Parameterized Algorithms
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Lecture-42
Algebraic Techniques: Polynomial Method

Welcome to the last lecture of week 10, we up until now we have seen inclusion exclusion based algorithms we have also seen an algorithm space matrix multiplication, but today we are going to talk about polynomial methods. So, to do that, let me introduce some of the basic concepts that we will be encountering in this lecture.

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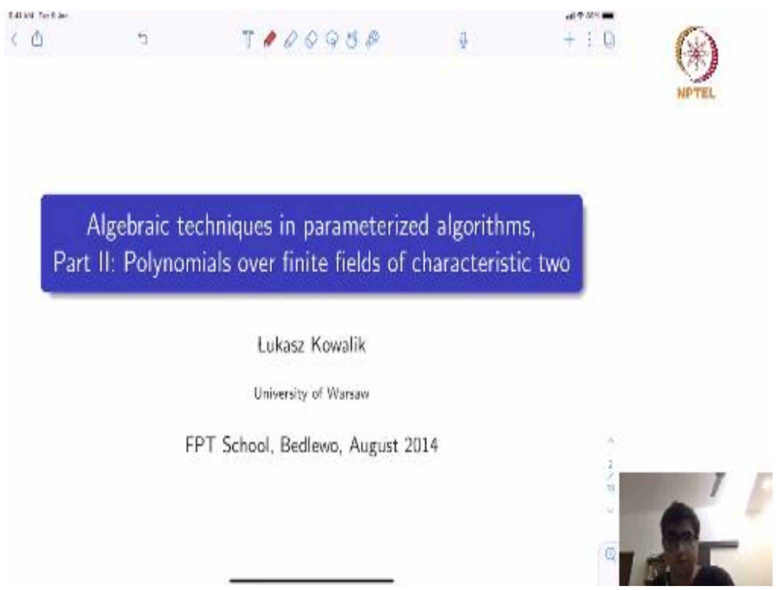

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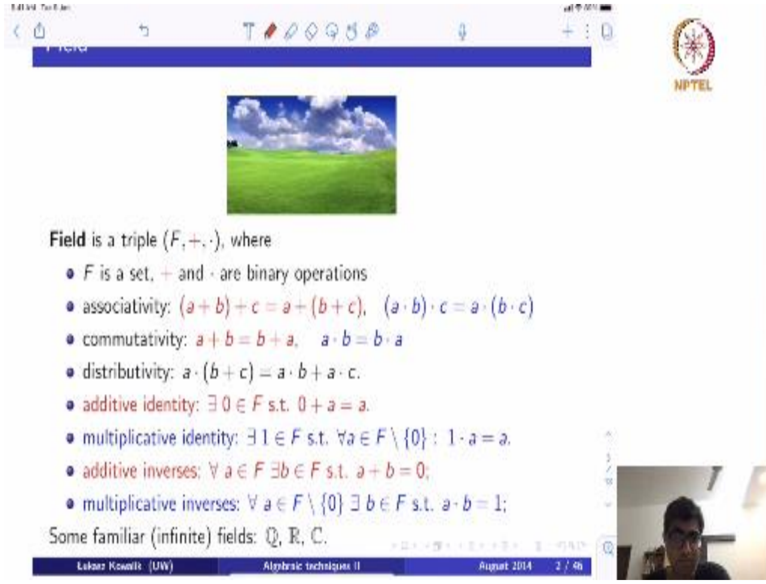

Before the slides are by Lukas Kowalik from an **(())(00:41)** school, but that will serve our purpose. So, the concept we need is the concept of a field. So, this is not a course in math's or this, but I still let me tell you what fields are basically it is a triple. So, field is basically set which are be stored with a couple of operations some may be set some multiplication, that is what is generally called but they are basically some binary operators and they satisfy some property like some associative property, commutative property, distributive property.

And there is this notion of additive identity which basically tells that there are a multiplicative identity basically that if you multiply with that, then you will get the same element and the similarly additive identity is like 0 which if we just add it with you get the same element. So, there are some small number of properties that the set needs to satisfy and there is also a notion of multiplicative inverse which basically means that for all a in F - g that exist b in F so that a dot b are equal to 1.

So, for example, if you take rational numbers, real numbers or complex numbers, then they form a field. But if you look at integers then they do not form a field because if I give you a number 2, there is no number in integers that if b if you multiply with that it gives you 1. So, while integers are not a field, but rational numbers are field because given any number 2, I can give you 1 2, 1 / 2. If you give me 1 / 2 then I can give you 2 so on and so forth as a multiplicative inverse.

So, this is a very good exercise to check that fields, such as rationals, reals, complex satisfy this property. So, you could look at this concept also from Wikipedia or from web.

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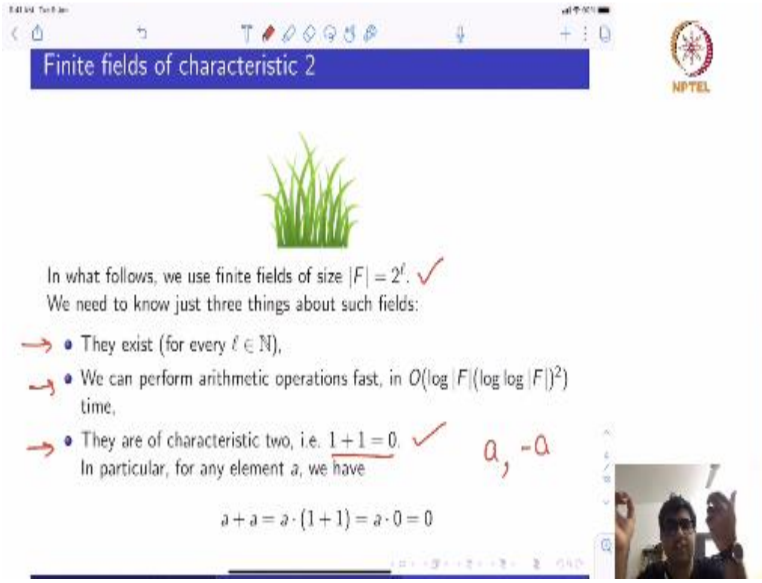

But, these are infinite field meaning their sizes are like Q, R and C are of infinite size. But the kind of field we will be working with will have size 2 to the power l and the only things that we; need to know about this field is the following they exist for every l in natural number. So, there exists a field of size 2 to the power l for every l in natural numbers, so that is the first thing that we need to know. And we can perform arithmetic operation very fast in this field. So, arithmetic operation only depends logarithmic in the size of the field. So that is the second thing.

And thirdly, which is most important is that they are of characteristic 2 that is 1 + 1 in this field is equal to 0 which implies that if you take any element and you add with itself then this is 0. So, a field is called characteristic r, then you have to add a r times to get 0, but the kind of field we will be talking about field of characteristic 2. So, basically in particular, just like if you take any element added twice then it is going to be 0. So, basically in this field if anything; occurs even number of times then they will evaluate to 0.

So, this is the 3 things important to know that we will be only talking about field of size to power l, they exist for every l in natural number, we can perform arithmetic operation very fast in this field. In particular, like in logarithmic the side of the field, they are of characteristic 2 that is 1 +

1 = 0. And in particular for any element a, we have that a + a like or in other words, for every number a, -a is additive inverse or a = - a rather. So, it is additive inverse is itself. So, these are the few things that you need to know they exist.

So, our finite field of size to power l exists. They exist for every l in natural number we can perform fast. And if I take any 2 element if I take any element and add to itself then that will cancel out or that will become 0. So, this is all that we need to know

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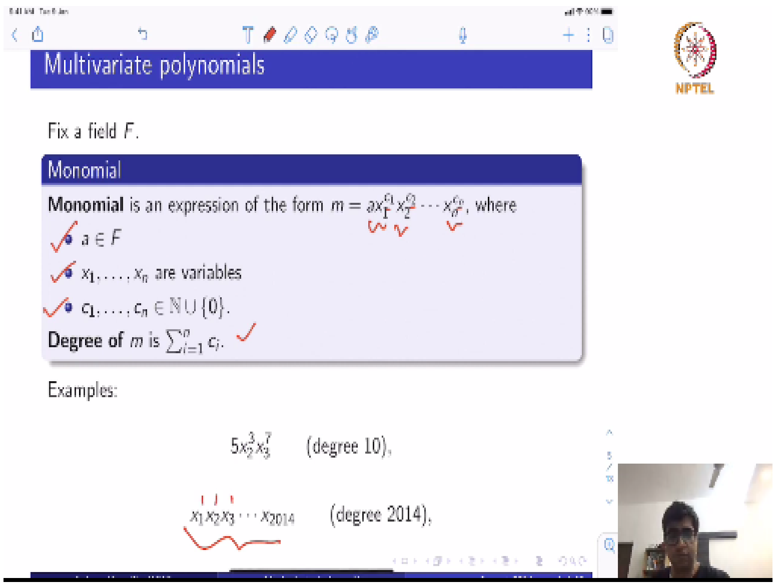

So, another concept that I need is of multivariate polynomial. So, you fix a field F then monomial is an expression of the form ax 1 to the power c 1, x 2 to the power c 2, x n to the power c n where a belongs to the field, x 1 to x n are variables and c 1 to c n are natural number union 0. So, then what is the degree of a monomial you just sum the coefficient and sum the powers of these x i this a is called coefficient. So, degree of m is basically summation of the power. So, example look at 5x 2 to the power 3 x 3 to the power 2 what is degree 7 + 3 10 and what is your degree here it is 2014 because everyone has 1 has bar, so in this monomial.

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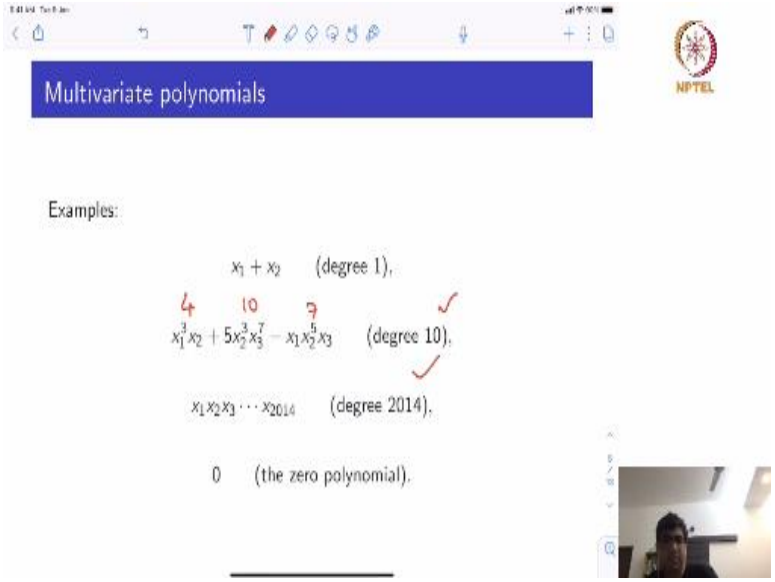

And what is a polynomial? It is basically a finite sum of monomials. For example, so, polynomial is an expression of the form you like so, you have a different term and for every term you could have a monomial, you could have a coefficient coming from your field. So, here this is nonzero only for a finite number of tuples. So, we are not talking about polynomial with infinite number of monomials. So, those are very different terms different name.

But in polynomial it is just a finite number of terms in the summation and what is the degree of this polynomial? Degree of a polynomial is a maximum degree of it is monomial. So, whatever is the maximum of the degree of each of these monomials that is a degree of a polynomial. So, for example, x 1 + x 2, what is his degree is 1, x 1 to the power 3 x 2 what is his degree? So, here the monomial has degree 4 this is 7 3 10 this monomial has degree 6 + 1 7, so, maximum is 10. So, similarly, this is just one term and 0 has a zero polynomial it has a degree 0.

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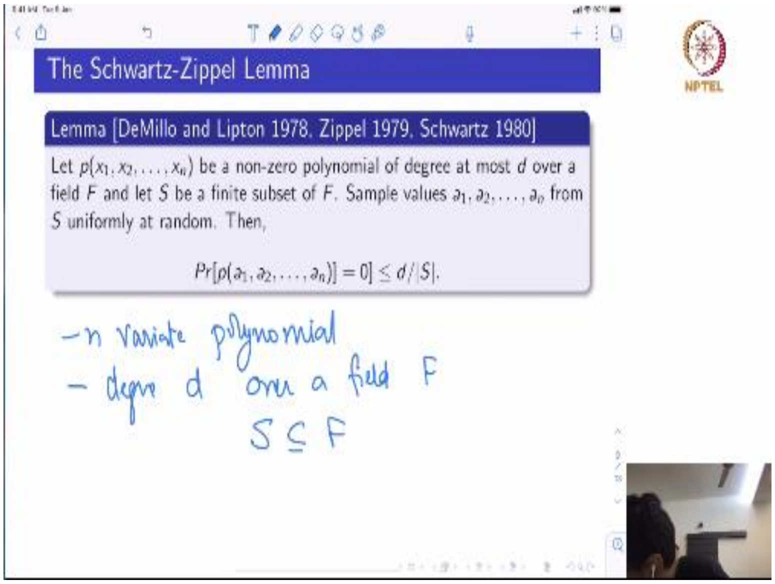

So, why are we interested in multivariate polynomial? It is because of what is known as Schwartz Zippel Lemma and this is at a lot of randomized algorithms or even otherwise. So, what does it say so, let us try to understand. So, suppose you are given a nonzero polynomial of degree at most d over a finite field F. So, suppose you have a polynomial p x 1, x 2, x n, so, this is n variate polynomial So, what is given to us is an n variate polynomial.

So, what is given to us is n variate polynomial, what is its degree? Degree is d and degree d over a field F now what you do? You fix some S subset of F this is finite. Now, from this you sample in values so, now what we are going to do?

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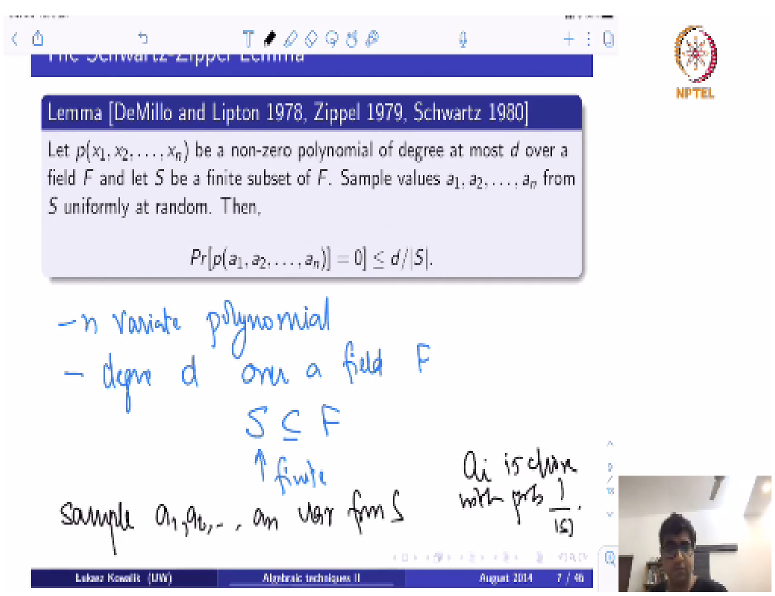

We are going to sample a 1, a 2, a n uniformly at random from S that is a i it choose n with probability 1 over S this one. Then what does it tell us? Then it tells us that now, what you do is that now you evaluate.

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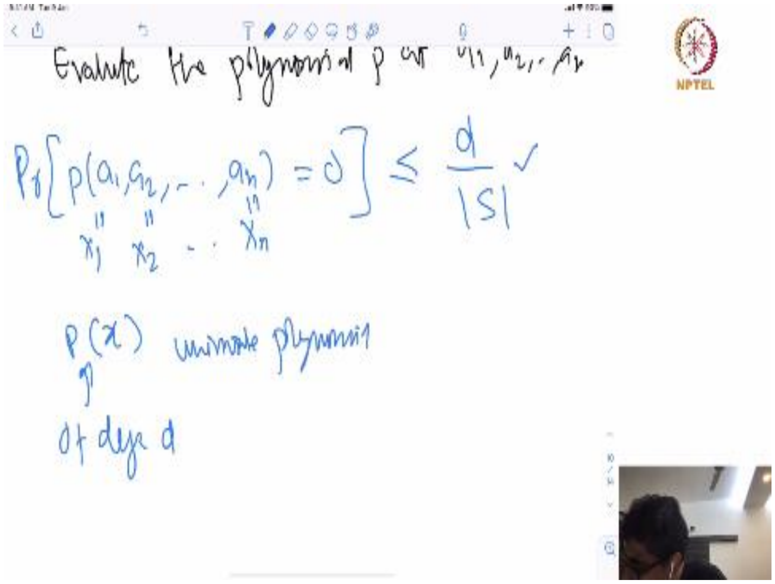

Now, what are we going to say? We say now let us evaluate the polynomial p at a 1, a 2 dot dot dot a n. So, now let us look at p a 1, a 2, a n. That is you substitute a 1 for x 1, you substitute a 2 for x 2 dot dot dot x n, a n for x n, and you elevate this point. So, this is what it means, then what is the probability? So, then it says look probability that this is equal to 0 and look what are we assuming it is a non zero polynomial, it is important.

So, what is the probability that this is equal to 0 is upper bounded by d times mod S. So, if the polynomial has small degree then what it says to you that look I have a polynomial it might be very big polynomial or something right but suppose I can evaluate at a random selection of point then the probability that look what happens that it is it could be that the polynomial already was 0 then of course it will evaluate to 0. But polynomial may not be nonzero identically but because of our selection of a 1 to a n this is like I have evaluated it to 0.

Then the probability that it is evaluated at 0 is at most d times S. Now, the people who are aware of field theory or something should realize that suppose p was just a polynomial of degree d and like it is a univariate polynomial.

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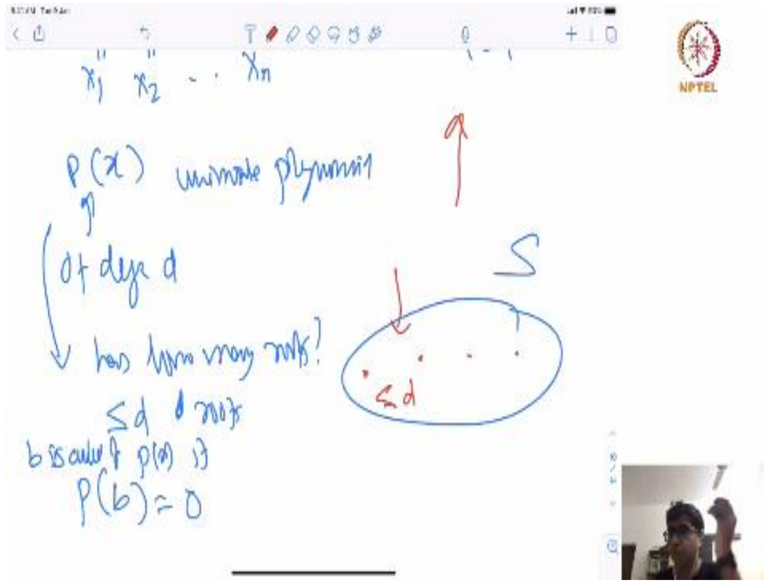

Then this has how many roots think for a minute for. So, it has at most d roots. So, what is the meaning of roots? Basically b if called root of px if P of b = 0. So, we know that it has at most d roots then definitely if I pick a set S then at most d points from here, if I pick that on that evaluation P of this will be 0. So, if S is very large compared to d then chances of hitting these roots are very low. So, definitely those people who know this theorem will know that for a univariate polynomial this is obvious.

So, what is a good thing about this proof is that it generalizes this property of univariate polynomial to a multivariate polynomial. So, for a univariate polynomial the proof follows from this what is called fundamental theorem of algebra. So, that are from the fact that univariate polynomial of degree d has at most d roots. So, if I take a set S which is very large and randomly pick a point then chances of hitting the roots are at most d over mod S. And this is generalization to not a univariate polynomial, but a multivariate polynomial.

So, what is Schwartz Zippel Lemma says very simple let p be invariant in variable polynomial of degree at most d over a finite field and then if we pick an S if you pick a finite subset of F and then sample a 1 to a n the values for x 1 to x n uniformly at random and evaluate the polynomial at these points then probability that we started with a nonzero polynomial and we get a zero polynomial is proportional to d divided by S where d is the degree.

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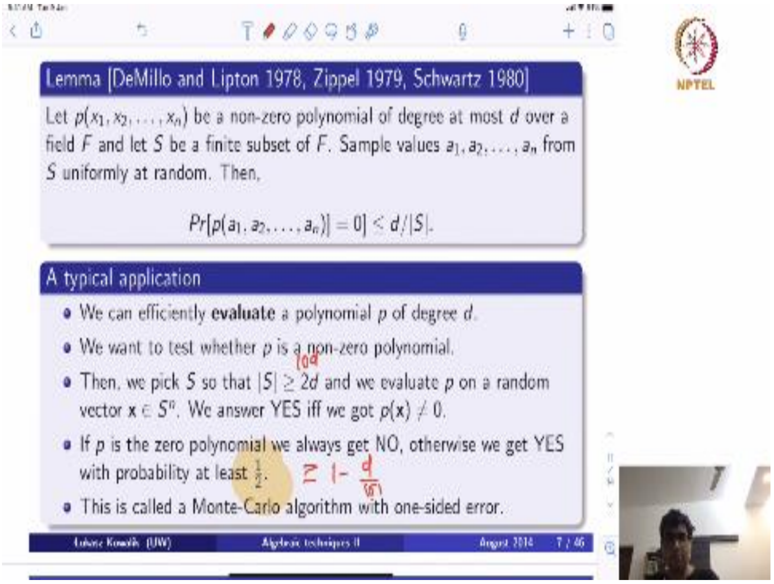

So, this is something so, what is a typical application of these are we can efficiently evaluate a polynomial p of degree d and we want to test whether p is a non zero polynomial or not then we pick S so, that the cardinality of S is at least 2d and we evaluate p on a random vector x in S n we answer YES if and only if we got px is not equal to 0. So, if p is the zero polynomial we always get no otherwise we get yes with probability at least half and you can boost this probability.

You can boost this probability to anything like for example, if I would have picked up S not 2d say 10d then this will be how much d over 10, so, this will be 1 - d over S. So, it is at least will be 9 over 10, If you choose 10d, 1 over 10, So, 9 over 20. And this is called Monte Carlo algorithm with one sided. Why it is one sided? Because look at one side if it is zero polynomial you will always answer zero polynomial. It is only when you are given a nonzero polynomial that you might say it is a zero polynomial, but that happens with very low probability. So, this is a typical application.

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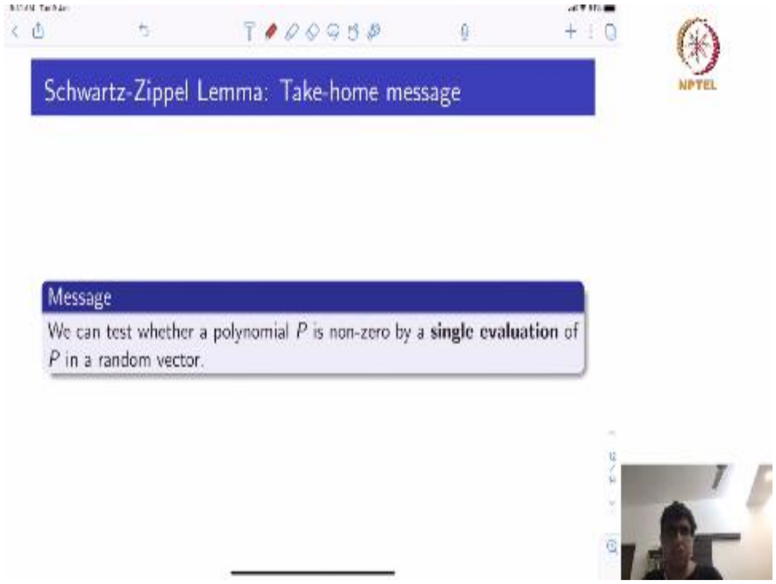

So, for example, so what is your take home message you can test with a polynomial P is a non zero by a single evaluation of P in a random vector. So, this is a take home message that you should take about Schwartz Zippel Lemma.

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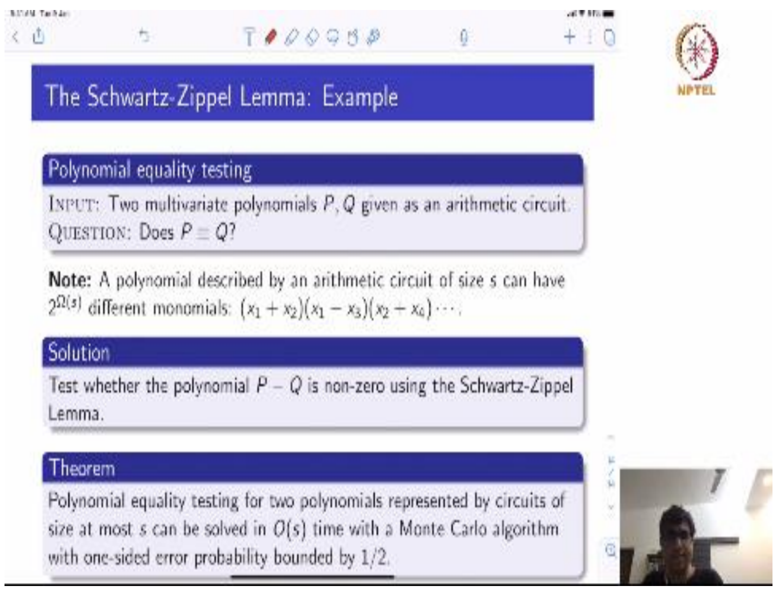

So, suppose very simple, so 2 multivariate polynomials P, Q given an arithmetic circuit, and you want to does P = Q, why? Because a polynomial describe an arithmetic circuit of size S can have exponentially meaning, so you could have a small circuit, so if you do not know about these things that is perfectly fine. But, look, if you look at a big, big, big monomial, like big polynomial, you might have a very succinct representation with a circuit.

And you can also evaluate this succinctly then maybe writing down all the terms and then evaluating could take you lots and lots of time and checking whether to follow given 2 polynomials P and Q, writing them time explicitly could be very, very huge. So, you do not want to like write them explicitly, but how can you check whether 2 multivariate polynomials P and Q given an arithmetic circuit, or a very succinct representation looked very polynomial very small. How can you check whether they are equal? That is very easy.

So, you test that polynomial P - Q is nonzero or not using the Schwartz Zippel Lemma, if P = Q, then P - Q which is zero polynomial, but if they are not equal P - Q is a nonzero polynomial. So, now what you do look at a P - Q, you have computed circuit P – Q and then you based on this degree, you pick up a subset S, you do a random evaluation and you can say so, basically it allows you to test whether 2 polynomials are equal or not, at least in this example in a very simple in a polynomial time, so that is all. That we will talk about Schwartz Zippel Lemma and now we will move to an algorithmic application of this.

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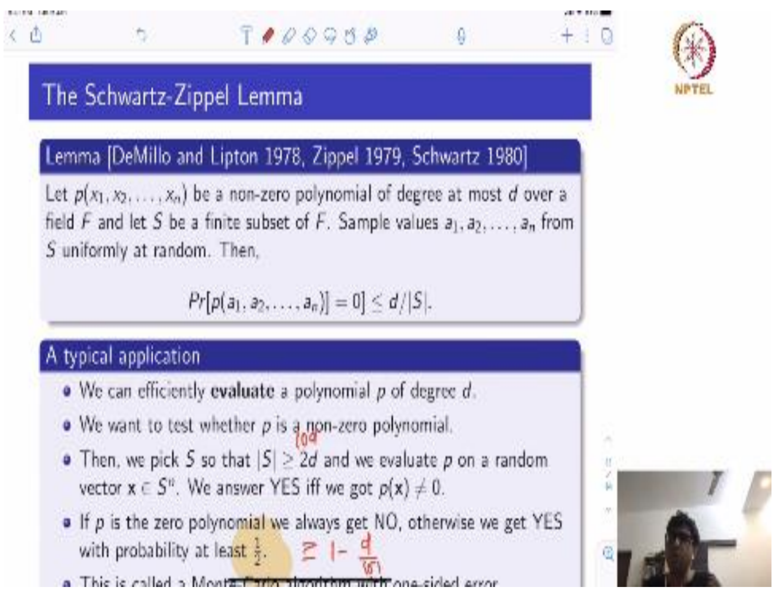

So, the tool is the following that if you are given an n degree n variate polynomial of degree at most d over a finite field and you pick up a big finite subset F of S and if we sample values from this uniformly at random, then the probability that if we evaluate the polynomial at these points it will evaluate to 0 given though you have started with nonzero polynomial is at most degree divided by the set S. So, we will talk about an algorithmic application of this.

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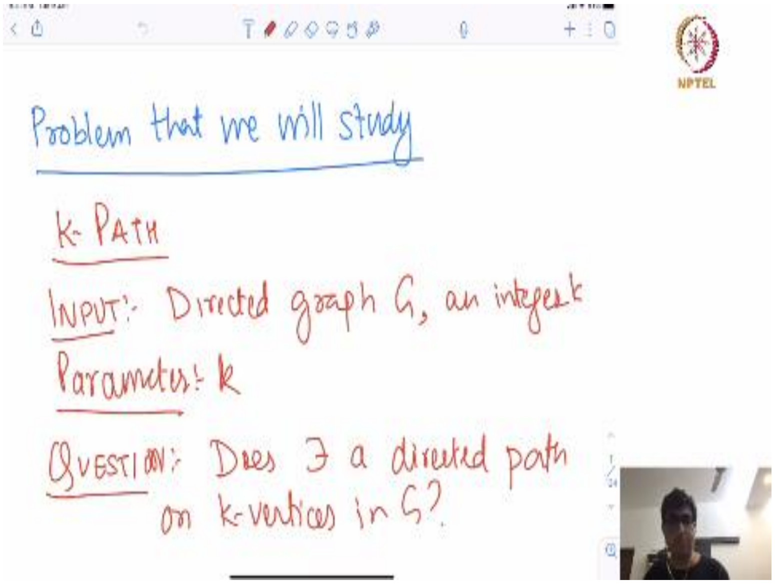

So, the problem that we will study for an application is will apply this machinery are the Schwartz Zippel Lemma based algorithm to design a much faster algorithm for K path, but deeply talking about directed graph G and later we will see how we can do this problem on undirected graph. But for now, we will be explicitly talking about directed graph. So, what is an input? You are given a directed graph G and integer K, your parameter is of course K. And your question is, does there exist a directed path on K vertices in G.

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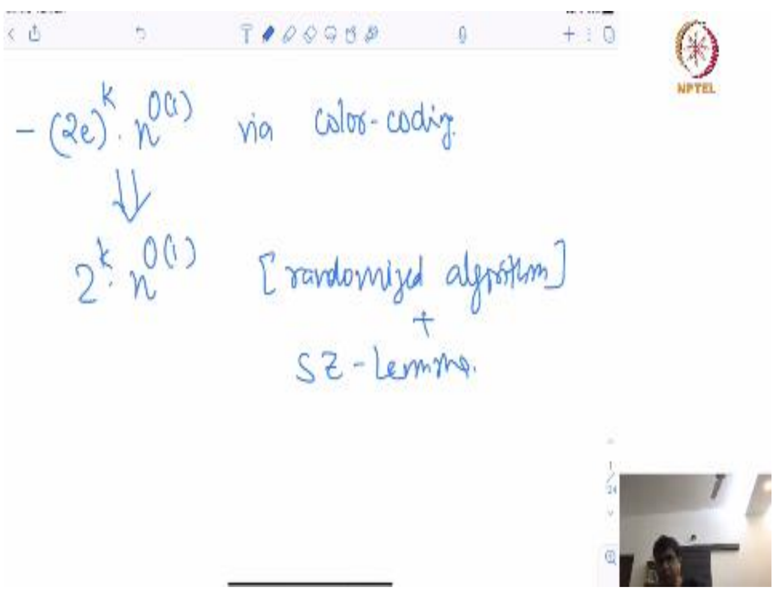

So, up until now, you have seen an algorithm with running time 2e to the power K n to the power O of 1 via color coding for this. So, in this lecture, we will try to outline an algorithm with running time. But a randomized algorithm and of course we will apply Schwartz Zippel Lemma. So that is a goal. So, the goal is to study a directed K path problem, K path problem in a directed graph and design and randomized algorithm. And design a randomized algorithm for this with a much more efficient running time and we will to do this we will use Schwartz Zippel Lemma. So, let us just get started.

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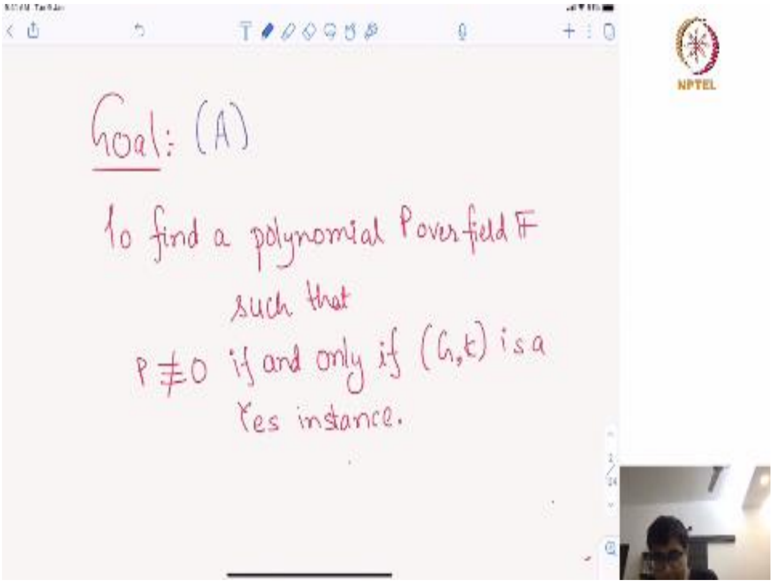

So, our first goal will be so now that we would like to apply Schwartz Zippel Lemma. So, first of all we have to cast our problem or we have to find a polynomial P over a field F and we want to say look, this polynomial does not evaluates to 0, if and only if G, K is a yes instance. Because to apply Schwartz Zippel Lemma.

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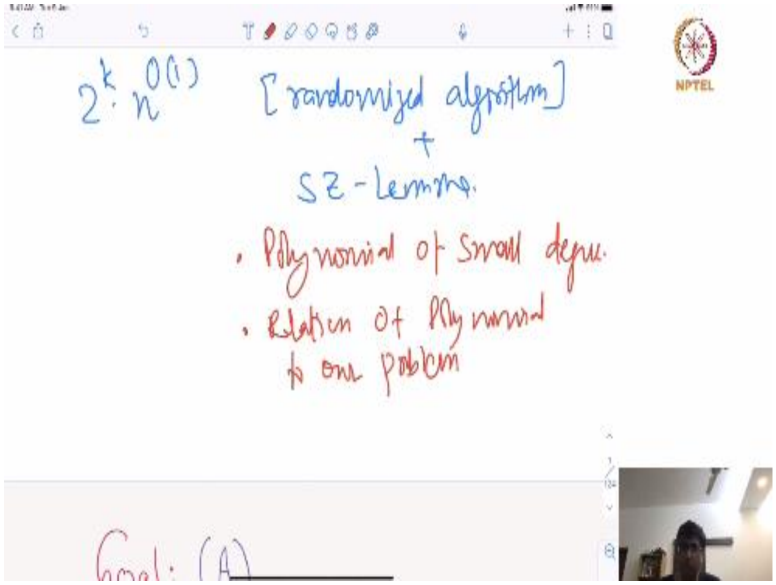

So, what are the ingredients to apply a Schwartz Zippel Lemma, we need polynomial of small degree, whatever that means and relation of polynomial evaluation polynomial to our problem. And then because if there is some relation or to a polynomial of small degree then we will do an evaluation this polynomial and based on evaluations answer we will say something.

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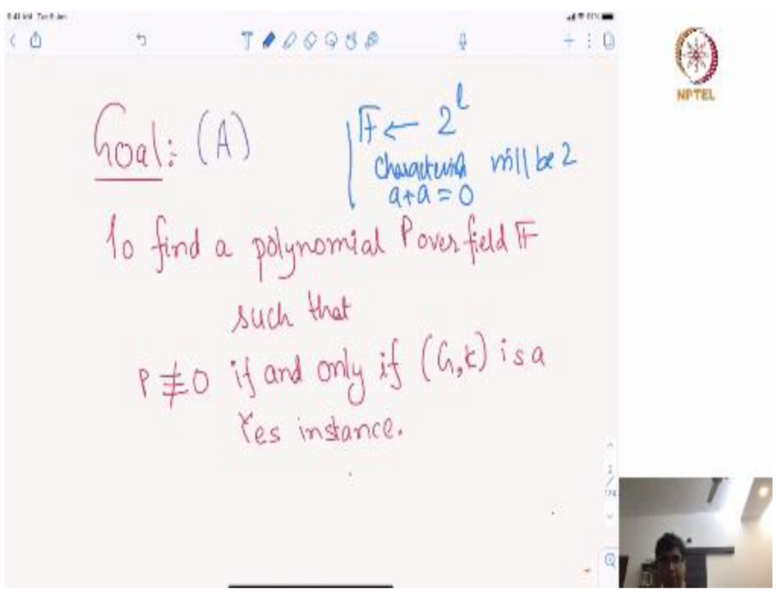

So, what I want to say that you want to find a polynomial P you will say look P is not equal to 0 if and only if G, K is a yes instance. So, in our case what it will be that I will make a polynomial says that this polynomial is not equal to 0 if and only if G, K. G has a directed path on K vertices so this is it. So, I want to find a polynomial P over a finite field F and in fact let me tell you straight. That the finite field F that we will work with will be of size sum 2 power l.

And its characteristic will be 2 which means that you take any element a + a. And you add it to itself it is going to evaluate to 0. So, this is the kind of field we are going to talk about. So, the nature of field is fixed all we need at this point of time is to find a polynomial such that this polynomial is identically nonzero if and only if G does contain a directed path on K vertices.

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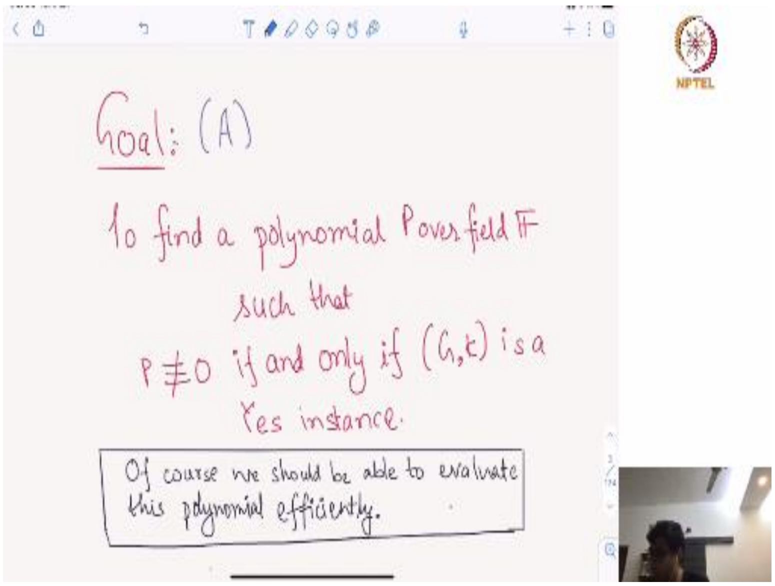

But of course, the one important requirement of this polynomial is that we should be able to evaluate this polynomial efficiently. So, for example if I could write this polynomial succinctly as an arithmetic circuit are some ways to evaluate this polynomial very efficiently. And this notion of efficiently will is what will determine the running time of this algorithm.

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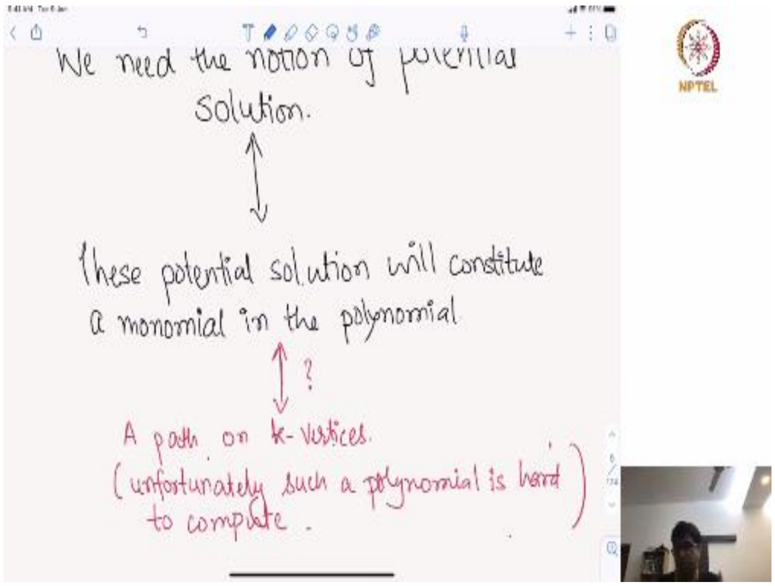

So, we need a notion of potential solution. Why, because this potential solution will constitute a monomial in the polymer. As I look, this is your particular this could be a potential solution, this could be a potential solution, this could be potential solution and for each potential solution, we will create a monomial and we will say look, because in this polynomial well some of we need to capture these potential solutions. So, what is work first like; so maybe what a potential solution of course, a potential solution the path on K vertices. A potential solution is a path on K vertices.

But unfortunately, such a polynomial is hard to compute because in some sense it is like up you are talking about come finding an algorithm for finding a directed K path. And directed K path itself is constituting like a polynomial itself has a monomial for each only K paths. Then that kind of polynomial looks a bit difficult to or hard to compute. So, this is just a very big overview.

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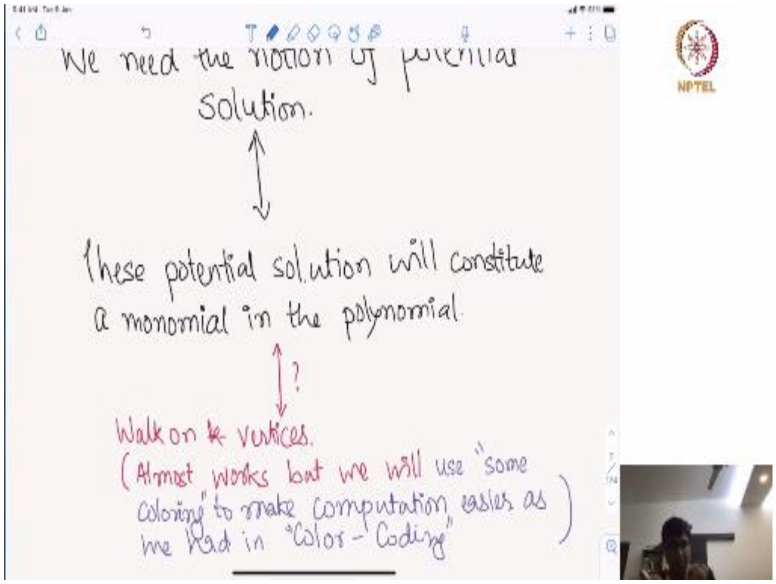

So, notice that when we were talking about a Hamiltonian Path algorithm using inclusion exclusion, we talked about walk on K vertices because what is in walk on K vertices? Walk on K vertices is K vertices he says that vertices could repeat and adjacent like consecutive vertices have an edge between them, but they may not but in walk if the vertices are not repeated then they constitute a K path. So, it was like a kind of slightly we relax this idea from path to walk because walk was easy to compute.

This notion almost works, but we will use some coloring to make computation easier as we did in colour coding. So, we will not only need walks but we will need some colorful kind of walks because that will help us to compute these paths like computations will be efficient. Because remember colour coding how did we do because it was not easy to compute things properly. So first, when we did colouring and then we were looking for a colourful path it allowed us to do some set of dynamic programming.

So, all these are in compassed into one straight application is that is like real code all this like coloring idea as well as relaxing idea in a monomial itself as we will see in a minute.

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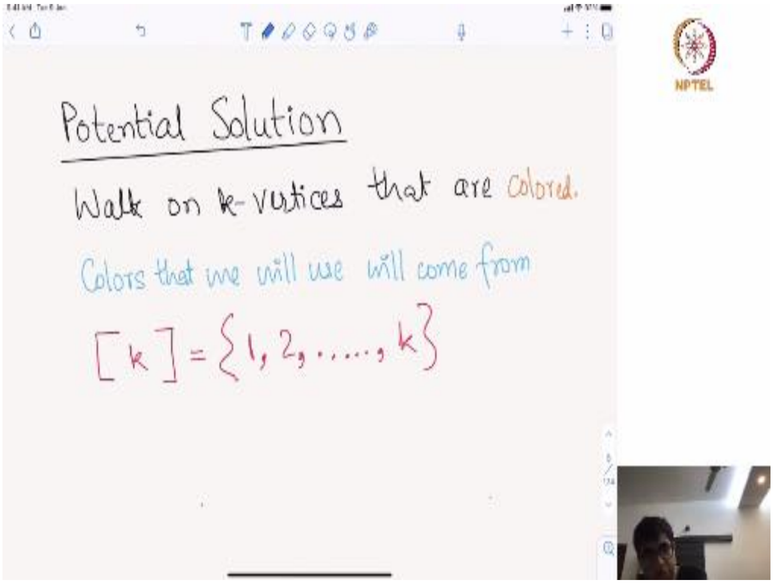

So, support whatever potential solution the walk on K vertices is that are coloured. Colours that we will use will come from 1 to K. This is exactly like color coding that colors are always used from 1 to K to colour K vertices distinctly. So that is way of this come. Colours so walk on K vertices so over what are a potential solution walk on K vertices that are coloured.

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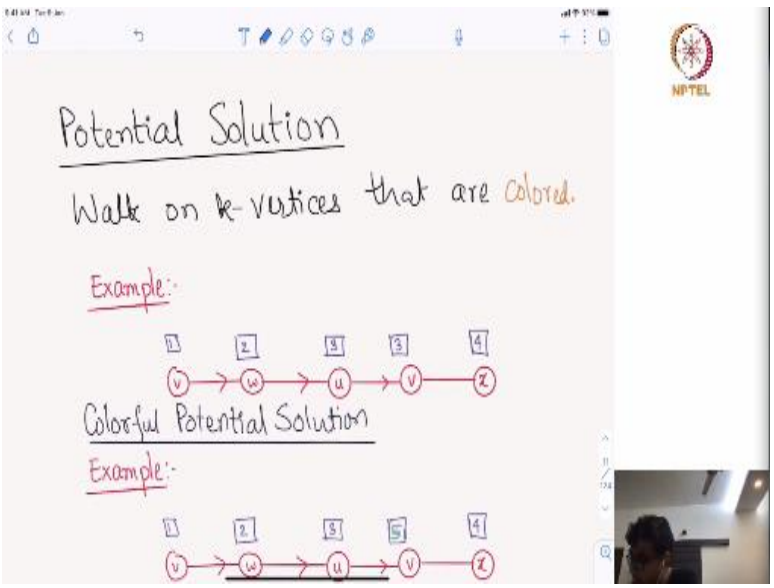

For look at for example v w u v x so, it is a walk because this is vertices and we are like 1 2 3 3 4 is a color. So, this represents color and this represents walk. And what is the colourful potential solution? A colourful potential solution is the one where if the vertex occurred second time you assign a different color. So, since it is a directed key walk you have a notion of first vertex, second vertex, third vertex, fourth vertex, fifth vertex every vertex should be assigned distinct color.

Every occurrence of a vertex should be assigned distinct color or other walk if you look at the K word look at the first vertex of the second word they all have distinct colors the colors do not repeat. So, here look at this the same this is not a colorful potential solution because on u and v we have given a different or same color, so you should see a distinct from the color coding. Color coding; we first assign colors to all the vertices that is not what we are doing now. So, because once you have assigned a color to a vertex then same color but look at v here it has been colored 1, it has been colored 5, so one should not confused with that idea.

So, first let us just worry about this so what is a walk on K vertices that are colored. So, you are given a walk and colors just you assign color to this occurrences of vertices they are colorful if each occurrence if each vertex and each of their occurrences get distinct color. So, basically this is what is the colorful potential solutions.

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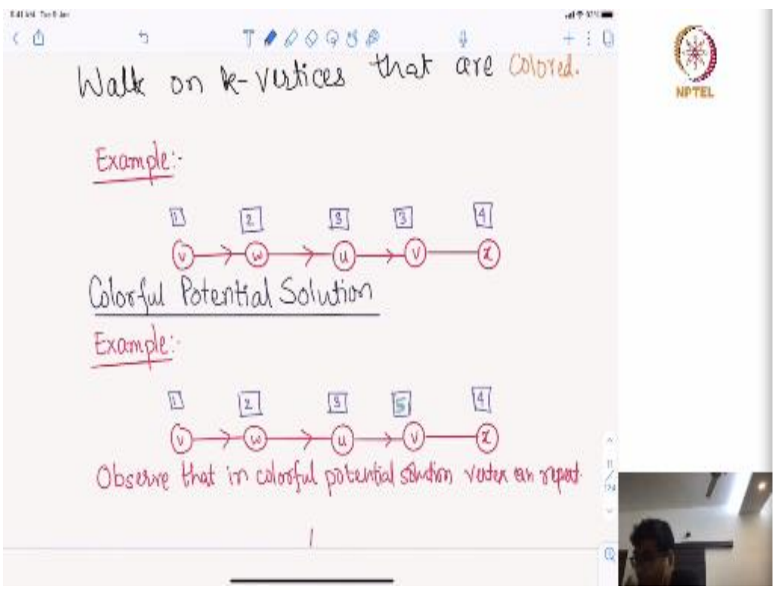

So, although that this is important point in colorful potential search solution also vertex can repeat right because ultimately we are still talking but walk. It just said that every occurrence vertex gets different color, so when you are watching this video, pause for a second and absorb this definition.

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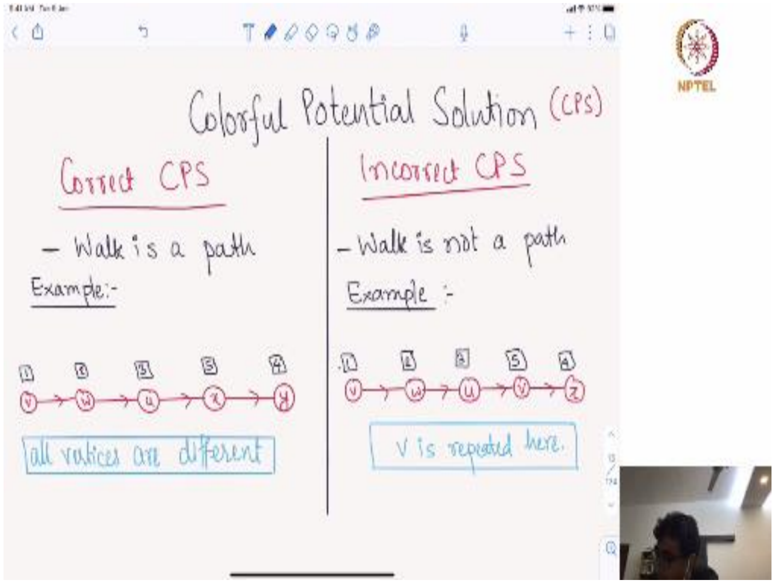

So, now we are talking about we will generally most of the time we will talk about colorful potential solution. Now, so we will denote the colorful potential solution by CPS. So, what is an incorrect CPS walk is not a path. So, for example look at this v w u v x. So, v is repeated here, v is repeated here. So, all though the colors are distinct since vertices are repeated it is not a path. So, what is the correct CPS walk is a path.

So, now we will want to talk about colorful walks and it is an incorrect colorful potential solution if walk is a walk and it is a correct colorful potential solution if walk is a path, that is every vertex that occurs on the walk are distinct. So, for example v w u x y so, this is a correct colorful potential solution, this is a incorrect colorful potential solution. So that is all what is different here v is repeated here. So, again so these are the 2 notions that we will use going forward.

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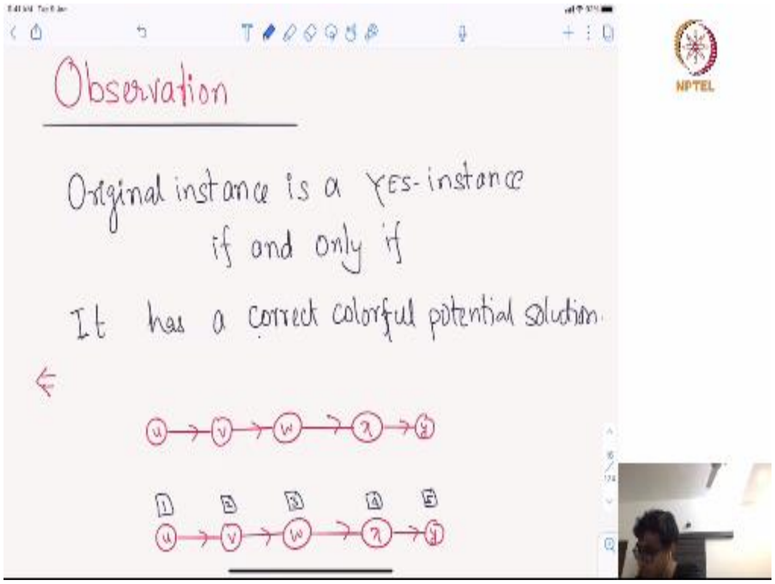

So, what is an observation? Original instance is a yes instance if and only if it has a correct colorful potential solution why follow the action is very easy. You start suppose you had u v w x y then just give 1 2 3 4 5, so you like just give distinct colors. And if you have a colorful potential solution then you know that you are talking about K path. So, this is a very simple observation but, it is a useful observation to have. So, if you have a colorful potential solution which is here, just forget this color and what you are left with is a K path.

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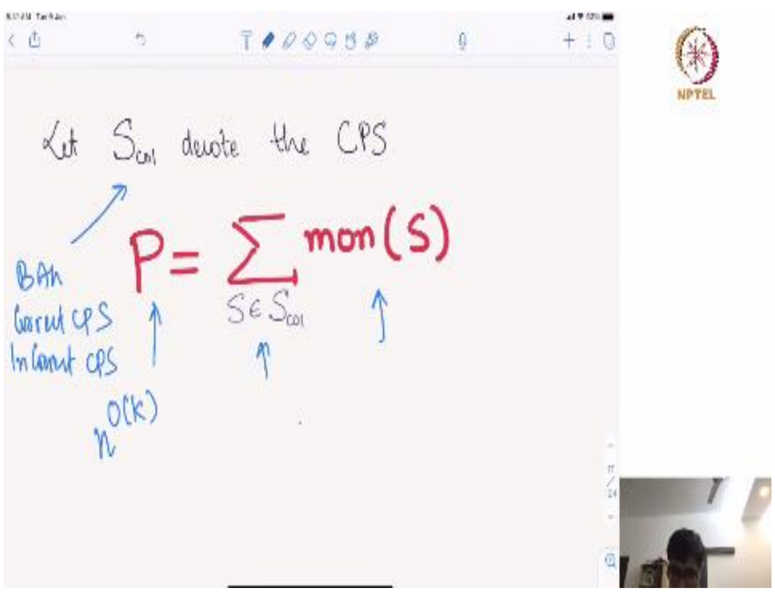

So, what are we going to do? So, our polynomial is going to have as I told you before, for is called denote all colorful potential solution, including incorrect as well as correct potential solution. Correct colorful potential solution and correct. So, S col denotes the colorful potential solution. Both correct colorful potential solution as well as incorrect colorful potential solution. And what is by polynomial? Polynomial is very simple. It is a summation over; you go S and S col and a monomial with respect to S.

Now notice that S col could have like this number of monomials in this polynomial could we have n to the power O of K size. So, we are making a polynomial, but we cannot explicitly write down completely and evaluate it. Whatever that polynomial we have not talked about it, but just by the look of it, it should point you that it is a very big polynomial. So, we cannot write it down explicitly. But we need to find ways to evaluate P if effectively so that we can apply Schwartz Zippel Lemma.

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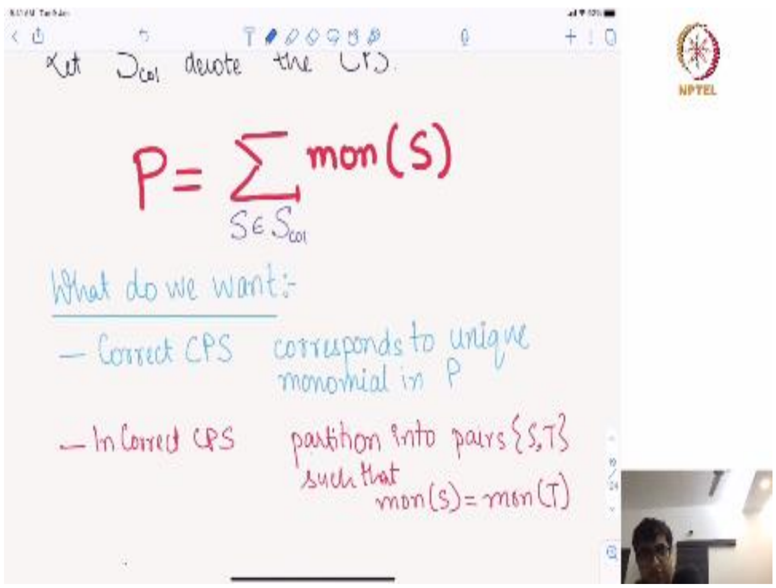

So, what do we want? We want correct CPS corresponds to unique monomials in P. So, if we have a correct colorful potential solution, they should correspond to a unique monomial in P and for incorrect CPS colorful potential solution, we would like to partition into pair S, T. So that monomial of S is equal to monomial of T why it is important to do that well.

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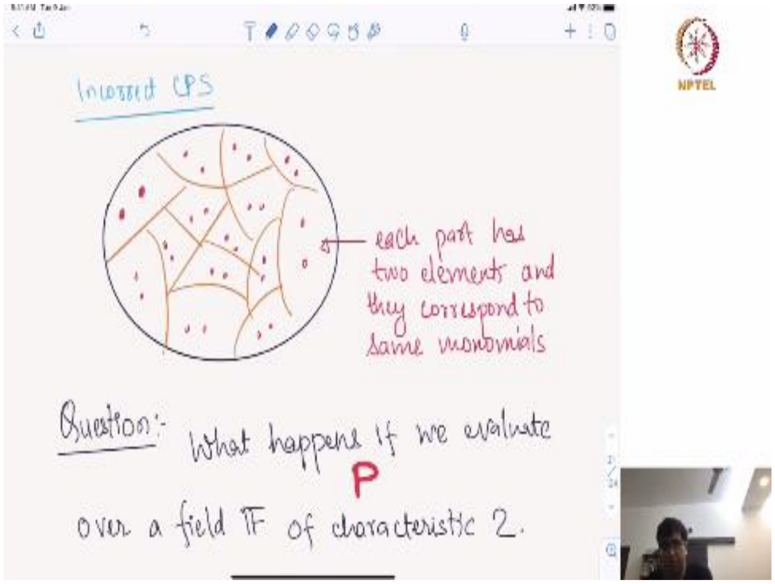

So, incorrect CPS you would like to partition them into 2 elements each and each part has 2 element, they correspond to the same monomial. So, if they correspond to same monomials then what happens in the like for these 2 types you have same monomial into 2. So, evaluate this P over a field F of characteristic 2 then because of the coefficient 2 what is going to happen? Look x 1 to x n and you will fix this numbers, you chosen this number a 1 to a n now they are same monomial. So, as a number they will be same, but counted 2 times. So, over a field of characteristic 2 all this will evaluate to 0.

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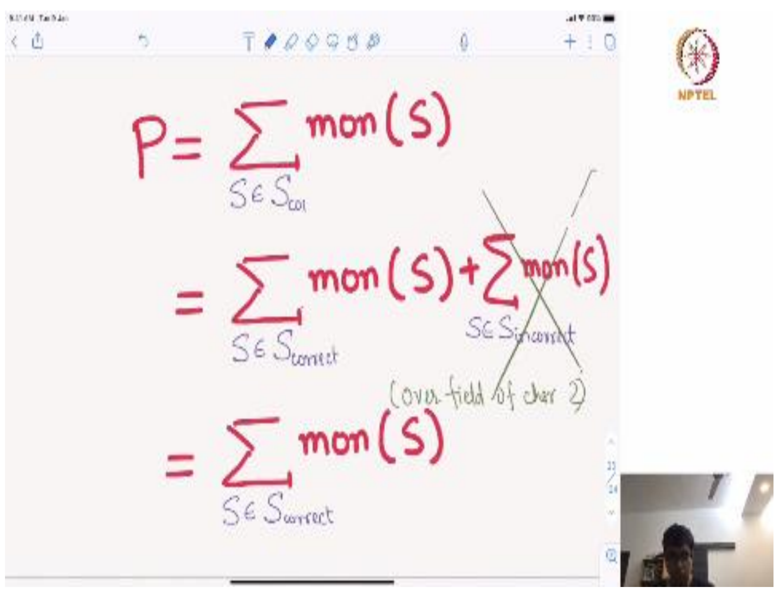

So, if we have S in S correct S in S incorrect. Over a field of characteristics to do once evaluate over this because you have paired them up this will vanish this will go away, so, the only thing which will be left in S in S incorrect. So, this is why we are using a field of characters to nullify the effect of incorrect CPS. And this is the way generally you use that you look at a larger object where you embed the right object, and you saw that look everything else when I do over the field of characteristic true will cancel out each other.

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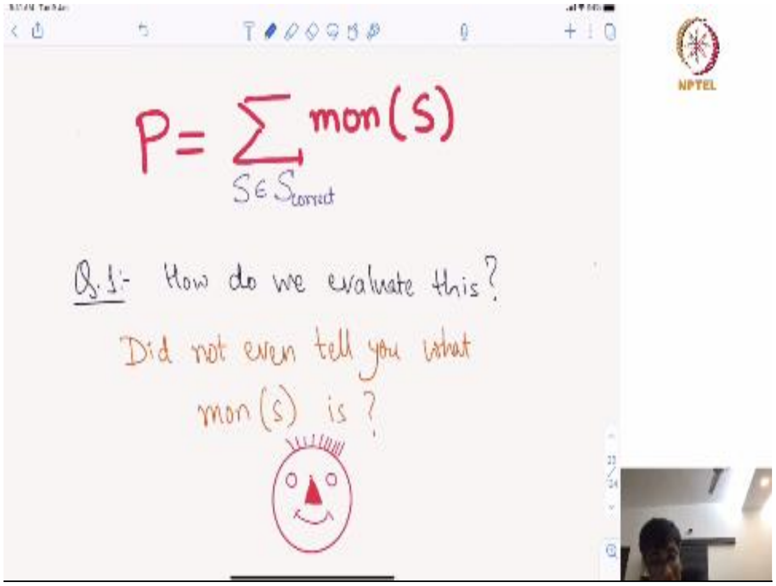

So, how do we evaluate this? That is a question but first of all, we have not even told you what monomial of S is. So, let us remedy that situation.

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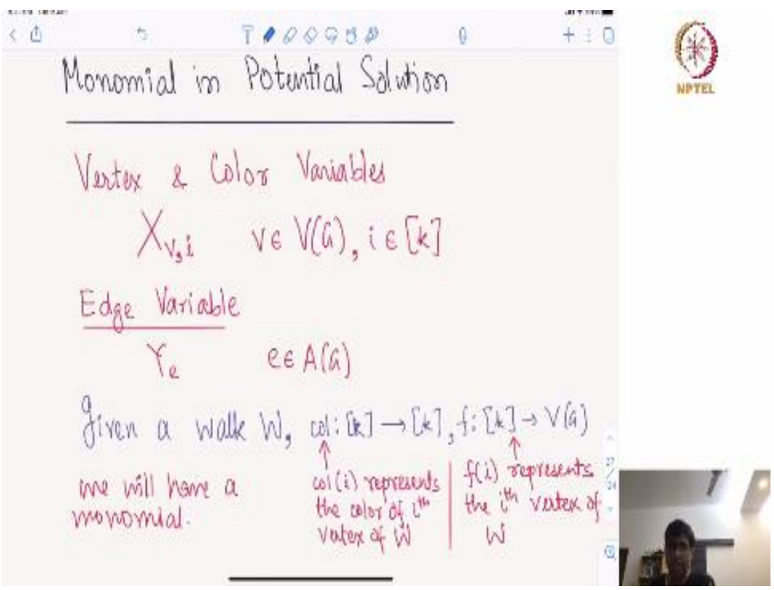

And let us define the monomial in potential solution. So, we are going to have a vertex and color variable. So, what is the vertex and color variable? So, for every vertex v we are going to have a variable X v, i v in vertex set i in 1 to K and we are going to have an edge variable Y e in A of G, it is a directed graph remember. So, AB is not same as BA. So, for every vertex how many variables we have? K variables we have, a vertex name and the color it is a sign and how many other edge variables we have like I mean.

So, total number of variables we have is nK + m where m is number of edges or arcs and n is a number of vertices. Look at and you are given a walk W we will have 2 functions a color function and a vertex function let us call it. What is the color function? Given a walk W we will have a monomial. So, given a walk W a coloring function and another function f we will so what color of i represents the color of ith vertex of W. And what is f of i represents ith vertex of W. So, this is a map from 1 to K to 1 to K.

But this map is from 1 to K to V of G. So, it will tell us what is the fifth vertex? Fifth vertex is the vertex V, what is f of 7 it is seventh vertex. So, seventh vertex is y or whatever. And what is the coloring function we will say for the first vertex colored is y, for the fourth vertex color is red or so on and so forth. I will give you an example in a minute. So again, spend a minute. And look at what is our vertex and colored variables what are edge variables and what will our monomial consist of? Monomial will consist of 3 things a walk, a coloring function and a vertex function.

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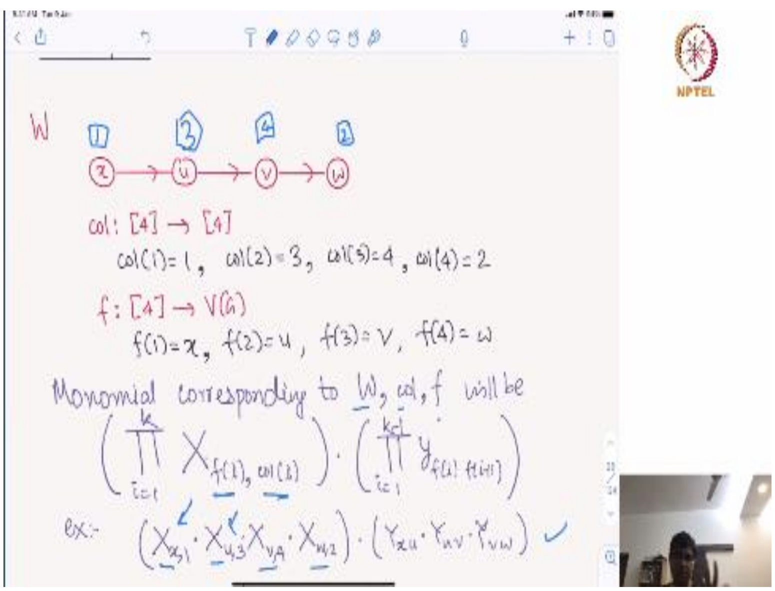

So, here is an example, so, look at x u v w, what is my coloring function? So, coloring function is function from 1 to 4. So, suppose color 1 is 1, color of 2 is 3, color of 3 is 4, color of 4 is 2. So, basically it will mean that and what is a function from 1 to 4, f of 1 is x what is a second vertex is u. So, f2 of u what is the third vertex is v, so f3 is V what is the fourth vertex so, f4 is w and what a color? So, this first vertex has been assigned color 1.

Second vertex has been assigned color 3, a third vertex has been assigned color 4, fourth vertex has been assigned color 2. Now what is a monomial corresponding to w a coloring function and f will be fine. So, it is a multiplication of vertex color variables on this. So, i = 1, What is this? It tells you what vertex it is and what color it has been assigned. So, for example, what is the first vertex x what color it has been assigned 1 so X x, 1 what is a product. What is the second vertex?

It is a u and what color it has been assigned 3 so u 3, what is the third vertex v and what is the color assigned 4, what is your fourth vertex w and what is the color assigned 2. And what is i = 1 to K – 1 edges from f i, f of i + 1 that is it. So, Y xu Y uv Y vw so, this is a monomial which corresponds to a walk coloring function and a f function or the vertex representation function.

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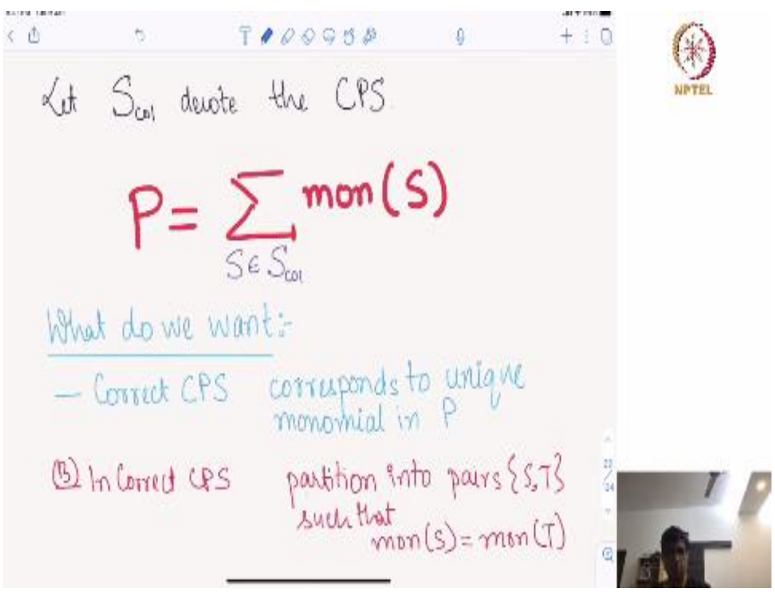

So, let us call denote the CPS and this what we wanted correct CPS corresponds to unique monomial in P incorrect CPS we should be able to partition into pairs S, T such that monomial of S is equal to monomial of T so, let us try to do that.

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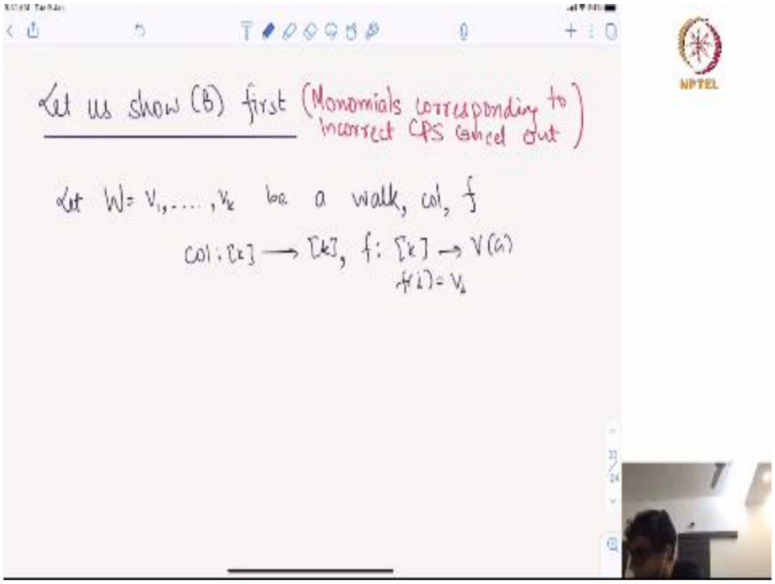

Let us show B first monomials corresponding to incorrect CPS will cancel out. So, let us W be v 1 to v K a walk coloring function and f and color of is K to 1 to K and f of K VG, f of i it is a v i.

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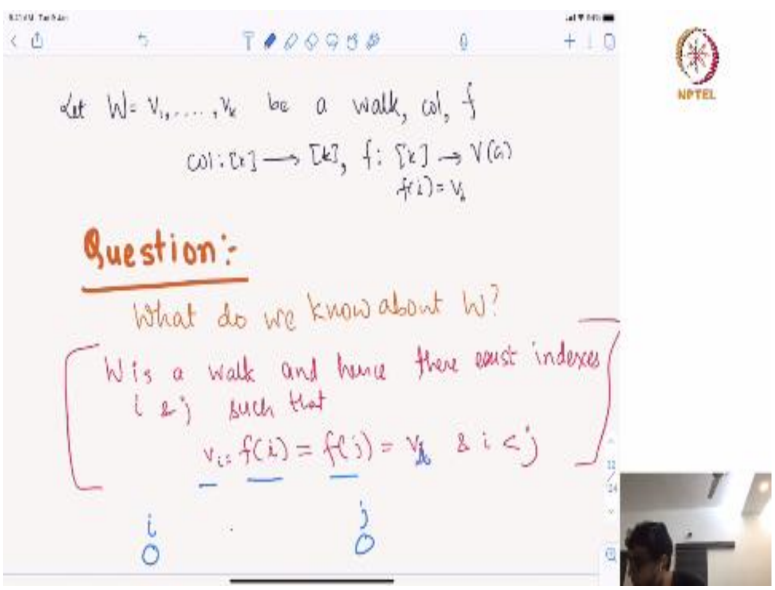

So, question is what do we know about W? W is a walk and hence there exists indices indexes i and j such that, what is the index i and j such that f of i is f of j which is v i. So, basically I can find an index i and j such that look at f of i the vertex which is assigned at a f of i look at the vertex which is assigned at a f of j they are same because W. So, look this is where it comes like so it is colorful, but it is an incorrect CPF. Incorrect colorful potential solution means a vertex is repeated. So, I can find an index i, j vertices are same that is it and i strictly less than j.

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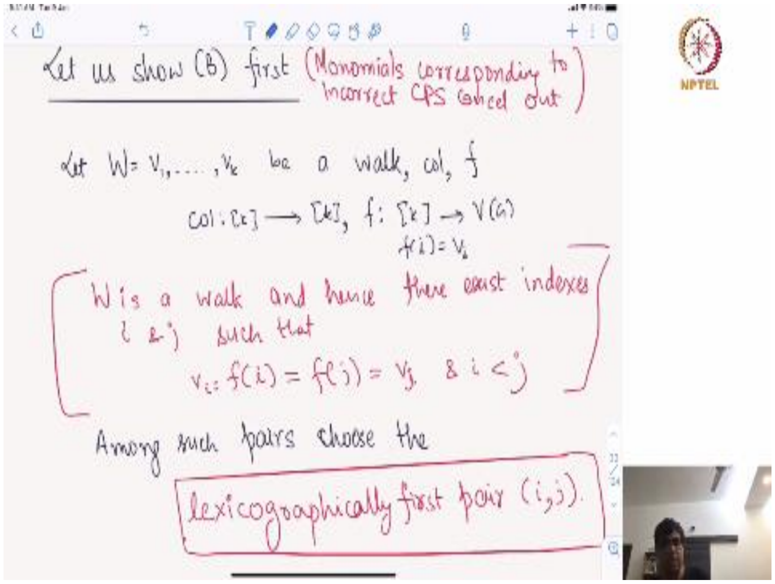

So, what are we going to do among such pairs we are going to choose lexicographically first pair i, j. I will tell you in a minute what I am write among all this.

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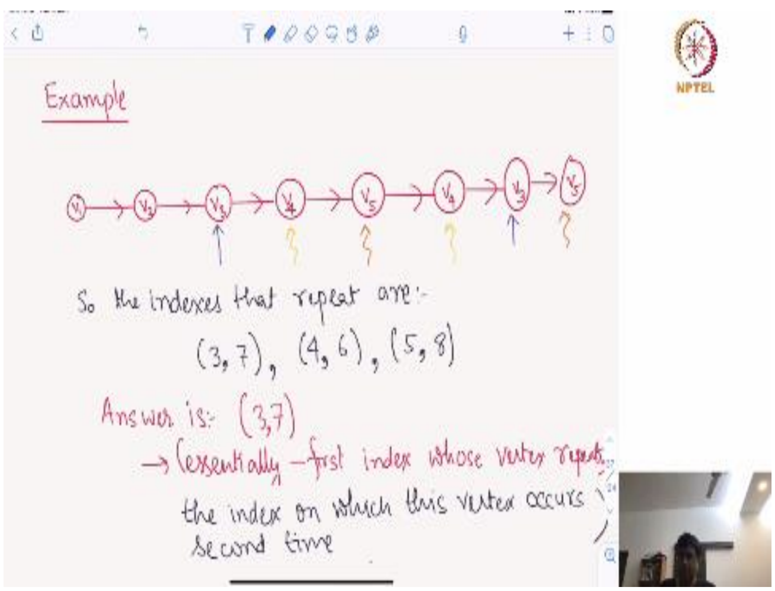

So, basically, what it means is that look at this example v 1, v 2, v 3, v 4. Look v 3 is repeated here, v 4 is repeated at here, v 4, so what are the indexes you are going to get? So, the indexes that repeat are 3, 7 4, 6 and 5, 8 so maybe if some other like, if say for example if 3 would have been occurring 3 times then, it would have been 1 like for all 3 pairs similar so among all such indexes that repeat.

What is about lexicographically smallest is that? Lexicographically smallest means that for the first coordinate, you choose the smallest. And then you choose the smallest among the second coordinate. So essentially, what it means that first index, whose vertex repeat that is the first coordinate. And the second index on with this vertex occurs second time is what basically is not basically is what lexicographically first object will be. What Look at the first vertex which repeats so that is your first coordinate like look at that index the first.

And look at the first time it repeats like the first time it repeats, that index corresponds that come i comma j corresponds to lexicographically first pair.

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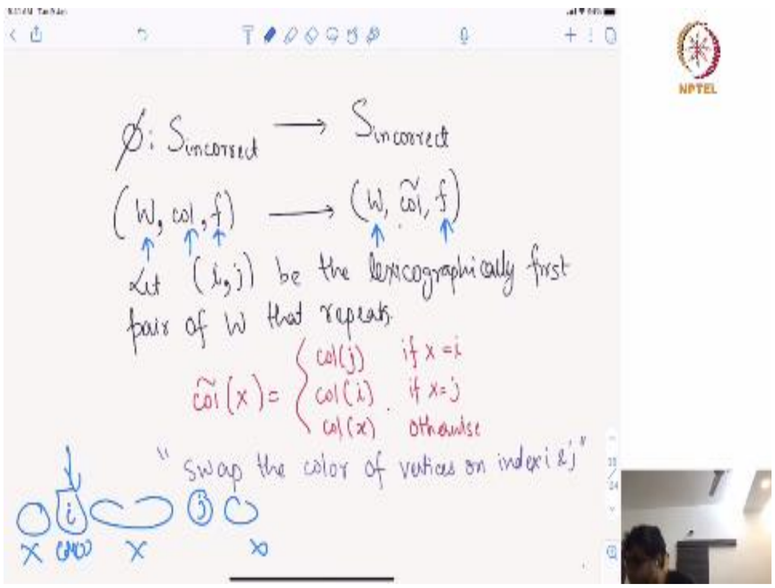

So, now I am going to give you a map from S incorrect to S incorrect. So, what is S incorrect consists of W, a coloring function and f function. So, I am going to give you a map so W remains the same, f remains the same, just a coloring will change. So, let i, j be the lexicographically first pair of W that repeats. So, what is the new coloring function? Look so, what happens, so here, it is i and index j. So I go, I say look for all these indexes here and here, keep the same colors.

But suppose I came at index suppose x is i, then I assigned this guy new color which is the color of j. So, this is like becomes color of j and then I go to I so, if x equal to j which means this I assigned a color. So, basically I swap the color or vertexes on index i and j. That is it that is all, So, What does coloring function did? It looked at the sphere and it just swap the colors of these vertexes.

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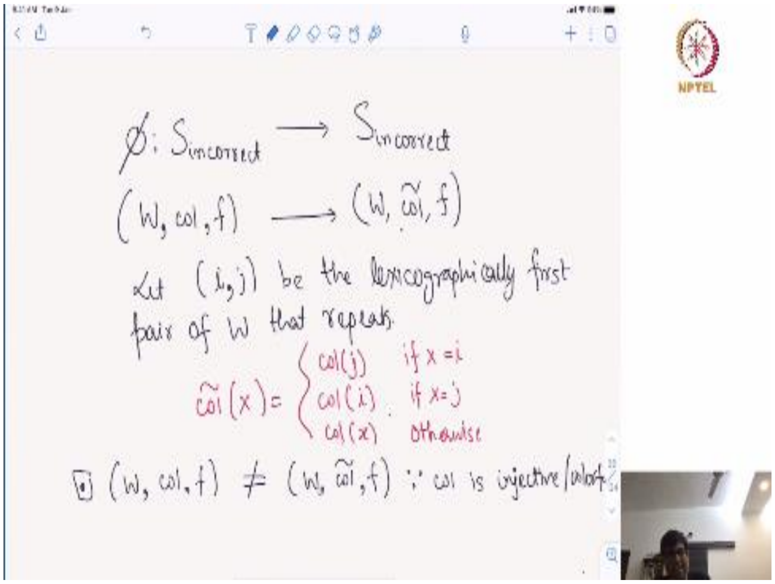

First of all, notice that W col f is not equal to W col tilde f, because since coloring is an inject shape or colorful, this is similar view. So, I mean, you have just swapped it so coloring functions are not going to be same, but f and W are same because the walk is same if assigned the same word. This is the first one.

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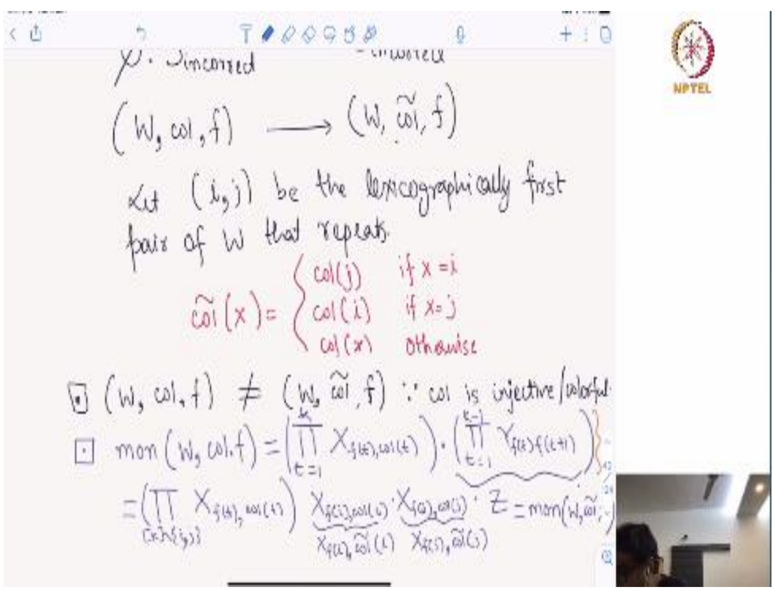

But now let us write down their monomials, let us write down their monomial. So, what is a monomial of W col f is like t 1 to K X ft and the color of t and the edges. Let us call this edge Z because edges on this walk are not going to change. So, it write it but now let us from this let us remove i, j this then what is this X f of i col i, X of f of j col j, but notice what is X of f of i col i it is same edge but what it has become in next. Now, you have to replace this with f X f of i col tilde i in the new one and this with col tilde j. But what is this? This is this and this is this right. So, this is same as monomial W, col tilde, f.

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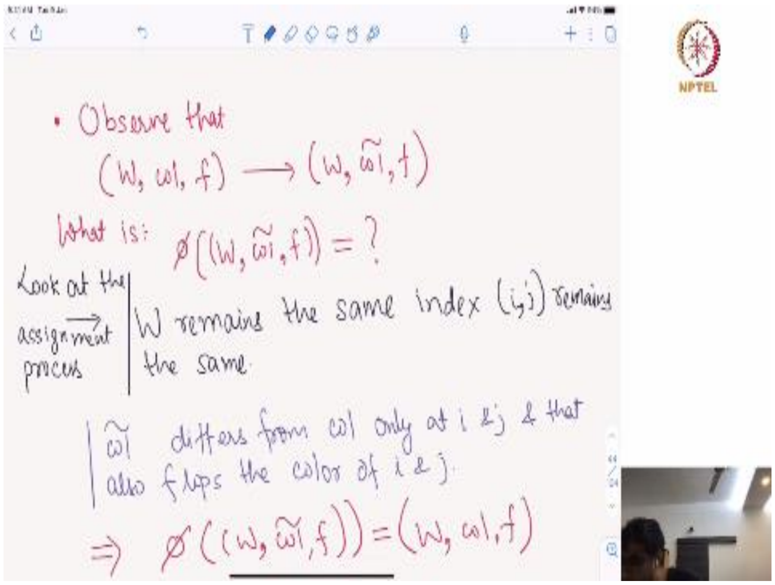

So, what can we observe? We can observe that W col f is maps to W col tilde f but let us ask our self what is the phi of W col tilde f. Notice that lexicographically pair will not change. So, again if I apply phi to this you will get back the col function, look at the assignment process W remains the same index i j remains the same then what will phi do coloring differs from colors only at i j and that also flips the color of i and j. So, what you are able to get is that phi W col tilde, f is W col, f.

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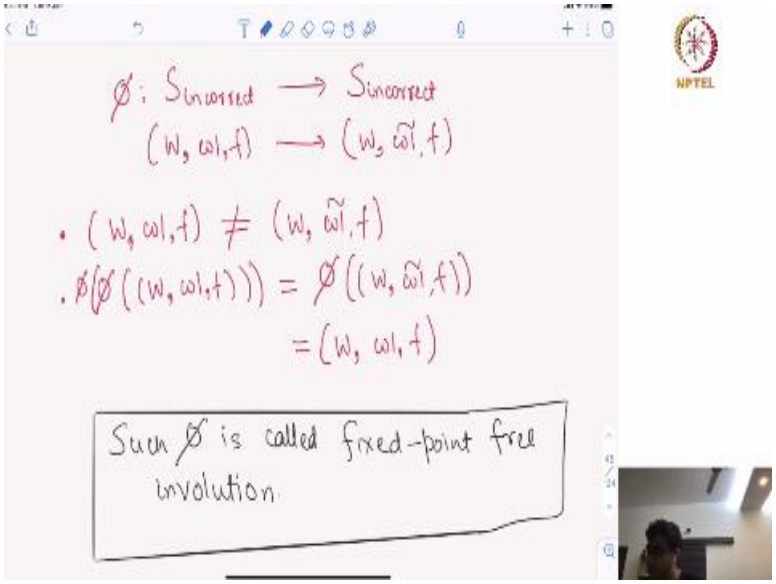

So, what does it implies now, let us look at this W col f is not equal to W col tilde f. So, if you look at phi functions you apply it phi to twice you will get. So, first I apply phi I get col tilde you again apply phi you get back the same thing. So, these kinds of functions are also called such phi called fixed point free convolution. So, such phi is called fixed point free convolution

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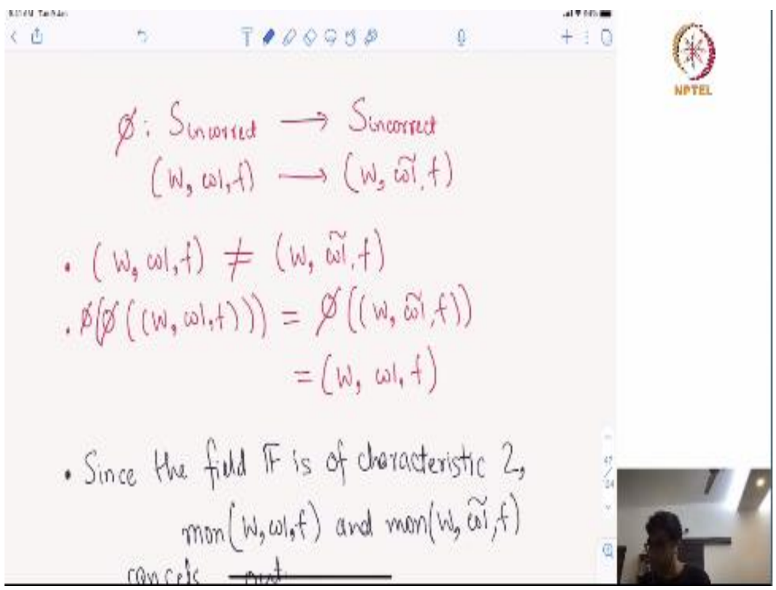

Since the field F is of characteristic to monomial W col f and monomial W col tilde f will cancel out, because they have the same one on it. So, now what we have been able to achieve? We have been able to achieve the pairing function. Look at this if you have an incorrect CPS then we can pair them out and over field of characteristic to they will cancel out each other.

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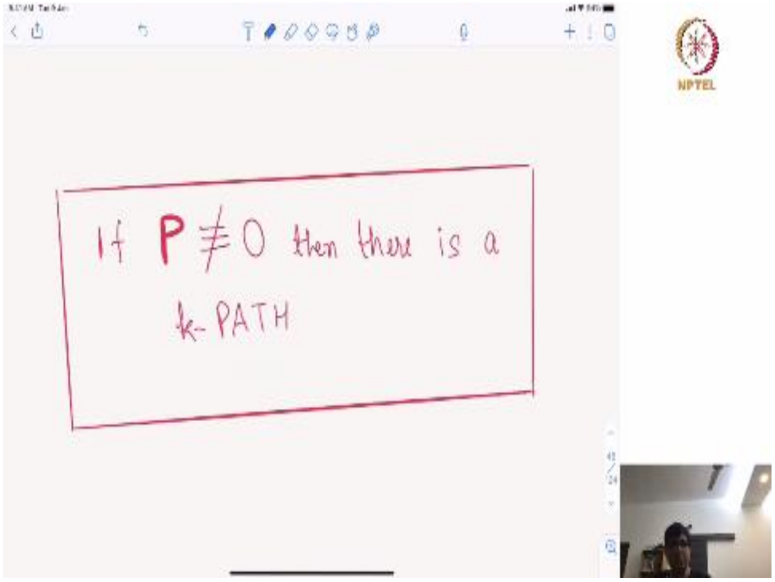

So, what we have known, if P is not identically 0, then there is a K path. So, let us look at that P. So, if P is not identically 0 then there is a K path because we know that given any correct CPS corresponds to a path because of our power path property, but because all the incorrect CPS will cancel out each other.

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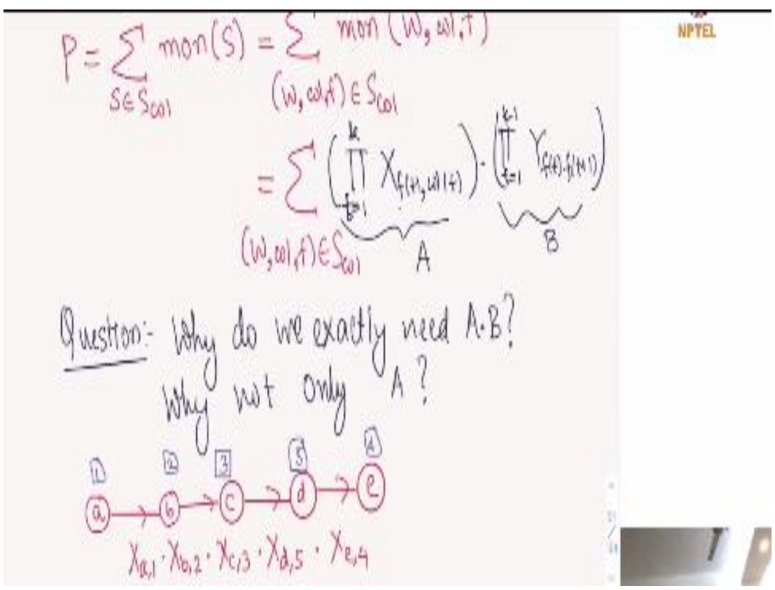

Now let us look at P. So, P is S in S col monomial S and this monomial function. So, I have just explicitly written down the monomial corresponding to this. Question there are 2 things I would like to say. So, look, we have been able to show to you that if P is not identically 0, then there is a K path, so let us look at this P. Question why do we exactly need A, B? We had this A and we had this B. Why not only a like just like this edge variable vertex variable why vertex color well why is there a reason for it also.

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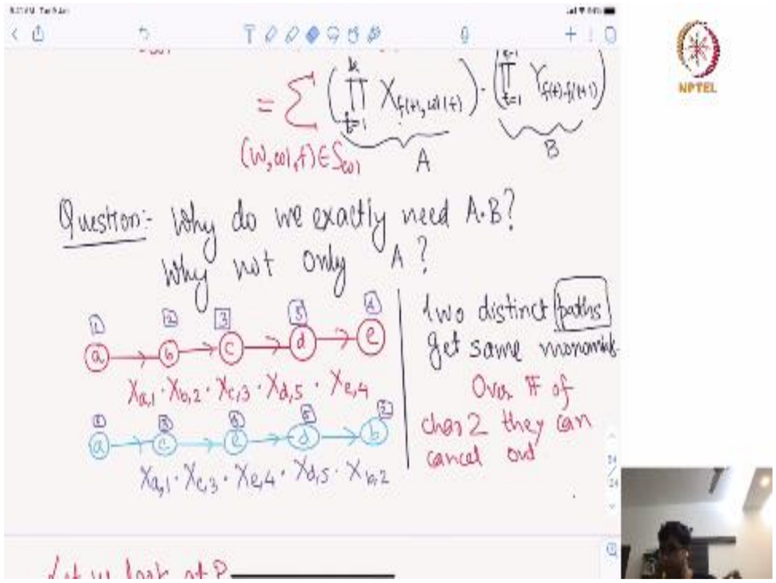

So, look at this a, b, c, d, e and so let us write down the color 1 2 3 5 4. So, you get this vertex. Now, let us look at a look at another path on the same vertex at a, c, e, d, b and just change the like just take the same coloring, so like one assure the c assigned 3 here, e was assigned 4, d was assigned 5. So, now if you write down these 2 monomials, they are same, 2 distinct paths get same monomial which implies that if I do a field of characteristic 2 they will cancel here.

So, what is happening is that look at same set of vertices, there could be 2 different paths on them first of all. There could be 2 different paths on them just the ordering of the vertices could change because look at a click you have n factorial ways of getting a Hamiltonian path. Now, if you just also fix the colors on these guys like same vertex gets the same color just at whichever occurs then these 2 guys will multiply and give you the same monomial but since these 2 are distinct path adjacent are distinct and they will differ. So, that is a reason why.

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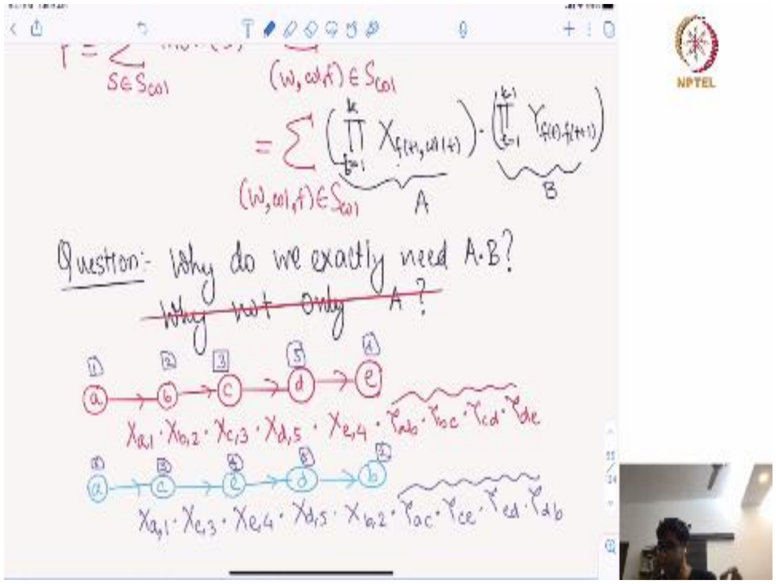

So, now if you add a then they are Y ab Y bc Y cd Y de it is a different and now they are distinct. So, you do not want correct CPC correct like correct colorful potential solution to cancel out each other and this is why we have added this edge variable because if 2 distinct paths their edges are definitely distinct. So, you can never cancel out them.

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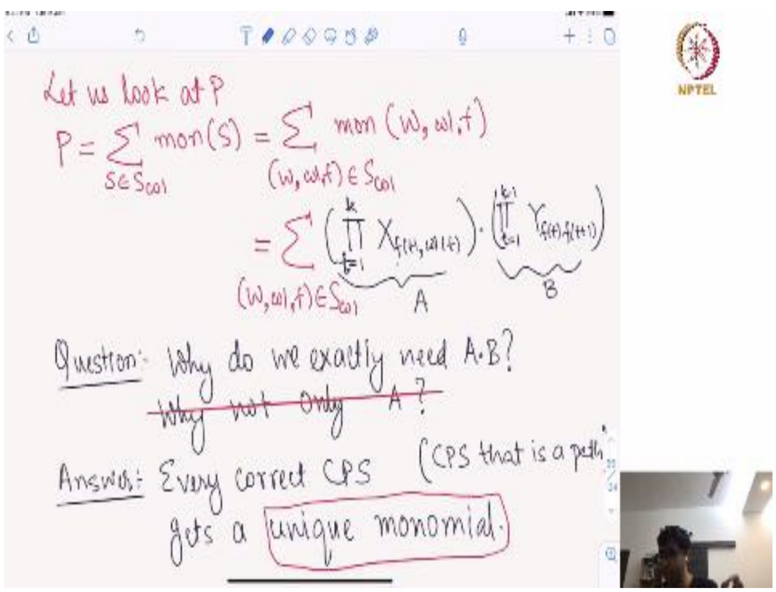

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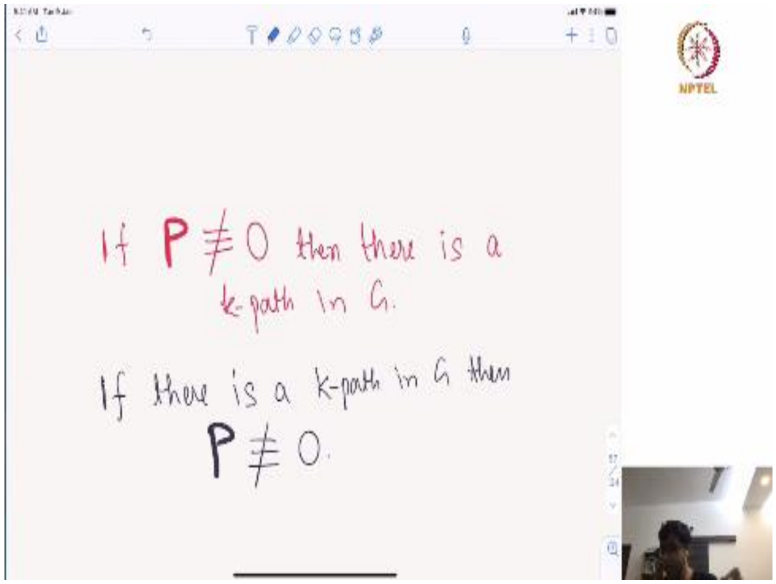

So, which implies that every correct CPS that is a path; gets a unique monomial. So, now what we have shown that, if P is not identically 0 then you know that the K path in G and if there is a K path in G then P is not identically 0. So, this is what we started our so, we have been able to obtain a polynomial which is a property of this polynomial that if P is not identically 0. Then there is a K path in 0 and if there is a K path in G, then P is not identically 0. So, P is not identically 0 if and only if the K path in G. So, that is it.

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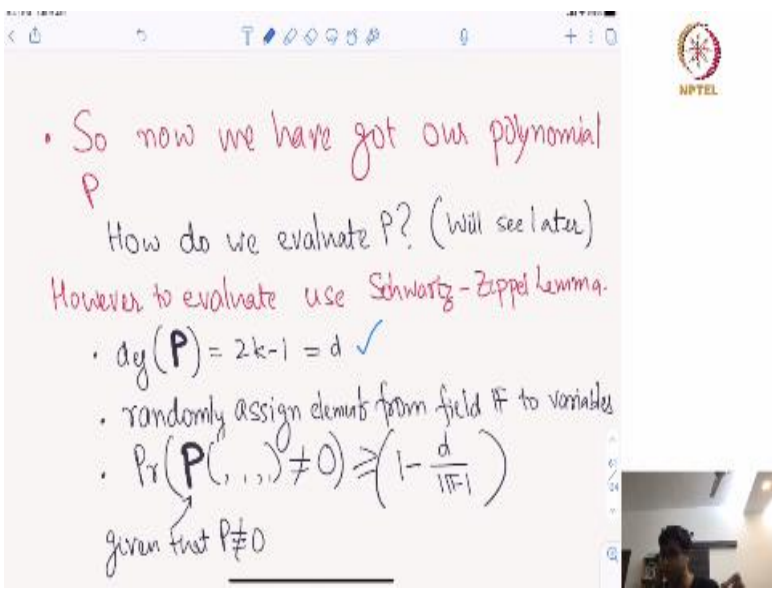

So, now we have got our polynomial P. The question is how do we evaluate we will see this later. However, to evaluate, we are going to use Schwartz Zippel Lemma. So, what is the degree of this polynomial? It is just 2K - 1. Because look at any monomial it consists of K variable vertex variable and K - 1 edge variable, so 2k - 1. So, what we are going to do randomly assign elements from field F to variable, so that probability of this is not equal 0 is at least 1 – d over F given that P is not identically equal to 0. So, this is what we are going to use.

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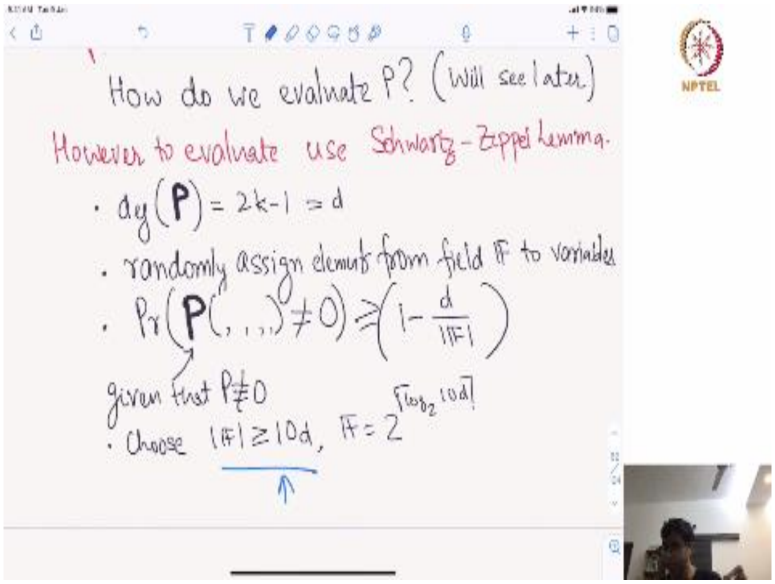

So, we are going to choose field F of size more than say 10 d. So, the field F l will be like roughly log base 2 10 d.

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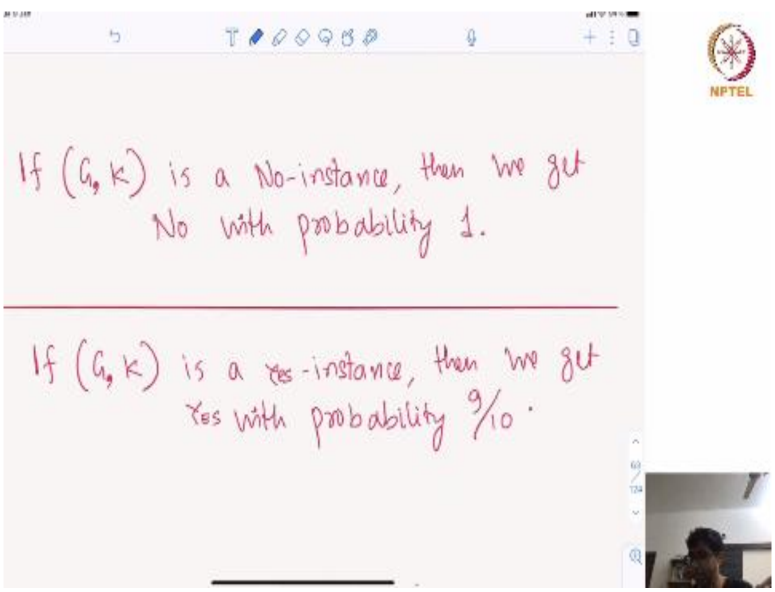

So, what is now if G, K is a no instance then we get no with probability 1 so, if you do not have a path then you do not have path if G, K is a yes instance then you know that this polynomial is not identically 0. And if it is not an identical 0 then if you do this random evaluation, then you will say yes with probability 9 over 10. So, that is it. So, now this is how we are going to do like, how do we do it, we will come to this but this is how we will do evaluate, we will apply Schwartz Zippel Lemma. We know that degree is 2K - 1, we choose a field F of size at least 10 d and we will do this, so far so good.

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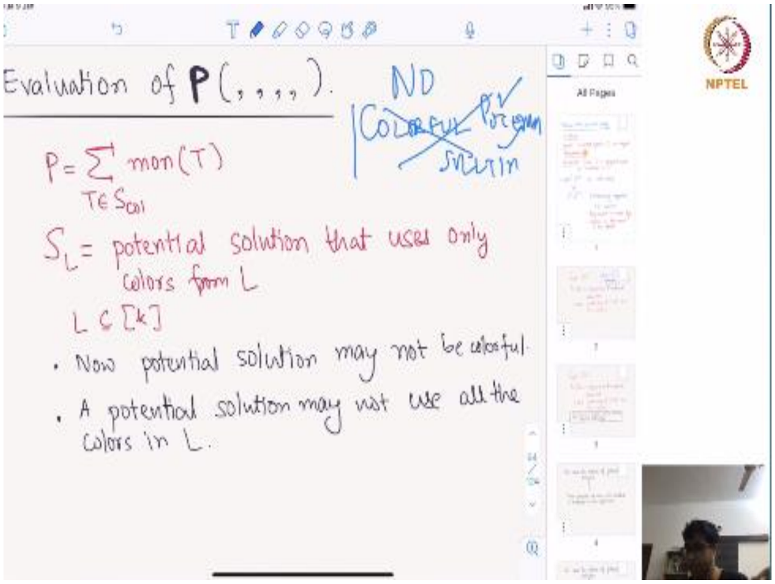

Now let us talk about evaluation of P. What is P? Remember P is summation T is called monomial of T. So, this is what our P was. So, now we are going to do something more like what is S of L? It is a potential solution that uses one of the colors from L. Now we are not talking about colorful potential solution. No, we are talking about potential solution. So, the point potential solution are a walks first of all. We are not even talking about incorrect potential colorful potential solution or correct colorful no we are just talking about potential solution. So, potential solution if you recall was nothing but walks. Let us just go back.

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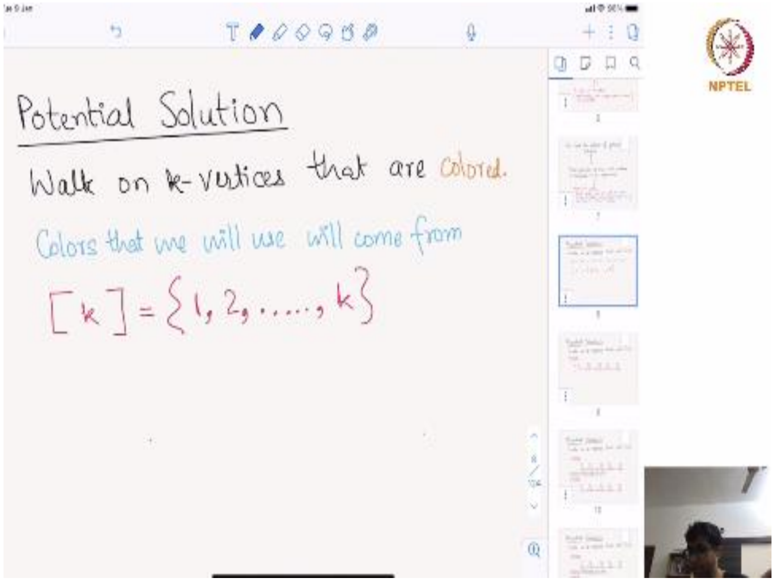

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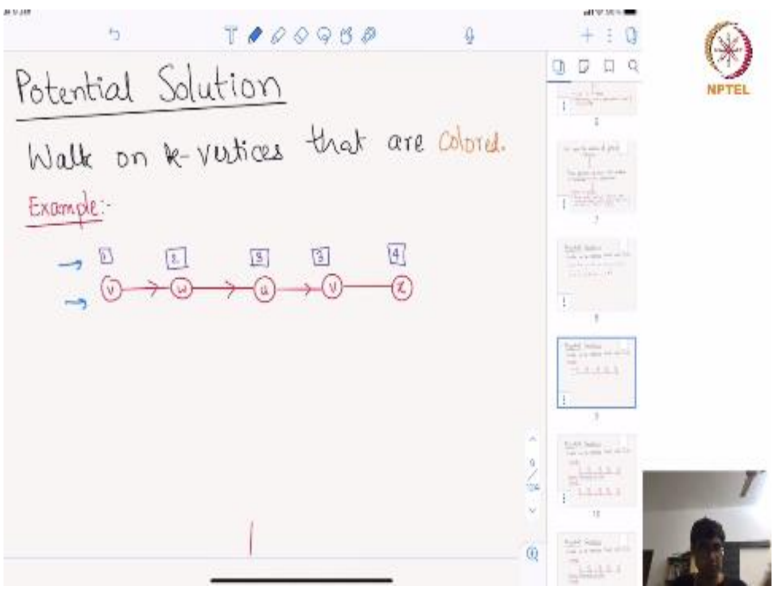

Walk on K vertices that are colored that is exactly what so. So, this is a potential solution right look here vertex are also repeated and colors are also repeated. So, this is a potential solution and we had refined that to talk about a colorful potential solution and then good and bad.

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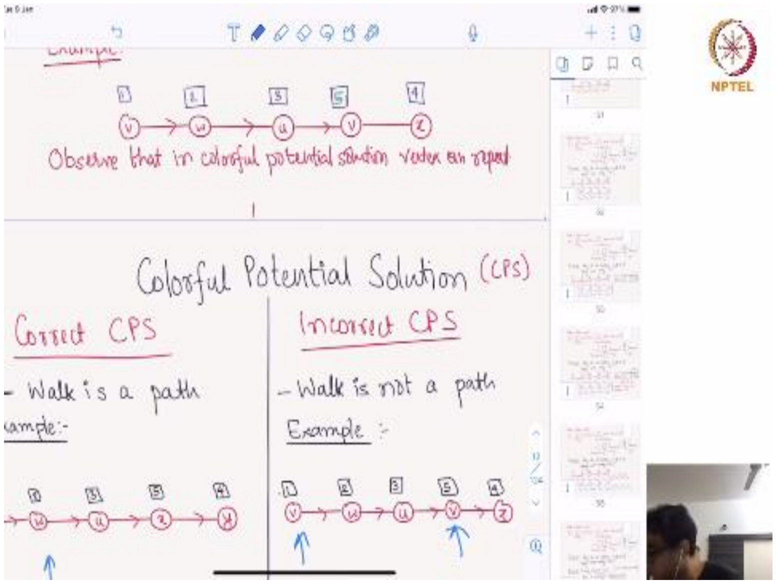

So, just like so, we have come back to just talking about potential solution which is nothing but walk where vertices are colored

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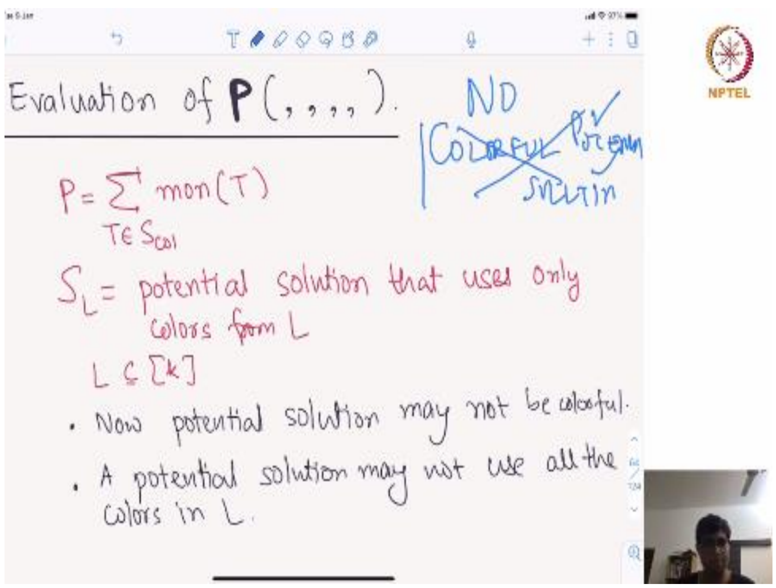

Now potential solution may not be colorful. So, a potential solution may not use all the colors in L. Only thing we are saying that look at this, like what is the S of L those potential solution where the set of colors used is a subset of L that is it. So, potential solution may not be colorful, a potential solution may not use all the colors, this what S of L means.

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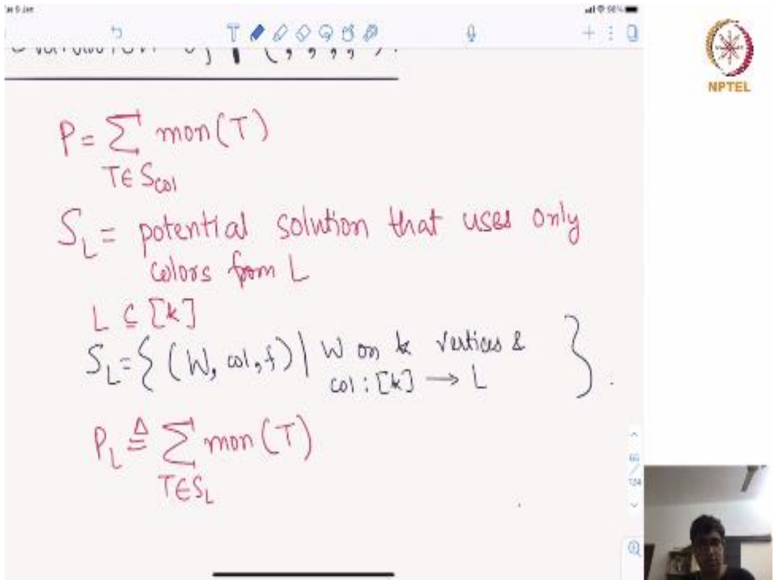

So, what is S of L W col, f W on K vertices and col function is from 1 to K to L. So, what is P of L? It is by definition T in S L monomial T and we know how to write monomial corresponding to S L.

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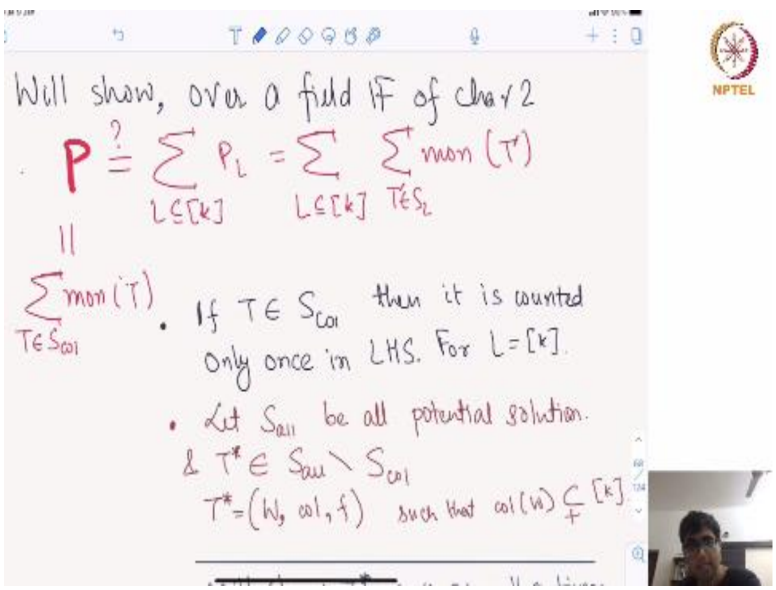

So, let us try to see what this P corresponds to. So, let us ask our self I want to say that the P which we made with respect to S col is summation L subset of K remember our definition of P was summation monomial T, T is in S col, S colorful. So, I want to say this is nothing but summation L subset of K P L and then this is just an extension of that let us see why. So, notice if T is in S col then it is counted only once in left hand side for L = K.

Because what is the property of S col? It is colorful potential solution. It means all the K colors are use it could be it may not be path but it could be walk but all the K colors on use this is the definition of P which became of here now if T and S col then it is only in counted for L, L = K only once in LHS for L = K.

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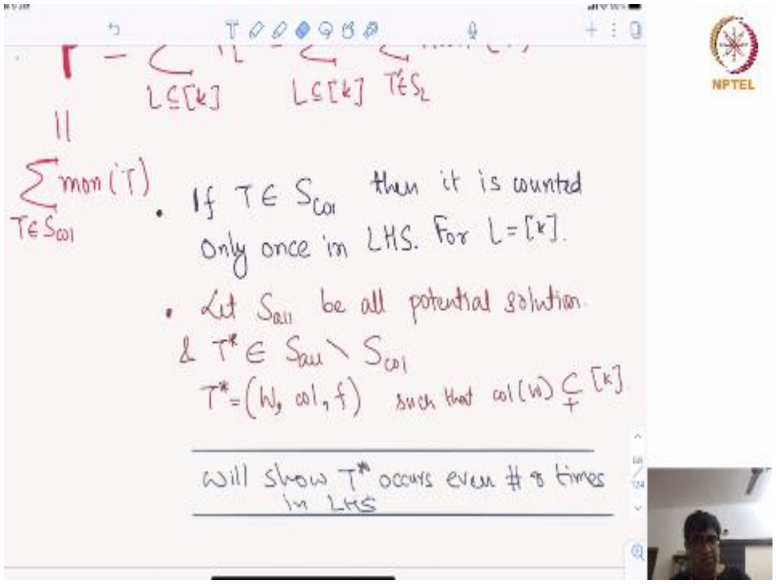

So, now let S all be all potential solution and T star when it is all minus S col. So, T star is W col f such that col of W is not is a proper subset of 1 to K. We will say that T star occurs even number of times in left hand side like T star occurs even number of times. So, when you sum them up the monomial corresponding to T star is even and they will contribute 0.

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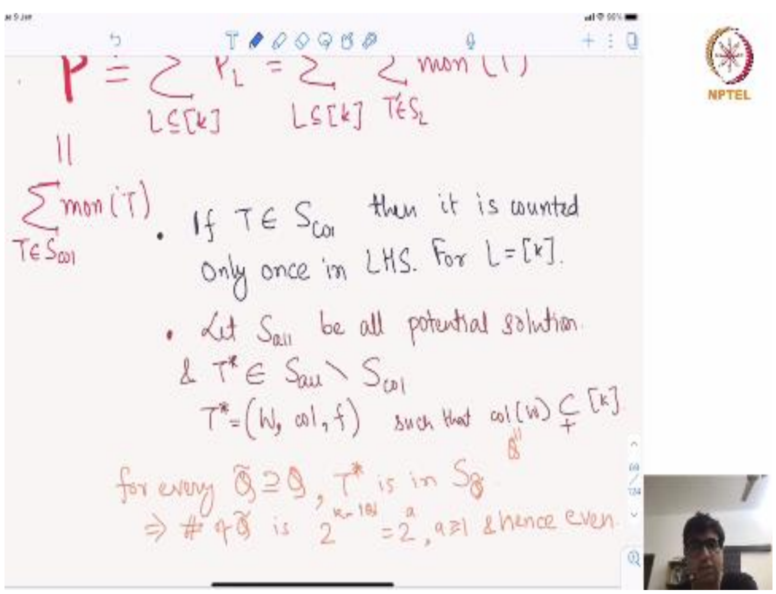

So, now look at just ask yourself, look at suppose the col of W is Q just color Q. Now, look at any Q tilde which is a superset of Q then T star is going to appear in S Q tilde. So, now you ask yourself how many Q tilde there? It is 2 to the power K - cardinality of Q which is 2 to the power a is greater than equal to 1 because this is a proper subset and hence this is even.

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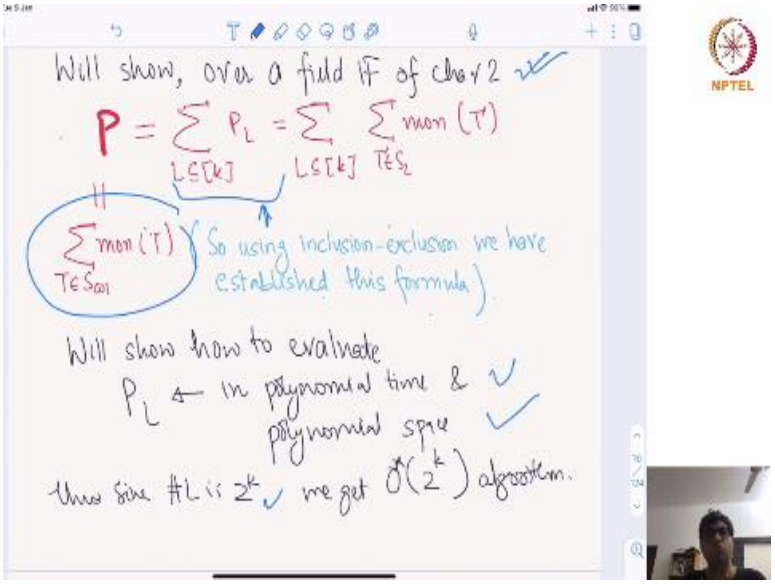

So, this is very much similar to using inclusion exclusion which and we can establish this formula this is over a field F of characteristic. This is very important otherwise we cannot say this. So, what we have been able to show that summation T in S col Mon T is summation l subset of K P L where P is overfilled of characteristic. Now we will see how to evaluate P L so rather than evaluating this we will try to take this definition to evaluate P efficiently.

So, now look at it I am going to tell you how to compute how to evaluate one particular P L in polynomial time and polynomial space. Now how many terms are there? So, the number of terms is like 2 power k so that will automatically by O star 2 power K algorithm that is 2 power K Poly K Poly n and poly space. So, now we have established this formula. All that remains to show is how to compute the evaluation of P L.

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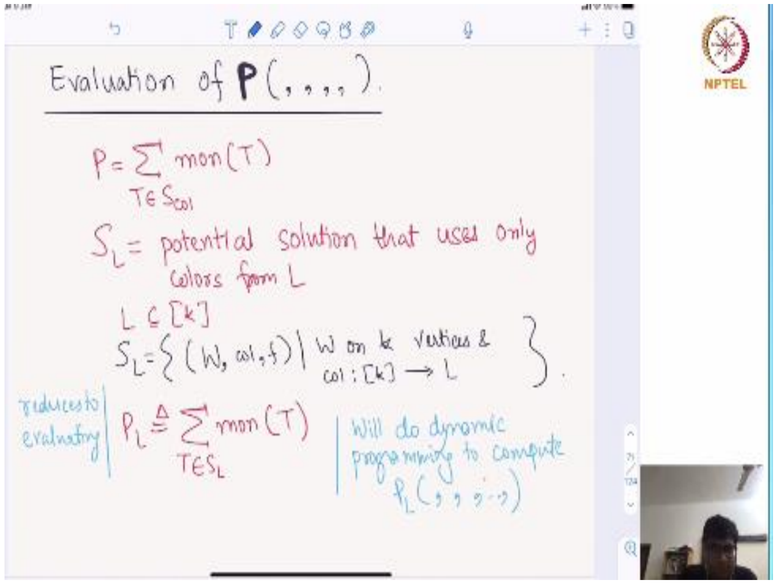

This is a simple dynamic programming algorithm. So, let us spend few minutes on that. So, evaluation of P is this so S L is potential solution that uses only colors from. So, S L is this so we have reduced is to evaluating P L by definition T in S col mon T and we will do dynamic programming to compute. So, you know that what is the dynamic? So, you know that these variables have been assigned numbers like excise now you want to evaluate P of L on these points.

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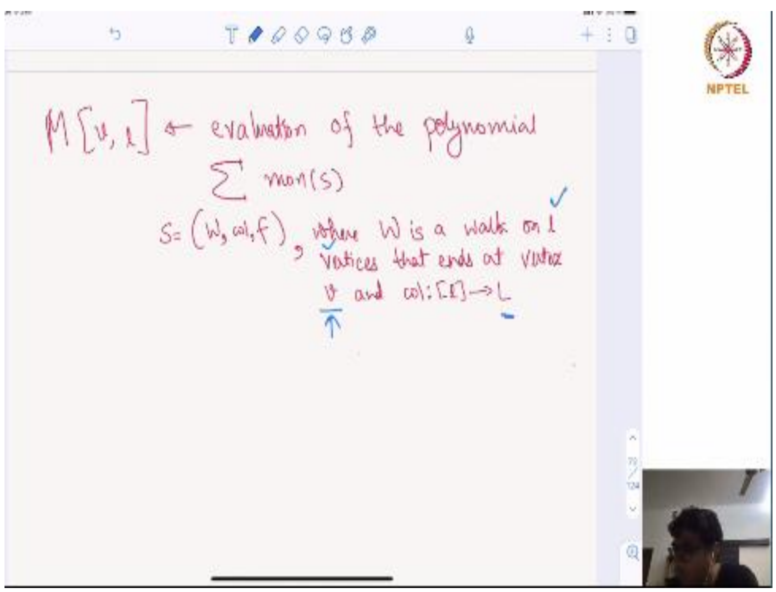

So, we are going to have M v l so, we are going to keep up 2 dimensional array which will keep the evaluation of the polynomial mon S going over their W the walk on but this is only going to keep an evaluation of a part of my polynomial what is the part of my polynomial? W is a walk on

L vertices that ends at a vertex V and first L vertices uses color from 1 2 that is it. So, what is M v l going to compute? It is going to compute evaluation the polynomial mon S going to W, col f and W is a walk on L vertices that ends at the vertex v and this.

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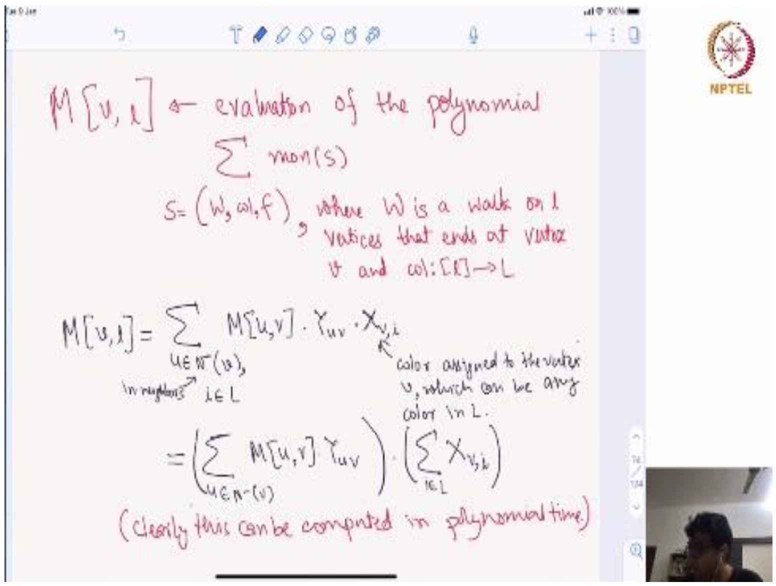

So, now what is M v l? So, we are going to write this so look if I am trying to construct a path like compute all walks of L vertices that ends in v it means I must end in some u which is in neighbours of v. So, summation u in neighbours of v and after that I have to take this edge u v so, this is an edge u this is how your polynomial is and it is a summation over this and summation also what and also this last vertex could get any of the color assign. So, you also have to in the put in the summation i in L X v i.

But what you have computed previously you already have computed so I can write this is nothing but this into summation X v i in L. So, clearly this can be computed in polynomial time you already have computed this you just put up like you put up this substitute all this M u,v summation, just put up all this substitution and you can compute this polynomial time. So, this is a very simple evaluation, you can do this and you will be done.

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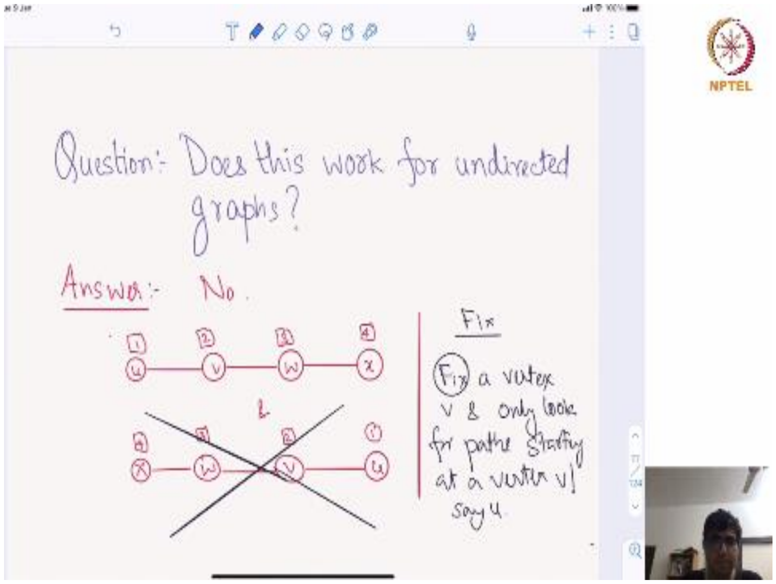

Now the question is, so just one point I would like to make before we end this, does this work for undirected graph? Answer is no, why? Because look at u v, w, x 1 2 3 4 x, w and x w we just change it and change the color. These parts in their reverse will cancel out, but that is not a problem. Because what do you do? You fix the vertex v. And you only look for paths starting at a vertex v, see you so you fix one vertex and say, Look, I am only we will start looking at a vertex which starts at u. Then this proposition will not happen.

So, this is how we can make with just this small fix, you can make this also work for under it. That is all will be in this week. And that is it. Thank you very much.