Parameterized Algorithms

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Lecture – 40
Algebraic Techniques: Inclusion Exclusion (Hamiltonian Path)

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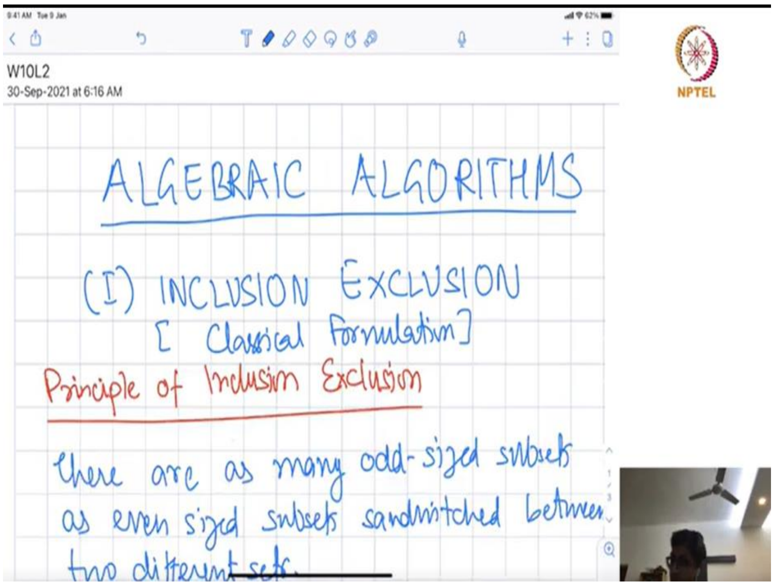

So, welcome to the second lecture of this week and this lecture we will talk about algebraic algorithms again, we will talk about again inclusion exclusion give another example, but we will talk about slightly classical formulation. So, if you recall from the last lecture that the non-classical formulation that we saw is that there are as many odd size subsets and even size subset sandwiched between 2 different sets for odd subsets of T and then we derive this formula – 1 to the power of T - S R S T is evaluates to 1 when R and T are equal otherwise, it evaluates to 0.

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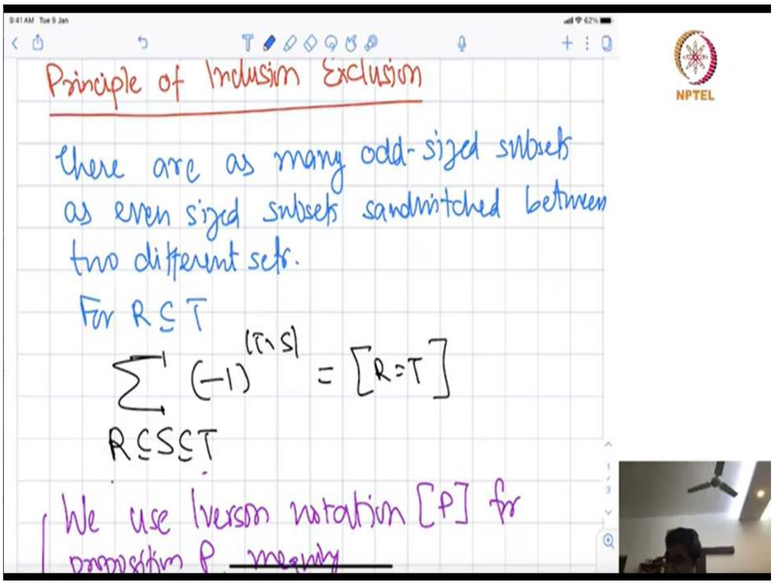

Because, look at the number of sandwich sets of even size and look at the sandwich sets of odd size. So, even size sets are going to contribute one to the left-hand side some and odd size sets are going to contribute - 1 and since they are equal they are going to cancel out each other than that is going to be 0. So, this is a classic but this is not how the formulation of including generally taught.

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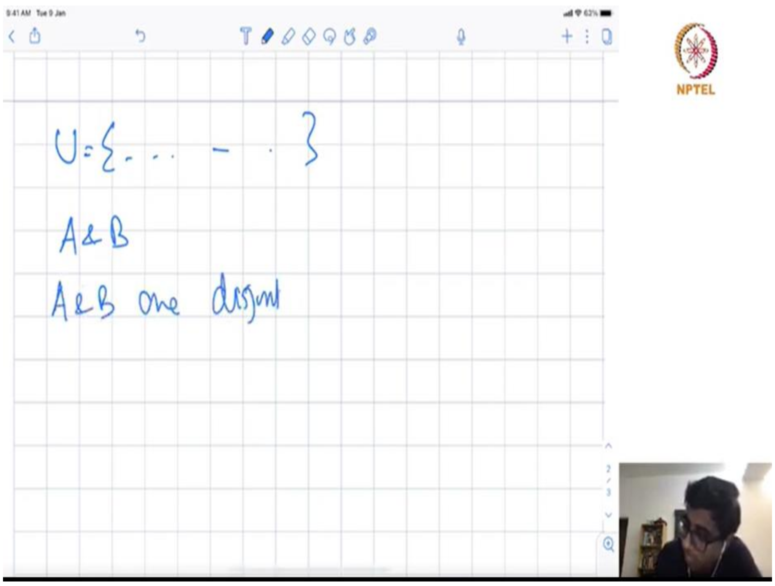

So, the generally taught is that we are given an universe U of some objects. And now, suppose, you have 2 sets A and B right and A and B are disjoint then all we can say that A intersect A union B is = A + B but A and B need not be disjoint then all we can say that A union B is at most cardinality of A plus B because you could have sets say A B, but the intersection object is counted twice right both in A and both in B. So, the basic formula that we could come up with you have seen this A union B just simple in diagram exercise.

But they can also generalize this to 3 sets. So, this is A this is b and this is C. Now, we would like to compute the value of A union B union C. But now notice that there are lots of over counting going on right? What is an over counting going on? Over counting going on is that these elements are counted for this set this set and so on and so forth. So, you try to subtract them out. let us see this elements.

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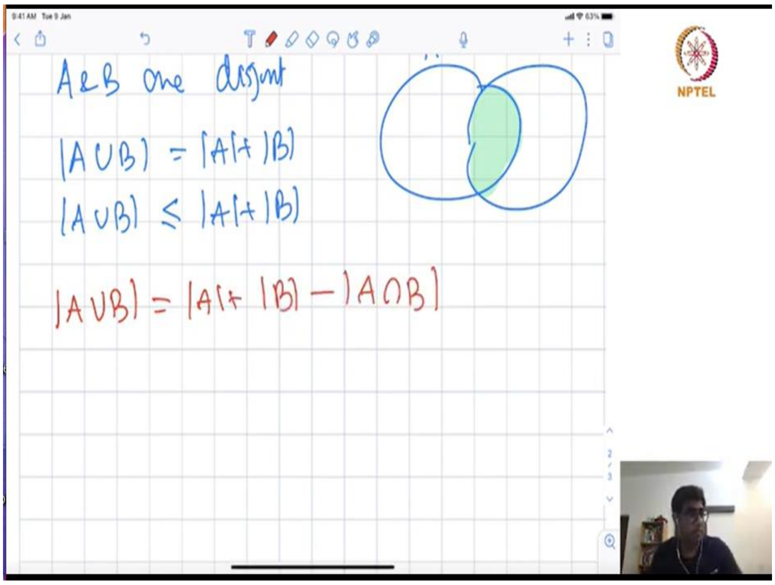

Now, I also need to subtract out this element. So, this is given by B intersection C, but now also need to subtract out. So, B intersection C but I am Now notice that these elements are subtracted exactly once because they occur in A and so yeah, so these elements are subtracted exactly one because they are common only between A and C. So they will one day be sorry this is not into A intersection C similarly these elements are subtracted again. How many times?

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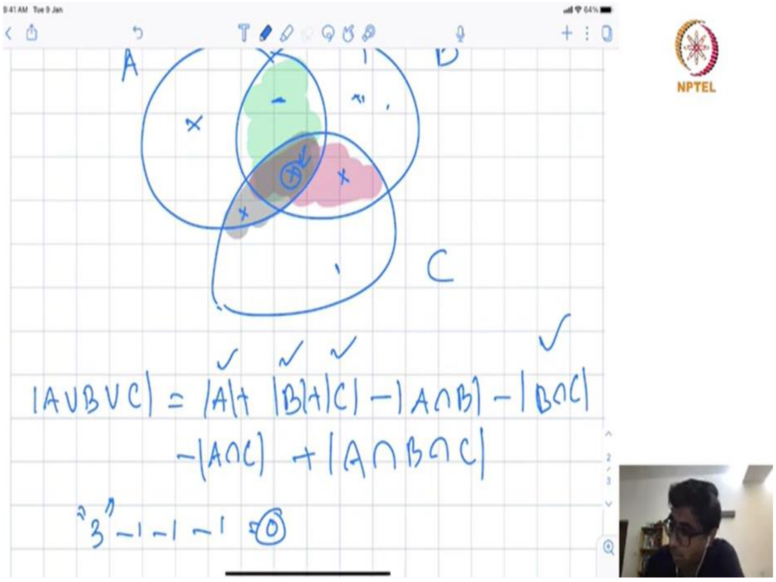

Only one time just B and C. So this is fine. This green elements similarly they are subtracted only one, but look at this common A intersection B intersection C, How many times have they been subtracted? So, these elements but counted 3 times so let us look at A B and C. So, look at these elements, these elements are counted 3 times each element one in A one in B one in C. Now, this element is subtracted how many times?

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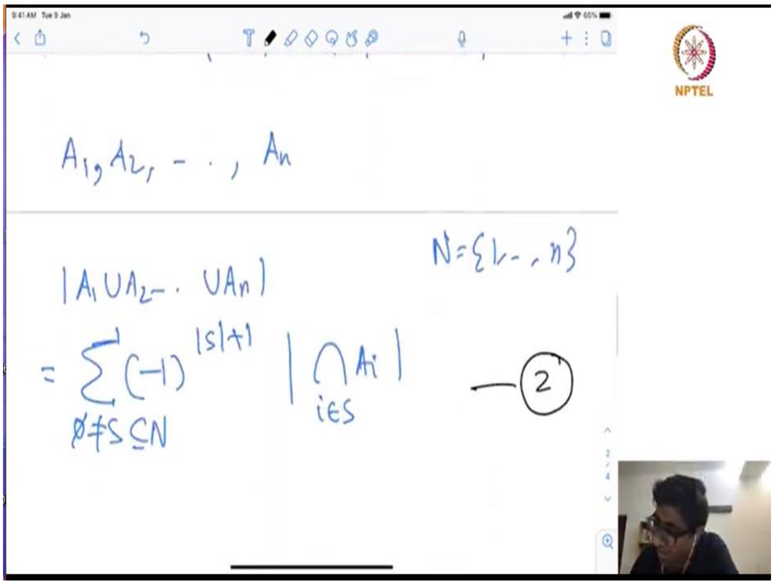

Let us look at these elements are subtracted when you subtracted A intersection B - one time, these elements are again subtracted when you subtract B intersection C, so another - one time and these elements are again subtracted when you did A intersection C. So, you have not at all counted them. So now you need to add A once more. So, this is a formula and now this formula can be generalized to not only 3 sets, but these can be generalized to further.

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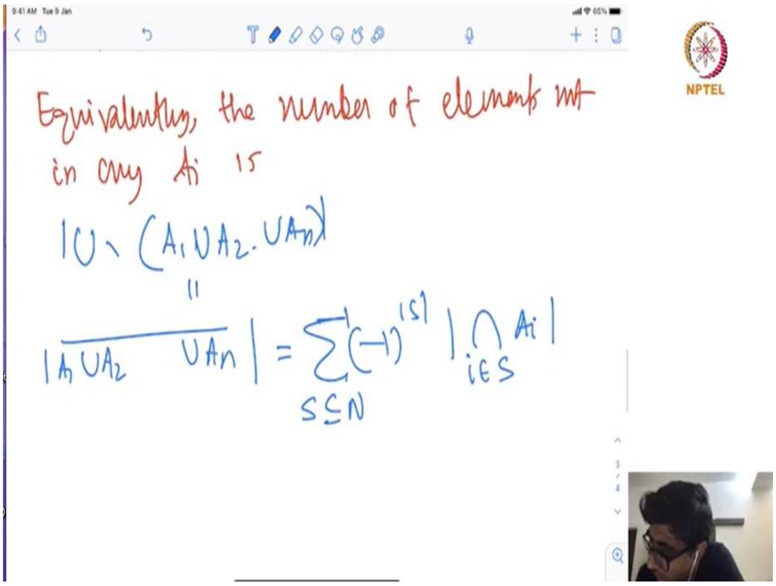

So, notice that right hand side contains basically of all possible intersection the sets we have and the sign depends on how many sets it intersects. So, now generalize this to suppose you are given over this universe not 1 set, but you are given sets A 1 A 2 dot dot dot A n, n sets are given. Then we can generalize this and we can get this A 1 union A 2 union A n is nothing but summation.

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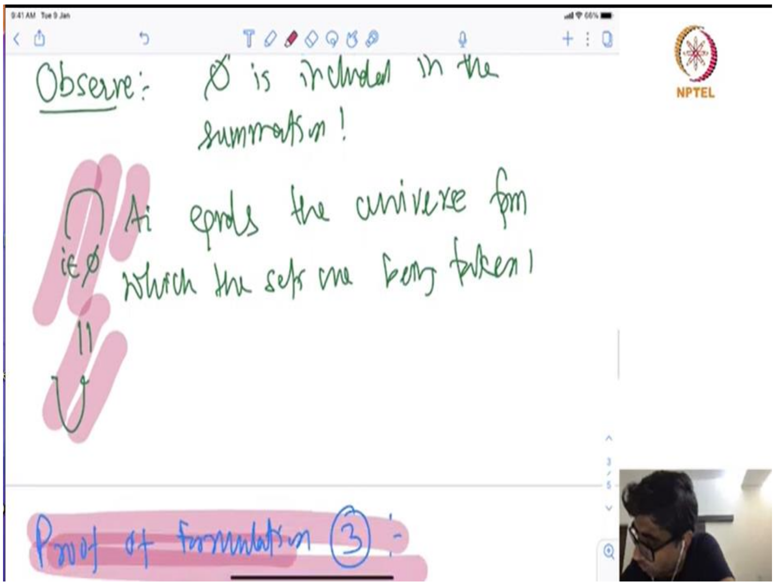

This is nothing but summation going from phi not equal to and let me say that N is this index at n. So, phi not = S subset of N - 1 that is a sign coming in mod + 1 intersection Ai i in S. So, basically look at S. So, when will the set look at S = Singleton so, then they are going to contribute + 1. So, why did you add + 1? Because you notice that the alternation your of sign so, once I said all sides contributes +1 and all even set size contribute - 1.

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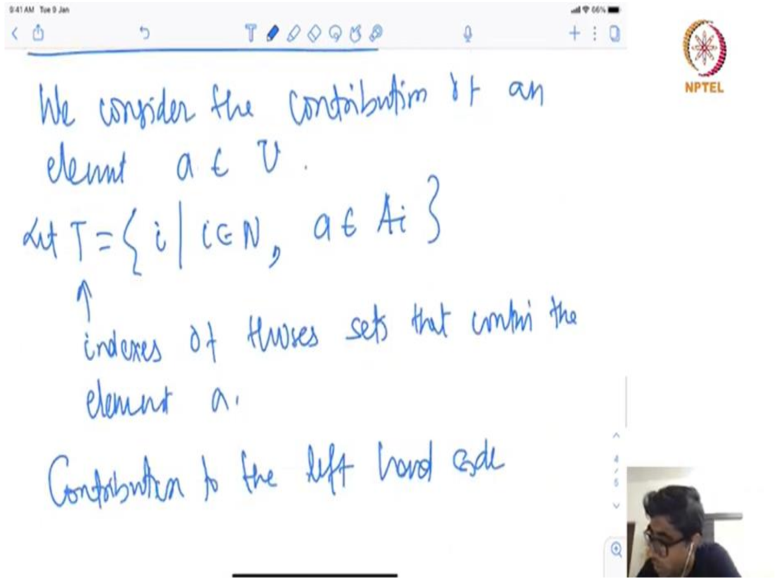

So, if I would have just written cardinality of S for S subset of N, then if it even said that contribution would have been positive, but we want that to be negative, so, we just added + 1 to this. So, this is the formula. This is the classical formulation of inclusion exclusion that if you are interested in counting the union. So, this is one formulation but the more algorithmically useful formulation just follows again from the Venn diagram computation is the following.

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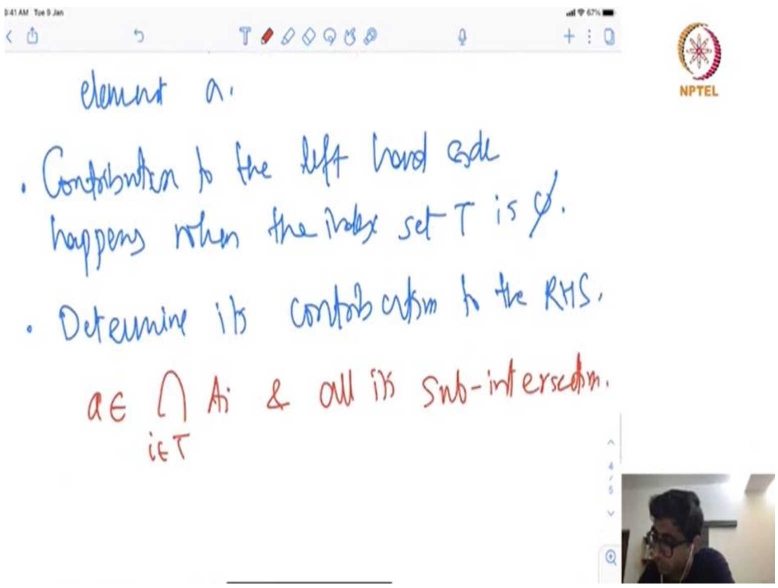

So let us call this 2 actually. So, this is our second formulation, let us call the non-classical formulation that is first formulation we saw and there is another formulation who just achieved by the following. So equivalently we can say the number of elements not in any Ai. So suppose now, we were counting all the elements that occurred in the universe, but suppose I wanted to count U minus this.

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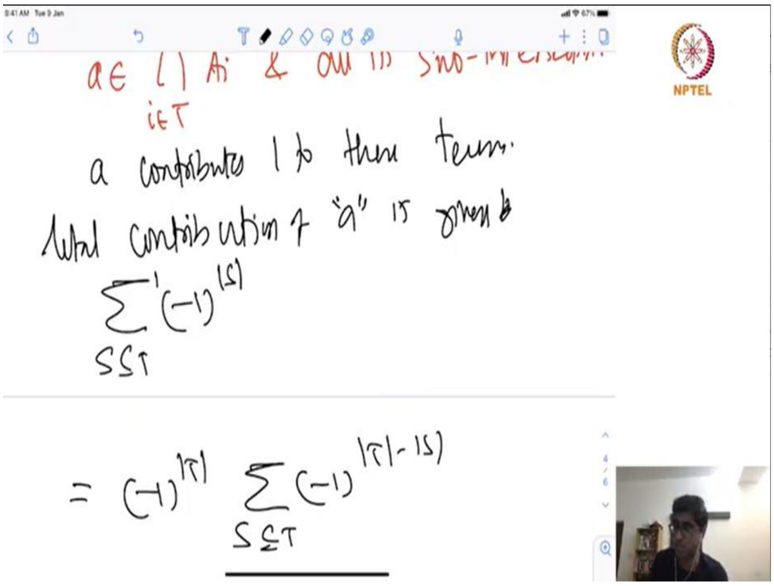

So, I wanted to count the cardinality of universe – A 1 to A n which is nothing but this is same as A 1 union A 2 union A n complement and we will prove this. S subset of N – 1 to the power S i in S Ai. But there is a usual convention notice that I wrote here phi S was not in phi but here phi is included. So it is an important observation to make phi is included in the summation and what is the meaning of this?

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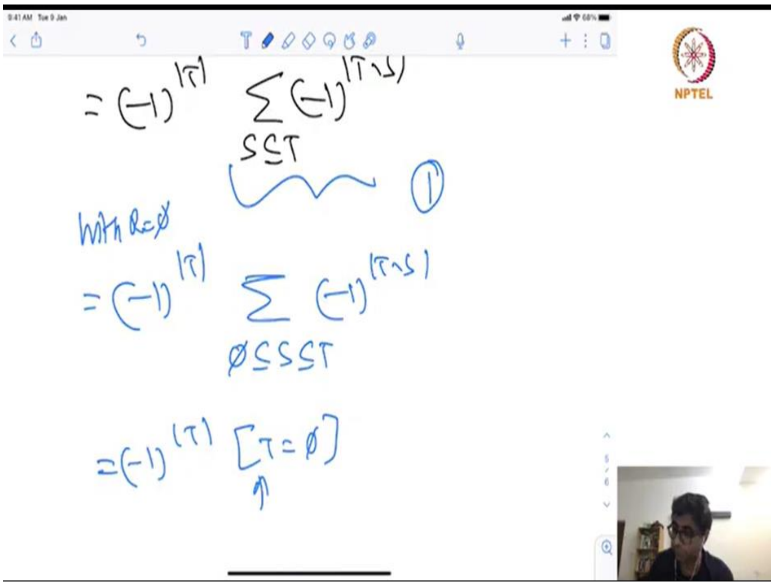

So, i in phi Ai the convention is this equals the universe from which the sets are being taken. So, this is a usual convention. So, in our case this is equal to U this is an important observation. This is an important something that we need to remember. So, by including this now let us try to prove the, let us give the proof for let us call this formulation 3. And the usual way of showing any such proof is very simple is that you look at particular element.

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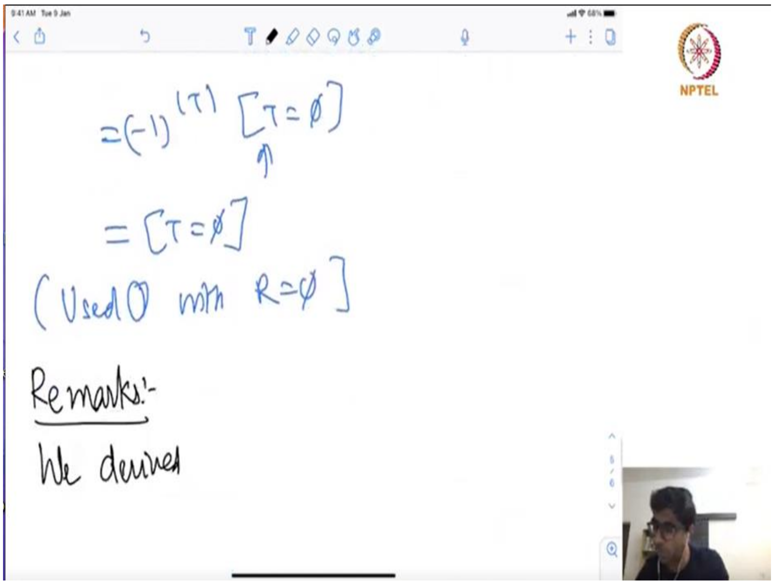

And you see what is its contribution the left side what is its contribution in the right side. So, we considered the contribution of an element a in our universe and how do we go about proving this very simple. So, you look at this index at T. So, what is this i so, this is an i, i belongs to N the index set and what is the property a belongs to the set Ai. So, T is precisely all those sets Ai which contains a. So, what is this T? Basically, indexes of those sets that contain the element a.

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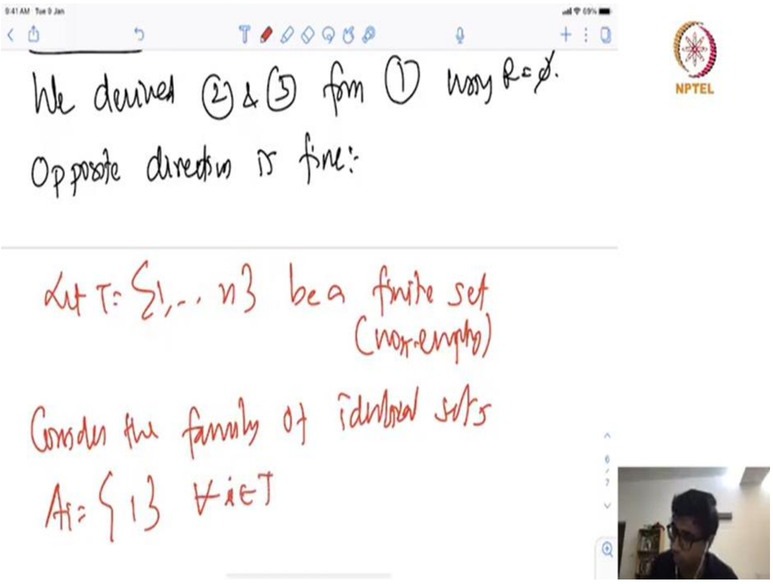

So let us ask ourself. Contribution to the left-hand side, right? Let us ask ourselves think for a minute, when are you going to contribute to the left-hand side? only when you do not appear in either A 1 A 2 A n, or in other word, contribution to the left-hand side happens when the index set T is empty. Now let us try to determine its contribution to the right-hand side.

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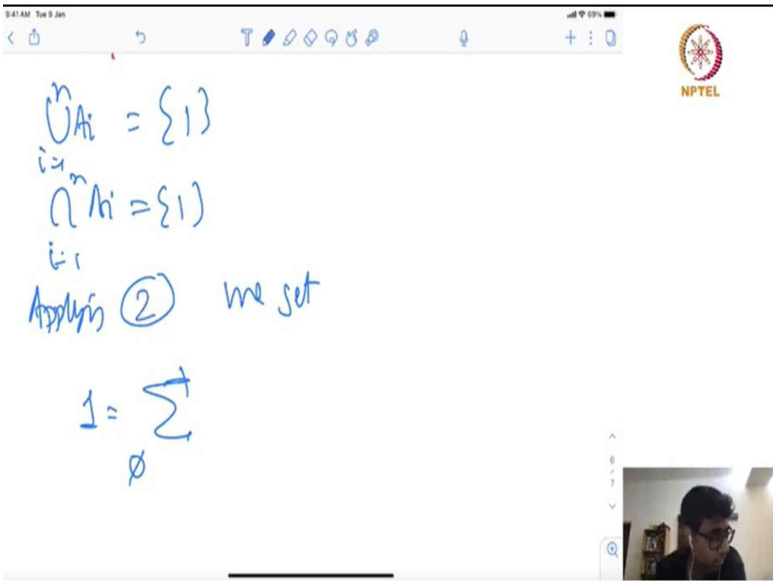

Let us try to contribute it is a count. So where does a belongs to so a belongs to i in T Ai and all its sub intersection. Notice, so what is right hand side? It is all this Ai in which like for all index set their subset that intersection. So now I asked myself, I know that a belongs to this index. So, it means a is in all this T. So, a will appear in every subset of T because like in the intersection of these elements.

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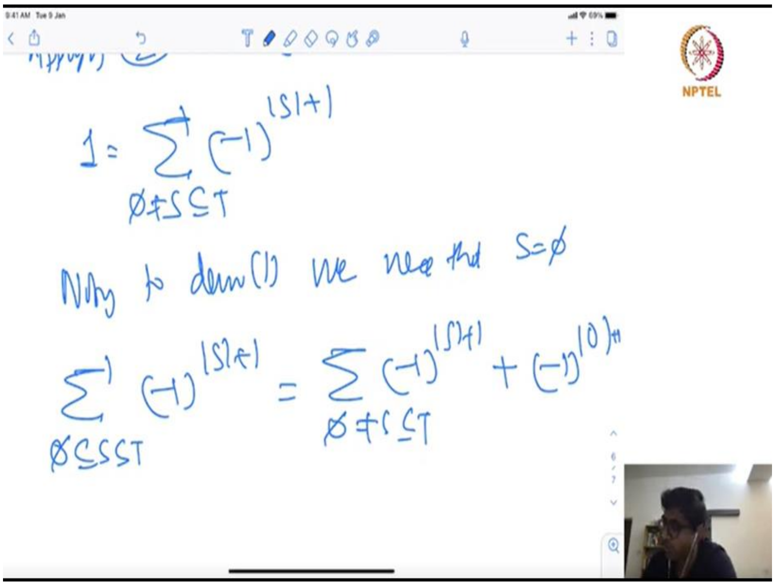

So, for every S subset of T. I know that a belongs to its intersection including phi because phi is nothing but U itself. This implies a contributes 1 to these terms. So, now the total contribution of a is given. Why? Let us check right summation S subset of T - 1 cardinality of S this is it because all we are trying to see is the contribution of a. So, a is going to contribute like 1 to all of this, but now you also have the set sign.

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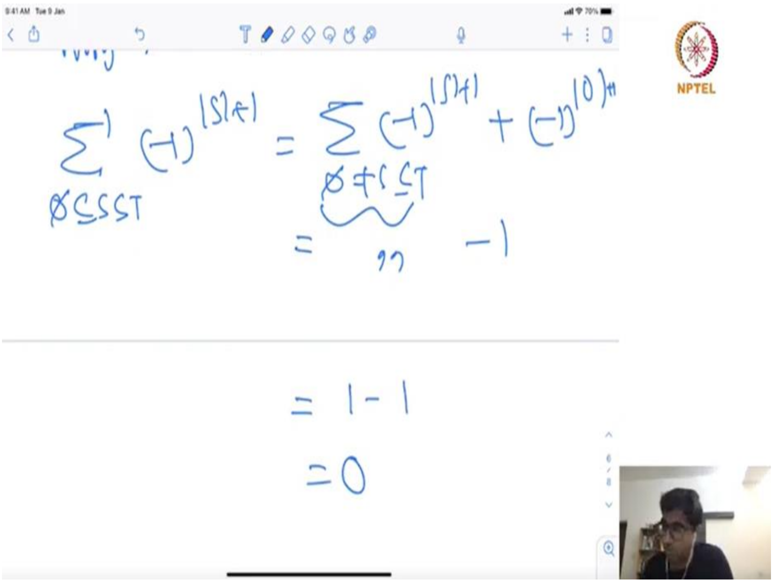

So, now, this is = - 1 to the power T just multiplied - 1 to the power T and + T. So, + T has gone inside So, this will come out. Now, does this is exactly the expression 1 and we know this is = - 1 to the power T. And when is this term going to contribute? So, how I am going to apply this. So, I am going to apply 1 with R = phi. So, I could have written this summation phi S subset of T.

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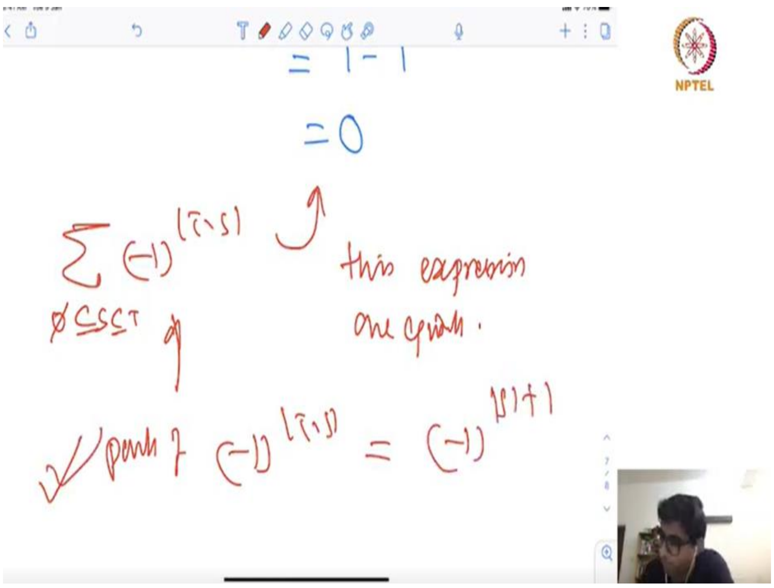

So, if you apply this then this implies what? - 1 to the power T = phi. This is that notation. So, this is = - 1 to the power T T = phi. So, T = phi then this is going to be contribute 1, So, T = phi so, this is going to contribute 1. So, T = cardinality of T 0. So, - 1 to the power 0 is 1. So, this is 1 and if T is not = phi this is going to contribute 0. So, this is great. So, this is we are going to get T = phi.

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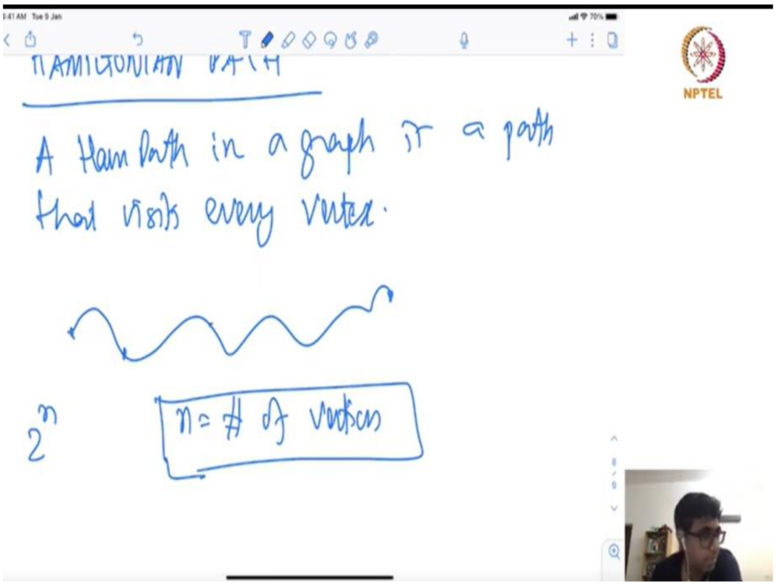

So, basically to derive this we used 1 with R = phi. So, what did we learned? So, we learned that an element a its contribution to the right-hand side if 1 if and only if a if the index set T is empty. If index set T is non empty then a contributes 0 to the right-hand side. So, now we know that we have picked up an element and we have shown that its contribution to the both sides is equal and if we have shown that its contribution to the both sides are equal then we have shown the equivalised

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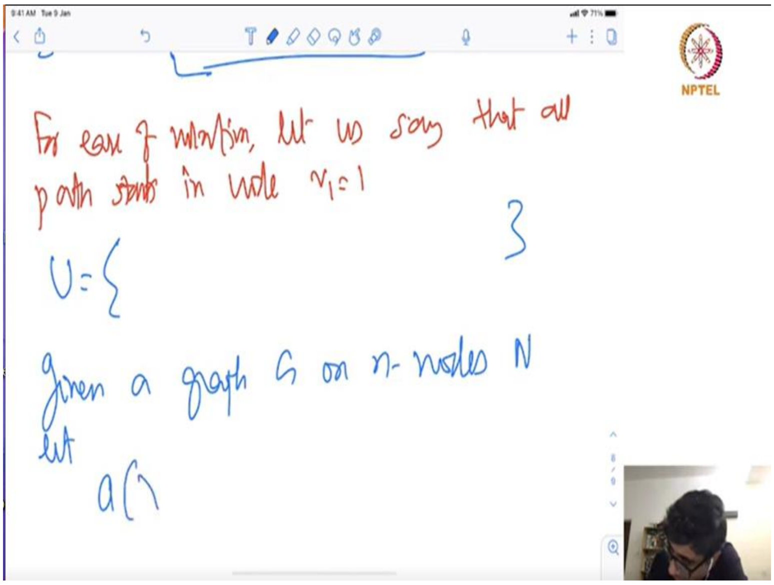

So, look at what does this 3 tells us? That look if you wanted to talk about some properties A 1 to A n, if you wanted to count elements that do not appear in this set then this particular formula works. I will show to you how actually we use this in algorithm in a context of

Hamiltonian cycle and enhance for other problem. But like notice or like this, so some remarks are in order. Look this and this follows from just De Morgan's law.

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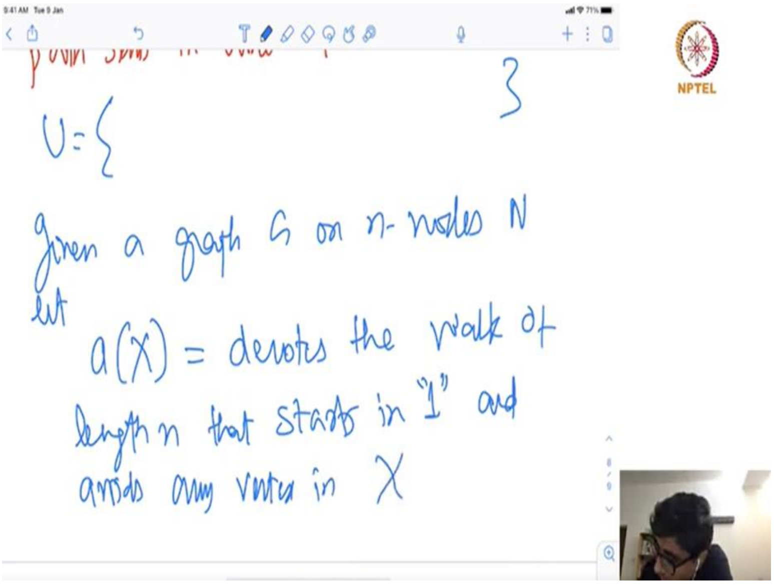

So, this is nothing. This is this 2 and 3 are exactly the same formulation. If you apply the De Morgan law you will get exactly this. So, I am not going to prove why 2 and 3 are equivalent, but 2 and 3 are just straight forward. So basically, what we derived? 2 and 3 from 1 using R = phi and let us so now that even opposite direction is fine. Why? So let us, we are going to take, let T be a non-empty finite set and T = 1 and now we are going to configure the family of the identical sets.

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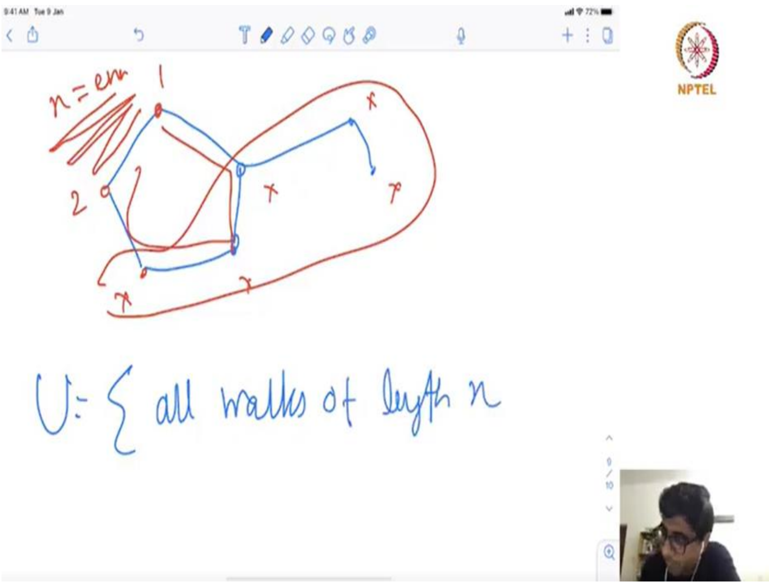

Consider the family of identical sets. What is this? So, Ai = 1, for all i in T so I mean basically I am fixing your index set. So, now, I should have started with, so basically what I am saying that you fix up U some universe and you come up with the sets A 1 to A n where A n is like each Ai is just contains element 1. So, now what we know about both these things that all the union of i = 1 to n is one intersection and in fact for every subsets of T the intersection and union is 1 so now if we apply 3, applying 3 what will we get?

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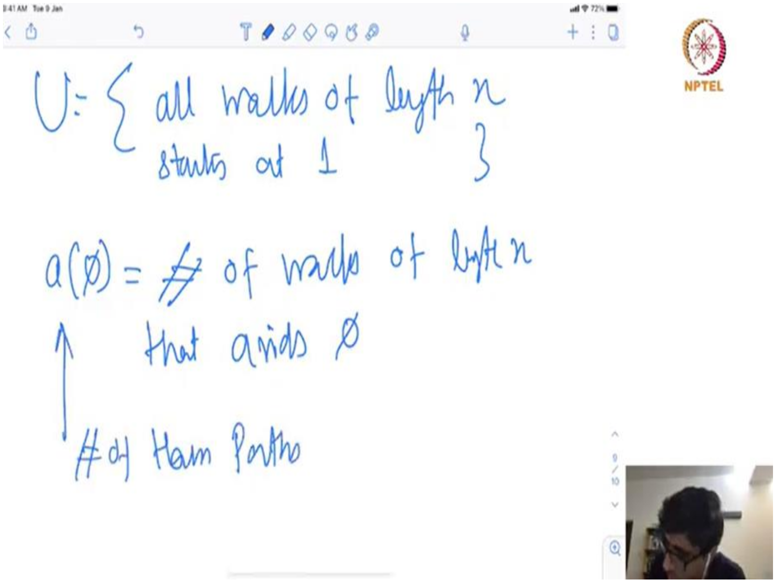

Applying 3 on this sets we will get 1 is equal to summation phi. let us not apply 3 let us apply 2. So, applying 2 soldiers to remember the 2 was A 1 to A n is this. So, basically A 1 to A n is just 1 and each Ai is also going to 1. So, this is basically – 1 mod S +1 and the summation. So, 1 = phi is not equal to phi contain inside this. Now notice to derive 1 we need that S = phi. But what is S = phi?

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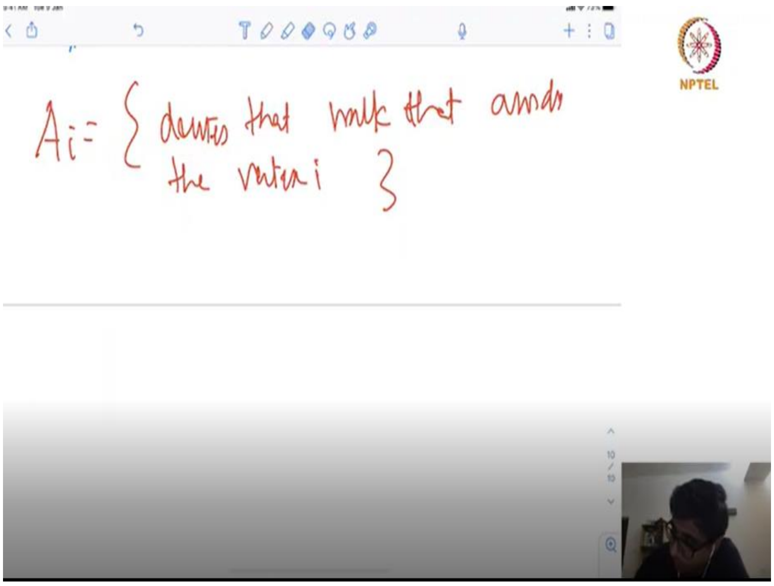

Well, this I can write this - phi if T - 1 S + 1 = phi not = S subset of T - 1 mod S + 1 + - 1 to the power set which is 1. We also have 1. So, - 1 to the power 1 this is so this quantity and – 1. So, what do we get? So, this quantity equal to minus 1. So what are we going to get is this is equal to 1, so, 1 - 1 which is equal to 0.

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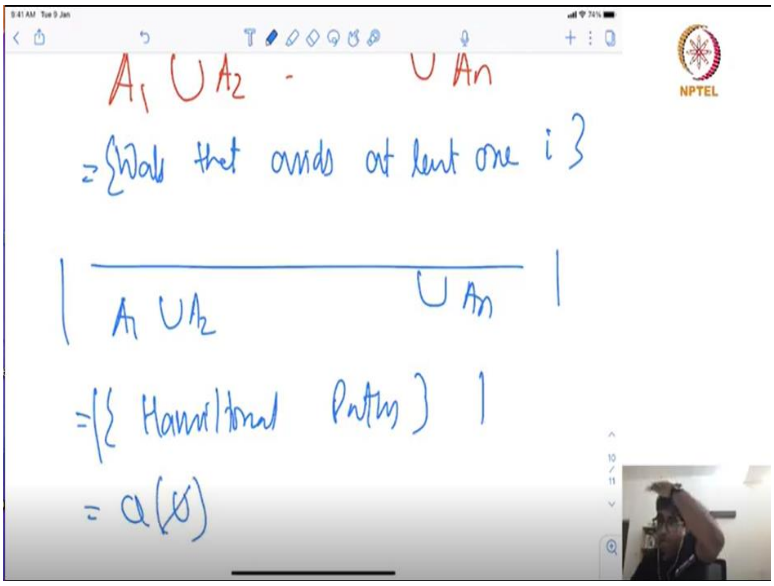

So, be applying to we got this and now to derive 1 we started this is what we were interested in. Right now, you might ask but this is not what we were interested in? Let us look at so this is what we get. But we were interested in – 1 T - S. This is the expression we were interested in. And now I show you this and this expression or equivalent. Prove it. So, you can show that the parity will not change parity of – 1 T - S is same as for each

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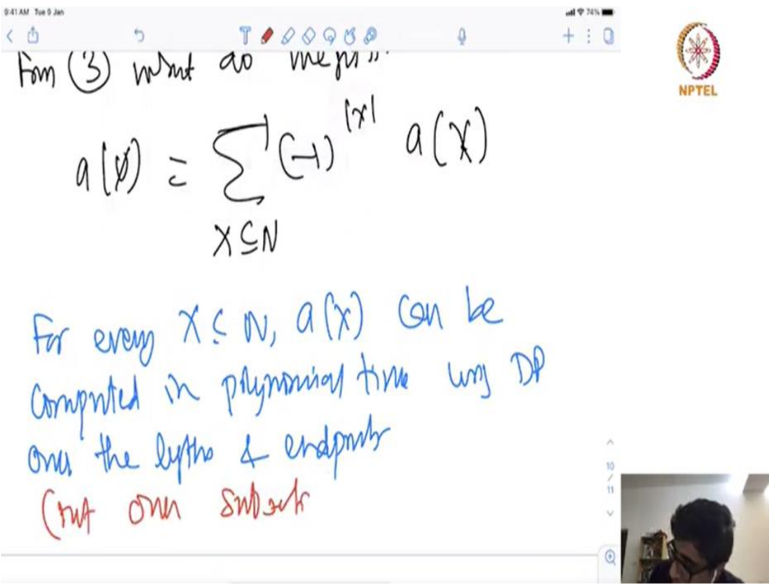

This is exactly the same. So, once you have done this. So now that we have shown this classical formulation of inclusion exclusion, that if you are interested in counting intersections of elements or counting union of elements or complement of it, then this is the way to derive the formula. How do we use algorithmically?

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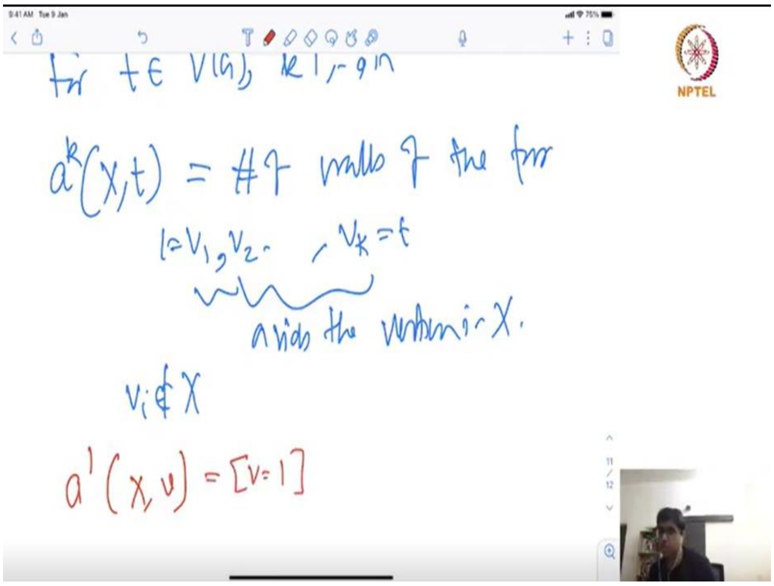

So, to use algorithmically I am going to explain to you via an example. And we are going to derive an algorithm for Hamiltonian Path say Hamiltonian Path or Hamiltonian cycle. So let us say we are interested in Hamiltonian Path. So, what is the Hamiltonian path? Ham path in a graph is a path that visits every vertex. So that is if it is every vertex that is it. That is important point. So, it is basically a path this is that visits every vertex.

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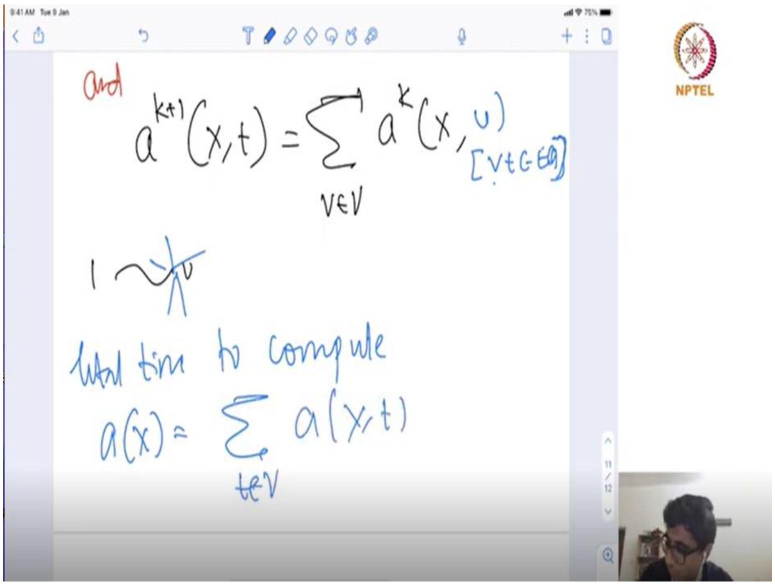

It is well known that this is a classical hard NP hard problem and so, what we are going to give now is 2 to the power n, n is number of vertices, and if the number of vertices. And let us say for ease of notation, let us say that all path starts in node v 1 = 1, this is just notation. And now what I am going to decide? So, the way you are going to work out these things is, what is my universe? So, my universe is going to be something which you be slightly easier object to count. Now what I am going to do.

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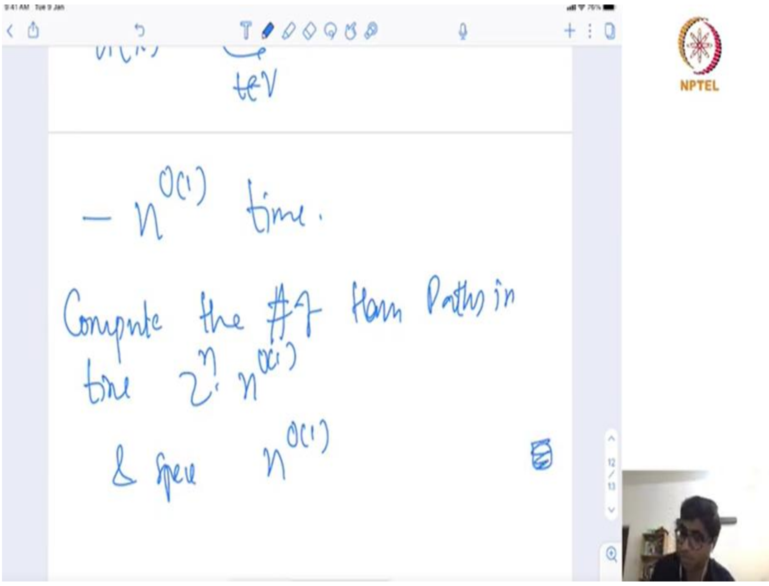

So, now suppose you are given a graph G on n nodes say n let a of X. What is a of X? It denotes the walk of length n that starts in first vertex the pre specified vertex and avoids any vertex in X. So, for example now notice if I say give me an walk of length and say n is even between a vertex 1 now and that should avoid this vertex this vertex this vertex this vertex and this is nothing but I come I go I come and I go I come and go right.

So, this is in walk of vertex can be repeated edges can be repeat. But it is a path when every vertex appeared at most one. So, what I am saying that I want a walk of length n, so you start from a vertex could start at 1 and avoid any vertex. Next. So, I could have maybe gone like this but that does not avoid these set of vertices. So, when I say I want to have a walk of length n that avoids means I start from vertex 1 I take in like I just pick up some in edges.

But in a continuous fashion. What I mean so, if I pick up I start from vertex I go to some other vertex and then at that point of time, it is not that I have to avoid this edge I could pick up any edge starting from this vertex and move to another vertex then pick up that vertex avoid any. But if I have to avoid any vertex in X, it is just that I have to make sure that at any point of time I do not pick up a vertex in X.

I only take those edges that does not contain an endpoints of X. So, this is how we will denote. So, now, what is our universe is going to be? So, universe is going to be all walks of length n starting at 1. So, this is my universe. This is my agreement. Now notice what is a of phi? a of phi is set of what denotes the number of walks. So, what did a of phi set of for? Number of walks of length n that avoids phi.

So, now what is a of phi counts? This is nothing but precisely number of ham paths because before a walk of length n. So, number of walks of length n that avoids phi. It means this walk contains every other vertex and so, basically now it is a walk which contains every other vertex and since there it is of length n every vertex appears at most one time. So, this walk is nothing but a classical path. So, no as it will be repeated no vertex is repeated. So, a phi is a number of walks of length and that avoids phi.

Now we need to set up some sets over to U. So, what do we set up? We set up some sets. What is Ai? Ai is basically it denotes the walk that avoids the vertex i. That is it then you ask yourself what is a of phi then in terms of this? Well, look at let us ask ourselves first what is union Ai? A 1 union A 2 so, basically A 1 union A 2 are those walks. So what is A 1 it is those set of walks that avoids A 1 it is A 2 which is those set of walks that avoids element 2 so on and so forth.

So if I take the union, so look at A 1 to A n. What does A 1 to A n consist of? So, A 1 to A n is consist of all walks that avoids at least one i. So, what is a complement of A 1 union A 2 union A n bar is basically what will be there in those? So, this is basically is going to be Hamiltonian path. So, if I take the cardinality of this, this is this this is exactly what a of phi is. Now that we have set up everything.

Let us apply our formula the third formula that we derived. From 3 what will we get is the following that a of phi if X subset of N – 1 to the power X a of X. That is it. This is the third formula that we did. This is a phi - 1 to the power say our ai ai ns. So, which is basically what is U for us U avoid every element in S. So, which is nothing but directly cognitive a of X.

So, if we had some way to compute a of X we will be very good. So, how will we do? For every X subset of n? We can a of X can be computed in polynomial time we will see in a minute. Very simple dynamic programming. Using dynamic programming over the length. And endpoints not over subsets. That is important point whatever subsets. So let us see what I mean by this.

So, for every t in V G and the length can be 1 to n. k could be 1 to n I am going to say what is this a k Xt is number of walks of the form what 1 you will start in 1 V 2 dot dot dot V k. So, you will start in 1 you end in T this is what it tells us and you still avoids the vertices in X. So, none of these vertices appear in. So, that is basically saying that Vi is not in X. And how do we do this? So, we can so this dp we can set up. So, let us ask yourself a 1 XV. What is this?

This is exactly evaluates to 1 if V = 1 and 0 otherwise, that is it. And what is, suppose you have computed a k X t, so you can compute a k + 1 Xt, a summation over V in V a k. So, it is like you ask yourself, so I start from 1, I reach some vertex V, and then I take a edge starting at V. So how a walk of length k + 1 you start from 1 to some other vertex V with length k. We could very well be and then you take a 1 edge to t. So, this is nothing but V in V and a. So, you reached V and then what is the property V, t should be the edge.

So, the property V that hold is that Vt belong to mica and that is it. So, this is the way we can compute K + 1 and so you first compute a length like a 1 XV for every vertex then a 2 XV for every vertex then a 3 XV for every vertex so on and so forth. And this is how we can count all the walks starting from 1 to length. You can also do using what we call adjacent in matrix but we will not talk about. So that just the total time to compute a X is same as t in V a Xt. So, we can very well check that this is all polynomial term.

So, you can compute a X in polynomial term. I am not going to spend more time on this. But note this it is very good. So now you compute a of X keep it in and then you go for all X subset of N a of X like you go all subset X of N you compute a of X and then you do minus one to the power X. So, you can evaluate this some of the expression in actually 2 power n time into polytime and actually polynomial space. So, we can compute the number of ham paths in time 2 power n n to the power behalf 1 and space n to the power behalf 1.

So, Poly in space in 2 power n time we can compute number of Hamiltonian Paths. So, this was another example of inclusion exclusion-based algorithm. There are several modes but I will leave it to that and we will do some other tools or techniques in next lecture. Thank you.