Parameterized Algorithms

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Lecture - 43
Matroids: Representative Sets

Welcome to the last week of lectures on algorithmic sites of this course. In this week we will talk we would like to talk about weight rates but we will be talking about representative sets. It is a technique which was developed, it has been around for quite a while but it was only developed in a proper way in downtown 212 and it saw and it gave lots of interesting answer from 2012 to 2016 4-5 years it is also now also quite useful technique to have. So, we will talk about that.

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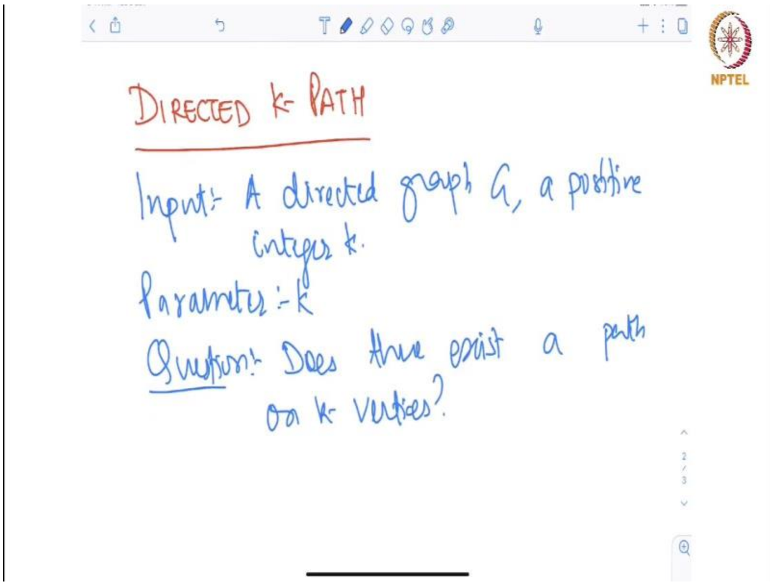

So, the problem we will be interested in is in directed k path and that is where we will did build our theory. So, we will be talking about directed k path. So, as you all know input is going to consist of a directed graph G, a positive integer k parameter is definitely k and the question we will try to address does there exist a k path or a path on k vertices rather path k vertices?

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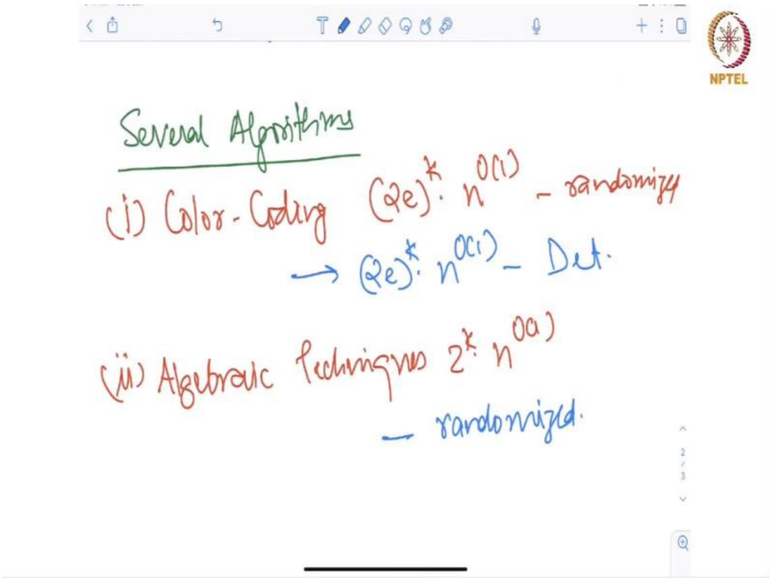

So, for this problem we have seen several algorithms so let us go back to what we have learnt, several algorithms we have seen. So, the first algorithm we would have seen is via colour coding that algorithm ran in time 2e to the power k n to the power 1 randomized, but you could also design a deterministic algorithm for this using hash functions. So, in fact let us say 2e to the power k n to the power bigger 1 deterministic.

And then last week of our lecture we saw an algorithm based on algebraic techniques with running time if you recall correctly was 2 to the power k n to the power 1 and this algorithm was randomized.

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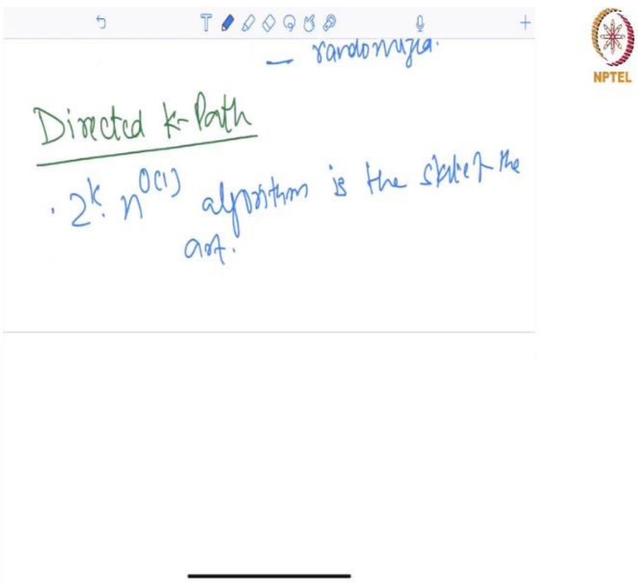

So, furthermore this is important to point out that for directed k path 2 power k n to the power big of 1 algorithm is the state of art.

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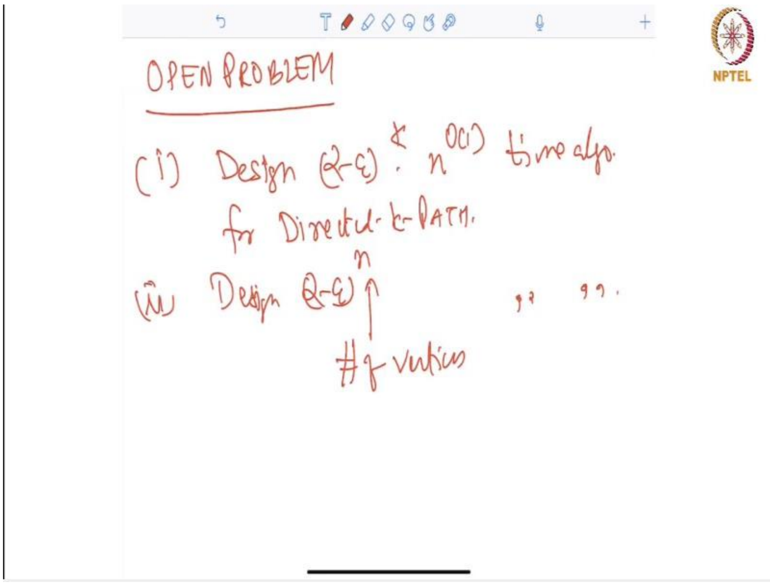

So, the open problem in the area is one of the good open problem is in the area is first design 2 - epsilon to the power k n to the power big O of 1 time algorithm for directed k path. In fact, what is not also known is to design 2 - epsilon to the power n number of vertices time algorithm for directed Hamilton in fact so this will be like director k path director. So, given that these are big open problem in the area.

I just wanted to mention but today we will not be talking about an algorithm which is anytime better than 2 power k algorithm but rather a different algorithm.

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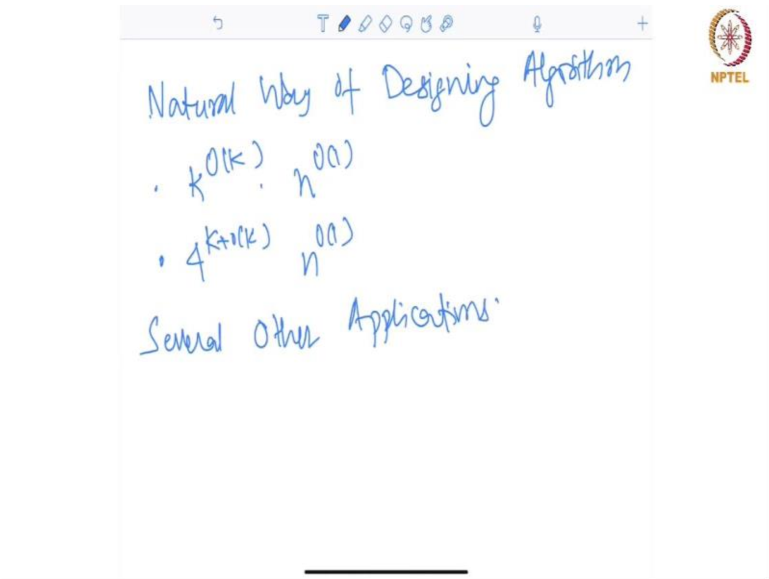

We are going to talk about a bit more natural algorithm let us say a bit more in my view a natural way of designing algorithm. The running time of our algorithm will be k to the power big o of k actually, but we will point at how this can be at least made to work in time at least say 4 to the power k + little o of k n to the power of big o of 1. I will point this out to you but besides those we will see several other applications.

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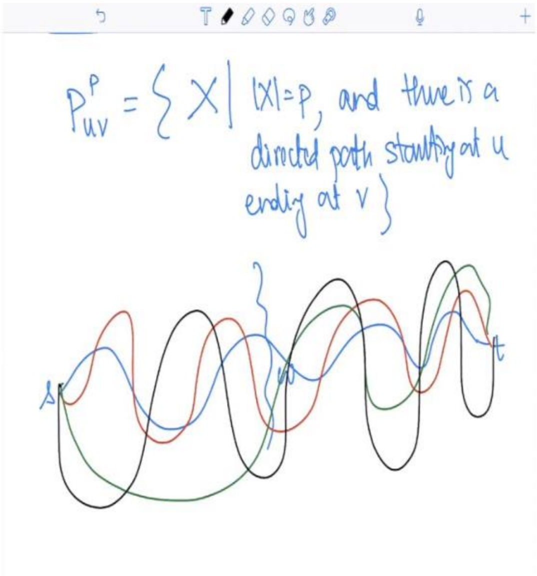

So, straight away jump to the technique, so let us try to under so for that we are trying suppose you are given a graph G and you are given a 2-vertex u and v and you are looking to for finding a path k. So, now I am going to define a family P uv and p. So, what is the P uv p this is basically let us say how to say this what is the right way of saying this let me tell you one second. So, this is basically consisting of set x cardinality of x = p.

And there is a directed path starting at I think let us say s t so what is this and starting at u ending at v. So, think of what is going to consist of so now suppose I have a u and or rather suppose I have s and t and I do have a path of say length k between them let us take some w right they are possible that there are paths this is one path directed path. This is another directed path and this is another directed path there is another directed path.

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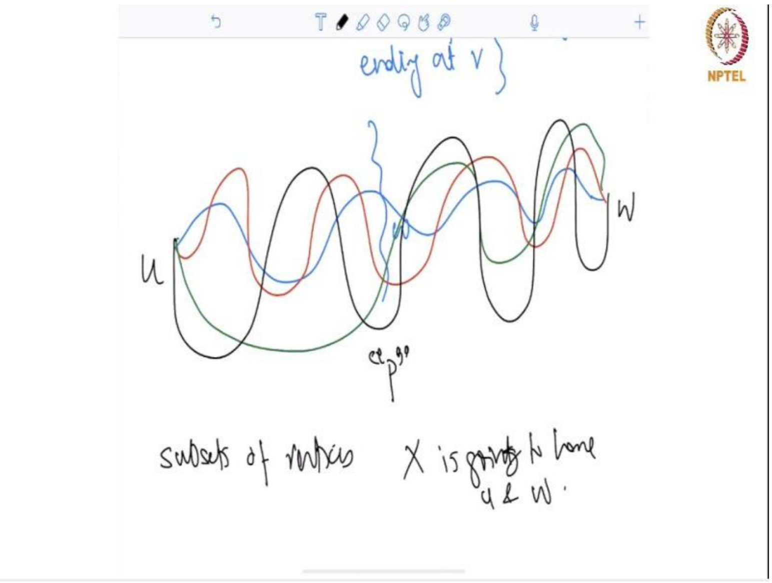

So, I fix an integer p and I ask how many paths or how many prefixes of paths or how many subsets of these vertices are there who could potentially reach to t. So, these are like a partial path these are p length partial paths. So, like they are not so but if you notice here rather than talking about path, I am talking about set because it is possible that to reach from s. Look at the first p length there could be a same subset but there could be different paths.

Because of the route we do not care about what route internal they but all I care about that if I start from s and i fix some vertex. How many waves can like what are the subset of vertices which could take me s to w right by a path. So, this is how a P p uv x is defined so what are these are basically if you notice these are subsets of vertices right and of course x is going to rather let us say u and w. So, subset of vertices x is going subset x is going to have definitely u and w first of all.

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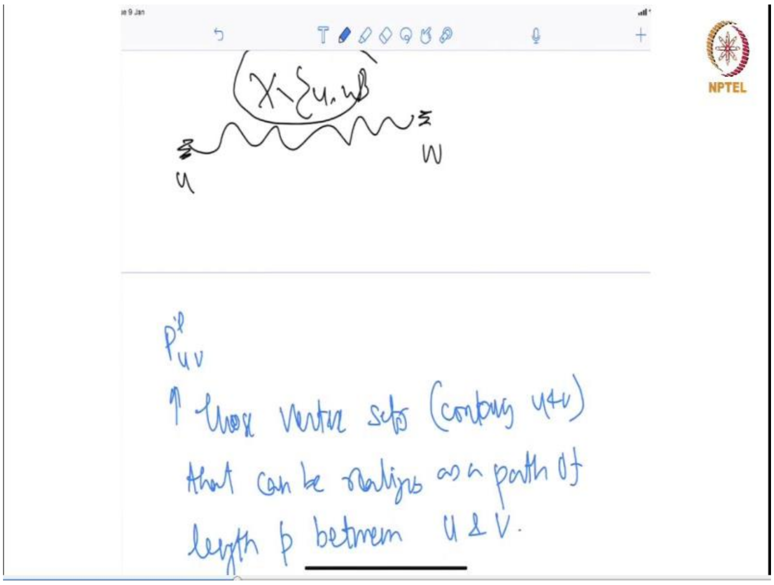

And so, these are those vertices so those vertices so you start from u end at w and look at so other vertices like x – u, w there is a path from u to w huge which uses all the vertices of x - u w and the I wrote x right not as a permutation because it is possible that the vertices of x - u w there could be several parts and then the vertices of this set could be occurring in a different sequence.

But we will but we are not caring about that we are only caring about sets. So, basically you can think of this we could think of as p length prefixes. So, that is the whole idea but we do not have to say this we could just say that these are nothing but what is this it is those vertex sets of course containing u and v that can be realized as a path of length p between u and v.

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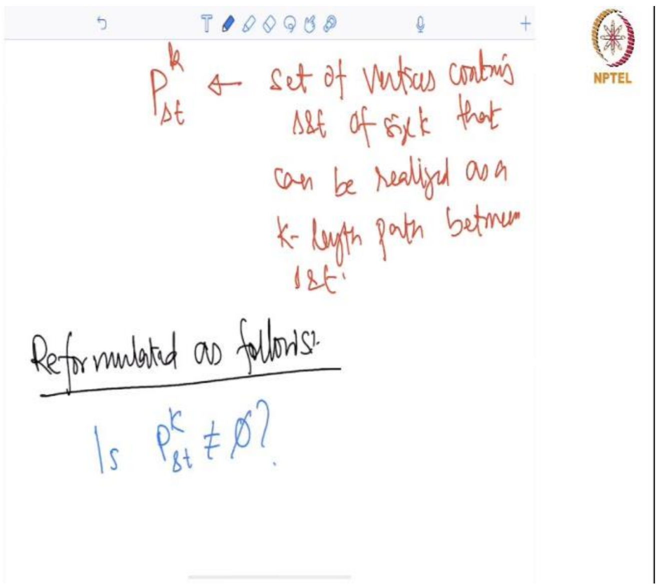

So, now let us look at P st and k what is this this is going to be set of vertices containing s and t of size k that can be realized as a k length path between s and t. So, what is our question so our question can be reformulated as follows; so, what is our question?

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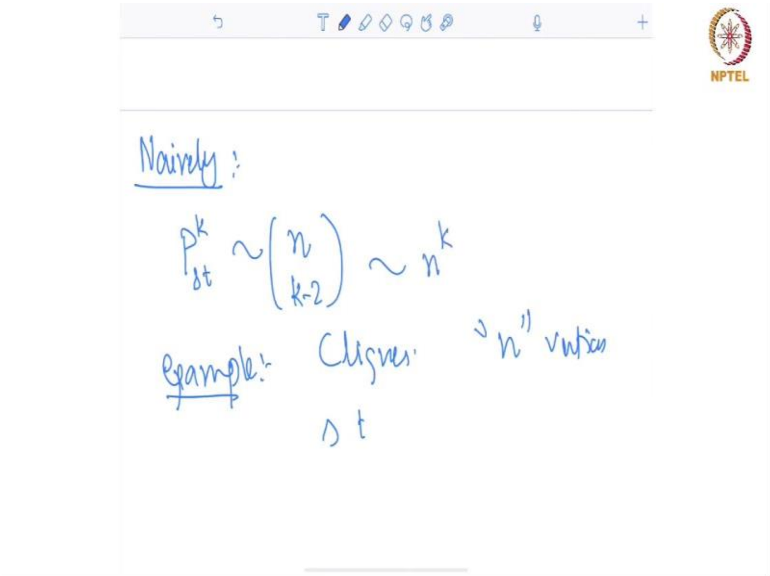

Our question is so, if it is non empty then we know that there is a path from s to t, fine but naively if you notice, naively what could be the size of this. Well, this could be as bad as in choose k - 2 example click. So, if you take a clicks on n vertices and you fix 2 vertex s and t then you look at any other k - 2 side subset you could. So, p k s t could roughly be let us say n to the power k.

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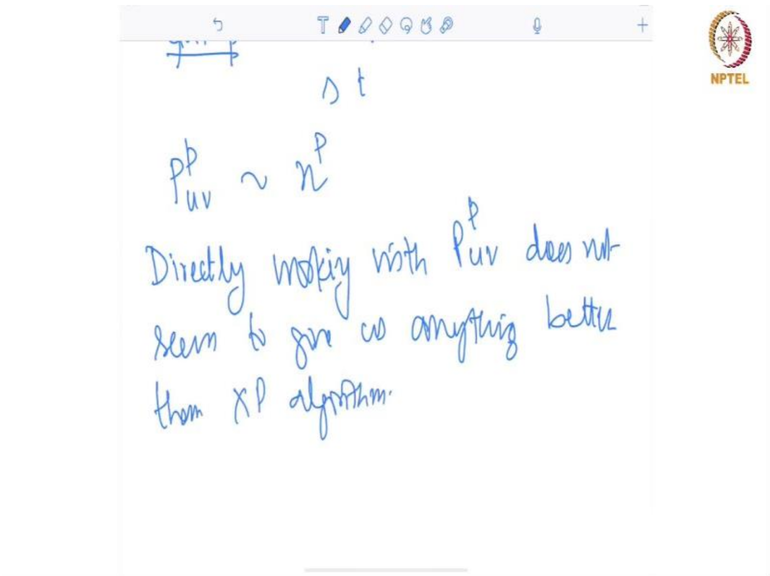

Or so, in particular u v could be as bad as n to the power p. So, directly computing this set direct computation, directly working with P p uv does not seem to give us anything better than XP algorithm or rather this is like a brute force or knife or whatever you want to call it.

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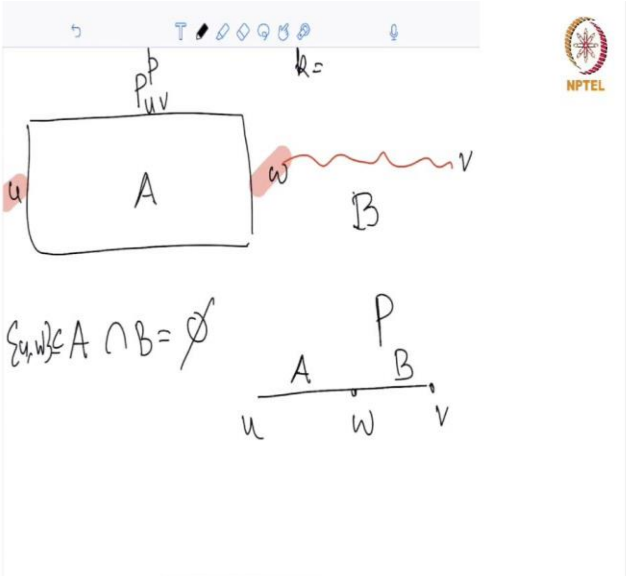

But so then, we ask ourselves question could be or rather before the freezer question. So, let us say let us look at this and here is w here is v and there are and suppose this is the set P uv and this is v. And there is of say path from u to v of length k right, but like all you see is this and this is some family which you do not see this is some family f or which is we know a why do I need to and suppose this is a this is there is a path of length but which is like;

So, this is fixed so every path in P p uv starts at u and w and everything and I know this, let us call this part of the path b and suppose there is a path A. Now I so what the property of this path is the first property is that notice that A of course which is u w is their subset intersection B = phi. So, for the path p that we have which starts at u goes via w and suppose this is P so the first part let us call it a and let us call it b.

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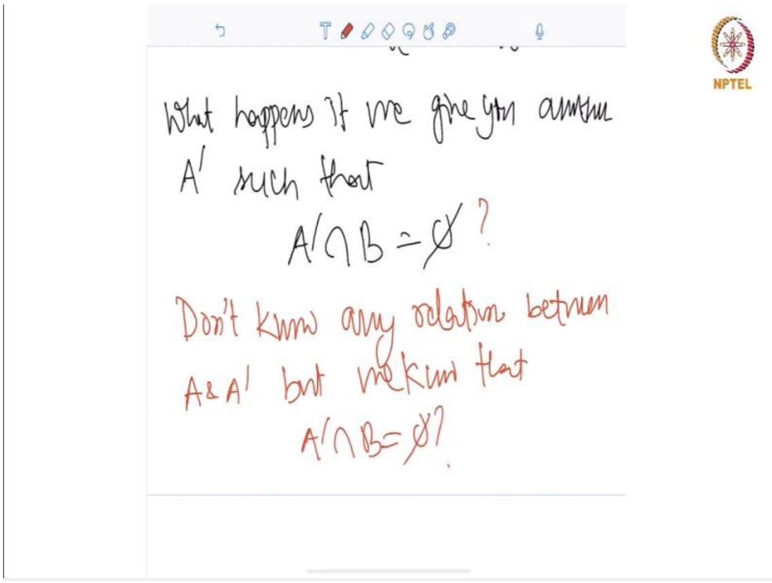

Now let us ask ourselves what happens if, we give you another A prime such that all I can say you that I do not know whether what I do such that all I can say to you that A prime intersection B is phi. All this is all that I can guarantee you, I do not know any relation between A and A prime but we know that A prime intersection B is fine. So, if I can give you so I do not know how to but suppose some magically I tell you that look I do have a set.

Some other set in I have said A prime but and I know that a prime intersection B is phi. So, what do we have?

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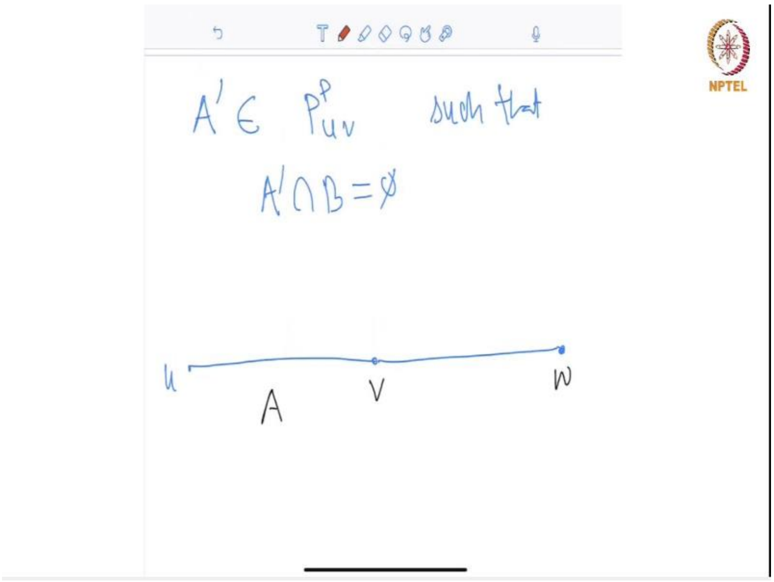

So, we have an A prime element of u v such that A prime intersection B is empty. So, at this point of time pause this video for a minute and ask yourself if, I did have such an A prime? Can I, is that can I do something with it? So, if you notice so I have this path u w u I think I have been this should have been v. So, sorry this should have been v my bad this would have been;

So, this is fine, this is fine, this is fine, this is fine and this should be this is w this we can draw it. So, you gave me such u v w, now I know this is A but you also I know that A prime is in P p uv.

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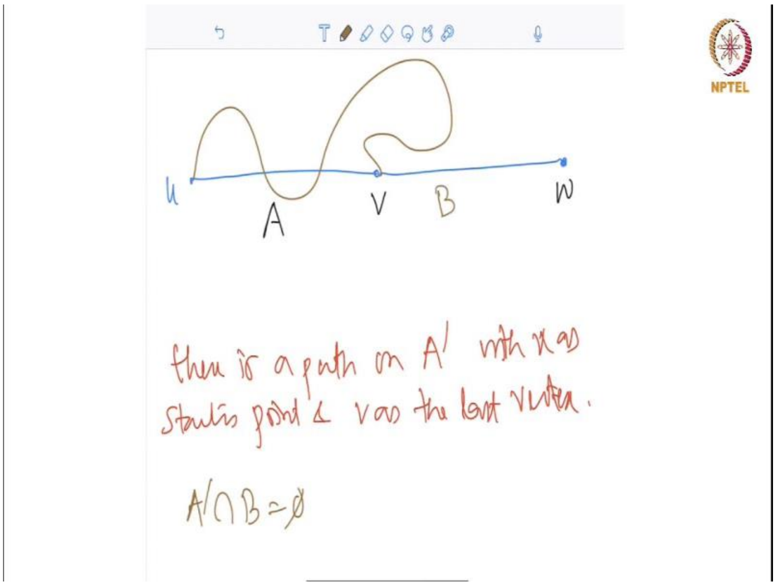

It implies that there is a path on a prime with v with u as starting point and v as the last vertex, great. Let that path P let that so all I know at this point of time that look at that path so there is a path on set a prime. So, there is a path with this and notice because that path A prime intersection B is phi and this is B right now let us call this brown path and blue path.

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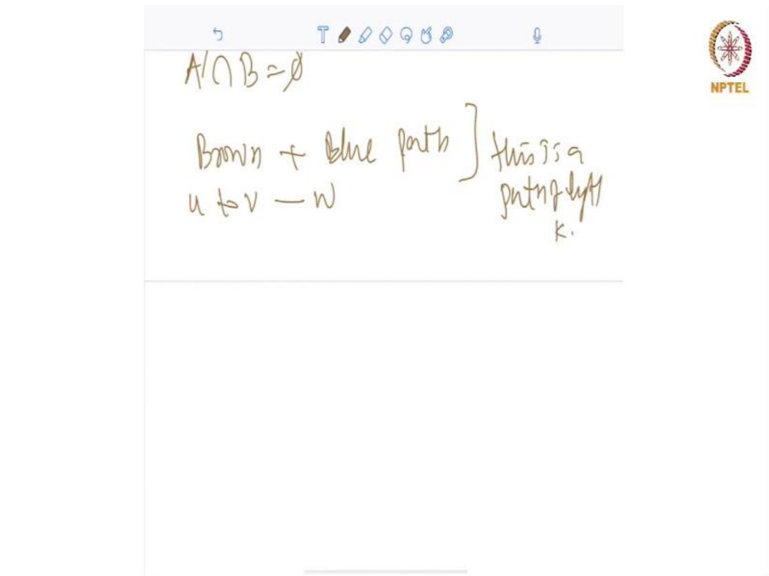

So, now if you contact me brown plus blue path, so brown path you go from u to v and blue path you go from v to w this is still a path of length k this is a path of length k. so, to have a path of length k at any point of time I do not need to like I mean this tells us that look if I have a set so look what we had we said look in P p uv, I am trying to keep all partial paths of length p.

I know that I would like to keep all those sets which could potentially be grown to or that or rather that way can be could be.

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So, the question be phrase question, question could be find a set or family. So, this is the setting which we are in so now just to phrase this question nicely let us make it slightly like let us put it in abstract setting.

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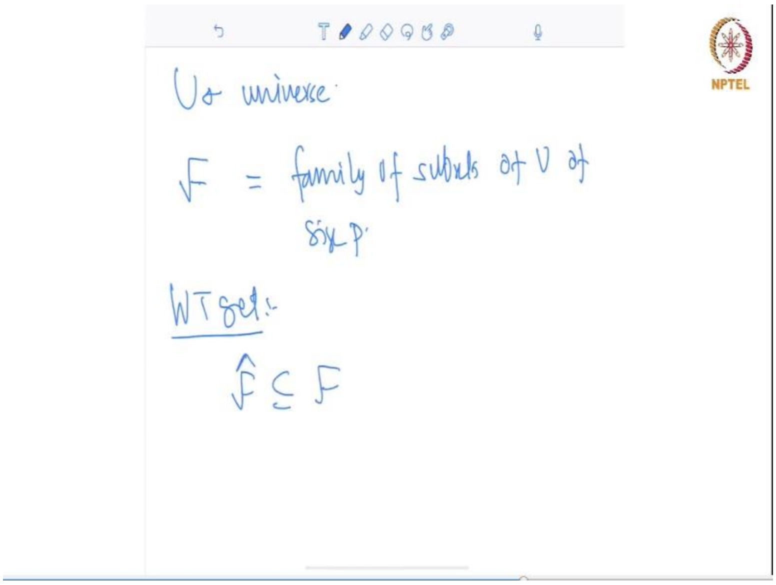

So, what is my abstract setting so I have universe and I have family of some say I have F p. Family of subsets of u of size p this is what I have, rather this is my f and what I would like to keep what want to get is a F hat which is a subset of F with a following property.

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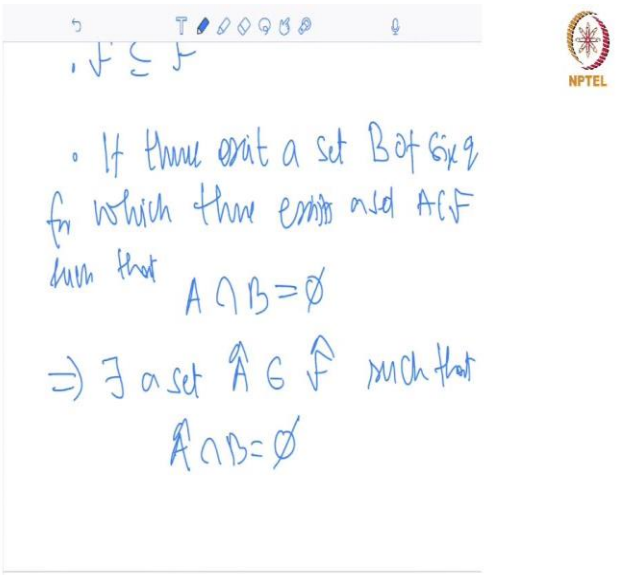

For every for or rather I want to have this property that if there exist a set B for which if there is a set B of say size q size q for which there exists a set a in our family such that A intersection B is empty. Implies that it just set a hat in f hat such that a hat intersection B is empty and so we are we will not talk about so. So, huge universe and q is an integer so let us try to understand what happens.

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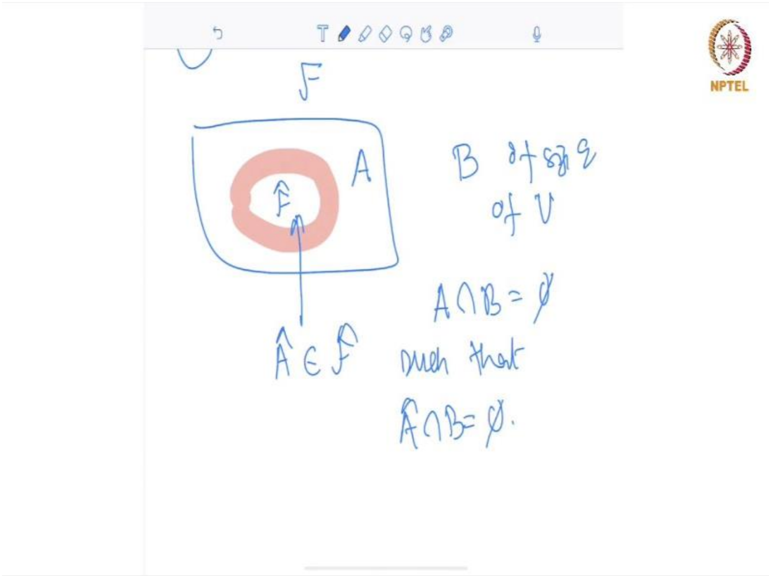

So, basically what happens that here your universe and suppose this is your family f so what you want to achieve is a smaller sub family here which preserves disjointness in what sense. So, this is like f hat so what I mean by this so suppose you give me a set B from some set B then I say I do have a set a such that A intersection B is phi. So, for example for every set B of size q of universe if there is a set A in your family.

Which is disjoint from this then you need to guarantee there exists an A hat in f hat such that A hat intersection B is fine. So, that is the kind of property we would like from f hat right and so if we if so, the reason why I put it into this is because this is the same kind of situation which our path problem was motivating because for the path u will be vertex set of g v g and what is my F is like P p uv.

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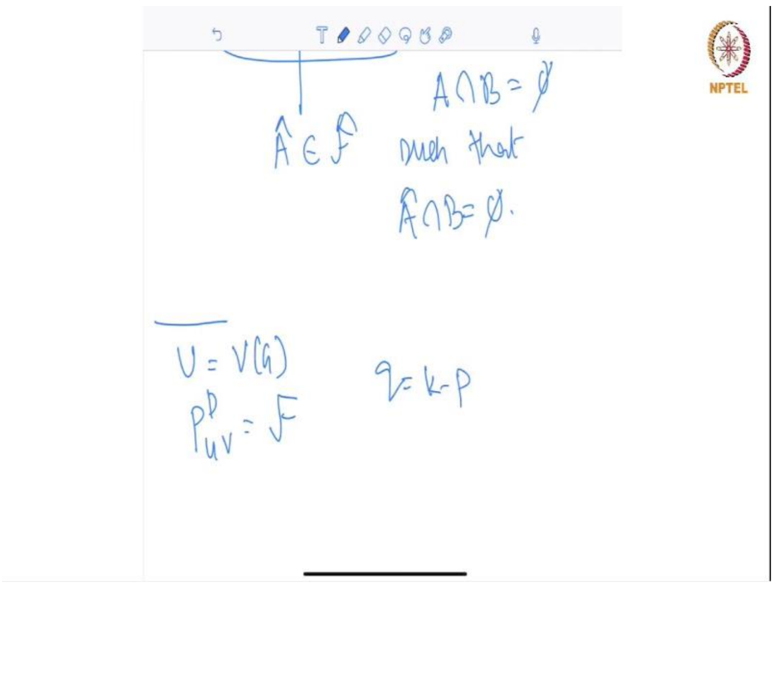

So, just two, so abstract to this thing so in path case you would have been equal to V G and P p uv would have been our family that is it right and q would have been k - p that is it. So, that is it.

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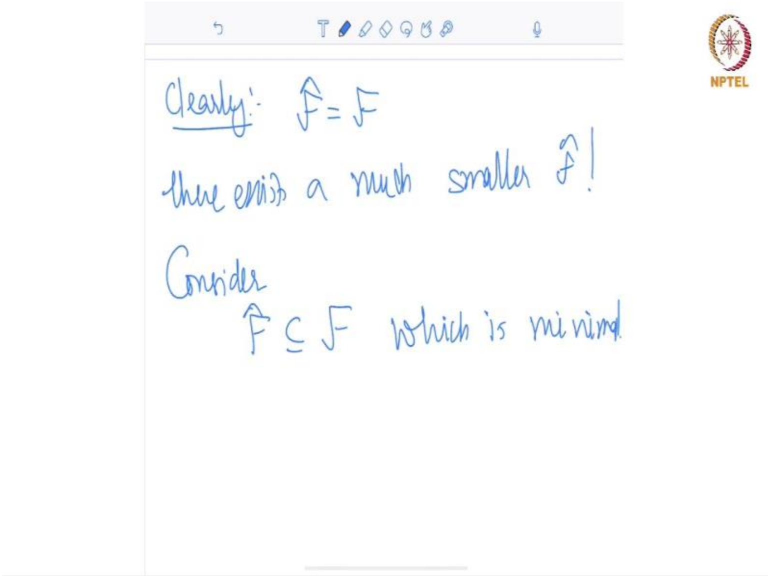

And so definitely clearly for F hat we can take F but we will show to you in a minute that there exists a much smaller effect. There exists much smaller effect and so there is this much smaller effect so just to do that let us say consider f at subset of f which is minimal.

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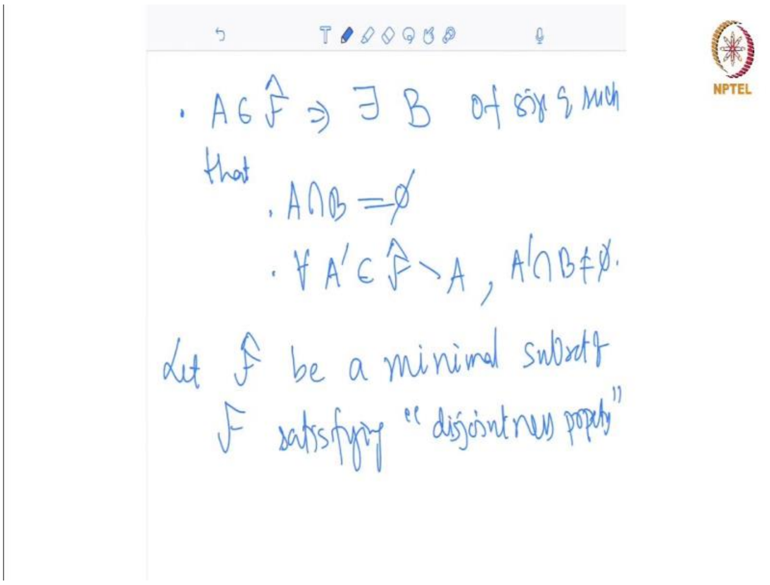

What is the meaning of this? Let us try to understand. So, what is it what it will be meaning of minimal? So, meaning of minimal should be right suppose A is an F hat implies there exists B there exists B of size q such that A intersection B is empty and for all A prime in F hat if had minus A A prime intersection b is not equal to right. So, in in other sense I have this A because there is a set of size q with which this set is disjoint.

And every other set that I have kept in my family they are not destroying me they intersect with it right because then I know that because for every set we if some other set would have been disjoint with like disjoint then this A is of no use because for every set B for right for which right. So, right so what is the meaning of this okay so the meaning is that look I am going to keep a in effect.

Because if there is a b of size q such that A intersection B equal to 5 then for every A prime in this set, A prime intersection B is not equal to 5 meaning the set of objects that I have kept has the following property is that for every set that I have kept there is some set which it is disjoint with, but it intersects with every other set. So, this is what I might mean by minimum.

So let us consider so suppose okay so let F hat be a minimal subset of F, let us call say satisfying disjointness property let us call it disjointness property minimal subset of this.

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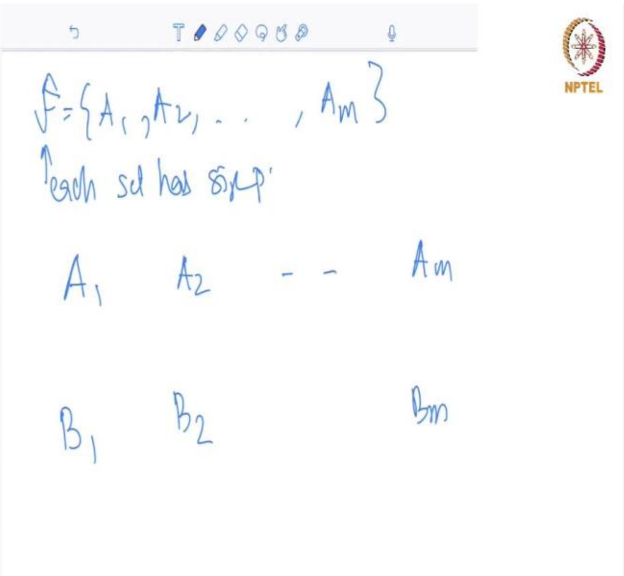

So, and let us say if hat is A 1 A 2 A let us say m. So, the m sets, now because they are disjoined and always recall that each set has size p but they are minimal. It means for A 1 I will be able to find B 1, for A 2 I will be able to find B 2 for A m I will be able to find B m and what will the property.

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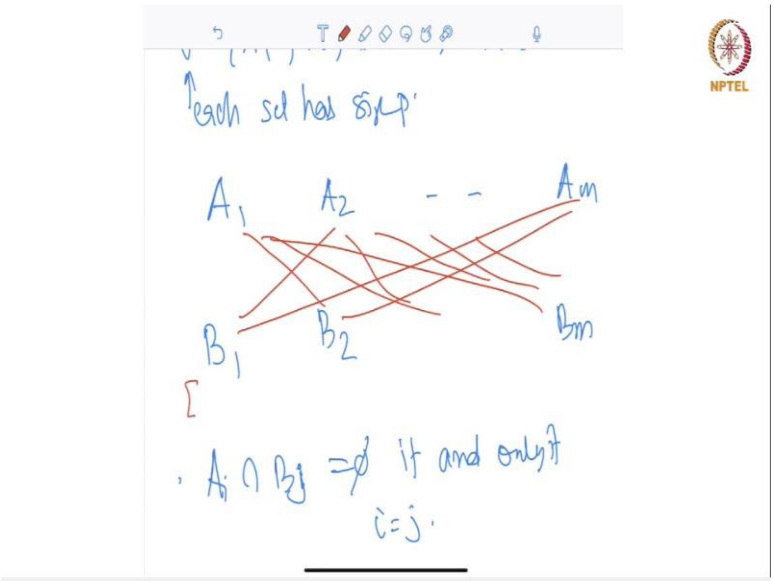

Ai intersection Bi is empty if and only if A i B j = m t if and only if i is = j. So, A 1 and B 1 do not intersect but A 1 intersects with everybody else so if let us draw an edge if they intersect. So, then A 1 look at A 2 there will not be an edge look at Am right except this right. So, this is how it is going to be now you ask ourselves right we took any minimal set right any minimal set has a property that for each A i.

We could find a set B i such that that bi intersects with every other A i every other A j except the A i, so we have the property that we have these pairs of sets A1 B 1. And what is size what is the and what is the property of each of these B i's that each of the B i's has size some set q.

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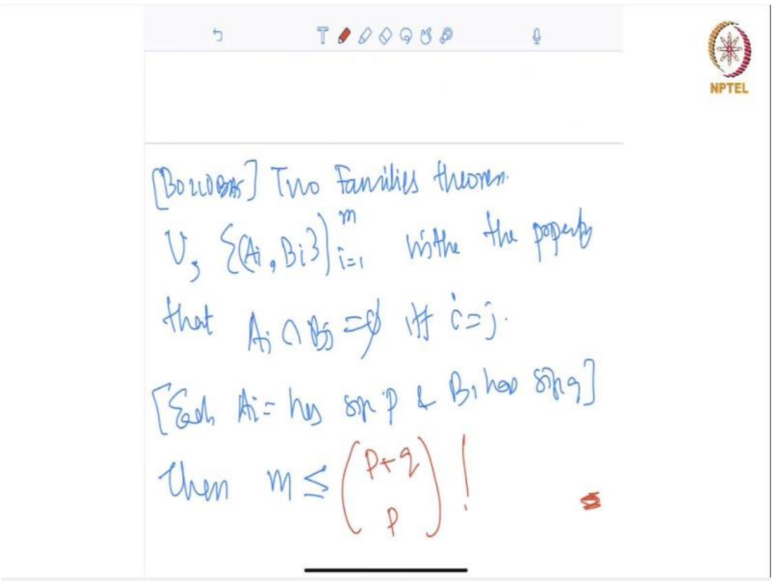

So, this is another so this is like a what is called two families theorem, this is exactly precisely what is called two families theorem. What is two families theorem? So, the two families theorem is so we have universe we have pairs A i B i = A i B i, I like some m I going from 1 to m and what is this all cardinality of A i is p all cardinality of B i is q. So, notice that we were trying to do this and we say look they must satisfy this property.

So, now let us is equal to q and A i intersection B j equal to phi. So, let us say this is not clearly written so let us this brings us to what is called classical Two Family's Theorem of Bollobas, it is also called Two Family's Theorem. So, what are given to us we have given to universe and pairs of sets Ai Bi, 1 to some m sets with the property that A i intersection B j = phi if and only if i = j of course each Ai has size p and Bi has size q.

So, this is what then m is less than equal to some number r b will get that so in fact what we will show that m is less than equal to p + q choose p. So in fact the number of such pairs that we could obtain is upper bounded by a function of p and q it does not depends on the size of the universe. Then, it would imply that F hat has size p + q choose p and so that is great right because so now let us try to prove this.

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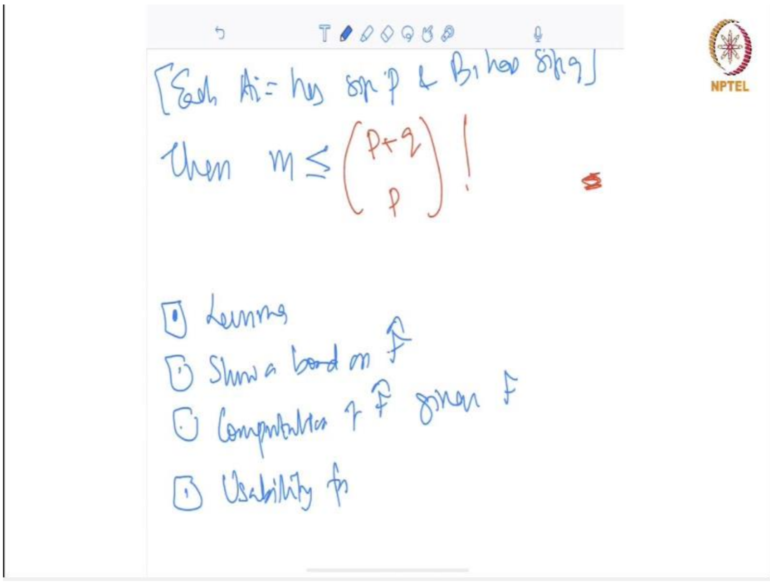

We so, now what we will do in the next lecture that now that we have set up all these things in the next lecture, we will first prove the Lemma. So, a bound on F hat and then we will consider a question about computation of F hat given F and then usability for parameterized algorithm. So, that is what we will do in the next lecture.