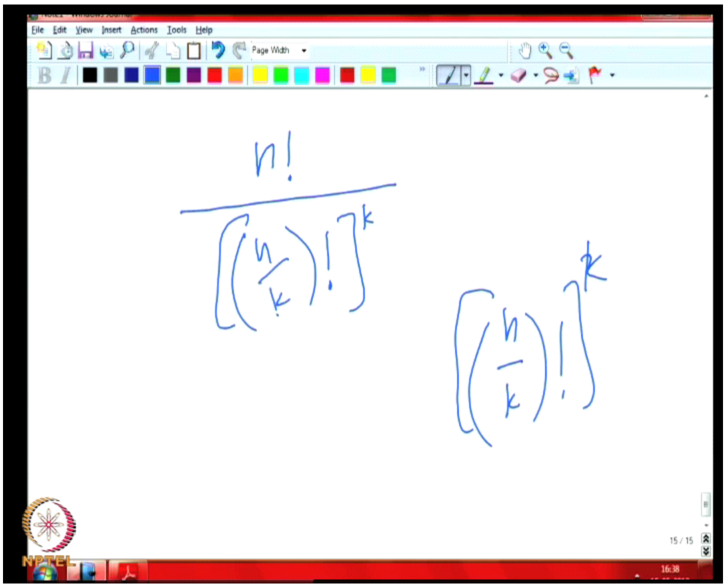

Therefore, the total number again we can use division principle here to infer that the total number of ways you can place n objects, n balls and distinguishable in k unlabeled boxes. Unlabeled means, they cannot recognize the box; one box from another such that each box should get the same amount is this n 1 factorial. So, it is maybe you can set n by k factorial. N by k is a individual. Now, n by k factorial, all raise to k, right? So, the reason is that this is again just division principle.

We have seen that when the boxes are assumed to be undistinguishable like each arrangement here, valid arrangement here would give rise to k factorial different arrangements in the earlier situation. That means, earlier situation was when the boxes were distinguishable. You can just assume that the boxes are placed in a row and the now the objects are placed inside the boxes. Now, without changing the content of the boxes, we just replace the boxes. The order of the boxes had changed, right? That is the same thing for in the second case where the boxes are unlabelled, but different thing in a case when boxes are labeled.

Therefore, division principle says that on one side, we have this unlabeled box case and the other, we have the label box case. From the label box case to the unlabeled box case, we can define a function. The function is defined like if this arrangement correspond to this arrangement, then we map it with that and it is definitely add to one function, where d is equal to n by k factorial. N by k being the number of things which we have given to that to the square, right, to the k, right? So, the number of partitions will be equal to that. So, this is one different case we can remember.

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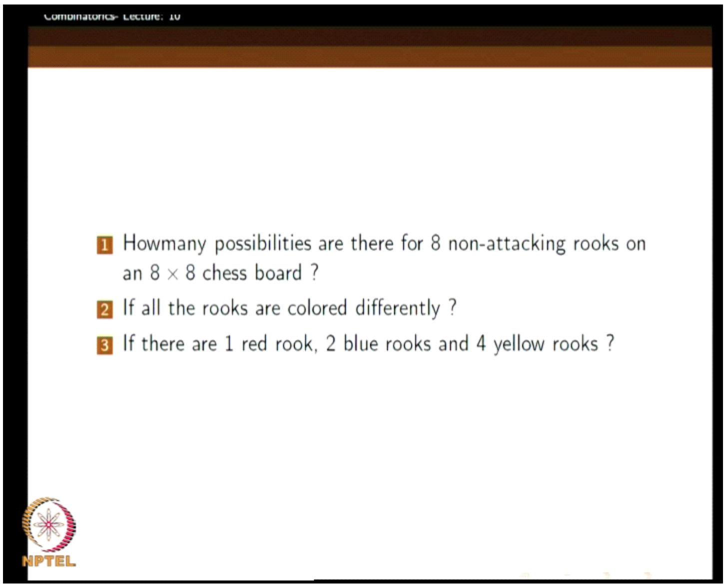

Now, this next question we are considering at this. So, these questions were about the chess board. It is about the rooks which we use. It is a piece in a chess game, right? So, interesting question this is.

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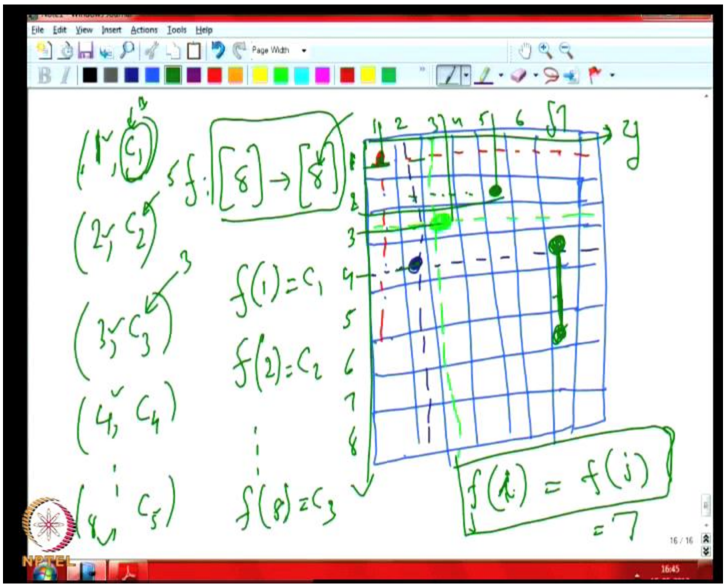

So, we consider an 8 by 8 chess board. 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 7, 8 and a 6, 7, 8. So, the rook means, suppose if I place a rook, then this can attack anything on this row, anything on this column, also a naught. Now, no one attacking configuration means, you cannot place another rook here, right? This is not allowed. One rook is placed here, another rook is not. So, you can place another rook, say may be here, but once you place this k here, this rook here, so it will attack anything which is placed here or here, right?

Now, in any other rook if you want to place, may be you can place it say here. Yeah, because neither this attacks, this attacks, but then once you place this, this will attack this row and this row, right? Now, if you want to place another one, how many rooks you can place like this? You cannot place more than 8 because you know there are only 8 rows and if you place 9 by pigeon hole principal on the same row, 2 rooks have come and then they attacking each other. You can place 8. It is not difficult to show the configuration where you can place 8, but what this problem is. The question is how many ways you can place 8 rooks on the chess board in non-attacking configuration in such a way that they do not attack each other.

So, this question first of all we notice that in each row, we can only place one rook. So, this is one first row, this is first row, this is second row, this is third row, this is fourth row, this is fifth row, this is sixth, seventh, eighth. In each row, we can place one rook. So, if you want to describe the positions or the rooks, we can say in the first row, this column, which is the column. So, I can just say j one-th column, right or maybe I can say column number c 1 and similarly in the second row. Now, in the first row, you can only place one rook. So, we just have to specify the column number. Then it was one itself, right something like that and the second row, you can specify the column number c 2. Maybe here, it can be here.

So, that is 1, 2, 3, 4, 5. So, this can be 5 in the third row that is c3, column number c3. Here we have placed it as third column, right? This is 3. Like that in the fourth row, we have c4, column number 4 and so on or eighth row, we have c5. You see that we could have, you can easily notice that it is the function that has been written here because see function from 8 to 8, right this is some function from 8 to 8. Why it is a function? Because for each of, if this eight numbers, 1, 2, 3, 4 up to 8, we have a value assign to it, right? So, f of 1 is equal to c1, f of 2 is equal to c2 and f of 8 is equal to c3. So, for function, another thing we want is that everything has one value and also, everything has only one value. So, that is also true here, right? We will never know that in a given to i-th, though we cannot have more than one rook.

So, it is only the function that rook, the positions of the rooks is specifying a function. See any configuration which is valid will define a function. What kind of a function is this? The function, see for instance, if this function is essential as if it is plotted here, right? So, looking from one, so this is plotted here. So, the function value is plotted here. This is like height 1, 2, 3, 4, 5, 6. This is the y axis, this is x axis, something like that, so 4, this 2, 3, 3 and 1, 2, 5 like that. The interesting thing is that given i and given h j f of i and f of j, can they can be equal? No, because in that case, see suppose f of i equal f of j is equal to say 7. That means in the seventh column, you will see two rooks placed, maybe something like this which is not allowed because these rooks will be attacking each other.

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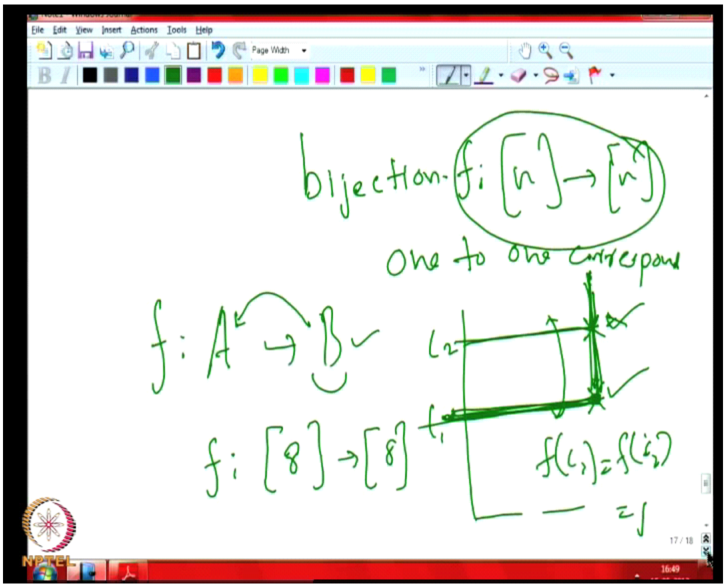

So, this is also not true. Then what kind of function is this? It is a function from 8, this 8 to 8 and then 1, 2, 3, 8 to 1, 2, 3 up to 8 and no number is getting mapped to two different numbers. Sorry, no two different numbers are getting mapped to the same number. That means, it is a bijection. It is a 1, 2, 1 correspondence. It is bijection or sorry, it is a 1, 2, 1 corresponds 1, 2, 1. That means, it is a function from a set to set for anything for any function from a 2. Bright will say any number in B has exactly one pre-image. It does have a pre-image. It has exactly one pre image and it does not have more than one pre-image. That is what is happening here. Just both situation we see that if you take any i, so it has an image. That is why, it is a function and if you take any column that is only one rook sitting there. Therefore, that will define it is pre-image uniquely, right?

So, that is why, it is a bijection. Actually, any bijection from 8 to 8 would give a valid just position. Why? Because you know it is a bijection. That means i-th row if you look f of i is uniquely defined. Only one value is there sitting there and any column if you consider j-th column, you cannot have a 2 rook sitting in the column because if they were two, then that means that two different i 1 and i 2 are such that f of i 1 is equal to f of i 2 is equal to j, right? Two different rows are getting the rook on the same column number, right? So, that is not allowed, so the same rooks and the column number. That means, this bijection says that it is not allowed, because you know the same thing will not have two different pre-images bijection.

So, the bijection from 8 to 8 corresponds to the number of, sorry a rook arrangement of eight rooks on the 8 by 8 chess board, such that they do not attack each other their wise non-attacking position and reversely conversely. That means, if you have a non-attacking configuration of eight rooks on the chess board, then that would correspond to a bijection from 8 to 8. So, we have this 1, 2, 1. I mean, this is a bijective proof that this one and the same, right? So, 8 to 8 we can of course generalize it to an n by n chess board because the same argument says that if you have a n by n chess board and then n rooks have to be placed in a non-attacking configuration, any such configuration will correspond a bijection from n to n, right and reversely, any bijection would give us a configuration of n rooks on the chess board such that they do not attack each other, right?

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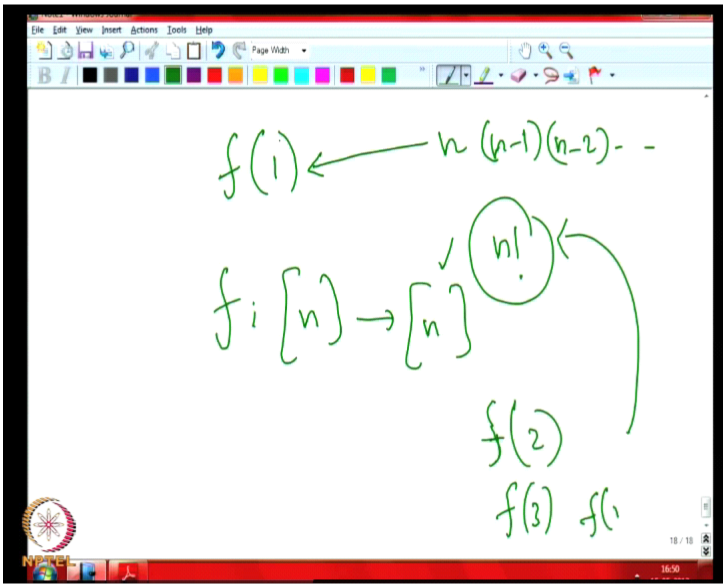

Therefore, we are actually counting the number of bijections and the number of bijections is easy to count. What is that? For instance f of 1. How many ways you can fix a value for f of 1? There are we are telling about f number of objection from n to n, right? So, there are n possible ways to fix it because the n values here, but the only thing is, once you fix a value for f of 1, f of 2 can be assigned a value only from the remaining n minus 1 thing. So, n into n minus 1 possibility for fixing values to f of 1 and f of 2. Now, f of 3 can take values from the remaining n minus 2 things. So, there are n minus 2 possibilities and so on. That means, by the time we will fix value for f of 1 also.

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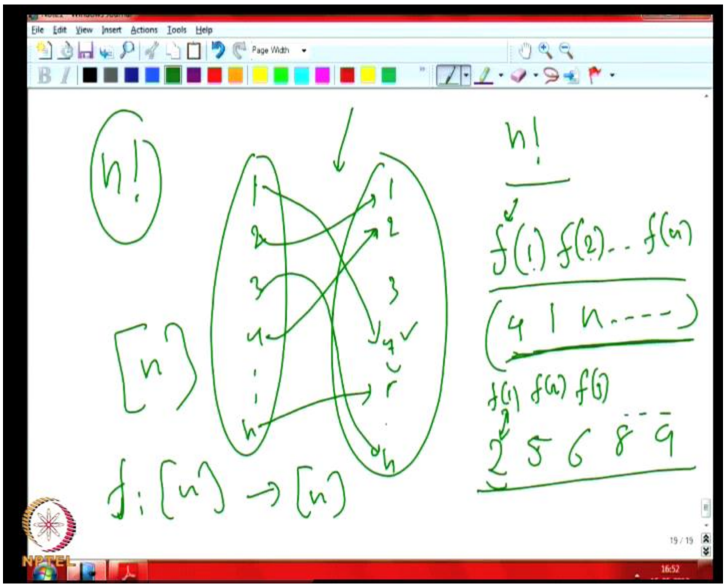

We get this number n factorial, right? There is n factorial possible. So, this tells us that probably some relation with permutations also. Yes, it is because you can see the bijection. You can draw like this, 1, 2, 3, 4, 5. So, this one is given this value. Two is given this value, three is given this value and for something like this, some figure. I mean, I can draw this, sorry not like this. It is a bijection. So, yeah such that each object here gets say unique object here. So, this is the image of this. This is the pre-image of this thing. If you look from here, there is only one pre-image for each of them and everything gets a pre-image that is the picture of a bijection will look like this.

As you mention, this corresponds some rook configurations, right non-attacking rook configurations and this is actually n factorial. This tells us that maybe this has something to do with permutations. Also, it has something to do with for instance, if you write f of 1, this is like this f of 1, f of 2 up to f of n in a sequence. How will it look like? So, f of 1 is 4, right? F of 2 is 1, f of 3 is n and like that, nothing will repeat because these are bijections, right? For instance, once f of i writes some value here, no other f of i dash can write the same value, right? Therefore, this will be a permutation, the way it is written, right?

Each bijection will give us a different permutation and reversely if I have a permutation like, for instance this is a permutation 2, 5, 6, 8, 9. I can say that 2 is f of 1, 5 is f of 2 and this is f of 3. Clearly I can define a bijection using this, looking at this permutation, right? So, this is a 1, 2, 1 corresponds between the permutations and bijections. Also, essentially it is a permutation. The bijections from n to n is a number of a corresponds as a permutations of n. So, therefore, there is no surprise that it is n factorial. So, we see that this n factorial corresponds to the number of permutations of the set n and also, it is a number of bijections from the set n to n and then it is the number of non-attacking rook configurations n, rooks on a n by n chess board, right?

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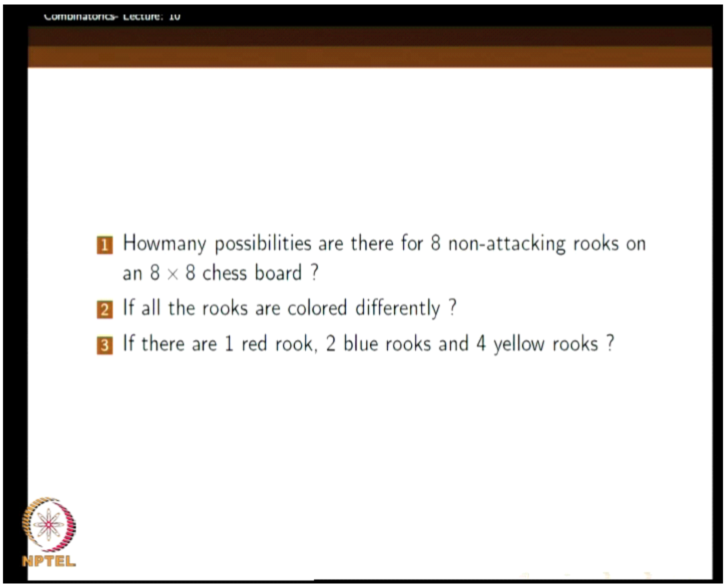

Now, we will ask a new question. I mean for instance, we just setup this rook problem to ask, I mean the following question. So, this is a question. We want to ask, suppose if all the rooks are colored differently, then how many configurations are there? What do we mean by these things? For instance, looking back at a problem, I just draw a small chess board, right? It is not necessarily 8 by 8. It is just say n by n chess board, right? Now, this n by n. Now, I am saying that I am placing a rook say one here, say one here, one here. This is known of attacking configuration, right? So, we can put a little more on here. Yeah, maybe this is not necessary, right?

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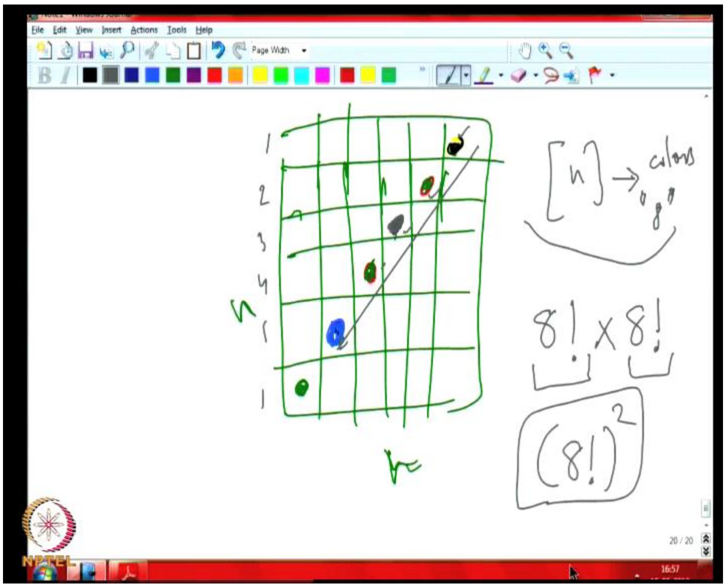

So, this is a non-attacking configuration of the server, but now we see suppose, this rook was green, this was red and this was blue and this was yellow and this was brown and this was black. These are all we can identify which rooks and then if I have placed this rook here and see, for instance if I had placed this blue rook here, so blue rook here and red rook here. It would be different or if I had brought this yellow guy and placed it here, right and put that black, sorry black rook here. So, this put that black rook here. That is different, sorry black rook here. Yeah, that will be different, right?

So, how many ways you can do? We have basically eight factorial ways to place the rooks. Now, for one placement of the rooks, we can play with these colors. I mean, this is like for this position, this is the first position, say in first position, second position, third position, forth position because in every row, we have only one position. This is the first, this is second, this is the third, this is the forth. Whatever you see on the first position, first row is the first rook, second row what you see is the second rook and so on. Now, you can assign the colors to that from you can say from one ton, right? We have a mapping from the colour, set colors. There are how many colors? There are eight colors.

Suppose, now all the rooks are different colors, so which colour should go to that position. That is what we are asking. First row rook which colour should get? So, this is essentially again the bijections. All possible bijections will give you that. So, there are eight factorial ways to do this thing, eight factorial ways to position the rooks. Now, for one positioning, we can assign the colors. Assign the colors means, we can put, we can decide to put which color here and which colour here. That will be another eight factorial.

So, that is either we can say that we are permuting the colors. Here, all the eight colors. So, we read from here to here. It is an ordering of the colors, red. Eight colors or we can say that it is a bijection from 1 to n, the set 1 to n to the set of colors, right? So, f of 1 is a particular colour, f of 2 is a, f of i is a particular colour, f of 5 means that colour of the rook which appears on the i-th row, it is only one rook which appears on the i-th and that can be any colour, right? So, there are again eight factorial ways. So, it is essentially eight factorial square is the number of possible arrangements.

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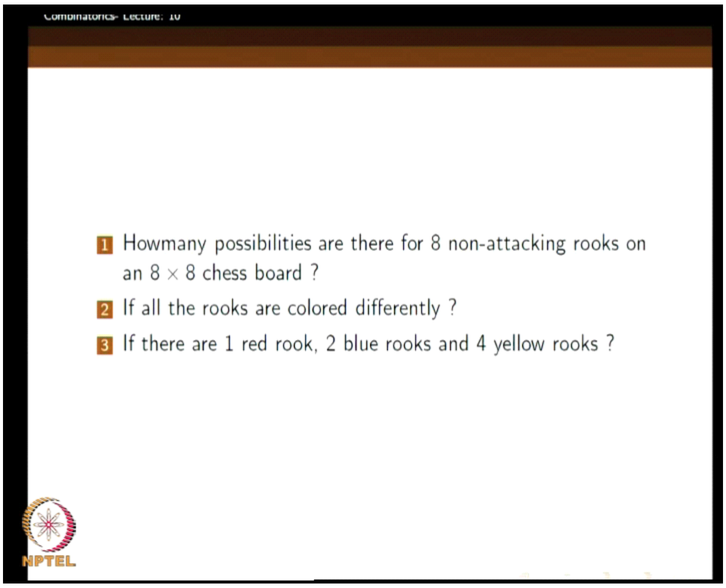

Now, the next question is exactly I mean again suffocate the question. Suppose, now we say that I mean, initially we considered a situation where all the colors were same. Actually, we the rooks are indistinguishable. In the second question, we considered the case where all the rooks were of different colors. That means, any two rooks can be distinguished from each other. Now, we say there are one red rook, two blue rooks and four yellow rooks. That means, some of them are distinguishable, means two blue rooks, between them we cannot differentiate that it is just two blue rooks. How will you know which one is which, but between a red rook and a blue rook, definitely you can distinguish.

How many ways you can do this thing? Again, basically the arrangement of the rooks and first arrangement of the rooks, this row which column, I mean the first row which column should I select for the rook to be placed? The second row, which column should be placed that I can again do in eight factorial ways? As we discussed earlier, colors does not make any difference there, but once we decide a position, the colors make a difference. It is essentially the colors, this is a multi set.

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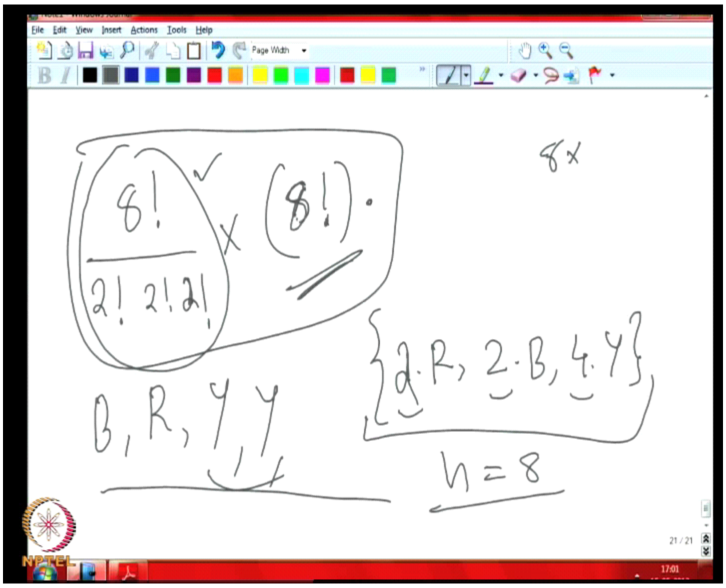

Now, earlier it was all different colors. One red, one blue, one green like that. Now, it is one red, two blues and four yellows, right? This is what we have told, right? On the problem, the two blues and four yellows and total number n is actually 8. 4 plus 8, sorry actually yeah.

So, I took actually I mean, say let us say two red rooks. It should be two red rooks. Let us say two red rooks, two blues, so that n becomes 8. So, we consider 8 by 8 chess board that is 8 by 8. I wrote eight factorials, right? Now, how many ways we can assign the colors in the sense that deciding which colored row, which rook of which colour should go to the first row, which rook of which colour should go to the second row, rook of which colour should go to the third row? This is essentially a permutation of the colors because row 1, row 2 is you can assign. The first row got blue colour, the second row got the red colour, red rook, the third got the yellow colour, rook forth row again got the yellow colour rook.

So, therefore, this is a permutation of the colors just that some colors are repeating. So, it is a permutation of the multi set, this multi set, right and we know how many ways we can do that because 8, that is eight factorial because n equal to 8 and n 1 equal to 2, n 2 is equal to 2 and n 3 equal to 4, that is two factorial into two factorial into two factorial. e have to multiply this with this, right because in each, basically we have eight factorial placements.

When we forget about the colors, we have eight factorial permutations, for each eight factorial non-attacking configurations with colors. We can make many changes. I mean with placing the rooks on the same position, but we can change the colors. I mean colors means we have to decide which colour should go to which row. In this many ways, eight factorials by two factorials, this is what the final thing. We told the permutation of the multi sets, right?

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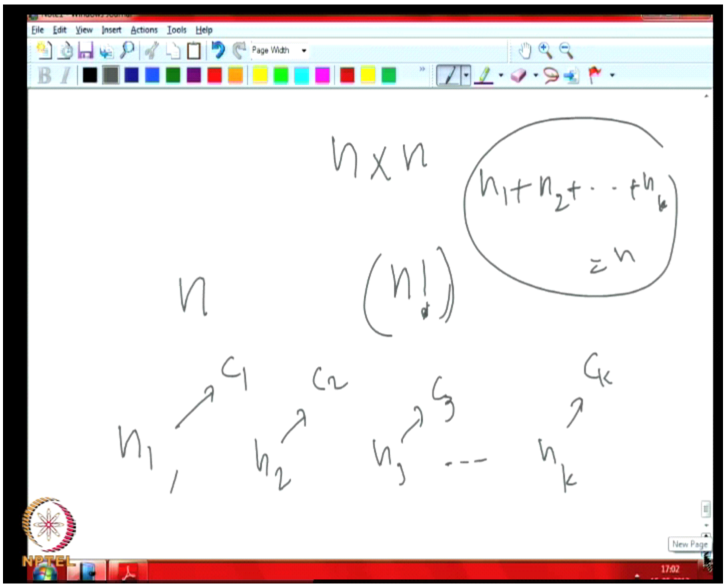

Naturally, we can very easily generalize it to the n case. Suppose, there is n by n chess board and there are n rooks placed in a non-attacking configuration. Of course, we can do it in n factorial ways as we have already argued out.

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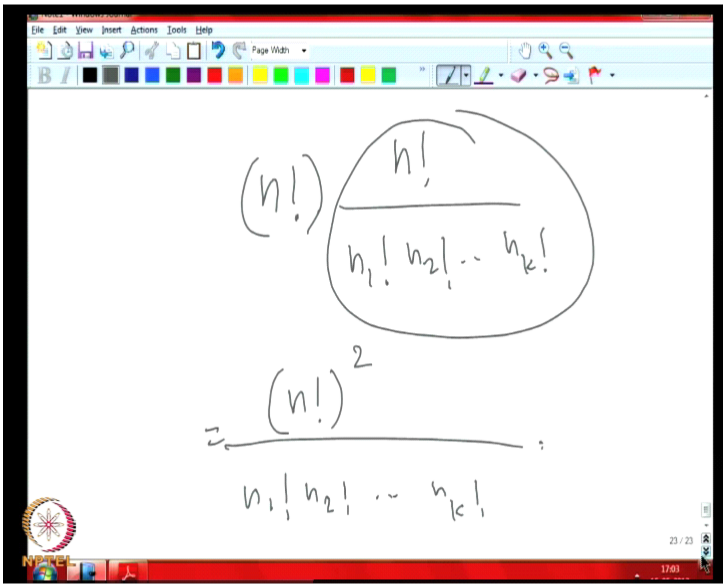

So, now there are n 1 rooks of colour c 1, right and n 2 rooks of colour 2, n 3 rooks of colour 3 and n k rooks of colour k, right? So, such that if you add up these things, these numbers, these are repetition numbers, right? This is n k will give you n. There are n rooks, right? They should add up to n. Now, how many ways you can do this thing? Of course basic arrangement is n factorial and for given one arrangement, the colors can be assigned or means, you can decide which rook of which colour should go to which row.

In n factorial by n 1 factorial into n2 factorial into c n k factorial because this is the number of permutations of that multi set, right and this is equal to n factorial square divided by n 1 factorial into n 2 factorial into n k factorial. So, this is the number of ways to place n rooks on n by n chess board in a non-attacking configuration. When n 1 rook is of certain colour, n 2 rooks of a certain different colour and n k rooks are of a k-th type of colour, so we will continue in the next class. Thank you.