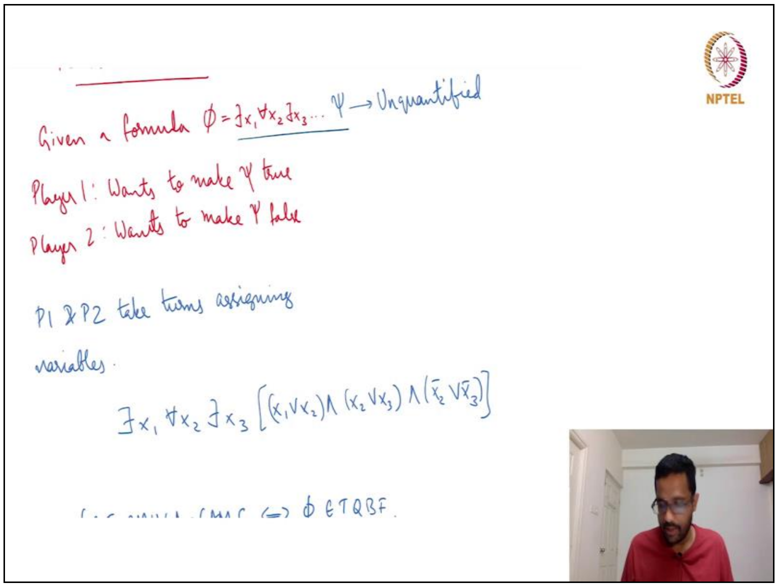
Computational Complexity
Prof. Subrahmanyam Kalyanasundaram
Department of Computer Science and Engineering
Indian Institute of Technology, Hyderabad

Lecture -20
Games and PSPACE Completeness

(Refer Slide Time: 00:15)



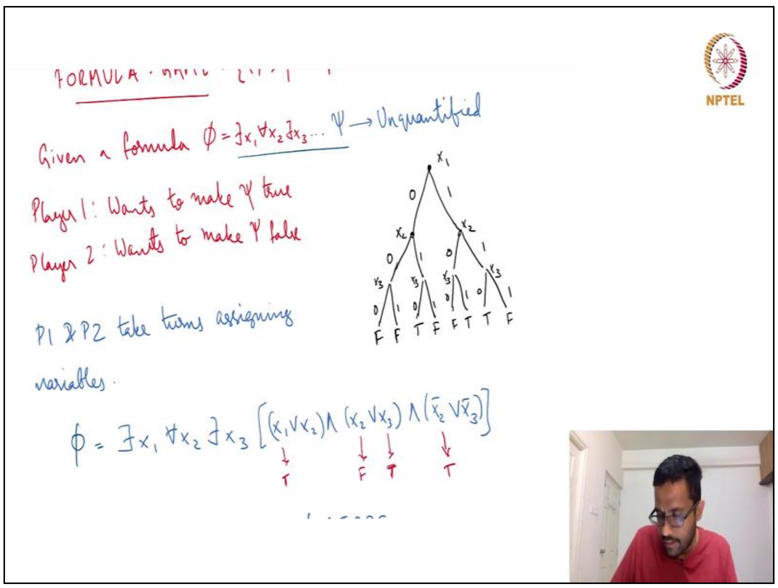
Welcome to lecture 20 of the course computational complexity. This lecture is going to be short but interesting. So, we are in the last lecture we saw what is PSPACE completeness PSPACE completeness is a notion of completeness for the class PSPACE. So, a language is PSPACE complete if it is in PSPACE and all the languages in PSPACE A all the languages A and PSPACE are reducible to that language in polynomial time.

So, today we are going to see how certain languages based on games are PSPACE complete. So, in the previous lecture we saw that the language TQBF true quantified fully quantified Boolean formula is PSPACE complete. So, the first language that we consider in this lecture is what is called formula gain and it is very closely related to TQBF. So, what is the formula game? It is that there are 2 players let us call them Alice and Bob or player 1 and player 2 and they are given a Boolean formula with many variables and they take turns.

So, the variables are x 1 x 2 and. So, on they are labeled and they take turns trying to assign assign the the true and false values true and false values to the variables and of course it won't be a game unless they have conflicting objectives. So, player 1 wants to make the formula true and player 2 wants to make the formula false. So, just to be more formal. So, it is a form it is a game like this. So, phi is a quantified Boolean formula like this there exists x 1 says that for all x 2 there exists x 3 something some something psi and where psi is an unquantified Boolean formula.

So, they are both trying to assign values to x Boolean values to x 1 x 2 x 3 etcetera such that player 1 is trying to make psi true by setting the values player 2 is trying to set psi to be false.

(Refer Slide Time: 02:43)



So, player 1 wants to make it true player 2 wants to make it false and they take turns assigning the variable values. So, let us try to see this one formula let us say phi is this that exists x 1 such that for all x 2s this is the exists x 3's as if this is true let us see what happens in this case. Now let us say if let us say player 1 sets x 1 to false. So, now that means that the first now this is a CNF form. So, this the first clause is already has to be made true. So, player 2 can now player 2's objective is to make the entire formula false.

So, player 2 can immediately set x 2 to be false and then the entire formula is false. So, player 1 should do something smarter player 1 should set x 1 to true. Now once player 1 sets x 1 to true.

So, again one just to just recap what I said if player 1 sets x 1 to false then player 2 can set f x 2 to false and then the entire formula is false. So, that leads to a lose for x player 1 and win for player 2. Now if player 1 sets x 1 to 2. So, the first clause is already true now what can player 2 do let us say player 2 sets x 2 to 2.

So, then this is the second clause is already true now and part of the third clause is false because x 2 complement goes to false now but then player 1 gets to choose x 3 and he can choose x 3 to be false. So, that x 3 complement is true. So, the entire formula gets evaluated to true. So, player 2 if he sets x 2 to true then that does not lead to a win for player 2 that leads to win for player 1. Instead if player 2 sets x 2 to false second clause is now not true but third clause is set to true because x 2 is false.

Now all the player 1 has to do now player 1 again has control over x 3 but he does not need to very worry about clause 3 he all has to do all that he has to do is to set clause 2 to true for that he can simply do that by setting clause x 3 to be true. So, now it is very clear. So, if x 1 is set to true then. So, player 1 has a winning strategy here he sets x 1 to true. Now depending on what player 2 sets the value of x 2 to be he can choose x 3 to be the opposite of that.

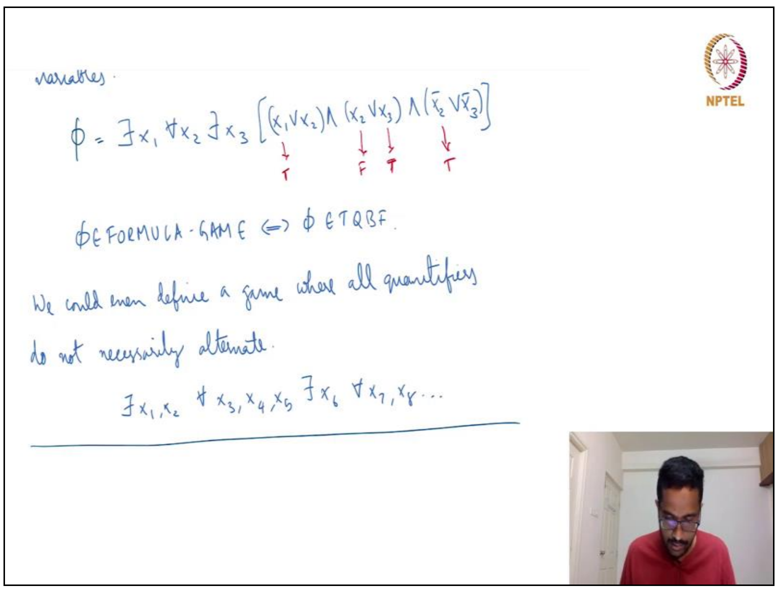
So, if his if player 2 sets x 2 to false then x 3 is a to true and if player 2 sets x 2 to true then he can player 3 can set x 3 to false. So, either way there is a win for player 1. And there is a win for winning strategy for player 1 means player 1 can always win which means there cannot be a winning strategy for player 2 if I can always win playing this game then my opponent will whatever he or she tries cannot win the game.

So, now this is the game and maybe it is uh. So, now you can see when x 1 x 2 x 3 are kind of alternating between player 1 and player 2. Now this is exactly like form for TQBF. So, we are just asking whether this fully quantified Boolean formula is true or not. So, that is; so, what is formula game? Formula game is set of all again I have not formally defined it formula game is a set of all Boolean formulas phi where player 1 has a winning strategy.

Player 1 has a winning strategy and player 1 has a winning strategy when this fully quantified Boolean formula is in TQBF because this fully quantified thing has to be true for player 1 to have a winning strategy. So, that is why the formula game is also. So, it is essentially equivalent to TQBF. So, this is also PSPACE complete it is essentially the same game but just being kind of interpreted as a TQBF was a something about Boolean formula but this is just more interpreted more as a game than Boolean formulas.

So, maybe just to gives a bit more insight let us try to see this like game tree. So, initially x 1 is going to be set. So, it could be set to 0 or 1 or true or false and then x 2 can be set to true 0 or 1. Then x 3 can be set to 0 or 1 and now we are at different places. So, if x 1 and x 2 are both false then I think as we discuss initially both these lead to false, false meaning player 1 is losing player 1's goal is to make it make the formula true if x 1 is false but x 2 is true again this is something that we did not explore x 2 is true.

(Refer Slide Time: 09:11)



Then now again there is a meaning it depends on what x 3 is if x 3 is 0 or false then the formula is correct is set to true otherwise it is false and if x 1 is true then player now x 3 is set to just opposite of x 2. So, this is false this is true and this is false and this is true. So, now sorry this is the opposite itself in for player 1. So, now you can see this game tree and see what is happening. So, there's another way to explain the whole thing again.

So, player 1 wants to go to a place such that no matter what player 2 does player 1 has a winning strategy. So, let us see. So, if player 1 had taken this path this path x setting x 1 equal to 0 then player 2 could lead player 1 to x 2 set by setting x 2 to 0 to this point now no matter what player 1 does it leads to false it leads to a lose for a player 1. So, it is like this you could you could for these levels just above the leaf.

Now that we know that the last turn is by player 1 you could see what. So, if player 1 gets it in this node this node over here it is a false meaning whatever player 1 does he cannot win but if he gets it here it is a true maybe I will use another colour. So, this is false and this is true because from this there is one path and here also true and here also true from these points there is one path but look at this now look at the at player 2's turn at x 2 now maybe I will use another colour green.

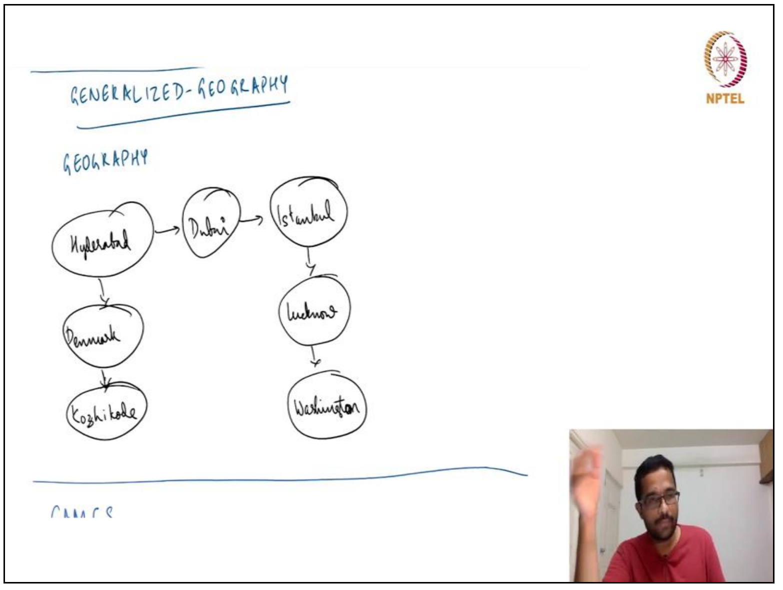
Now over here it is a win for player 2 because he can guide player 1 to the false place. So, this is also a win for player 2 which is false in our terminology. So, false true is absolute win for player 2 win for player 1 is relative to player 1 player 2. But this the left the subtree sorry it is a lose it is a win for player 1 because no matter what player 2 chooses here. It leads to win for player 1 and at the top it is a win for player 1 because now player 1 gets to choose.

So, now there is a false and true he can always go to the true. So, now this is how you evaluate a game tree. So, at the at the place where player 1 has to decide he just needs one path of a victory at the place where player 2 has to decide player 1 if player 1 is to win all the paths lead all the paths should lead to victory for player 1. So, so that happens in this tree. So, this is a victory for player 1 is there a way to modify this.

So, that you can think about it. So, this is how you view games as trees again these are simple games which you can compute and this is the formula again and just one more point one more final point here we had players second turns x 1 x 2 x 3 and so, on but it need not be turns like that. So, player 1 could have one or maybe 2 variables with 3 variables 2 variables player 1 place followed by 3 variables played by player 2 player followed by x 6 by player 1 and so on.

It could be like that also you could get a corresponding 3 there also but then more choices will be made by one player. So, it will be a bit different but it is still the general principles of what I said just continues to hold. So, this is how view you view games as trees. So, formula game is another free space complete game.

(Refer Slide Time: 12:55)



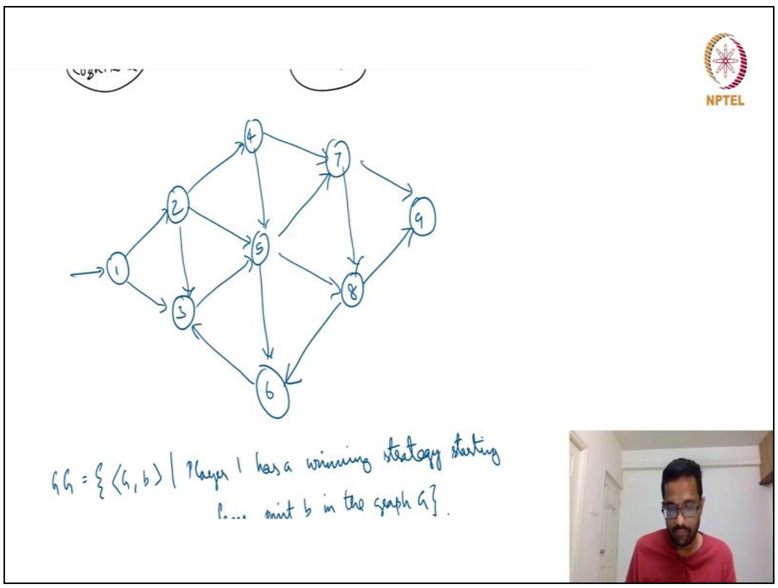
Another PSPACE complete game is what is called generalized geography. So, what is geography? So, geography it is kind of like antakshari that some of you might have played as kids. So, the goal here is to let us say player 1 says or let us say 2 players you could have more players but for simplicity let us stick to 2 players. Let us say player 1 says the name of a place. So, let us say Hyderabad now player 2 has to say the name of a place which starts with the last letter.

So, player 2 could say Dubai let us say now player 1 has to say a name of a place that starts with i. So, let us say player 1 says Istanbul now player 2 has to say something starting with L let us say Lucknow let us say player 1 1 2 1 2 player 2 has to start with w maybe Washington so on. So, this is just how it is. So, after player 1 says Hyderabad player 2 could have said say Denmark for instance and player 1 has to say something with k. So, he may say Kozhicode.

So, this kind of thing is what and this game continues still some player cannot say he cannot name any place with the assigned letter. So, this is the game generalized geography this is the game geography sorry this is not generalized the generalization has not happened yet. But this is a game but what is generalization the generalization is that now this is you can like i have drawn here you can view it as a graph.

And player 1 moves make a move then player 2 makes a move and finally the person who cannot make a move loses. So, the multi multiple ways to reach each place, so, let us say from Dubai you could have said India which is another name of a place or Indonesia. From Indonesia you could have said Afghanistan. So, from Afghanistan you could have said Newyork for instance. So, from Newyork again maybe again you could have said Kozhicode or Calcutta or whatever so on.

(Refer Slide Time: 15:54)

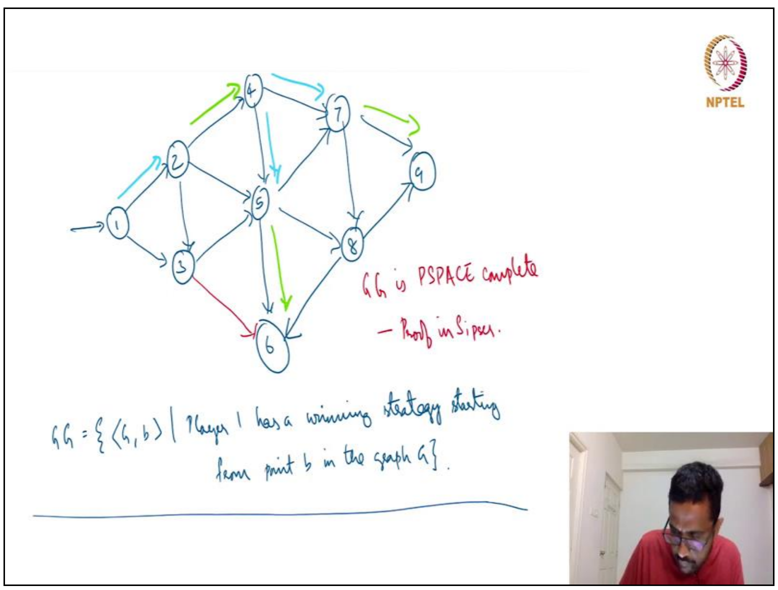


So, again there are different ways to reach the same place. So, the generalized version of this game is to is where you have again this is also there in Sipser. Again where you have some a graph one I am just reproducing the same example in Sipser. So, player 1 starts with 1 and it is his turn and he could I just draw the arrows and then talk. So, now let us say player 1 starts from 1. So, now in this graph the goal is to move leave a person without a move. So, player 1 can go to 3 and now player 2 can now there is only one move at from 3 there is only one outgoing arrow.

So, he has to go to 5 now player 1 can now go to six and player 2 is stuck there is there is no outgoing there is one outgoing arrow from 6 but that goes to a place that has already been visited. So, you again the name in the geography green game you cannot name a place that is already named. So here's 3 is already visited. So, that is ruled out. So, player 1 wins because player 2 is left without a move. So, this is a example of a graph where player 2 does not player 1 has a winning strategy.

So, maybe I just sorry write the for define the game formally generalized geography is a graph G and a starting position b such that player 1 has a winning strategy starting from point b at or in the in the graph G. So, this is the definition of the game. So, as I said if in this graph player 1 has a winning strategy the goal is to identify who has a winning strategy.

(Refer Slide Time: 18:59)



So, let us say let us say we change the graph a bit and we reverse this edge. So, now it turns out that player 2 has a winning strategy why because let us say player 1 plays to 3. Now player 2 can lead 1 to 6 and there is no outgoing arrow from 6. So, player 1 is has lost. So, this is why player 1 cannot play 3 first what if player 1 plays sorry player 1 plays to 2 if player 1 leads to 2 then player 2 can move to 4.

Now player 1 is again he is not stuck but he has 2 choices. So, player 1 could move to 7 or 5 if he moves to 7 then player 2 can push him to 9 without any exit point if he moves to 5 then sorry player 1 can push him to 6 without any exit point. So, this is a graph where no matter what player 1 does there is an x there is a strategy for player 2 to win the game. So, this is a this is not an instance of generalized geography or this is a no instance of centralized geography whereas if this red edge was reversed as I had drawn initially that is a yes instance.

So, again in games like this where the game is finite. So, here it is a finite game there is a finite sequence of moves it is a finite game where the win and loser win and loss are absolute. So, either a wins player 1 wins or player 2 wins there are no draws. In such games either 1 of the 2 players must have a winning strategy it cannot be that because you can look at the game tree and do the analysis that I said earlier like the game tree over here something like this.

You could do the analysis that i said earlier and determine who has a winning strategy. So, in games like this one player has to have the winning strategy and if one has a winning strategy 2 cannot have the winning strategy and vice versa. And this game of generalized geography also turns out to be NP sorry PSPACE complete and the proof is there in Sipser I am not repeating the proof here but it is kind of a long proof.

Generalized geography is PSPACE complete proof is there in Sipsir you can take a look and. So, this is also another game that is PSPACE complete.

(Refer Slide Time: 22:00)

So, now let me just talk about games in general. So again like I said earlier why do all the games have to have a winning strategy all the finite games. So, suppose the game tree is like this. So, like what I have drawn here in this game who has a winning strategy there is a very simple game with just 2 moves player 1 makes a move followed by player 2. So, player 1 where will he move? Let us say the W denote wins for player 1 and else do not lose for loss for player 1.

Player 1 will take the middle path this path because then whatever player 2 does whichever 3 next directions player 2 does player 1 is going to win. However if you change the sorry if we change the graph slightly very slightly let us say change this W to L now whatever player 1 does sorry whatever player 1 does whether he goes for the first let us see he goes here then player 2 can push him down here.

Let us say he goes to the middle path then player 2 can push him down to this down to this and if he goes player 1 goes here then player 2 can go anywhere let us say player 2 goes here. So, now in this case player 2 has a winning strategy. But in the earlier case when this L was not there instead it was w player 1 had a winning strategy. So, in general all the finite games where there are no draws one player has to have a winning strategy.

So, in that sense even games like chess are finite games even though chess has a possibility of draw they are finite games because there are rules of chess that prevent the game to but to continue infinite infinitely. So, for instance if a position appears 3 times then it is a draw or if a position does not appear 3 times then something has to keep changing. So, maybe after a while you run out of pieces. So, then the positions will not repeat.

So, the game by because of the rules is inherently finite. So, chess you can analyze it in this manner maybe the analysis will be more complicated because because of the fact that the number of positions to analyze is humongous even for a computer as of today. So, it is. So, complex that you cannot analyze that but in small games like tic-tac-toe. So you can certainly analyze and I think there are you could easily find the strategy perfect strategy if both player 1 and player 2 plays tic-tac-toe then it will be a drop.

So, this is about chess or any finite games they will have a you can study it but then the point with chess is that it is. So, complex it is it is even for a computer it is. So, hard and then it is an another altogether another thing for let us say there is a winning strategy for the white in chess. Now in an actual game in actual championship when a person is sitting across the table and across the table of another person and then you don't expect even if there is a waiting strategy it could be.

So, complex that somebody cannot remember all that. So, it is one thing to show that it is a win for white or win for black but then it is another thing to execute it but anyway as of now even computers do not know whether it is a win for white wind for black or draw. I think checkers has been shown to be a win for sorry a draw I think the game checkers where it is simpler than chess.

So, but still played on an 8 cross 8 board. So, games in general are PSPACE hard what do I mean by that like we said generalized geography is PSPACE hard like a specific game if I give you a graph a specific graph is you can decide but then in general this problem is PSPACE hard. So, what is generalized chess. So, as I said chess is a finite game 8 by 8 squares and fixed number of pieces etcetera.

But generalized chess's generalized chess is let us say played on an n cross n board and with maybe different rules for the pieces maybe more pieces etcetera. So, the question is given some n cross n board with some specific pieces is it a win for a white or black. So, this turns out to be PSPACE hard. The reason is what we said earlier for all these generalized geography formula game all that you could encode any given that you could encode the entire thing into a formula.

And then you are asking is there a move for player 1 such that whatever player 2 does such as there is a move for player 1 again says that whatever play player 2 does etcetera such that it leads to a win. So, this entire whatever is the n by n generalized chessboard you can you can condense it into a Boolean formula. The position can be captured into a Boolean formula and then this position now it just becomes kind of once you represent the win loss win losses it just becomes like a formula game kind of thing.

Again I am just speaking in a very, very high level it is it sounds like I am over simplifying things but this is a high level idea and indeed it has been shown the generalized chess is PSPACE hard and as are generalized forms of many other games. So, many games are PSPACE hard like many games when I say generalized games like not fixed like tic-tac-toe was a tic-tac-toe is a finite game. So, obviously you know the rule you know that it is the perfect strategy leads to a draw.

And like once you give a fixed graph its one can determine but the problem is in general if I am given a giving you a graph or giving you a board the problem of analyzing who wins or who loses that is PSPACE hard yes. So, that is that is about what i wanted to say in this lecture. So, just to summarize formula game is kind of identical to the TQBF and it is PSPACE complete generalized geography is a generalization of this geography game which is like contractually which is so, generalized.

The generalized form of which is PSPACE complete again I did not show the proof but you can read up the proof from Sipser. And in general I just talked a bit about games and why for finite games one person have to has to have the winning strategy as well if there are no draws and I said that the generalized games where the generalization of a certain game like generalized chess it has to be it has to be PSPACE hard.

So, this generalized chess is PSPACE hard sorry for the bad handwriting or PSPACE complete yeah that is all I wanted to say for this lecture. So, let us stop now, thank you.