Parameterized Algorithms
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Lecture – 36
Important Cuts: Enumeration and Bounds

Welcome to the second lecture of this week. And in this week, we now, we will be talking about the most important topic of this week or the important cuts. So, this is going to be our key concepts. And we will try to deliver that key concepts to you in a minute.

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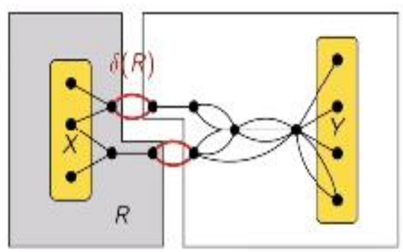
Definition: $\delta(R)$ is the set of edges with exactly one endpoint in R.
Definition: A set S of edges is a **minimal** (X, Y)**-cut** if there is no $X - Y$ path in $G \setminus S$ and no proper subset of S breaks every $X - Y$ path.
Observation: Every minimal (X, Y)-cut S can be expressed as $S = \delta(R)$ for some $X \subseteq R$ and $R \cap Y = \emptyset$.



So, let us recall some of the basic definition that we had, so definition is delta R remember is a set of edges with exactly one end point in R and a set of edges is a minimal X Y cut if there is no X Y path in G minus S and no proper subset of breaks every X Y path. And we also saw that every minimal X Y cut S can be expressed as set of edges leaving a set of vertices R. With is equal to delta R for some X subset of R, and R intersection Y is m t. So, for example look at these four edges they are like delta.

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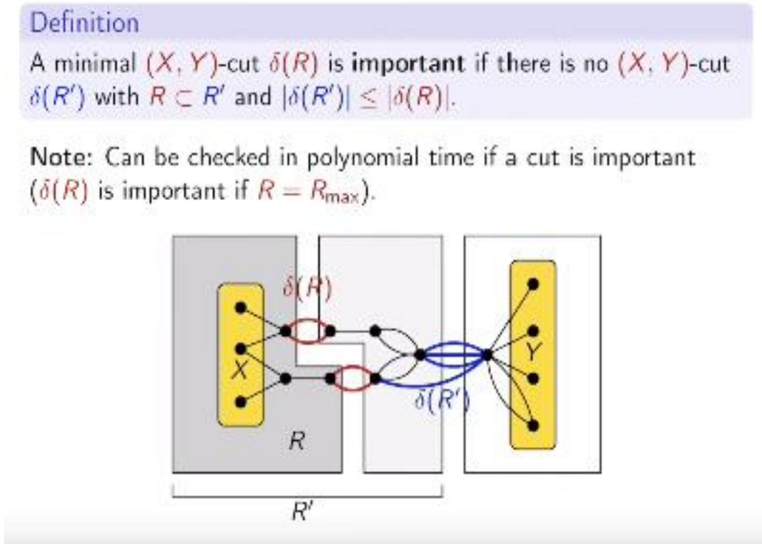

Now what is an important cut? So, this is our basic and most important definition. As it is like the idea is that look, I am going to call a cut important, if number of edges which are emanating out is as much as my current cut, but I am further from myself I contain more vertices than R itself. So, and a minimal X Y cut. So, let us just add, we.

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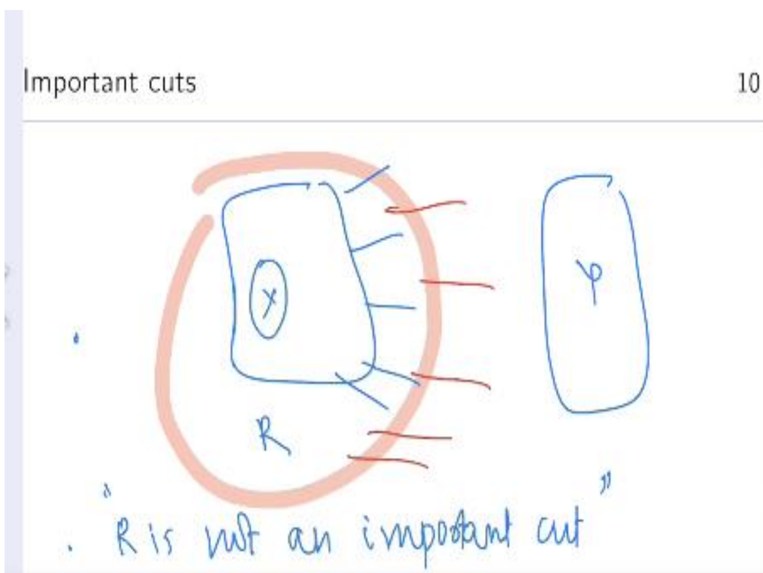

So, what is an important cut is basically imagine that I am going to tell you that look here is my X and here is my Y and this is my R. So, what is my prop, what do I want from this? I said look, or rather let us say R is not important what is the meaning of R is not an important cut? This is the way we will use it means I am going to tell you, if R is not an important cut. Then I can find another set what can I find another set R prime, and what is the property of R prime?

First of all, R prime is delta of R prime is less than equal to delta alpha. So, the number of edges which are leaving R prime is at most as much as R hat. But, more importantly secondly R prime intersection Y is fine so basically R prime is also an X Y cut. Or rather let us say, R prime is also an X Y cut. But more importantly R is a proper subset of R prime. So, I have been able to achieve the following that R prime is proper subset R prime contains our completely properly.

And the number of edges that you need to disconnect X from Y via R prime is at most as much you needed to disconnect X Y by deleting edges which were adjacent to 1 that is all. So, this is what. So, or in the other words, what is an important cut? A minimal X Y cut delta R is important if there is no X Y cut with the following property. You cannot give me another like if there is no X Y cut delta R prime with the property that R is a proper subset and delta R prime is less than equal to delta.

For example, look at this look at these four edges. Now, can you contain, so you have to ask is R an important cut? Well, look at this. Can I add this, if I add is delta R an important cut you have to ask this question. It may not be an important cut. Because, if I take this guy look at this cut what is the property? Number of edges are 3 and this if I am going to call it R prime contains R properly. Now you can ask yourself. So firstly, can we check if a cut is important.

We will see how to test whether cut is important or not in polynomial time, but for now let us just try to understand the concept of important cut. So, what is an important cut. A cut is important if I cannot find another cut but the property that the number of edges or the cut size is at most the current cut size and it properly contains. So, for example this R is not an important cut because this R is not an important cut because I can find an R prime with the property.

That R is a proper subset of R prime and delta R prime is actually strictly less than delta R, because delta R had four edges this has three ways.

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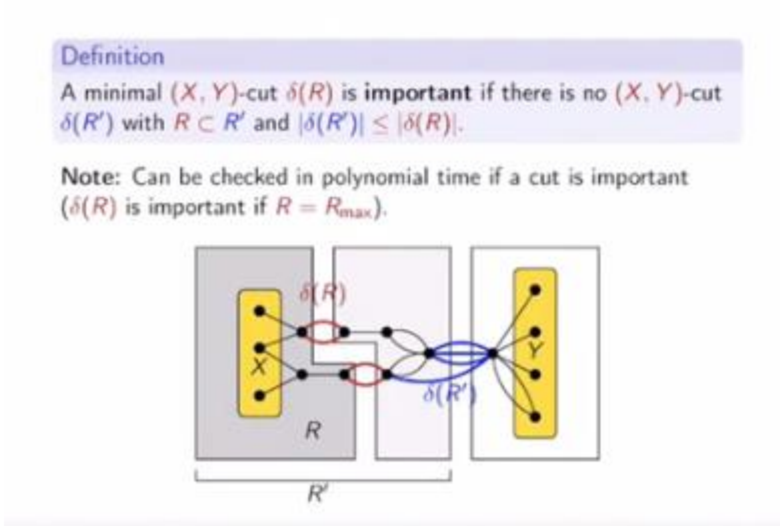

So, for example this if you take this R prime, I gave you another example but look at this R prime I could have taken even a bigger R prime. What is the property? Look definitely prime contains R as this it had 1 2 3 4 edges; it also has 1 2 3 4 edges going out. So, this is also important. But I told you another one. So, there are several reasons why, but what about this? Now let us ask ourselves. So, definitely this particular R was not an important cut, because we could find an R prime with at most four edges and properly containing R.

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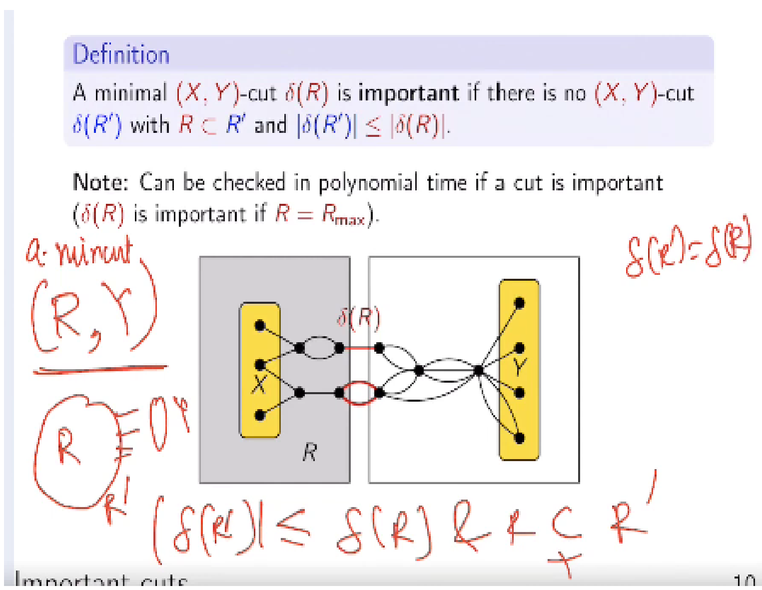

And remember always defined from Y which I am not, but what about this R? Now there are three edges, now notice if you add to R this vertices then that is not good for you, because then you will contain four edges. So, you like you take this you will still get four in it. I mean you take anything you will still get 4 edges. But, here we only had 3 edges. So, an important so; and for R to be not an important because we have to produce an R prime with at most 3 edges emanating out of it.

But, in R if you add any subset of these vertices you will get at least 4 edges leaving which implies that this R is an important cut. Now let us ask ourselves how will we check whether R is an important cut or not? That is very easy how to check and cut is important or not. So, how will we check and cut is important we will say okay fine. I am going to call a cut so suppose R, Y. So now look at just compute R, Y.

R, Y I am going to call compute a cut R, Y cut between R and Y a min cut. A min cut in R, Y what can happen? Either so if I compute a min cut then I will get some another R prime containing R and Y outside. And it is a min cut so notice I check its value if this value whatever this if the delta R prime is less than equal to delta R. and R prime and R is a proper subset of R prime then what we know? Then we know that there exists such an object.

So, for example so this is all that we need to do but you may ask well what about that if delta R is equal to actually like I mean that. Like so look at this delta R prime, suppose delta R prime is equal to delta R then what he could have returned? You could have returned, so let us ask just do slightly simply.

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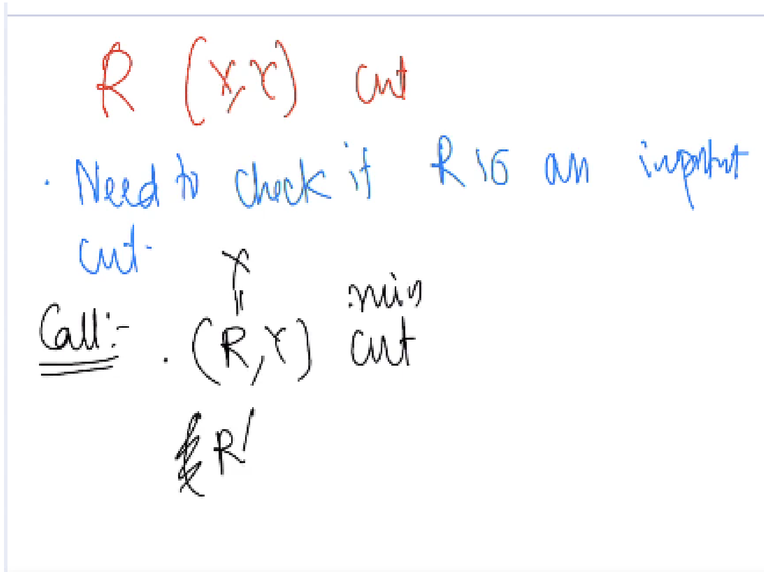

So, you are given an R and X, Y cut and what I need to do, if R is an important cut. So, what I am going to do is the following I am going to call, what I am going to call? I am going to find R, Y cut, so X I replace whether I want to find a cut between R and Y. And I am going to find a min cut. So, if the main cut is, so suppose I returned some R prime. What could happen here?

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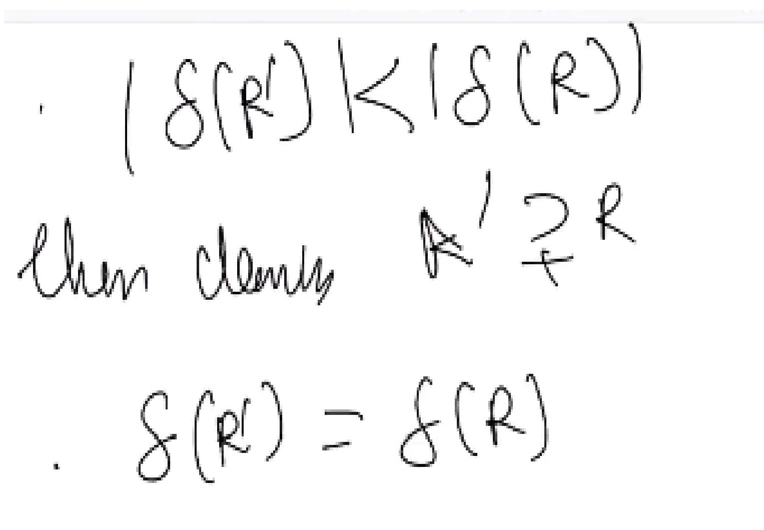

Notice, if delta R prime if delta R prime is strictly less than delta R. Then clearly R prime also properly contains R and you are done. But what could happen is that delta R prime is actually equal to delta R?

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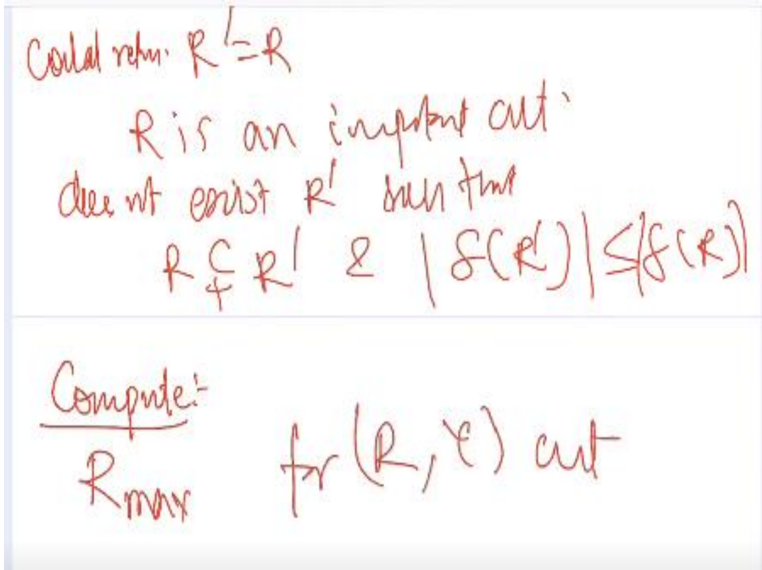

And in this case, you might just return R prime equal to R you could just return R prime equal to R. So, we but that does not mean that R is an important cut. Because, for R to be an important cut we have to say does not exist does not exist R prime such that R is a proper subset of R prime and delta R prime is at most delta. So, we know that we cannot have at most strictly because it is going to be equal so now what will be do?

You say fine. Compute R max for R, Y cut. What is the property? It is so you know that the mean cut between R and Y is same as the number of aj living now you want to contain a furthest cast a cut which is far this from this so R max, for like R max for RL cut, of size delta R.

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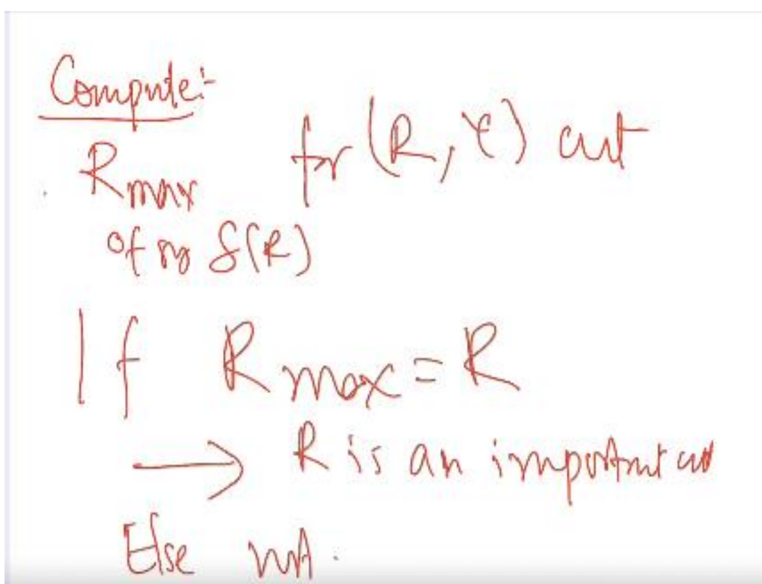

If R max is also equal to R, then you know that R is an important cut, else not. So, given an R, Y, I can check whether it is an important cut or not by computing R Y mean cut if it is size is strictly less than the delta R then you already know that it is not an important cut, but suppose this is equal then you compute, so you know that delta R is a min cut between R and Y, now you computed R max with respect to this delta R right of size delta R and if that you know that is unique

And you can find that in polynomial time we have seen, if that is also equal to r then only you know that R is an important cut, else we know that R is not an important cut. So, this is how we can check whether a given cut is an important cut or not. By computing, by just computing, what is that called by just computing a couple of min catalog running couple of min cut algorithm.

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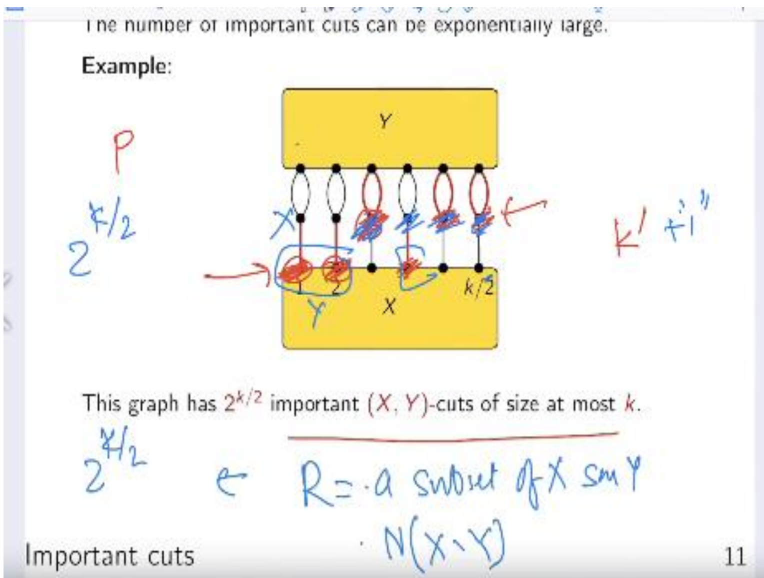

And let us see how many number of important cuts can be there? Number of important cuts can be exponentially large. So, suppose we have this is basically the whole graph so this graph has X there are K by 2 vertices and there is another K by 2 vertices and there are some middle vertices with path. Now what do you know, your min cut or rather and you are looking for a important X Y cut subside at most. What is an important X Y cut sub size at most K?

An important X Y cut subside at most K, I am going to tell you that you decide which vertex of this K by 2 of this path is going to be part of this and which vertex from middle word it says. For example, if I pick this, pick this, pick this, pick this, pick this and pick these vertices. Now what is your cut size? Because, look it is a K by 2 vertices the number of edges which are adjacent will always be less than K, say some K prime. Now, this is an important cut.

Because, Y is an important cut because to be not an important cut you if this is one word you cannot take the vertex of Y, because you have to be a disjoint from Y. So, then what does it mean? It means that you can only pick for the vertices that you have picked from X in your, so look at an important cut. Which consists of this I am going to call this an important cut. Why? Because, look the moment you pick anything from here the cut size will increase by +1.

Any moment you pick anything else cut sideways so just by numeric value it will not happen. So, harmony says this thing it is so number of important cuts of size at most K is upper bounded like is possibly you choose a subset from X and then the corresponding. So, it is since the size of X is K by 2 the number of important cuts is at least 2 like 2 power K by 2. Because, you just pick a subset subside this and then so basically R consists of, let us call it either a subset of X.

And say Y and secondly a subset of X say Y and neighbourhood of X. So, the moment you pick, so suppose this is my Y. So, and what else you have to pick R neighborhood of X -Y which is this vertex, that is it. So, how many such objects are there? Number of such R is upper bounded by 2 to the power K by 2 because the moment you guess a subset of X everything gets determined. So, the number of important cuts in a graph can be exponentially large as a function of K.

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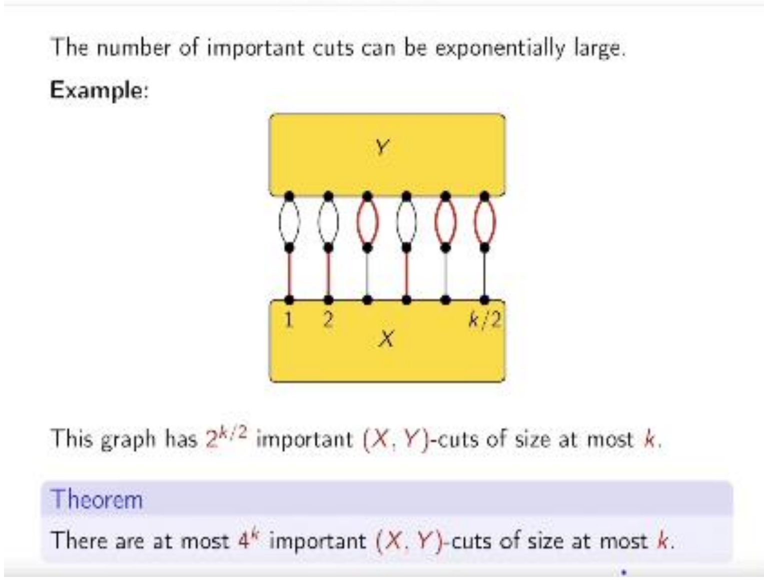

But, what we are going to prove that there are at most 4 to the power K important X Y cuts of size at most K and that is our important theorem. That the number of so here is a lower bound. But, what we are going to show the number of important cuts X Y cuts upside at most K is upper bounded by 4 to the power K. And towards that they are going to give you an algorithm.

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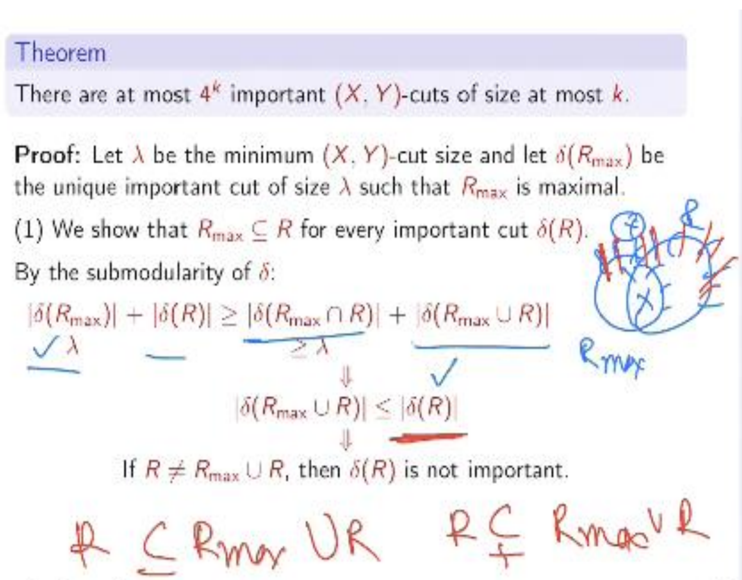

It is a branching algorithm with a very beautiful measure. So, let us see that branching algorithm, so we are going to show that a 4 to the power K important X Y curve subsides at most K. Notice, let lambda be the minimum X Y cut size and let delta R max with a unique important cut of size lambda so that R max is maximum, you know that there is a 1 R max which is maximum and its unique. So, we are going to show that that if you are looking for, so here it is my X and Y.

And this is my R max. We are first going to show that R max is a subset of R for every important cut delta R. Every important call delta R, R max is a part of this so that will be our starting point. So, let us first so this property again we are going to use some modality. So, what are we going to apply some modality? So here is your R max and here is your R and here is your X here the Y. So, we are going to apply some modulatory on R max and R.

So, by submodularity of R, delta R max + delta R is delta R max intersection R + delta R max union R. But what is the delta R max, it is a min cut. This is R max is this so this is min cut. So, this is exactly equal to lambda. And what is the delta R max intersection R this is also a cut between X and Y, so its value is also going to be greater than equal to lambda. It is at least greater than equal to lambda it could be even further but it is definitely greater than lambda. So, what will you get then?

So, then delta R max union R is delta R max - delta this so this is upper bounded by delta R. So, if R so what we know that the R max union R, look at this. It is at most delta R. So, if R is not R max union R then a delta R is not an important cut. Why? Because, what are we able to show? Look, so we are able to show to you that look number of edges which are going here is at most delta R, is at most delta. Then look at R max union R. R is definitely contained inside this.

So, if R is a properly contained inside R max union R, then that is a contradiction to the fact that R is R that then that is the contradiction to the fact that R is a, what you call important cut because, its value is less than equal to delta R. And it contains properly. So, but that is not happening. So, what the meaning of that?

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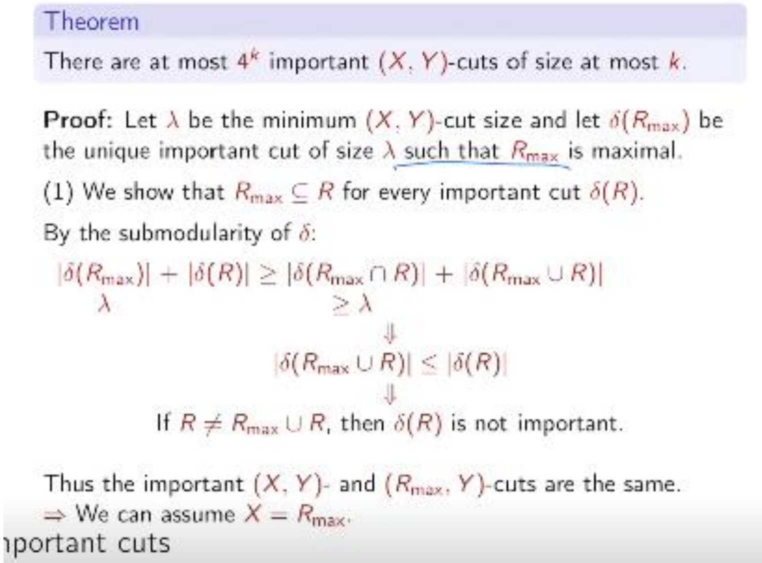

Thus, the important X Y and R max Y cuts are the same. and hence we can assume that X equal to R X. So, what are we able to show? So, what did we get a contradiction to we wanted to show that R max is a contained inside R. so, if R max is not contained inside R, it means there is some vertex here which will imply that R is a proper subset of R max union R and if R is a proper subset of R max union R then that will imply that that will imply that R is not an important cut.

So, by contradiction what we know that R max is properly contained inside R for every R.? So, what we know about this? Thus, we know that the important X Y cuts and R max Y cuts are the same. So, whether you are talking about important cut from X to Y or you are talking about important cut from R max and Y they are same. Because, R max is part of every important cut, and that is why we will assume that X is equal to R max.

So, we assume that X is nothing but equal to R in X. Because, our first lemma has shown that this is exactly, because lemma has shown that R is like R max is contained inside every so you are not going to lose out every any cut.

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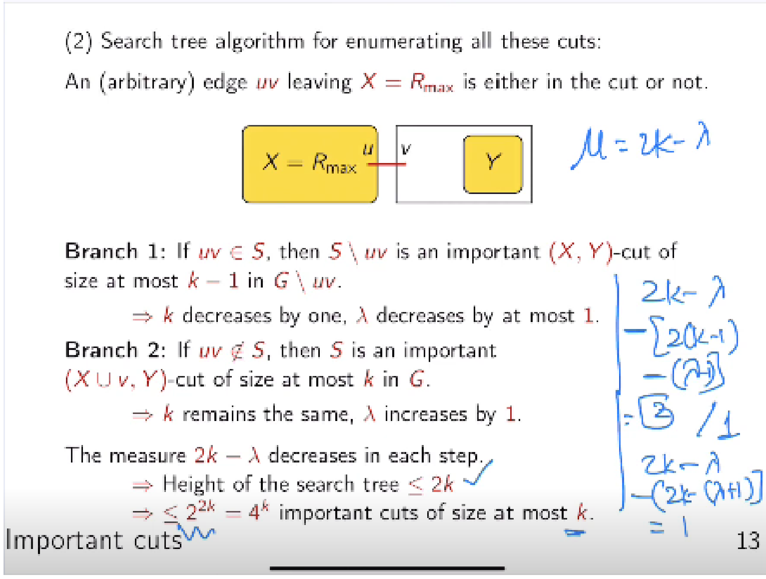

So, now I would search the algorithm for enumerating all this, so you look at this X and look at this Y. you pick up an arbitrary edge you be leaving X equal to R max. so what is the property? Either this is part of an important cut or it is not a part of an important? So, if it is not an part of an important cut then what can you say? So, first imagine that u v is a part of an important cut, if u v is not part of an important cut?

Then S minus u v, we will show is an important X Y cut, but what is its size now its size is k - 1 in G- U v. So, the what is my algorithm? It says okay fine. let us enumerate all important cut of sides k - 1 in graph G - U v and add U v to each of this cut that will consist of all important cuts that contains U v. Or we know that U v is not part of S. Then what I know, then S is an important X union V, Y cut off side at most k.

Then I know, that okay fine, u v is not in S. If u v is not in S in the important cut, then you are trying to get a cut of you put a V inside X, X union V and you are only locking about cuts of X union V and Y. Because, any cut that you are talking important and non-important is also have to contain both X union V. So, but it is cut inside that node K. So, we have been able to branch into stages what either this edge is part of my important cut or it is not part of the important cut.

If it is not a part of my important if it is a part of my important cut I delete and I look for a cut of size K - 1 find them all add uv to each of this. In the second branch where I am looking for an important cut that does not contain uv then I know that I am looking for an important cut of side X union V, Y. Now our question is to question why this branching is correct? It is exhaustive that is fine because the correctness, because white is exhausted.

Because, look at important X Y cuts either it contains uv then you are counting them in branch 1 it is not contains uv then you are then you are counting them or putting them together in a branch 2. So, this is exhausting this is absolutely correct. Why the branching steps are small? Why these things are correct? We will see in a minute. So, what happens in the first case? Look at definitely K decreases by 1.

But it could happen that the mean cut side also the min cut between X and Y also decreases at most by 1. It is possible. Because, like it can only decrease by 1. You cannot have min cut size suddenly because becoming lambda – 2, because you have just deleted 1. This is not possible then because then you take this lambda - 2 edges add that edge and that is an X Y cut of size at most lambda - 1 that is not happening.

So, it is possible that lambda can decrease min cut between X and Y could decrease. But, what happens in the circuit K definitely remains the same but, what is the property of R max? The property of R max is that if I add any other vertex to this the min cut between X and Y must increase. It must increase at least by 1. Because, if it will remain the same then R max is not the unique cut you could have to put the V.

And that would have been X Y cut off psi that moves lambda. So, even though K remains the same lambda the min cut value increases by 1 between X and Y. So, there is something are changing. Now see what happens? So, if I use the following measure 2k - lambda, then what happen s? So, look at this. So, it is 2 k - lambda to start with and the change in major is 2 times k – 1.

And - look lambda decreases by at most so if it is just lambda then you check 2k - lambda - 2k that will get cancelled and this is - lambda + lambda that will get cancelled and you will get - 2 + 2. So, in this case there is a drop of 2 but what could happen, that lambda could also decrease by

1 and in this case drop it at most. But, look at this case this is like 2k - lambda - 2k - lambda + 1, and if you do this this is going to come to 1. So, but what is the total value of 2k – lambda?

2k - lambda at most 2k. So, height of the search tree is at most 2k. And each branch such tree drops by at least 1. So, the total number of important cuts of size at most K is upper bounded by 2 to the power 2 power k which is 4 to the power k. So, that there are 4 to the power k important cuts of size at most K.

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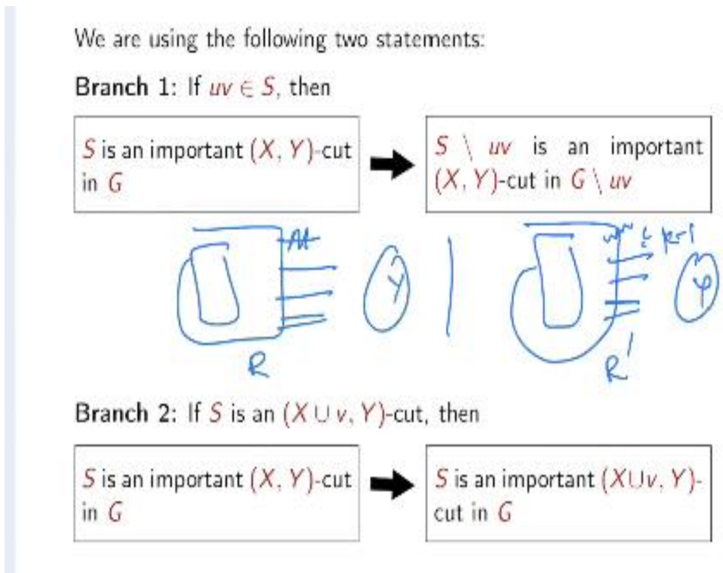

So, what are we using? So, now just try to get some feeling of the correctness. So, we are going to show to you that if uv is in s then s is an important X Y cut in G implies s - u b is an important X Y cut in G - U V. It is not true otherwise. So let us see, why this is true? So, look at s is an important, we will see. What happens? So suppose this is an important X Y cut then I want to show that is an important X Y cut in G – uv.

So, suppose this is an important this is R. And this is an uv and this is an important cut. Now, imagine that S - V is not an important cut in this. It means what I can find another R prime, which contains R properly and has at most K - 1 edges going out. Now look back at this R prime then this graph and this graph only differs by at most 1. So, maybe there is one more edge going out from R prime that is all that can happen.

But, then what does that imply? Then that implies that R itself are not an important X Y curtain G. Because, I could have taken R prime, as like because R prime would dominate R. Prime is a better cut than R. So, this is what it implies that if you give me an important X Y cut it definitely remains an important X Y cut in g - U V. So, if you are able to upper bound the number of important cuts in G - u v that will indeed provide an upper bound a number of important cuts in G that contains.

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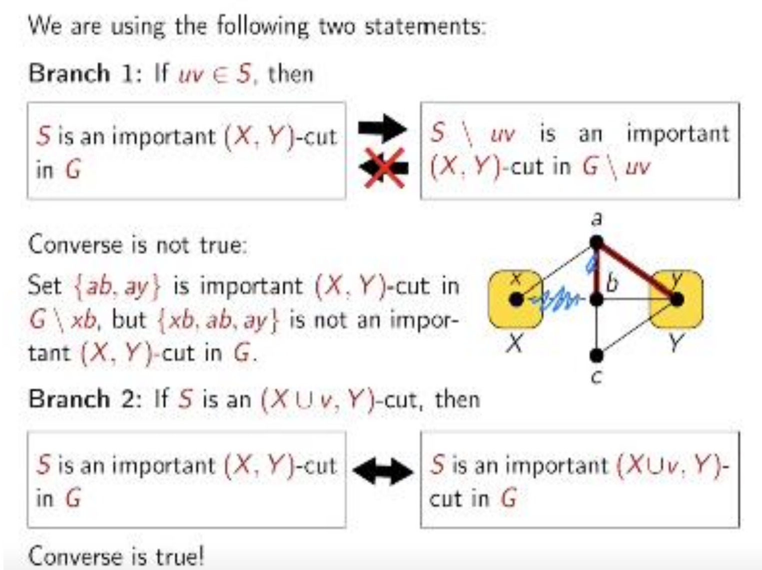

But there is another interesting thing converse is not true. And an important cut in S - u v may not be an important cut in original graph. Meaning, if you give me an important cut in G it is definitely an important cut in G - u v, but an important cut in G – u v may not been important cutting G. So, basically number of important cuts in G - u v could be much potentially much larger than important cuts in G itself.

But that is not a problem because we are only trying to get an upper bound number. So, for example Y. Look at this set a b and a y is an important cut in G - x v. If you delete G - x b, then this and this is an important x y cut. But, if you go back to the original graph, x b a b and a y is not an important x y cutting, they are not an important x y cut in j. So, a cut which is important here may not be an important thing.

But, what about the second case? S is an important x y cut in G if and only if it is an important x union v Y cutting. This is easy to show let us try to show formally. So, suppose S is an important x y cut in G. And we are in the premise that s is an x y b cut. We are only talking about those kinds of cuts.

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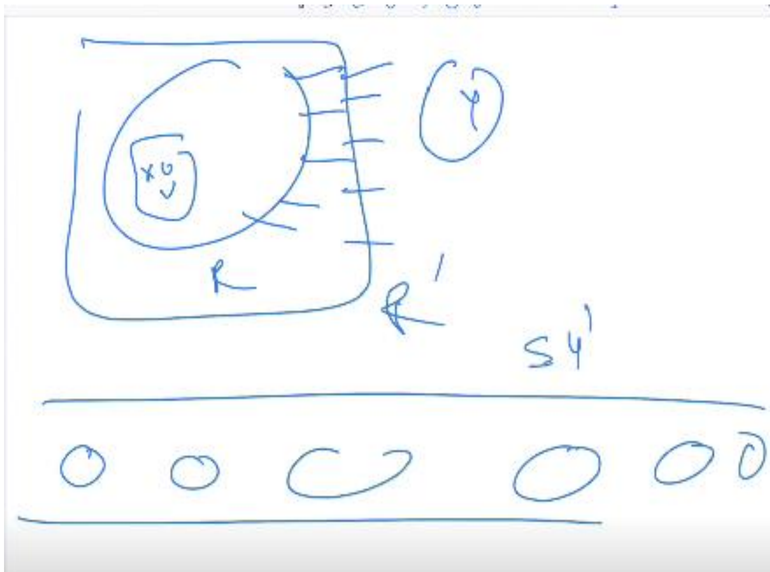

So, and so this is basically containing going to X Y V and Y outside. This is an R and it is an important, X Y coordinates. why is this not an important X Y cut in G? I mean, what does it mean it is not an important X Y cut in G. Well, you will say, ok fine I will go take more R prime number of edges at most K, blah blah blah and say but then I mean then that is contradicts the fact that R is an X U v union Y X Y cut.

And similarly, if S is an important x y b cut in G, then you can show that S is an important X Y cartesian, so this if and only. So, in this case you do not lose anything I mean you do not gain anything. Let us say and here you might gain many more things. So, what happens is that at the end you might get lots of cuts? You know that the number of cuts generated by this is 4 to the power k but you have to filter out important cuts.

Because, you know that you would have also got some chords because of the branch 1, and they not like they were just important cuts in G. You made them a potential cut of g by adding ub to it not all this may be important cut.

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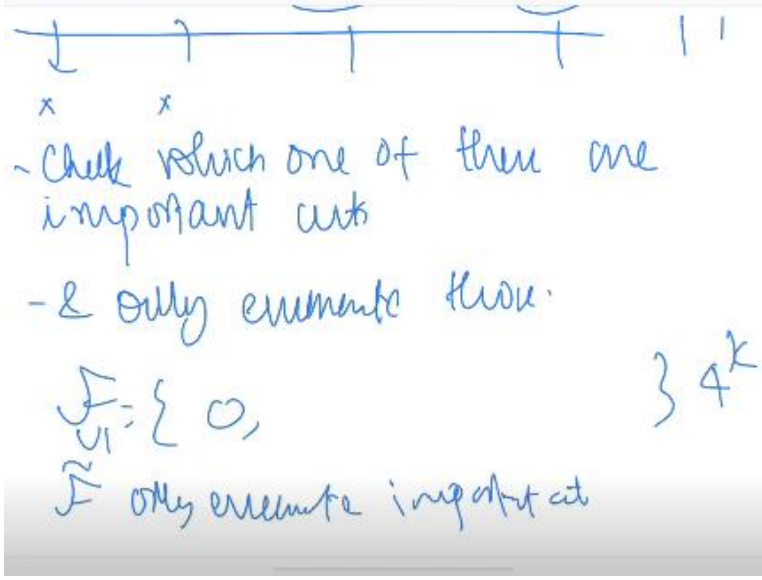

So, you check which one of these are important cuts. Check which one of these are important cuts and only enumerate things. So, you collect a family of sides say 4 to the power k that contains all important cuts and then you refine clean this up and only enumerate important cuts.

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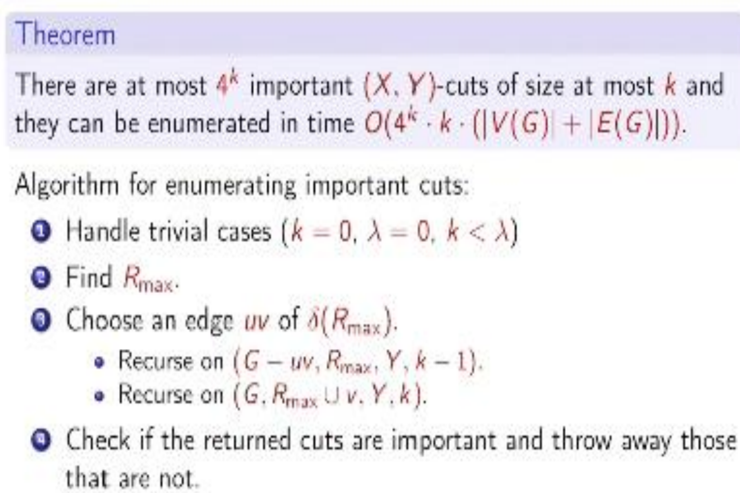

So, here is our algorithm handle trivial cases you find Rmax, choose an edge uv of delta r max you recurse G - u b r max y k – 1. Record some R max union v Y, k and check if the return cuts are important and throw away those that are not as I told you that some of the cuts that you could get may not be an important cuts. So, you have to test them check them and throw them out.

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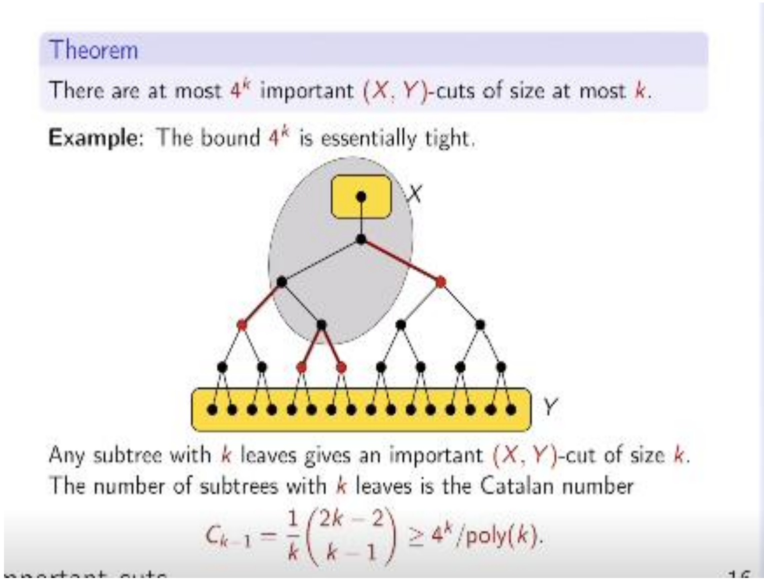

So, the last thing of this lecture that I would like to tell you that there are the just now so there are at most four to the power import X, Y cuts of size at most. We also saw a lower bound of 2 power we also saw a lower bound of 2 to the power k by 2. But, in fact we can actually get an essentially tight bound of 4 to the power k. How do you get this is again a very similar to the example we had?

You consist of binary tree such that number of leaves say y is greater than k that is it first time you get this this is this is all that you have. And you just add a pendant called it X and every leaf is your Y. So, here now look at a sub tree here with at most k leaves. So, this is buying any subtree with k leaves will give you an important X Y cut of size at most k y. Suppose look at this. How do you get? Look at the leave and delete these edges.

That is your cut edges this is delete distances right so how many edges you have? At most k sizes. But, notice that 2 for a set that contains this the moment you pick any other vertex number of edges that is adjacent will increase by at least 1. So, not so the moment you go beyond this set to slightly larger set the cardinality of the number of edges will strictly increase it and hence they cannot be important cut.

So every any subtree with k leaves gives an important X Y cut of size. How? Again, let me repeat again. You look at the leaves and the edges which are adjacent to these leaves that will be exactly at most that will be exactly as many as leaves, so they are bounded by the number of leaves. Hence they are number of at most k, and for any other important cut must contain this set no so the moment you pick any other vertex containing this the number of leaves, number of edges going across will increase by whatever it was.

And hence they cannot be an important cuts. And hence they cannot be dominating this, so every sub tree with k leaves gives an important X Y cuts of size at most k. And it is well known fact, I will not going to show this to you that the number of subtree with k leaves is a Catalan number. And it is given by 1 choose k 2 k - 2 k - 1 which is roughly at least 4 to the power k poly k. So, this also tells us that the number of important cuts in a graph is upper bounded by 4 power k and roughly that is a bound is tight.

Because there are a family of graphs and examples where you do have at least roughly 4 to the power k Polycarp divided by poly k number of important cuts.

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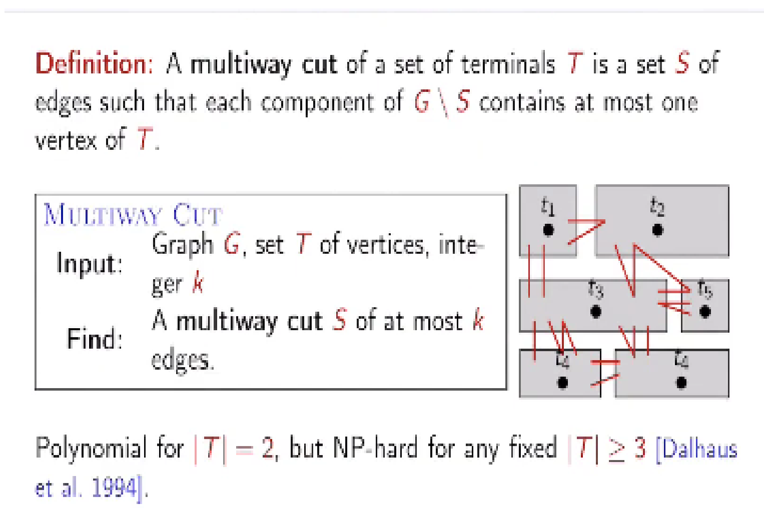

I think that is a good place to stop and in the third lecture we will see the first application of important cuts in a problem called multi-way cut etcetera, so we will talk about multi-way cut in the third lecture thank you for now.