Parameterized Algorithms
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Lecture - 35
Important Cuts: Basic

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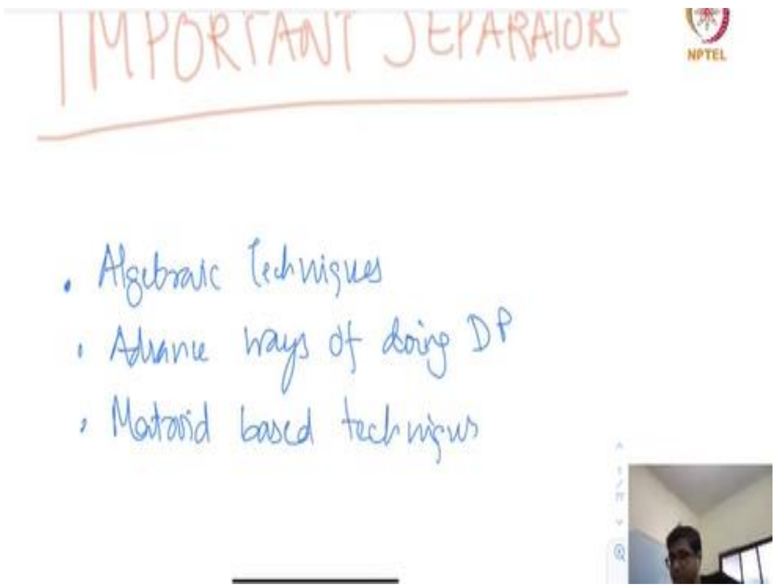

Welcome to the eighth week of the course. In this week we will learn some advances, the first week where we will learn some advanced tools in parametrized complexity. And in the next three to four weeks, we will see tools such as important separators, some algebraic techniques, advanced way of doing dynamic programming and some matrid based techniques can. So, let us start with an example.

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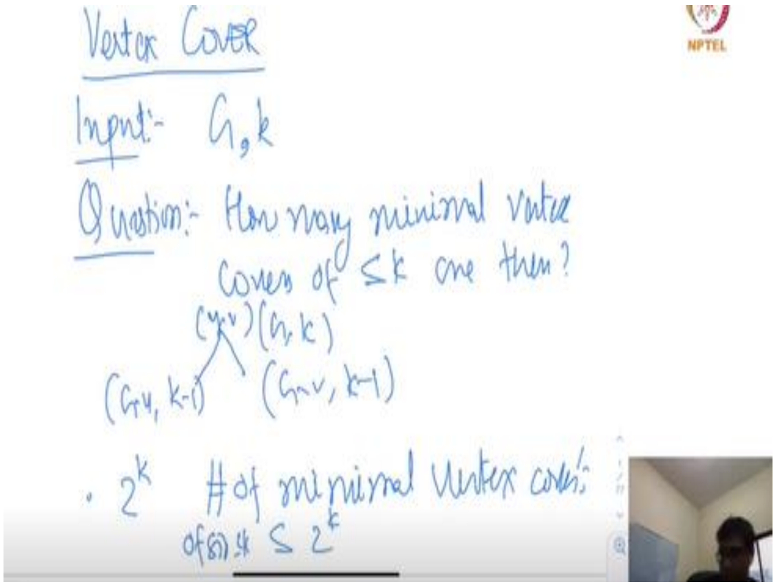

So, the tricks and techniques that I am going to teach this week is the most important separator. But before we even reach important separators let us talk about vertex cover. So, what is the vertex cover? So, vertex cover is like input is G, k or let us ask a very different questions. So, suppose if I asked you the following question, how many minimal vertexes covers size at most k are there? The 2 power k branching algorithm you take an edge u, v and you branch G – a, k - 1, G – v, k -1.

Starting with the v, k actually tells you that the number of such objects are upper bounded by 2 to the power k. So, the number of minimal vertex cover are at most 2 power k.

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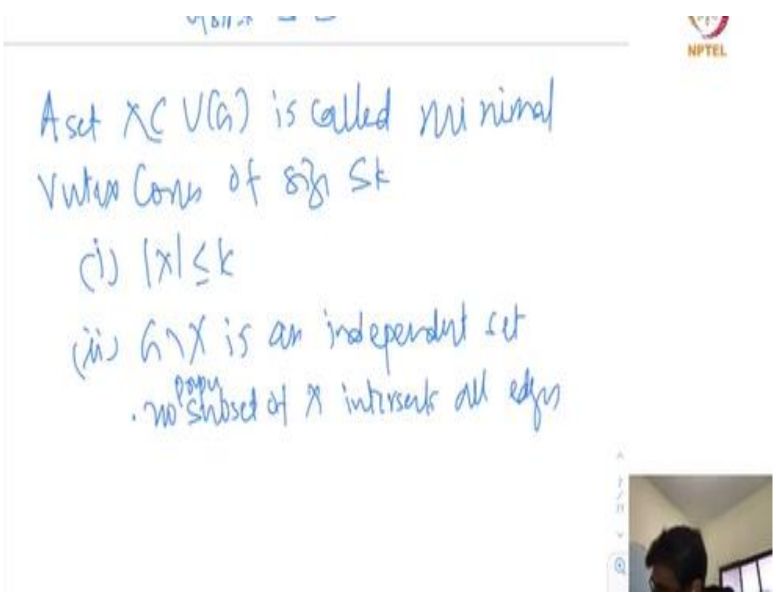

Now, you may ask what is minimal vertex cover? So, of course, minimum vertex cover of size at most k. So, what is the minimum vertex cover of size at most k? A set X subset of V G is called minimal vertex cover of size at most k. What properties is this? First cardinality of X is at most k secondary G - X is an independent set and finally no subset of X in fact proper subset of x, no proper subset of X intersects all edges.

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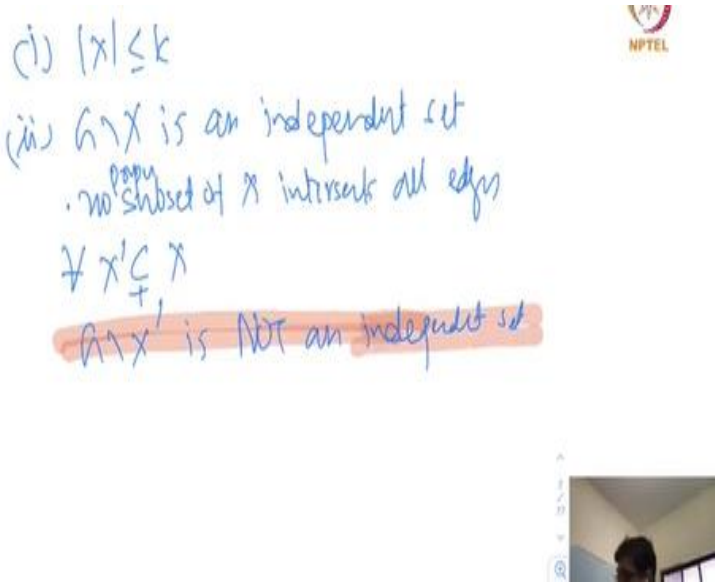

In other word for all X prime proper subset of X G - X prime is not an independent set. So, this is why it is minimal meaning no proper subset does the job of our requirement is not an independent set. So, we would like to ask for several problems the problem in between is about number of min cuts or number of minimal cuts between two vertices. So, if you recall correctly in the algorithm for all cycle transfers after doing it compression, we reduce the problem to finding or minimum cut.

But maybe for many we do not know whether I mean for a problem we do not know whether it contains some minimum cut but we might be able to show that there exists a min cut.

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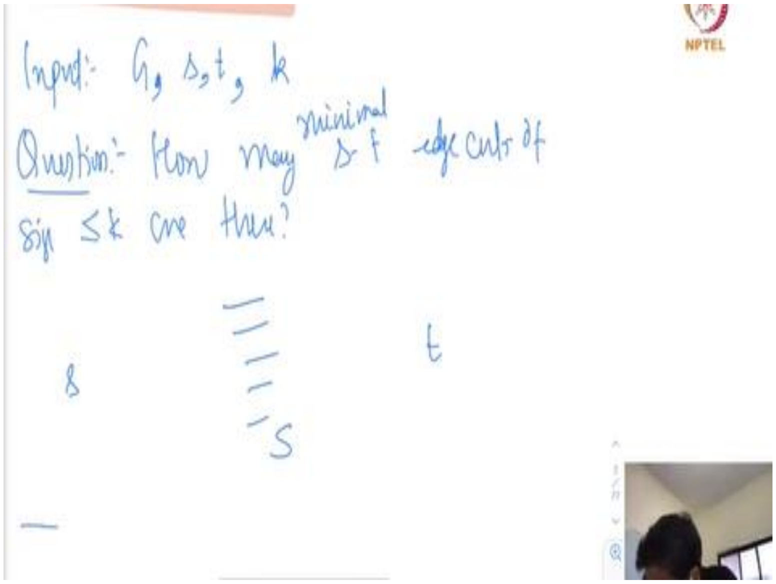

So, let us ask ourselves so, this is a combinatorial question. So, what is your combinatorial question? Say input is graph G, two vertices s and t and an integer k and the question how many s t edge cuts, how many s t minimal s t edge cuts are is minimal s t cuts of size at most k are there? So, what is the meaning of minimal s t cuts is like you are given a graph s t edge cut basically means to see some set of edges say S. So, moment if we delete s from G there is no path between s and t.

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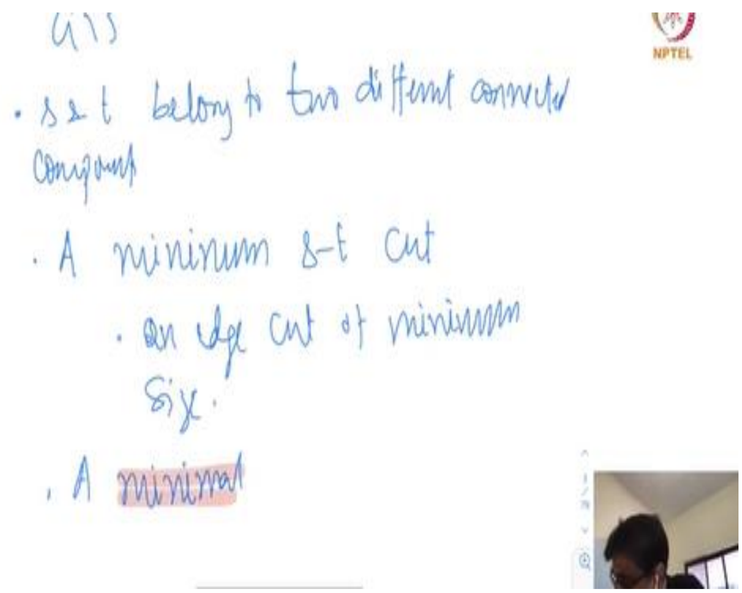

In other words, in G – S, s and t belong to two different connected components or 2 different connected components. So, such s are called edge cuts are and what is the minimum s-t cut and edge cut off minimum size capable. And what is the minimal s-t cut is first of all a minimal s-t cut of say size at most k is first of all. So, a minimal s-t cut say S of size k is basically first cardinality of s is at most k.

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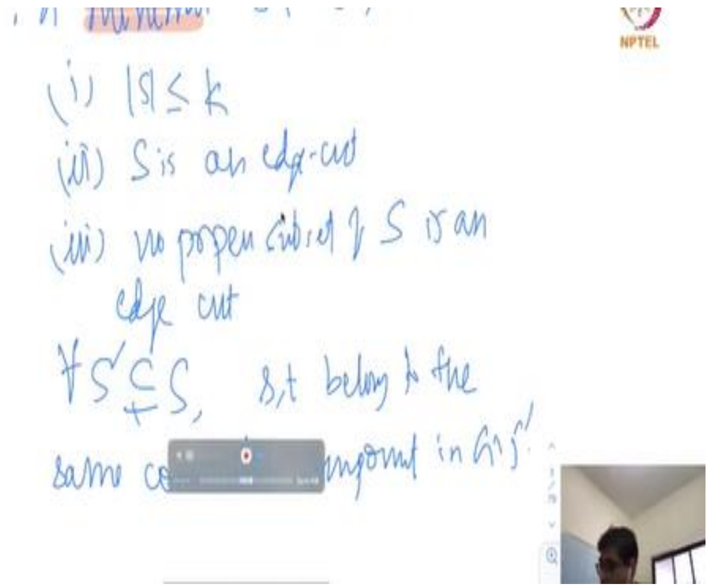

Secondly S is an edge cut and thirdly no proper subset of S is an edge cut. In other words, for all S prime proper subset of S, s and t belong to the same connected component in G - S that is it. So, these are called minimalistic.

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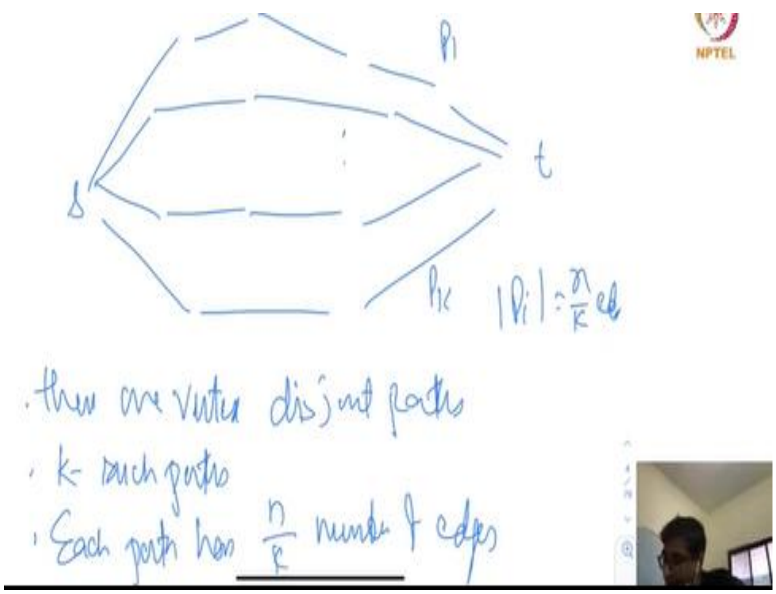

So, now if I ask our self in a graph how many minimal s t cuts have size at most k there? So, you notice if my graph is consisting of some s and t and suppose this is a vertex disjoint paths and how many such paths are there? Let us say, they are like k such parts and each path has n by k number of edges. So, we have path P 1, dot dot P k and each path has size say n by k in terms of edge. Now, notice these are edge destroying parts there are k edges destroying parts.

So, any mean cut or any minimal cut must contain at least one edge and notice for any minimal cut once you pick one is any other edge is redundant.

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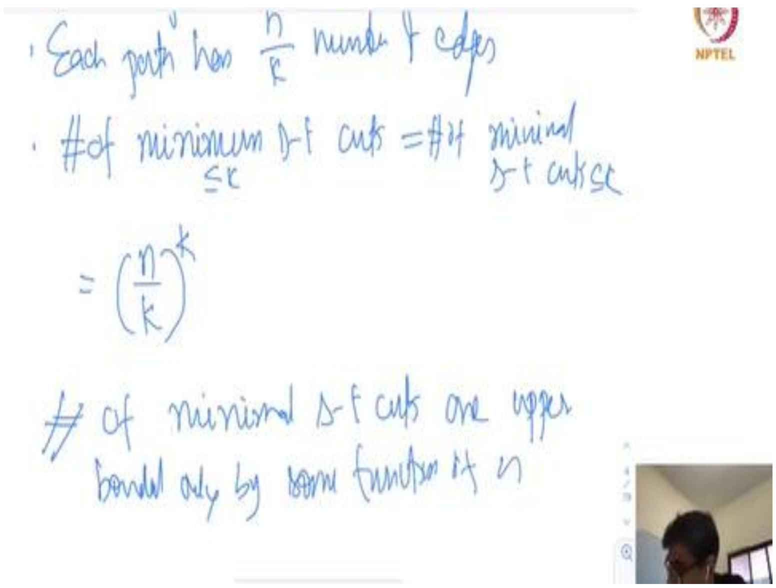

So, for this example number of minimum s t cuts is same as number of minimal s t cuts of course size at most k of size at most k in fact this is exactly equal to and this is obtained by what. From each path you pick you have a choice of any one at the moment you pick one edge as you are done. So, this is like we have a n by k to the power k. So, the unlikely vertex covered number of minimal s-t cuts are upper bounded only by some function of n.

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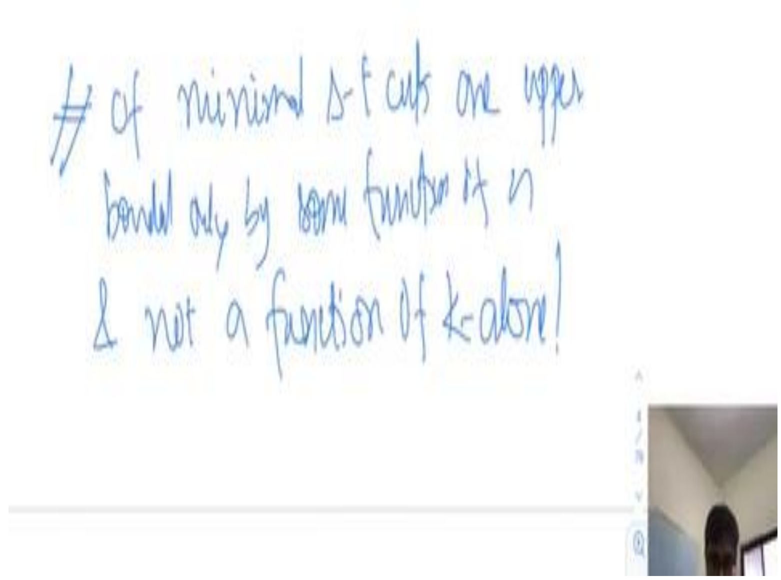

And not a function of k alone. Now you might ask where do we require this kind of like why do we care about minimal s-t cuts and holes. So, as I said to you before if you could have been able to bound number of minimal s-t cuts by just a function of k. For many problems, we could enumerate this minimal s-t cuts and reduce the problem to us with a smaller parameter. We will see some examples in this lecture.

But for now, all I wanted to tell you that the number of minimal s-t cuts of size at most k is not upper bounded by k. We could have also done the same thing with respect to vertices, but for this lecture series we will only be concentrate on edge cuts and we will not concentrate on vertex cuts.

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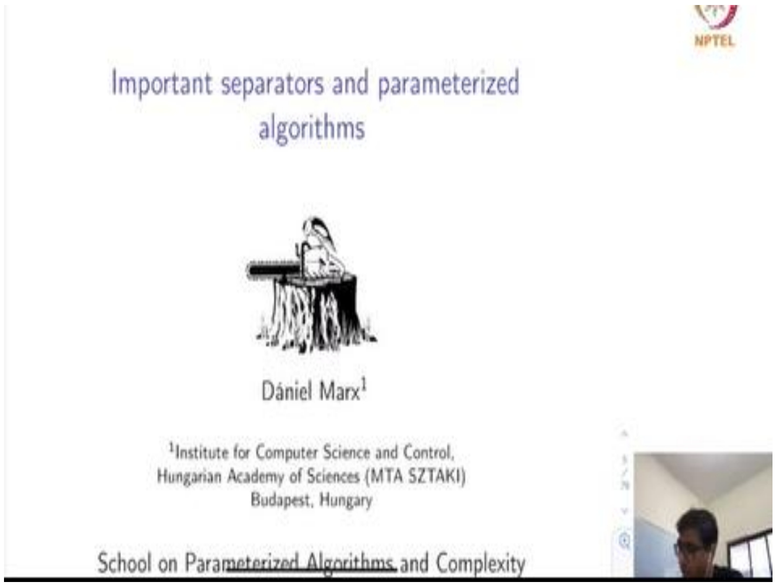

So, I will follow the slides by Professor Daniel Marx, it is an old slide and it is titled important separators and parametrize algorithms and so, have just kept it the way. So, that the proper connotation to his slides.

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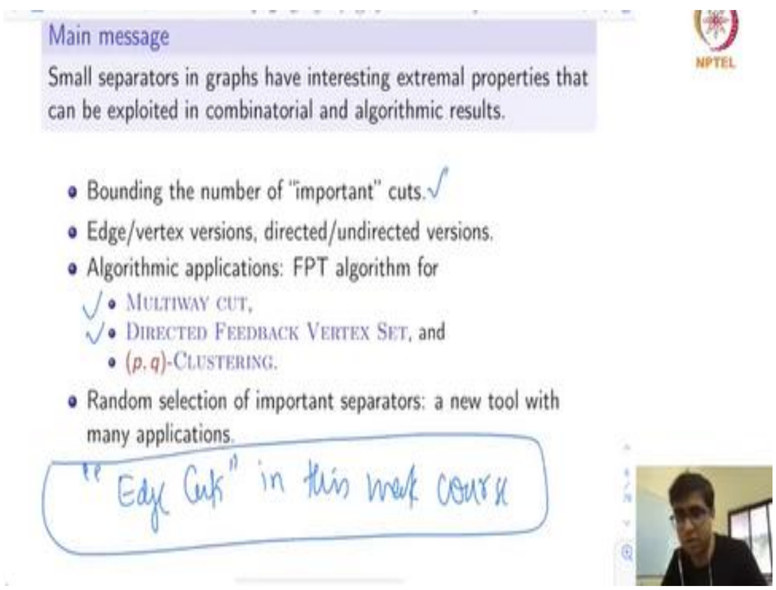

So, the main message we will try to give. Small it is separate us in graphs and interesting extremal properties and we could, exploited that for combinatorial algorithms. So, what are we going to tell you? We are going to bound the number of important cuts. So, we also need to define what is an important cut and how we can bound this and then show its application in designing FPT algorithm for the problem called multiway cut and directed feedback vertex set.

But it can also be applied in something called p, q clustering, but we will not be teaching you in this course. However, you can find that in the textbook. And this is based on what is a new technique which led to a several new algorithmic developments on the cut problems what is called random selection of important separators but again, we will not talk about it. And this notion of important cuts, I have only been talking about minimal separators in undirected graph.

But it could also be defined for directed graphs. It could also be defined for the vertex version in the sense that I have s and t and I would like to delete minimum number of vertices to disconnect s from t. And then we can ask what is the minimal number of vertex separators. But as I said before, we will focus on edge cuts in this week course. So, let me define.

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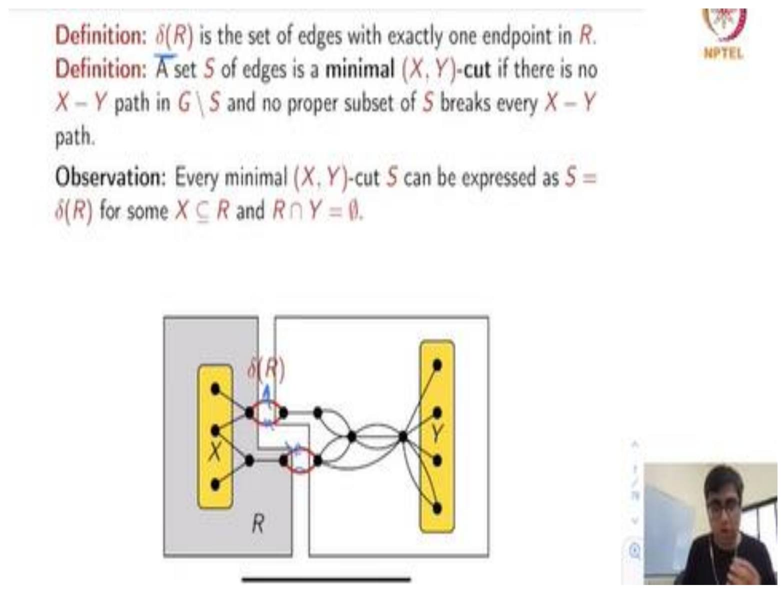

So, I am going to define a notion of boundary edges. So, what is the boundary? So, you give me a set of R vertices by delta R I will mean a set of edges that leaves R meaning it contains exactly one endpoint. So, for example, look at if this is my R, you see here the set of vertices that R then the red coloured edges are my delta R and a set S of edges is a minimum X, Y cut. So, X is the set of vertices Y is another set of vertices.

If there is no X, Y path in G - S and no proper subset of S breaks every X, Y path. So, what is the minimal X Y cut? I mean So, the way we defined before a minimal s-t cut now, s had been replaced by a subset of vertices called X, t had been replaced by a subset of vertices Y and our objective is to find a set of edges which disconnects X from Y. In other words when we delete this set of edges there is no path from X going to Y.

So, for example, if this yellow colour is X and this this yellow colour is X and this yellow colour is Y, then if I delete the edges going across this R then there is no path from X to Y. And an important observation is that every minimum X, Y cut S can be expressed as is equal to delta R for some X subset of R and R intersection Y is phi. Why? So, look at look at let us try to prove this lemma just to get a feeling for how these things are done.

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So, here it is. So, suppose after I delete G - S, how does the world look like? Here is my component that consists of X and everything else. But notice that this set of these components may not be containing X alone. It like so, this is basically minimum set of components that contains X. So, in other words let us see.

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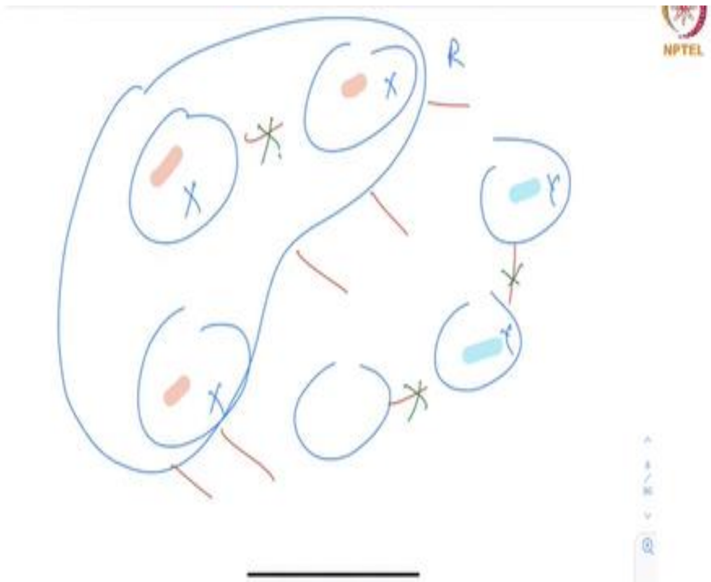

So, I delete G - S, how it will be? Suppose this is this and suppose what are these? These are vertices have say some piece of X, this is some piece of X, this is also some small piece of X, this is also a small piece of X, it is a small piece of Y. Now, notice ideally, I would like R to be this is my R. Notice that your minimal cut if it is a cut if it is a minimal cut so, if I take this you may ask, does a minimal cut like it is.

So, definitely I can disconnect X from Y by deleting edges incident in R. So, now you can definitely write but maybe this there is an edge here, there is an edge here, but notice then R is not a minimal cut. Because even if I delete these edges then then I have a set of edges whose deletion this connect X from Y which implies that whenever you have a minimal cut you can find a set R such that in that R all the vertices of X are there.

And the vertices of Y are in the complement and the set of edges which are looking for is the set of edges which are in the boundary alpha. So, you can prove this. So, I just gave you a small hint how to go about proving this observation. So, every minimal X, Y cuts can be expressed as like a set of edges emanating out of some R that contains X completely and which is disjoint from here.

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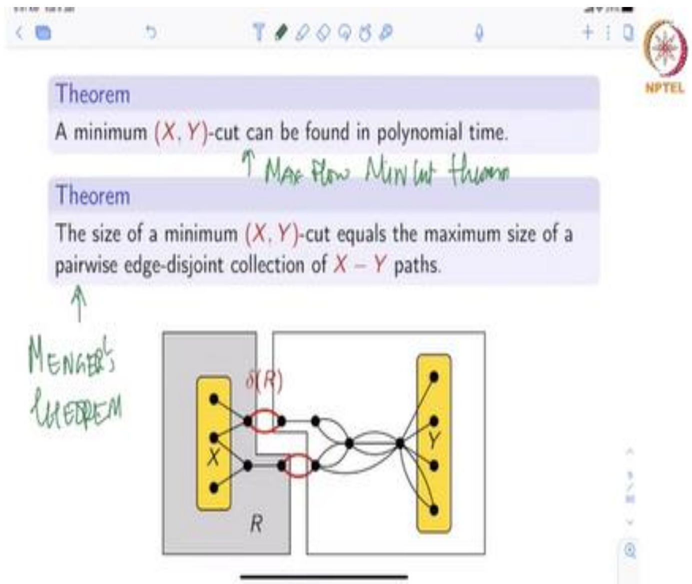

I need to the well-known theorem that a minimum X, Y cut can be found in polynomial time and the size of minimum X, Y cuts equal the maximum size of pair by disjoint collection of X, Y cut this is nothing but our classical Mengers theorem. And a minimum X, Y cut can be found in polynomial time and this is again using max flow min cut theorem. So, these are classical algorithm that is being taught in our undergraduate algorithm.

So, we will not be telling you the proof for this. But we will give you some idea how people actually go about proving it, because we will need some of these basic concepts also for the set of objects so, the tools which we need for our purpose.

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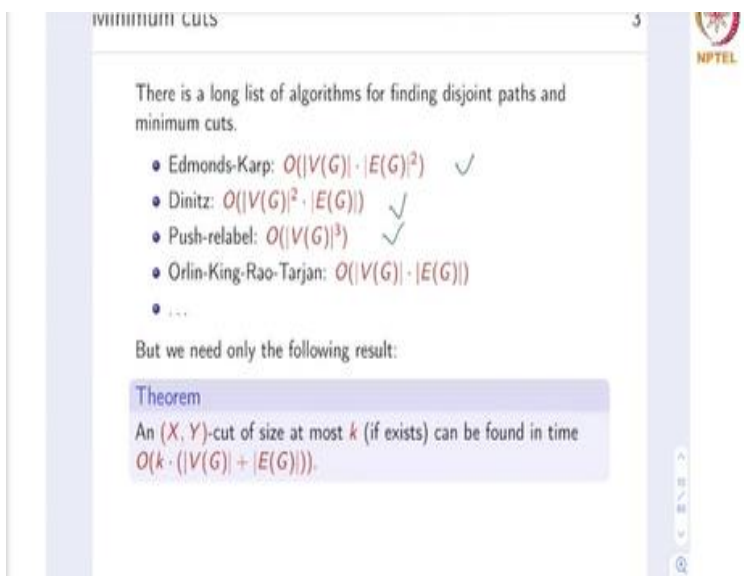

So, there is a long list of algorithms for finding disjoint paths and minimum cuts. There is an algorithm by Edmonds Karp, Dinitz, Push relabel, Orlin King Rao Tarjan. But for our purposes, all we need is the following; an X, Y cut of size at most k if exists can be found in time k times m + n.

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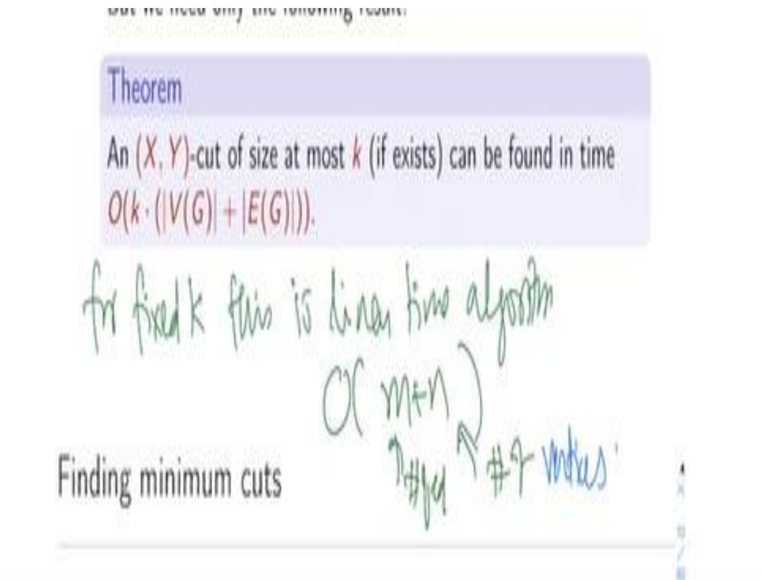

So, if you notice this is for fixed k this is linear time algorithm. And by linear what do we mean? linear we mean order m + n where n is number of edges, n is number of vertices, and k is the cuts. And how do we actually find such an algorithm very trial.

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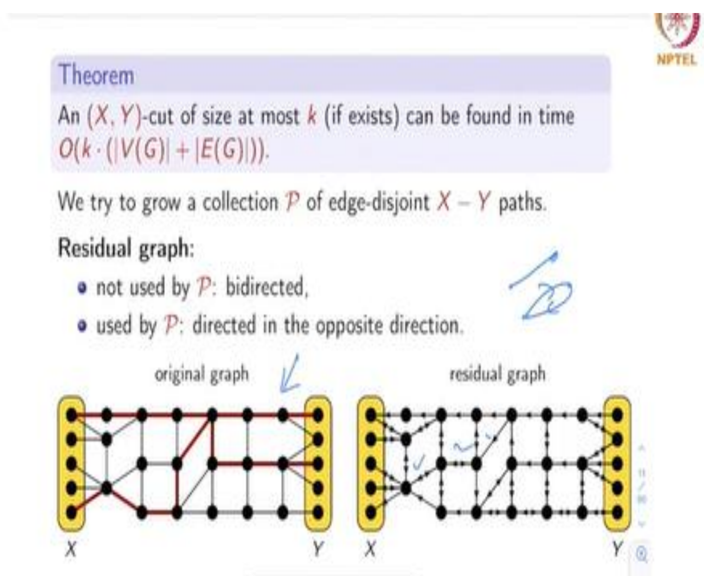

So, we tried to grow a collection P of edge disjoint X Y paths and at any point of time so, suppose difficult this is our original graph and we construct what is called residual graph. In the residual graph the edges which are not used are bidirectional meaning it can be used in both ways. So, you can simulate these unused edges by putting a two cycle or putting an edge in both directions. So, for example this, this, this, this and the edges which are used by your flow or used by your path, you put in the direction from Y to X.

So, for example, these edge on these red things are oriented from Y to X. So, this is what is called residual graph and you must have seen this in your whenever designing max flow min cut algorithm. So, at any point of time you are given some set of edges you construct this residual graph and then you try to see cut it does there is a path from Y to X. It means we can send an extra flow from Y to X augment like via this path. So, this is how it is done.

So, in this residual path we found a path from Y to X. What will we do now? After this, we will basically add this to the our current flow.

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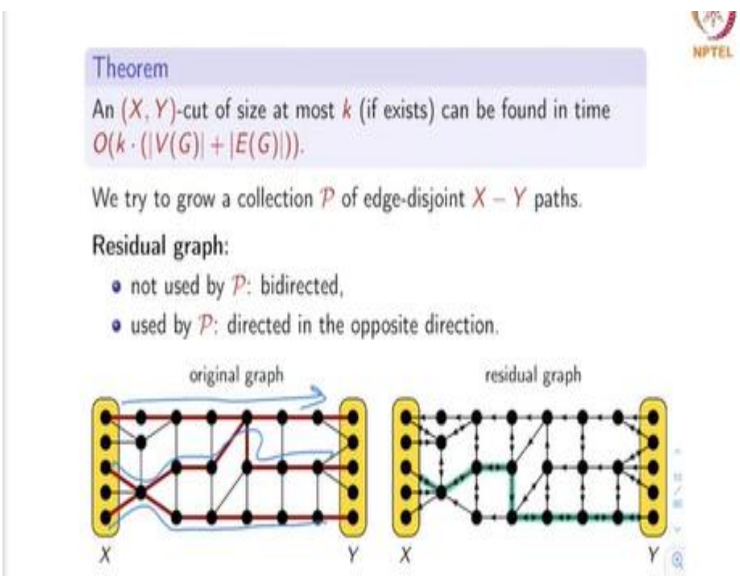

So, we took this and we added this. Now notice our flow has increased by one so, there is like one path here and this is an one path here and this is another. So, every time flow will increase by one and if there is a cut of size at most k by maximum and you can only do this k times. And since each of because we are one all we are doing is to find a BFS or DFS starting from Y to X, we can do this in linear time order m + n type.

Because all we are trying to do in this directed graph does the register path starting from a vertex in Y and ending in X. So, this is how you play around with the residual graph and just by k steps of this algorithm you are guaranteed to find either k is disjoint paths or you can find a cut of size.

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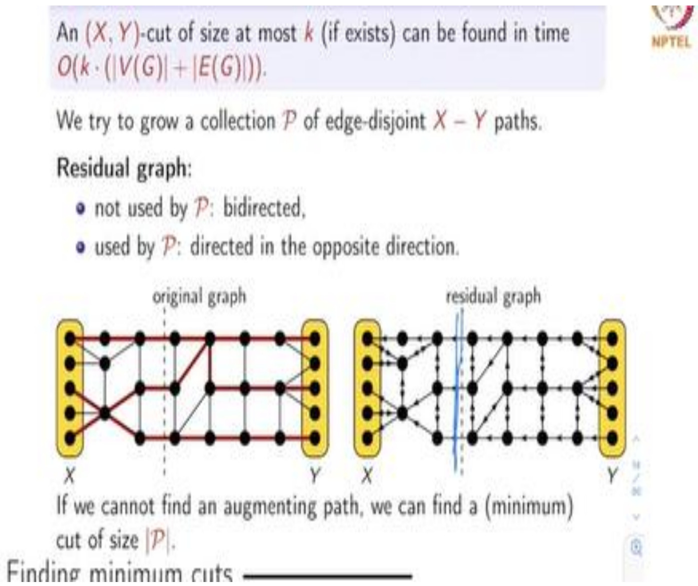

So, if we cannot find an augmenting path, we can find a minimum cut off size P. How can we find a minimum cut size P? That is very easy and there could be many. Starting from Y, you look at the reachability set, set of vertices that from starting from Y you can reach in the residual graph and the set of edges leaving that is your minimum cut. For example, here and if you go corresponding if you delete this set of edges then you cannot reach X to Y.

And guarantee of equality holds for major seven or by maximum minimum cuts. So, this is how we could find a min cut of size at most k in k into m + n time which is linear when k is fixed. So, but for our purposes because all we were looking for that there is a cut of size that most k are not and this we can do it in order k into m + n time I hope this is clear. The reason why I went slightly quicker; at this part because this is something which you may have seen in your undergraduate algorithm course.

So, all we wanted to tell you from this even if you did not understand the algorithm very well is that an X Y cut upside at most k exists can be fine in time this so, please take that. So, this is our take away point an X Y cut that at most k effect is can be found in time order k times m + n, that is it.

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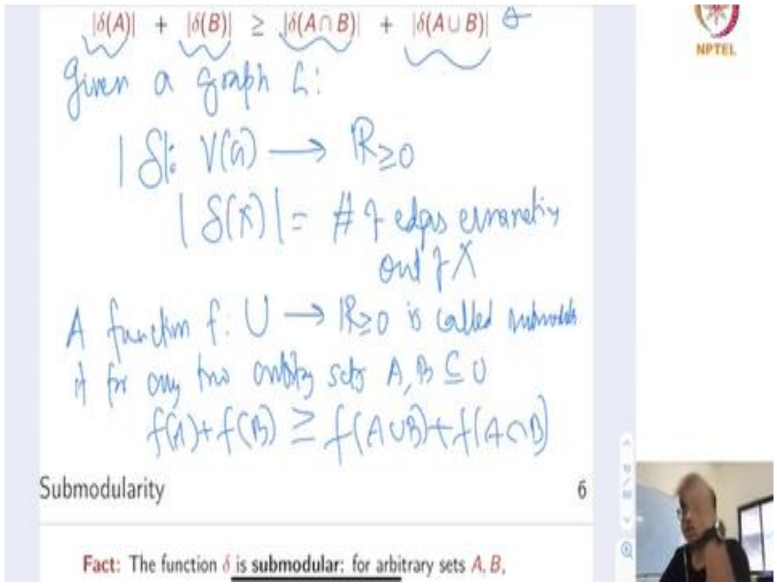

Now let us move. So, the notion which will be very important in combinatorial optimization and also in our lecture is what is known as the function delta is submodular or this function. What are that function delta if you recall correctly? The function delta was basically given a set R it is a set of edges with exactly one endpoint in R. So, the function delta some recall, given a graph G. What are the functions? The function was the delta function was in fact R greater than equal to 0.

What are the functions? Delta of X let us call this function. So, what is delta X is basically number of edges emanating out of X that is it. So, what is the meaning of the delta function if submodular? So, basically, we need to show that if you look at two sets A and B for arbitrary sets A and B we need to so, the contribution of A and the contribution of B is larger or equal to contribution of A intersection B plus contribution of A union B.

And so, this is a function satisfying these properties are called submodular function. So, what is a function? So, we can say a function f from some universe to say R greater than equal to 0 is called sub modular if for any two arbitrary sets A, B subset of U f of A + f of B is greater than equal to f of A union B + f of A intersection B. So, now we have to show that the set of edges the cardinality of set of edges emulating some set in the graph is a sub module function. So, this is what we will try to solve next.

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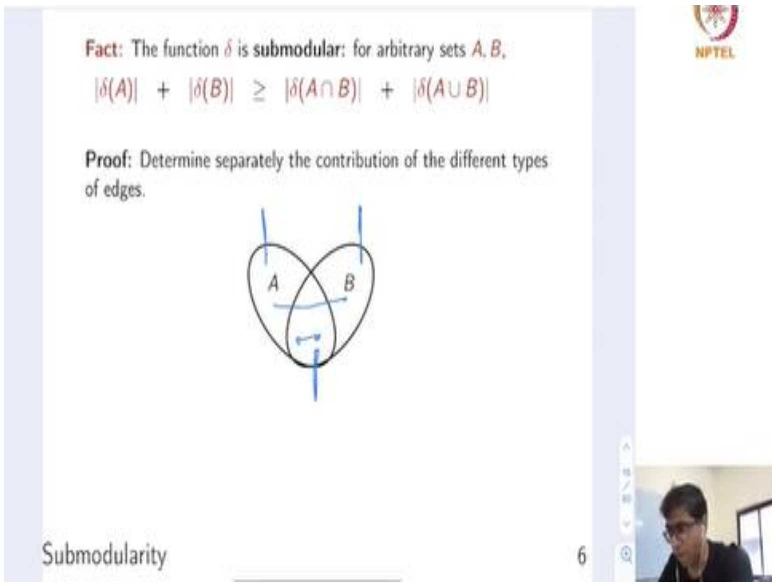

So, what I am going to do to show this we are going to determine separately the contribution of the different types of angles. So, suppose this A and B are there, A and B they intersect with each other. Now, where the edges could be? So, edges in my graph could be leaving A completely, leaving B completely, it could be going from A to B or B to A, it could be completely contained inside this.

So, the other kinds of edges are which could help us or it could be edges like this and let us see how they contribute to both sides.

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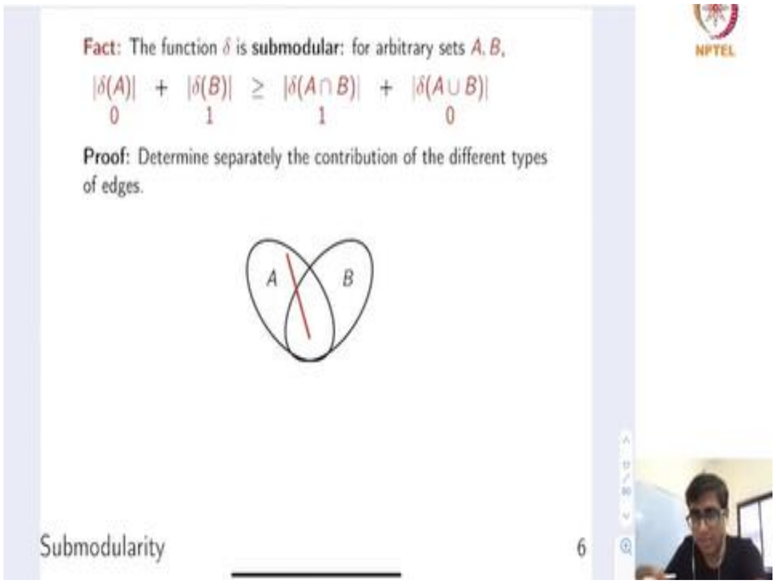

So, first imagine that there is an edge which is leaving B, but it is going inside it is not contributing to the delta thing. So, now what is the contribution of this? It is contributing to the partial of B is 1 but partial or the boundary of A is 0. But look at this A intersection B it is contributing to intersection B because it is going out from A intersection B to this. But it is not contributing to the partial of A union B.

So, and it is of this nature contributes one to the left-hand side as well as the one to the right-hand side and hence its contribution is proper.

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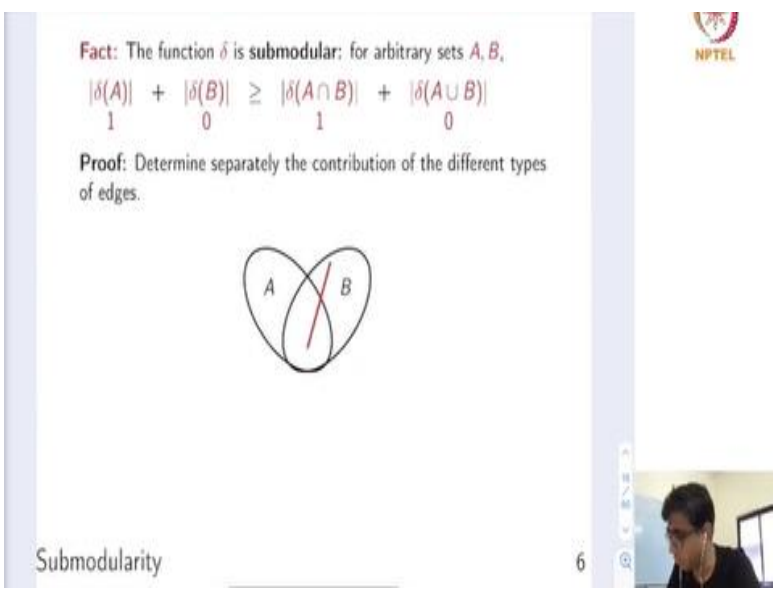

Now, similarly look at an edge which leaves from the intersection but now goes to the B side. Still now, it is contributing to the delta A side but it is contributes to the B side and similarly to A inter section B but not great.

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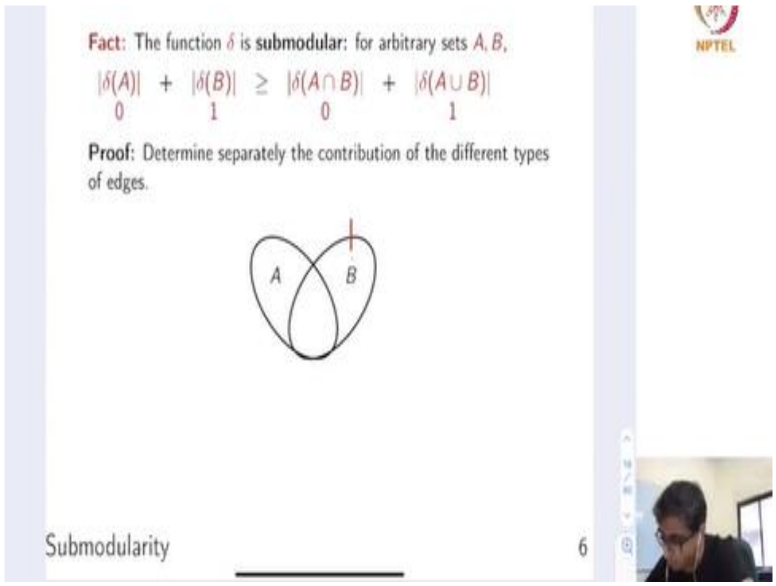

What about an edge which leaves B but it also its other endpoint is not in A. But then it contributes to A union B left hand side it contributes to delta. So, still the contribution of this is to the left-hand side is at least as much as its contribution to that right hand side.

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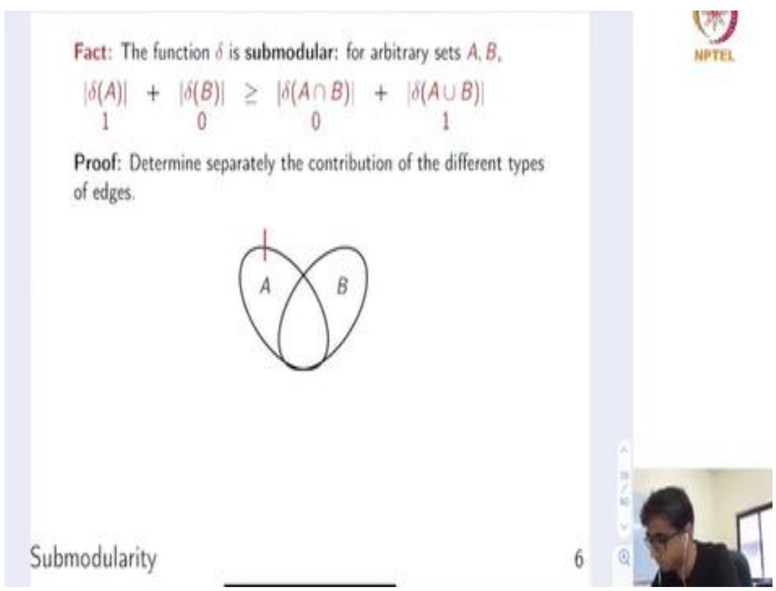

And similarly, for an edge A living from here and going completely outside will contribute to the delta A and to delta union B. So, left hand side contribution is still greater or equal to the contribution to the right hand side.

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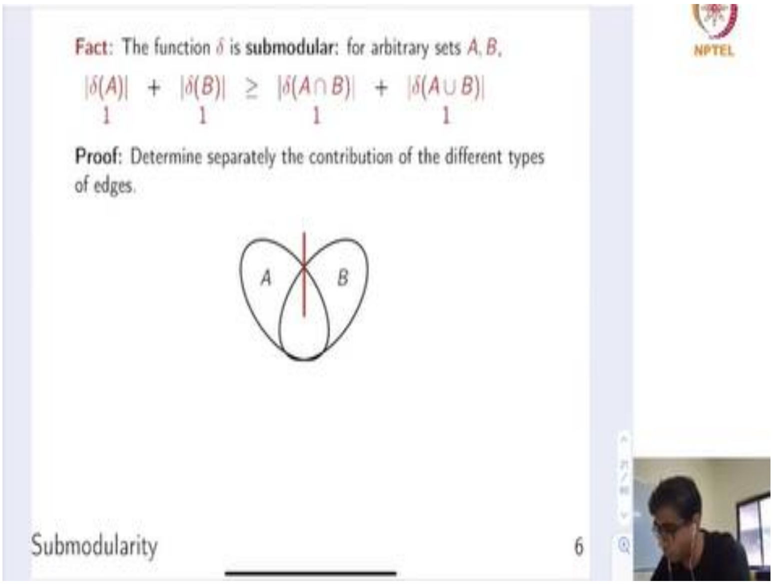

Now what about an edge which would intersect with one vertex in the intersection and it goes outside. Then you notice that it contributes to both A and B in the left-hand side as well as it contributes to the both A intersection B and A union B in the right-hand side. Again, the contribution to the left-hand side is at least as much as its contribution to the right hand side.

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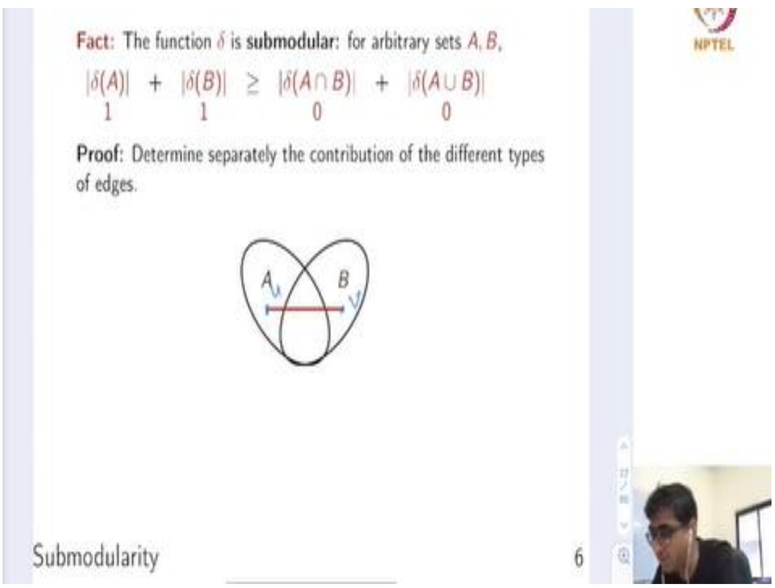

Now what about an edge which is like here? So, now what is the meaning of this so this is an edge with endpoint in u and endpoint in v. So, from the v is perspective, u is inside A, from the u perspective v outside, but it is not in A because not in intersection. Then again, it contributes to delta A and delta v, but it does not contribute to anything in the right-hand side. Neither it is A intersection B, neither A union B. And in this case, still left-hand side includes modal interactions.

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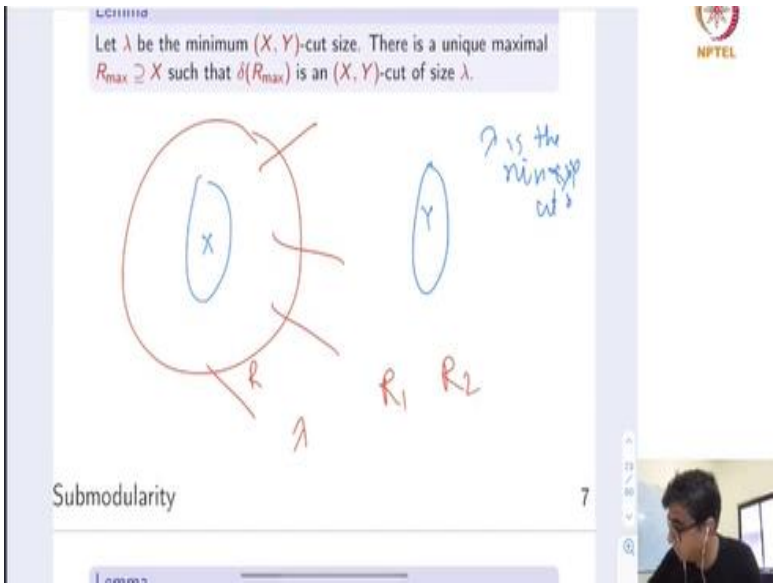

So, this just tells us that the function delta is some modular and now we will use this property of sub modularity to prove an interesting property about let lambda be the minimum X, Y cuts. Then the unique maximal R max which contains X such that delta R Max is an X, Y cut of size lambda. So, now what is the meaning of this? Let us try to parts this. So, here is my set here is a set X and here is set Y and lambda is the minimum X, Y cut size. But there could be many, many cuts.

So, what do you mean? So, this is an X and it is a minimum cut so, it is also a minimal cut. So, we can find a set R such that there are lambda edges leaving. But there could be several such Rs that will do the job for you. Now, what this lemma is tells you that if you have to R say R 1 and R 2.

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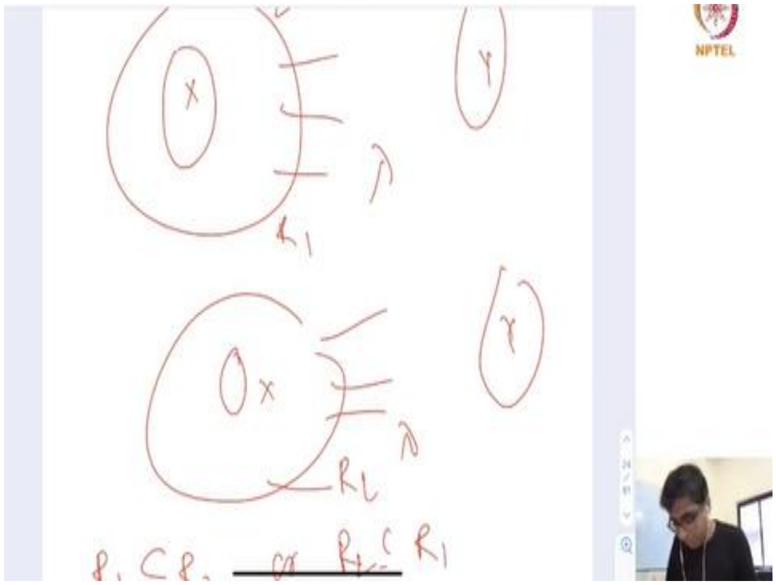

What is this tells us? That look here is an X and here is some R whose value is lambda. And of course, this Venn contains Y is completely outside and support this is R 1 and there is another one which is X and say R 2, number of edges going is still lambda. Then we will be able to show that R 1 is contained inside R 2 or R 2 is contained inside R 1. In other sense what is this lemma tells us? That there is a unique maximal R max that contains X.

So, I mean if there are two Rs such that they are min cut, the number of edges going out is min cut then one of them is contained inside which implies that basically what it means. So, suppose you took 1 which is contains the other. Now, what about R 2 and R 3, maybe it is contained inside R 3 so on and so forth.

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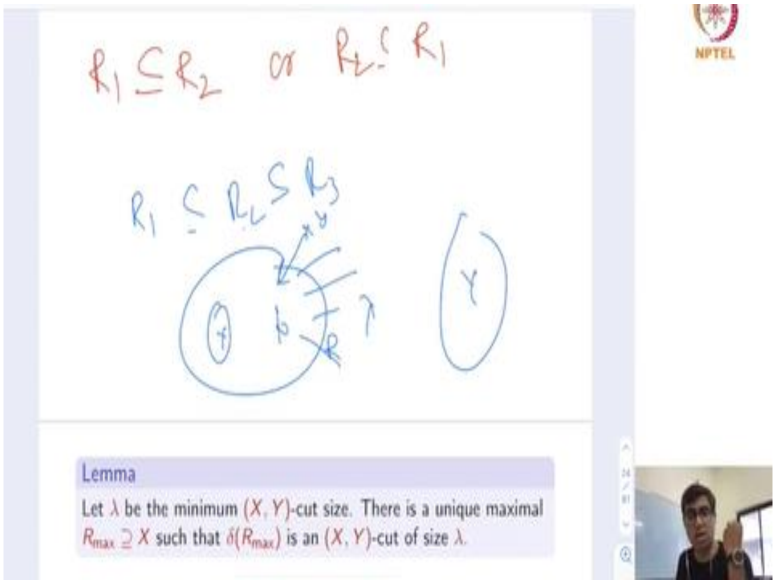

So, basically what I am trying to tell you that if you say that hey, look at this R and what is the property that this contains X? Number of edges emanating out is lambda and Y outside. Furthermore, you can say that look, I cannot add any more vertex Y to add without increasing the cut size between X and Y then such R is unique and we will call that R max. So, let us try to prove this.

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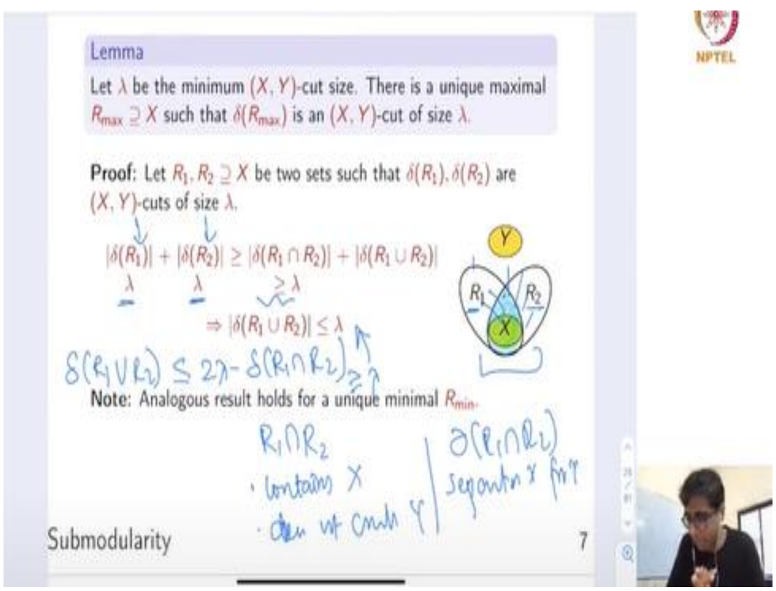

So, suppose there are two R 1, R 2 such that delta R 1 and delta R 2 are X, Y cuts of size lambda. So, what I needed to show to you that they contain each other, one of them contains each other. So, both R 1 and R 2 are X, Y cuts have size lambda. It means, look at this R 1 and R 2 is a set of vertices it contains X and Y is outside because that is the definition of R 1 and R 2. What are your definitions of R 1? That it contains X it does not contain Y, R 2 contains X does not contains Y.

So, if I take the union of R 1 and R 2 and then X will be contained inside the intersection and what now we know that the delta was it is a submodular function. So, now we will apply some modularity on the sets R 1 and R 2 then what we know delta R 1 + delta R 2 is at least delta R 1 intersection R 2 + delta R 1 union R 2. But what is the value of delta R 1 is lambda. What is the value of delta R 2 is lambda. And now, what is a value of delta R 1 intersection R 2?

So, what are transits? So, look at this delta R 1 intersection R 2. What is the property of delta R 1 intersection R 2? This is look at this R 1 intersection R 2, this size this part. What is R 1 intersect? Even if I look at R 1 intersection R 2 it contains so, what is the property of let us say write on R 1 intersection R 2 contains X does not contain Y. It means the edges which are adjacent it means delta R 1 intersection R 2 separates X from Y.

It means the number of edges adjacent to R 1 intersection R 2 can only be more than lambda. Because lambda is a minimum number of edges, we need to delete to separate X from Y and R 1 intersection R 2 is one separator. So, number of edges that are adjacent to R 1 intersection R 2 can only be more than lambda, it cannot be less than lambda which implies that delta R 1 intersection R 2 is also greater than equal to lambda.

Now, left hand side is like what? Two lambda is at least equal to greater than or equal to let us say what we can say we can say if we do this computation delta R 1 union R 2 is less than equal to 2 lambda - delta R 1 intersection R 2. Now, we know that delta R 1 intersection R 2 is at least lambda. This is greater than or equal to lambda which will imply that delta R 1 intersection R 2 is less than or equal to lambda.

So, what we are going to found that look I mean they are not maximal in the following sense. They are not maximal in the following set they are like in the following set. Because there is some other min cut which consists of these guys and it contains it is still. In other words, let us try to see what we what we prove just now.

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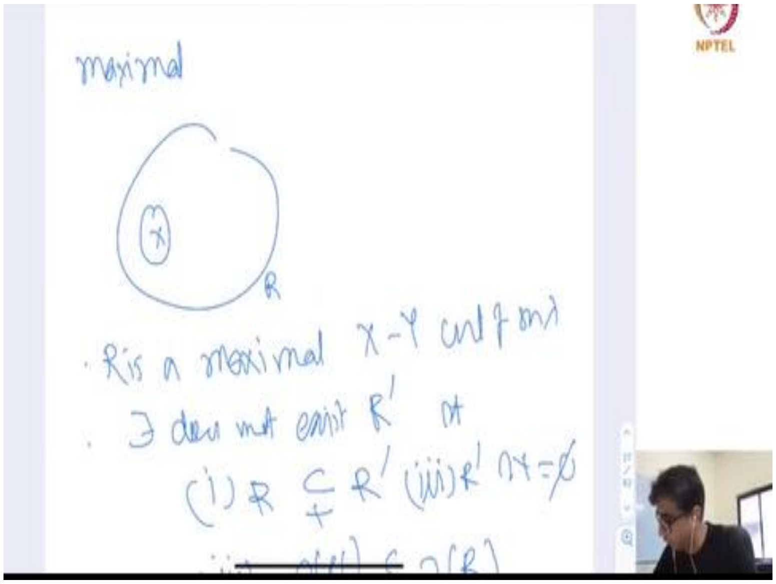

So, as I said before what is the meaning of maximum here. I wanted to say that look here is an X and here is an R. We say that look R is maximal X, Y cut of size lambda meaning there does not exist R prime such that first R is a proper subset of R prime. And second let us say lambda of R prime is less than equal to lambda that does not exist R prime such that R is subset of R prime. And also, we should write.

This is third point R prime intersection Y is ending. Such sets do not exist then we will call this in maximal with this property. Now, what we have shown to you is that look. If this is the definition of maximal you are trying to put then and you are only trying to put this maximal with respect to min cut size then there is you need maximum or with this property. So, you cannot have two R 1 and R 2 with the property.

I cannot extend R YX, other vertex such that the cut size still remains the same. Because if that happens then we could have taken R 1 in union R 2 which is a superset of both R 1 and R 2 and it is still disjoint from Y and the number of edges which are going across it is exactly it is at most lambda. So, if I am going to call you that R max, some set R is maximal with respect to set inclusion for a minimum cardinality lambda. Then there is a unique such set that is all that is terrible. So, what we prove is the following.

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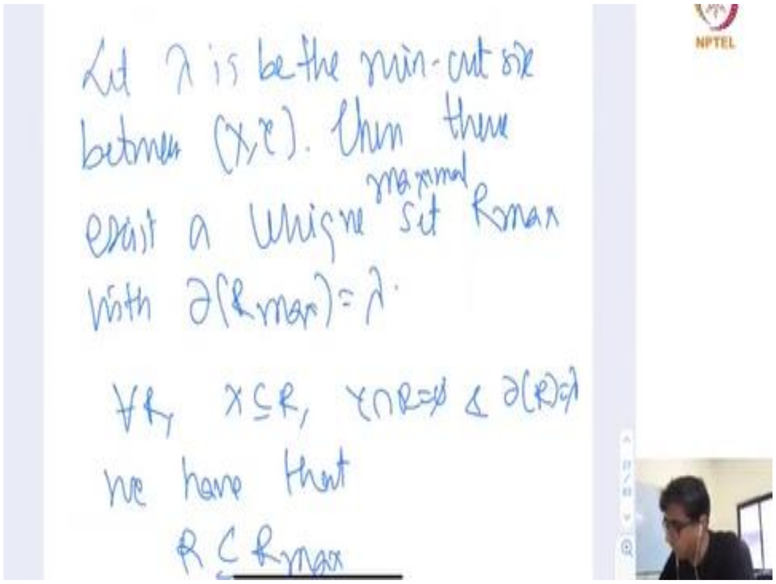

Let lambda, is be the min cut size between X and Y. Then there exists a unique set maximal R max with delta R max equal to lambda. So, in other words what it says, if you have some R 1 with the property that X is subset of R 1, Y intersection R 1 is empty or rather for every R such that what is the property, X is a subset of R, Y intersection R is empty and delta of R equal to lambda. We have that R is the subset of R max. So, any set as this has a property that they are contained inside R max.

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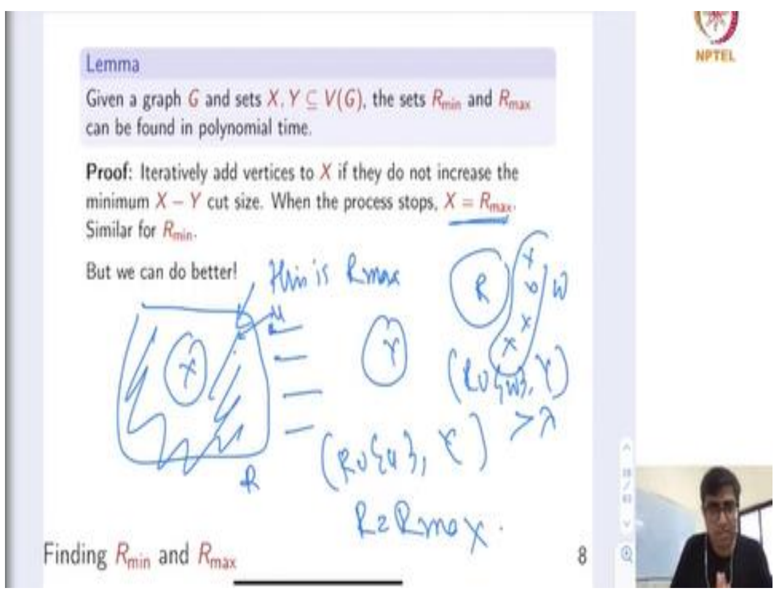

Now that also tells us an algorithm to find out, how? Given a graph G and sets X, Y we can also define the notion of R min and R max. What is R max? We will talk about R min later. R min is a smallest set that has this property the sense that R min is part of every R that has a part. So, first we will talk about X = R max. How can we find a R max? So, given a graph G sets X, Y subset of V G. And if you like to find the set R max, how can we form?

So, basically what you start with X and Y. You compute a min cut so suppose you got computed a min cut. Then the moment you computed the min cut; you know that this set of vertices is definitely contained inside that R max your consistency, because of the property. Now what you do, fine, maybe this is an R max or maybe it is not. If it is not what we do to the current R, you add some vertex u and check is the min cut increased.

Now, what will you mean by min cut increased? So, you will take, you compute R union u, Y. So, this is way you computed the min cut. Because you know the min cut between R and Y is lambda. So, now you take are union a single vertex put them together and say what is the min cut between this set and why. If it is still lambda then you know that Y, u is also part that. Because this set is contained inside that R max.

Otherwise, you forget about that and you try some other vertex. So, the ones you find are set R such that for every vertex outside W R union W, Y has min cut more than lambda then you stop and you can say that this R is our R max. So, I creatively add vertices to X if they do not increase minimum X, Y cut size then you stop and say this is my R max. And what is similarly for R min? What is an R min? So, first let us define what R min well mean.

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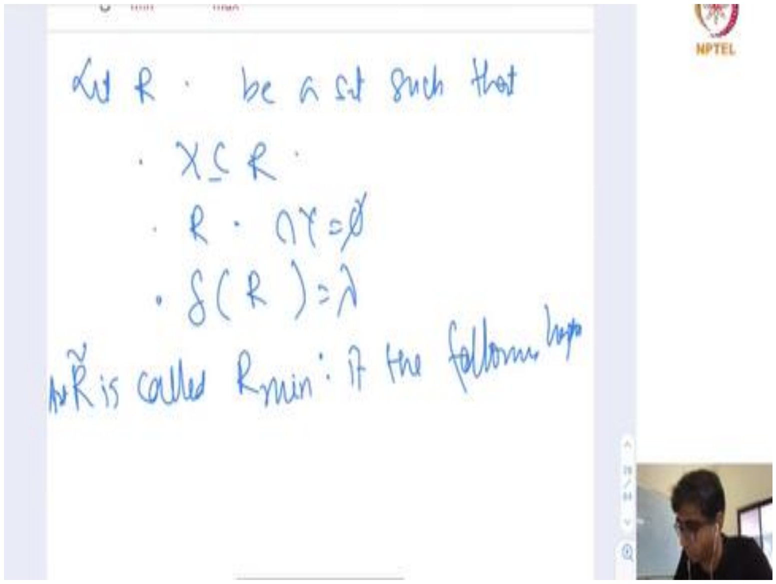

So, what is R min? So, one way of defining R min is that what we define R max. Let R min be a set such that X is contained inside R min, R min intersection Y is phi. What is the meaning of R min is? Set such that X is this R min is this. So, first of all and lambda of R min or delta of R min is equal to lambda. So, rather than writing, let R be a set such that this happens. R is called R min if the following happens. Rather R tilde, a set R tilde is called R min if the following.

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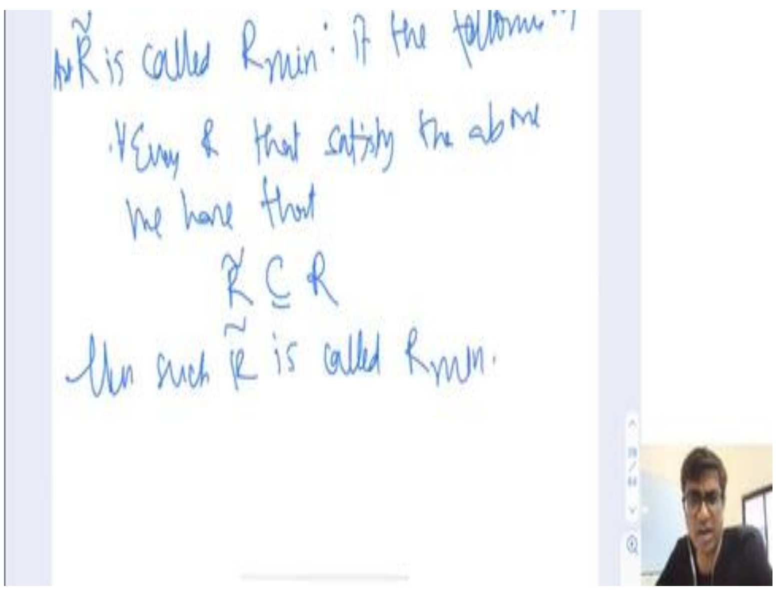

For every R that satisfy the above we have that R tilde is contained inside R. Then such R tilde is called R min. There are other combinatorial ways of looking at it.

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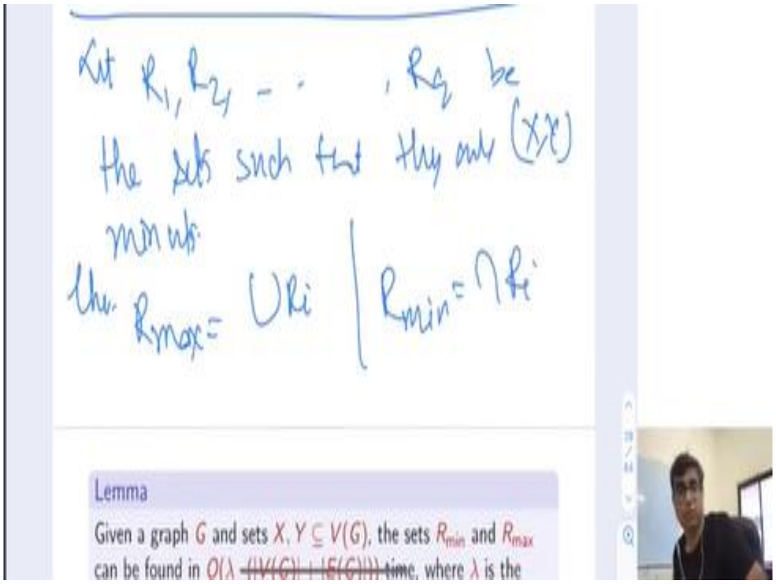

So, for example let R 1, R 2, R q the sets such that they are X, Y min cuts. Then basically R max is nothing but union R i and R min is intersections. So, this is how we can do.

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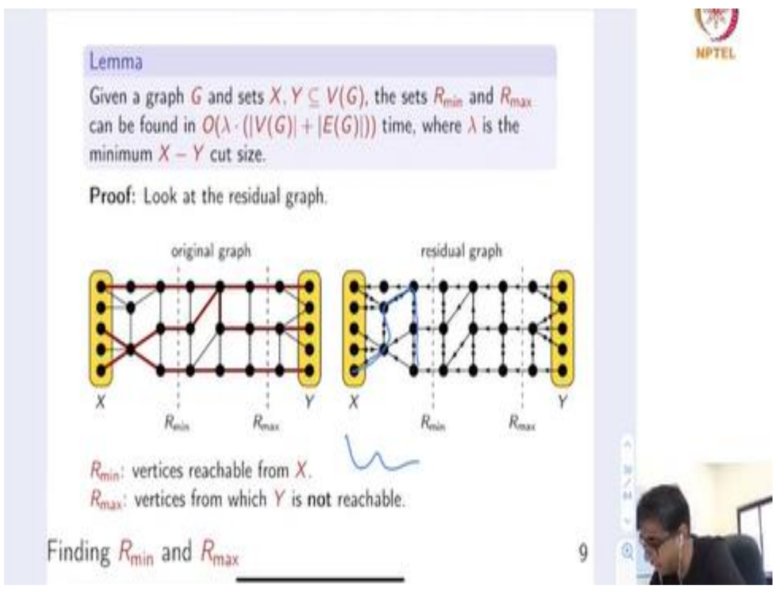

And there is another way of making If this. What is this? So, you look at the residual graph and you look at what is R min. Those vertices which are reachable from X and what is R max those vertices from which Y is not reachable. What are R max? Vertices from which Y is not reachable it. So, R min vertices those are reachable from X hey. So, these are the vertices which are reachable from X from, see I can reach.

I can reach all these guys and what is R max vertices from which Y is not reachable. So, which are the vertices which Y is not reachable? Well, I cannot reach from here as well as I cannot reach, from these vertices I cannot reach or these vertices I cannot reach or these vertices that I cannot reach Y. So, this is basically R max. This is R max this is R min. So, this is how also you can make R max and R min.

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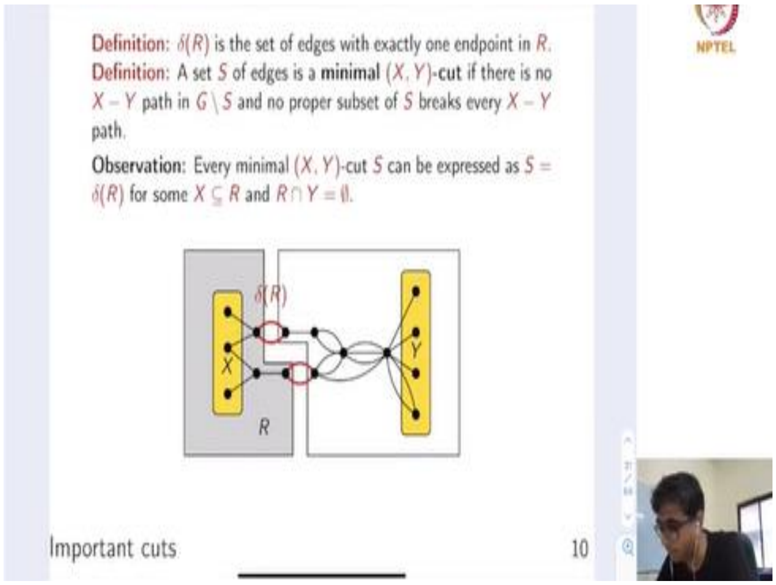

So, I hope we have talked about important cuts that extremal property. Now I am going to talk about important cuts and I am going to define important cuts.

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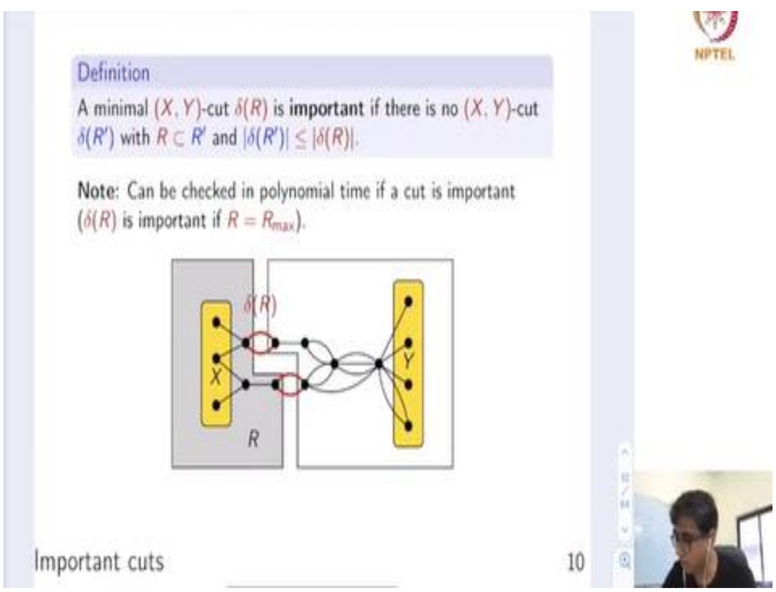

So, in the next class we will talk about important cuts defined what they are and how many of them are there. Thank you.