Database Systems
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Lecture-21
Proving soundness and completeness of Armstrong's Axioms

So let us continue our discussion on this topic of normal forms.

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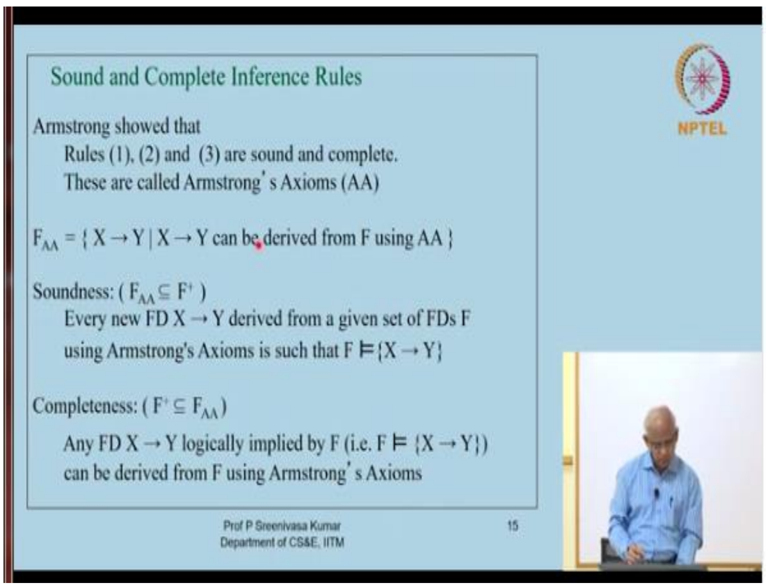

So the reference you can follow their (()) (00:26) this book and that will give you a different set of examples and all that. And the order in which these concepts are being presented to you in this slides is completely different from the order in which they are presented in (()) (00:45) and this thing is material is spread over 2 chapters there. I think 15 and 16, if I remember correctly do not go by my numbers.

But anyway, the concepts of spread to this one and this I am going to talk about the proof of soundness and completeness. That is actually not little bit of physical engineering will help (()) (01:21), what I was telling you, is that the proof that I am going to present today is not available in the mainstream textbooks, the ones that have given to you. But it is available in principles of database systems by Jeff Pullman very old book 70s book.

Our library used to have a few copies of that book our department library used to have book ok. So let us start we already slightly delayed ok, in the last lecture we were talking about the soundness and completeness of Armstrong's axioms. So basically, what we were discussing is that given some set of functional dependencies on a scheme. The set of logic; the set of functional dependencies; that can be logically derived from this given set of functional dependencies, as per our definition of logical derivations.

And the set of functional dependencies that can be derived mechanically using the Armstrong’s axioms they are one in the same. So that is what we are going to present today. So we call one of them as F underscore F double A, the other one has already been defined as F +. Now we are going to show that these 2 things are equal ok, so the way of showing 2 sets are equal is to show that one is a subset of the other and the other also is a subset of the first one.

So showing F underscore double A is subset of F + is what is called soundness. Because we are basically saying that whatever you can derive from F using the Armstrong’s axioms is always a correct functional dependency. It is always logically follows from F, whereas in the completeness, we argue that whatever can be logically derived from F can indeed be derived from using F using Armstrong’s axioms that is what completeness is.

So we will now look at the so this session is going to be a little bit you know dry and you know technical. So but follow me carefully, you will be able to understand this course. Most books which I have suggested to you for reading, skip this portion, skip this completely and then do not even state this set F double A, instead they will mention the Armstrong’s axioms and then say which Armstrong’s axioms can be used for, you know deriving functional dependencies. Whereas we want to be clear, that they are indeed same as the entailment relation that we have talked about earlier ok.

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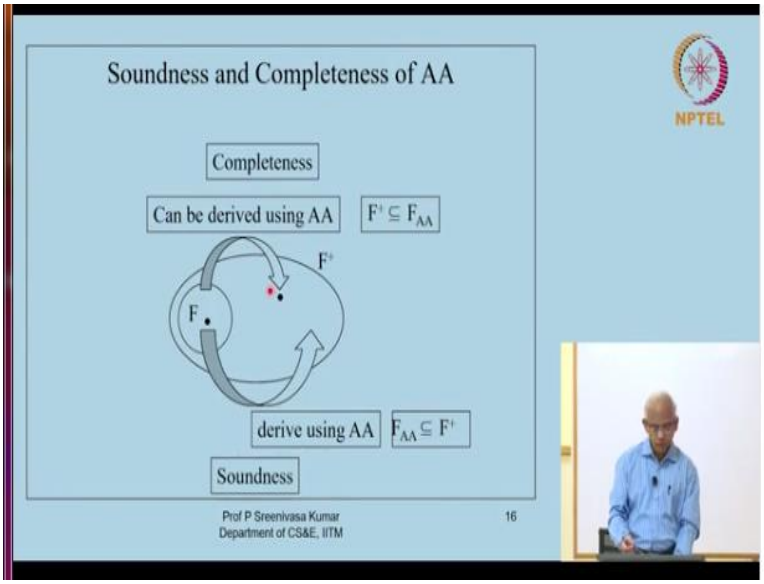

So let us go about this prove the soundness proof ok, here is the diagram that I have shown to you in the last lecture also. So whatever we derive using Armstrong's axioms is inside F + that soundness. And if you take something here and show that indeed it can be derived using Armstrong's axiom that is what is completeness ok, any questions so far, what we are doing ok. So out of these 2 the soundness is actually easier much easier soundness is much easier.

Basically why is it easy, is that if you are deriving using Armstrong's axioms, there are only 3 of them. So if we establish the correctness of these 3 axioms, then repeated application of these axioms will always give you correct new functional dependencies anyway, so that is the argument right.

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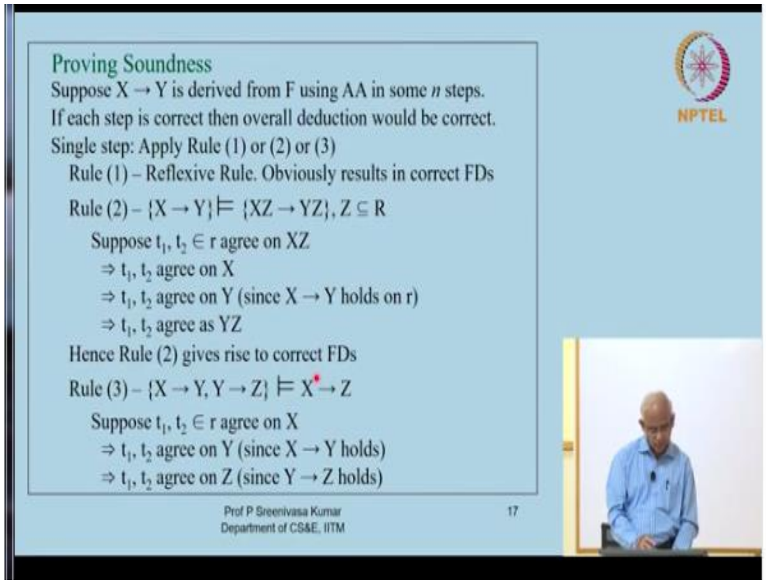

So I will show you that, so suppose X Y, X determines Y is derived from F using the Armstrong's axioms in some n number of steps. So if each of these steps is correct then obviously the overall reduction would be correct right. And each of the steps were using 1, 2 and 3 the Armstrong's axioms reflexive rule then they augmentation rule and the transitive rule RAP you remember that, so we are using these rule.

So, all that we have to do is establish the correctness of these 3 rules individually. And then we know that you know you can actually construct a more formal proof by using induction on air **o** ok, so I will skip that. But we will just establish the correctness of these 3 things formally again though actually while introducing this Armstrong's axioms itself a kind of told you as to why there are correct while let us just repeat that thing for completeness sake.

So rule 1 is reflexive rule, what does it say, it says that as long as the right hand side is a subset of the left hand side that is a correct FD it is actually a trivial functional dependency. So since it gives rise to trivial functional dependencies which is always true. So application of this rule will always result in correct functional dependencies, now rule 2 which says that augmentation. So what it says is if you have a functional dependency X determines Y, you can augment it on both sides by Z.

That means, you can add attributes Z to left hand side and then add same set of attributes Z to the right hand side and they will get a correct function dependency. So this is also actually we have established we have argued it but now let me I have written it down really now. So let us say suppose t 1, t 2, r in some instance and they agree on XZ they agree on XZ. That means they have the same corresponding values for attributes in X and Z right.

So if that is the case then in particular they agree on X because X is a subset of X Z, for in particular they agree on X. And if they agree on X we have given the X determines Y and so they will agree on Y and so they will actually agree on Y as well as Z. So hence you know applying this rule 2 always gives rise to correct functional dependency. Now rule 3 is transitivity transitive rule.

So X determines Y, Y determines Z entails X determines Z. So suppose t 1, t 2 that are there in a instance agree on X, then because X determines Y is given, they will agree on Y. And since Y determines Z is given, they will agree on Z. And so if you assume that they agree on X, we will be able to derive that they agree on X. So this establishes the correctness of the rule 3, so all these 3 rules will always give rise to correct functional dependencies and your repeated application of these 3 rules n number of times will always give you a correct functional dependency ok.

So whatever the rule that whatever is the new functional dependency that you are going to derive by repeated application of these 3 rules will be such that it will be entailed by the given set of functional dependencies. You can kind of establish it by induction because each step is can be proved ok, so do you agree on this now. So you want to recall what is soundness, soundness again is there are using Armstrong’s axioms.

So you may do it some in n number of steps, but each step is correct is what we are saying. And so at the end of all the n steps, you will only land up you will land up in always land up in F that is the argument F doing ok.

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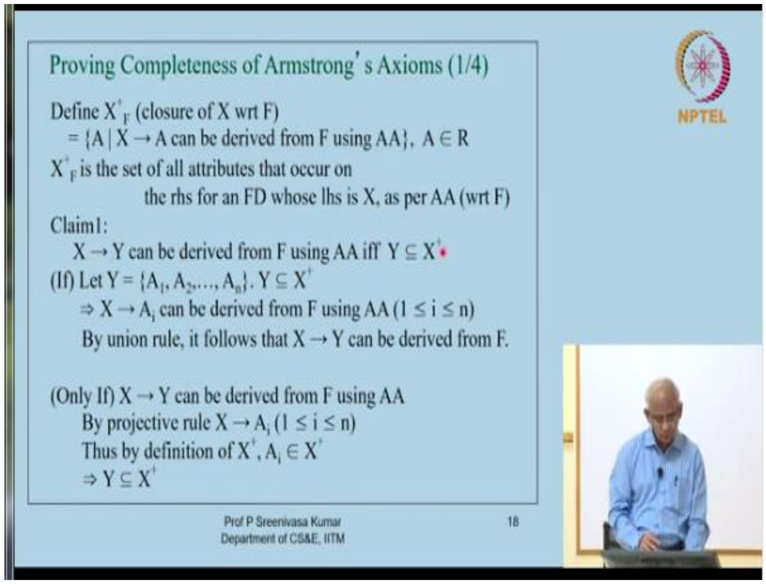

So to kind of prove the completeness, let us see what is the so we will need one important auxiliary concept for this particular proof. So let me set that up first we will define what is called attribute closure ok, X is a set of attributes remember that we are using the alphabets 1 the later part of the alphabet set as sets of attributes. So X is a set of attributes and so we will define so far we had defined for the closure only for functional dependency sets of functional dependencies.

Now we are defining it for a set of attributes ok, it is different, for a set of attributes let me define the closure of X with respect to a set of functional dependency ok, closure X. So what that is define how that is different is like this. It is the set of all attributes A such that X determines A can be derived from F using Armstrong's axioms ok. Now remember one thing that here I am not bringing in entailment at all.

I am just saying that these attributes can be derived from F using the Armstrong's axioms that means after some number of applications of axioms, you can derive X determines A. So if that is the case, then we will say that A belongs to the attribute closure of X with respect to F, because the F is in the background using only these functional dependencies in F we are deriving it. So we can also think of it as the set of attributes that occur on the right hand side of for any FD whose LHS is X as per this Armstrong's axioms ok.

Now we will use this we will make use of this attribute closure and then work out the proof of completeness ok. One of the claims that we can make out of on this X + is like this X now say we have X determines Y is some general you know FD. It can be derive from F using Armstrong's axioms ok, if and only if Y is a subset of this X + F they should be subscript FD also ok, Y is a subset of X +.

So how is this claim useful, so if you want to check whether X determines Y is derive from F using Armstrong’s axioms all that you have to now do is that if this is true is that just compute the attribute closure and then check if Y is a subset of that attribute closure. We will later on see how to compute attribute closures and how what will be the computational cost of that ok. So this is a very useful claim which is saying that the functional dependency can be there from using the Armstrong's axioms if and only if the right hand side is a subset of the attribute closure is disappear.

Now let us prove that, so these statements which have if and only if have 2 parts right. The proof has 2 parts the; if part right if and only if part right. So the; if part it assumes that X determines Y can be derived from F using Armstrong’s axioms. And then proves that Y is a subset of X + the only if part that is the other way around. So let A be let this Y be some attributes A 1 to A n oh I think I said it wrongly right. The, if part assumes that Y is a subset of X + and then proves that X determines Y can be derived from F using Armstrong’s axioms ok.

So that is why we have let Y equals A 1 through A n and we have given that Y is a subset of X+. So what does that mean Y is a subset of X+ that means every attribute here all of these A i's or such that X determines A i can be derived from F using Armstrong’s axioms (()) (17:12). And if that is the case for each of these i, if the individual attributes F X this X determines A can be derived from F using Armstrong’s axioms, this is by definition of the attribute closure.

Then we can basically make use of the union rule by making the using a union rule, what does the union rule say. We call that union rule, what it says is that if the left hand side the same as right hand sides are 2 different things. We can now derive a new functional dependency, where the left hand side is the same left hand side and then the right hand side the union of a (()) (17:50) that is the union rule.

So now that you have X determines A i individually, you can use the union rule and then simply you know derive this X determines Y. Y is nothing but this entire (()) (18:08) ok. So this can be by union rule it follows that X determines Y can be derived from, so basically the; if part is true. Now the only if part we assume that this it is in some sense the reverse of this argument X determines Y can be derived from F using Armstrong's axioms.

If that is the case, we can now use a projective rule and then say that X determines A i for each of these 1 to n can indeed be derived from Armstrong's axioms. So if that is the case, by definition each of these A i will belong to X+. So since each of these A i's belong to X + Y which is a same Y here is indeed a subset of X+. So it is actually not a very deep prove but it is a simple ok just that we just want to write down the proof.

Y is a subset of X+ is same as saying that X determines Y can be derived from F using Armstrong's axioms that is how the definition of X+ is set up ok. So this claim is easy, so we can remember that but only thing is we should remember this claim what does is it, it is giving remember this statement.

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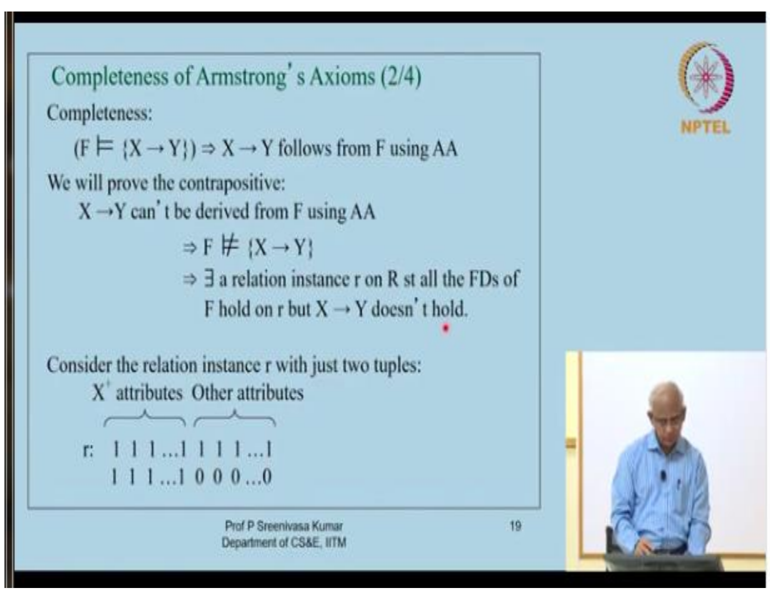

Now let us look at the completeness of this Armstrong's axioms. As a statement, let us look at that, what does it say, it says that if F entails X determines Y then it implies that X determines Y follows from F using Armstrong's axioms that is what we need to prove for completeness. Now let us write the; you have studied implications and mathematical ok how to prove integration, so various ways of proving implications.

Let us write the contrapositive of case what is the let us look at what is contrapositive; the contrapositive is equivalent to the statement you remember that right, a contrapositive is like this. If X determines Y cannot be derived from F using Armstrong's axioms, the negation of this implies F does not entail X determines Y the negation of this implies the negation of this is the contrapositive.

So that is what we written here X determines Y cannot be derived from F using Armstrong's axioms implies that F does not entail X determines Y ok. Now how do you restate this F does not entail X determines Y in terms of instances we can write it like this if this is the case if and actually we can even write this as if and only that there exists a relation instance R on capital R such that all the FD's of F hold on it.

But X determines Y does not hold on only then will say that X determines Y is not entailed by F right, there is at least one instance. Remember the definition of entails, says definition of entails. Let us recall that what it says is that if F entails X determines Y, if in any instance every instance where FD's of F hold, then it is guaranteed that X determines Y also hold ok that is what entails definition here.

Now in order to disprove that what we need to do is that we should exhibit one instance, there exists at least one relation instance R. Such that all the FD's of F hold on it but X determines Y does not hold on it. If that is the case then we are done right to show that it F does not entail X determines Y, is that clear, is that statement clear first. So this is what we are actually going to make use of in establishing the proof of the completeness.

So we are going to actually exhibit so this is instance a instance R constructed cleverly. Such that all the FD’s of F hold on r but X determines Y does not hold on r ok. Now this r is constructed in a interesting way this r is a table by the way ok, this r is accurate table I mean I need has only 2 tuples ok. So this is a theoretical tool right, so it has only 2 tuples and one of them is all ones the other one has all ones still some part and all zeros later.

And the attributes are grouped such that these attributes are all attributes of X +, remember what is X + is the attribute closure of X. So overall we can look at the attributes of R and then classify them as 2 parts right. Those that belong to X + do not thus do not belong to X +. So put all the attributes that belong to X + in the front and do not belong to X + later and then you know put all ones here and put all ones in the second row.

You have all ones for attributes that belong to X + and then zeros for other attribute that is all this is the nice a little instance. And then what we will actually the scheme a thing is that we will show that this instance r does the job for us. This instance r is such that all the FD’s of F hold on it but X determines Y does not hold on it. If this is the case that X determines Y cannot be derived from F using Armstrong’s axioms.

So under these assumptions if this is given to us we will show that this particular instance r is such that all the FDs of F hold on it but X determines Y does not hold on it, so that will complete the proof. So we only need to now prove to specific statements saying that this particular. So you do not ask me how this is particularly constructed. So there is a intuition behind how this proof needs to be worked out and so we have constructed this r has been constructed like this is that fine. So this is r is having exactly 2 rows the first row has all ones and I will repeat these r in other slides also.

So that you will remember what the r is, so now we only have to show this thing that all FDs of F hold on r and X determines Y does not hold on r, let us see how we can prove that.

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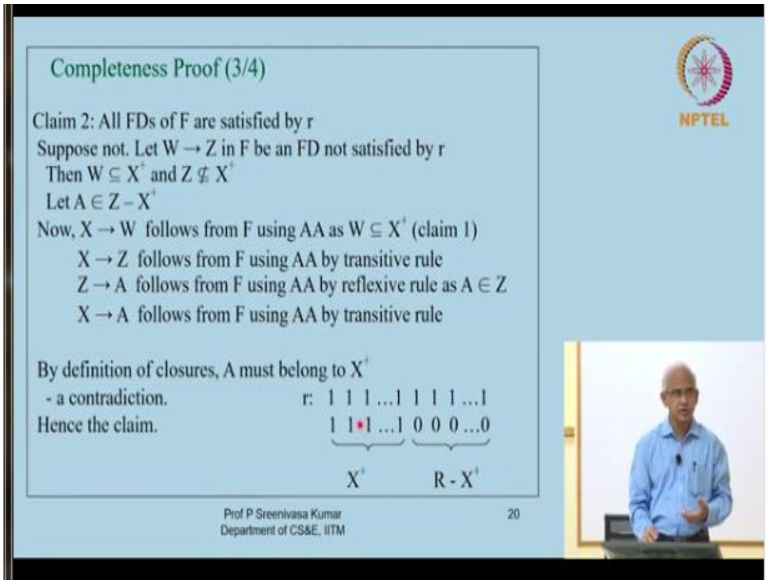

So claim 2 is that all FD's of F or satisfied by r, the r is dissipated here. Well, this is again proved by proof by contradiction it is proved by proof by contradiction suppose not suppose there some FD in F let us call it W determines Z is not satisfied by r ok. If it is not satisfied by r the structure of r is very simple it is like this right. If some FD is not satisfied by r that means what, this W the left hand side should indeed be here right, only then this only existing 2 tuples will agree on the left hand side.

So some FD is not satisfied by r means what that there are they exist 2 tuples that agree on the left hand side and do not agree on the right hand side right, that is what it is. So if W determines Z is not satisfied by r then W should be in this part, only then these 2 tuples will agree on left hand side right. And then the Z if Z also is here itself that Z means the set of attributes is right. These Z attributes are also somewhere here only then both tuples will agree on them because they all of them have an identical values 1 ok.

So this instance will end up satisfying W determines Z but we have so that is not the case right W determines Z is not satisfied by r. So W should be here and Z should not be here that is what we conclude. So Z is not a subset of X +, whereas W is a subset of X here ok, is that clear till that. Everything just logically follows from the definition of these thing, there is nothing surprising individuals steps ok.

So basically since we assumed that W determines Z is not satisfied by r it logically follows that W should be here and Z should not be here it should be somewhere like this, it should contain some attributes from R - X + ok. So let some attribute pick up some attribute X A that is there in Z, but not in X + that means somewhere here not all of them need to be in Z. But some of them need to be in Z at least one should be there in Z that we know because that should not be here completely.

So we know that at least one attribute exists which belongs to Z and which is not here, so let us pick up that attribute A ok. Now there is a sequence of inferences that we can make which will give rise by contradiction. So what is that sequence of the inferences we can make. Now since we know that W is a subset of X + from the definition of X + and the claim 1, X determines W follows from F using Armstrong’s axioms, the claim 1 basically said that.

Because if W is a subset of X + then X determines W follows from F using Armstrong's axioms, because of the claim 1 we have proved that ok. So X determines W is in our hand now, so X determines Z follows from F using the Armstrong's axioms by transitive rule why is that. X determines Z, W is there and W determines Z is already given in F ok W determines Z is already available in F and X determines W is there. So now X determines Z follows from here using the transitive rule, now A is a member of Z, remember that A is a member of Z and it is not a member of.

So since A is a member of Z, Z determines A follows from F using the reflexive rule because A is a basically member of rule. So now we have X determines Z, Z determine A and so by transitive rule X determines A follows from F using their Armstrong’s axioms ok. Now you can see the contradiction, the contradiction is coming because by definition of this closures, if X determines A follows from F using the Armstrong’s axioms which we have now shown.

Then A must belong to X + that is how the definition of X + is say that of. So we started off with A saying that they must exist an A which is in Z but not in X + and then about that A we are able to make a new conclusion that A must be in X +. So it is a contradiction to our assumption that such an A exists. So it all started because we made this assumption they saying that let there be some W determine Z which is in F which is not satisfied by r.

We make such an assumption, we will derive a contradiction and hence all FDs in F are indeed satisfied by r ok. So remember that derivations using Armstrong's axioms from a given set of functional dependencies here, you know do not take any instances into consideration at all. So our assumption that you know F is not satisfied by r or let W be not satisfied by r, you know does not played a role here in the sense of you know in driving using F ok.

Derivations of new functional dependencies from the given set of functional dependencies using Armstrong's axioms, it does not depend on any instances whether any instance satisfies this or not about. So you may be wondering that we are using it which is not satisfied by r but that does not matter as long as it is in F we can make use of it ok. So is this argument a bit technical argument of both of course is the argument clear.

That basically what we are saying here is that, if you assume make an assumption that W determines Z is an FD that is not satisfied by r we will be able to derive a contradiction saying that they cannot be any attribute A that belongs to this part ok.

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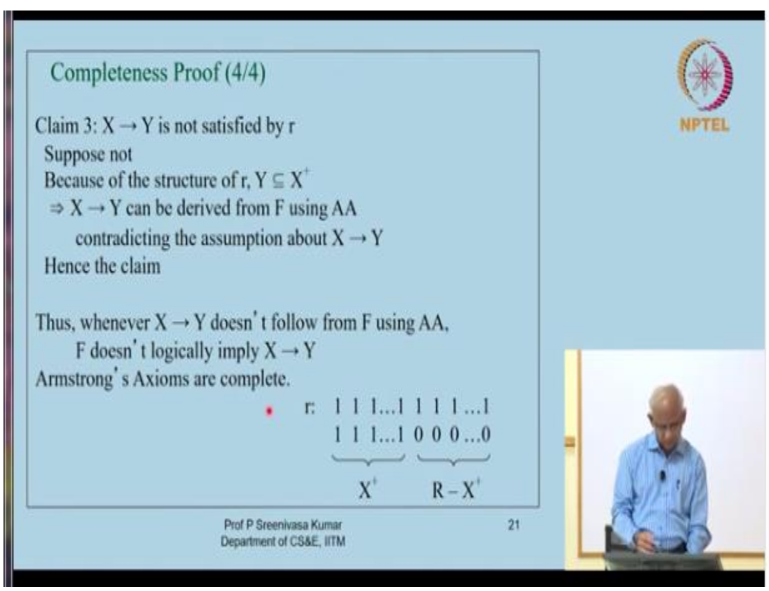

Now let us move on to claim 3 which is actually very simple this is much simpler claim, what this says is X determines Y is not satisfied by r this is not very surprising actually. Again let us do it by proof by contradiction suppose not let us say it is indeed satisfied by r ok. If it is satisfied by r because of the structure of r will be forced to conclude that Y should be a subset of X + that means this part.

Because only then the 2 tuples will agree on both X as well as Y only then both tuples will agree both on left hand side and the right hand side that this part has zeros right, you follow me. So if X determines Y is indeed satisfied by r if X is somewhere here you know if X cannot be here because X is supposed to be subset of X + right, X is always here only. And since X will always be a subset of X + remember that reflexive rule ok.

So X is here, now if Y is somewhere here then X determines Y cannot hold. Because the 2 tuples will not agree on Y attributes, so Y also is forced to be here only ok. So if Y is a subset of X + again because of the claim 1 X determines Y can be derived from F using the Armstrong's axioms, that is the conclusion we make, but this contradicts the very assumptions that. We made about X determines Y, what is the assumption about X determines Y, is something which cannot be derived from F using Armstrong's axioms.

That is the fundamental assumption about the X determines Y. So it immediately gives rise the contradiction and hence this is true X determines Y is not satisfied by r. So whenever, so this kind of completes the proof actually because we proved claim 2, claim 3 separately remember this thing. This is the assumption that we started off with X determines Y cannot we derive from here using the Armstrong’s axioms.

And then we showed that the exists relation instance r on capital R such that all the FDs of F hold on it and X determines Y does not follow. The claim 3 proved this X determines Y does not hold on it because if it holds then it contradicts this itself the assumption (()) (38:31) ok. So because of these 2 claims whenever X determines Y does not follow from F using the Armstrong's axioms, F does not logically imply X determines Y.

So this actually proves that, Armstrong's axioms are indeed complete. So soundness and completeness can actually be proved, so because the proof is a little bit technical and you know you have to go into the details of this thing. Many textbook actually gloss over this and then simply define that the Armstrong's axioms are there and you can derive new axioms out of 3. And then they are all logically correct that is what they claim ok.

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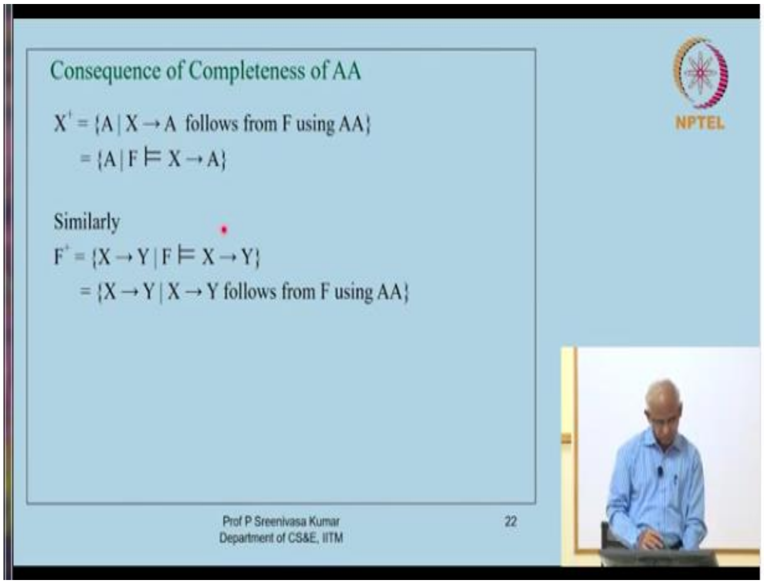

Say the consequence of the completeness of this Armstrong's axioms is basically that F + which is the set of all functional dependencies that are entailed by F right or exactly the same as the set of all functional dependencies that are derived from F using Armstrong's axioms. The rules whatever derived from F and what is logically implied they are same, so these 2 sets are same. So this we call it as F underscore W in one of the previous slides.

And in a similar way the attribute closure X + also is a set of all A that you know attributes are such that X determines A follows from F using Armstrong's axioms. It is the same as this a set of all A such that F entails A. Basically wherever follows from F using Armstrong's axioms is there we can simply replace by entails A because the rules are sound and complete.

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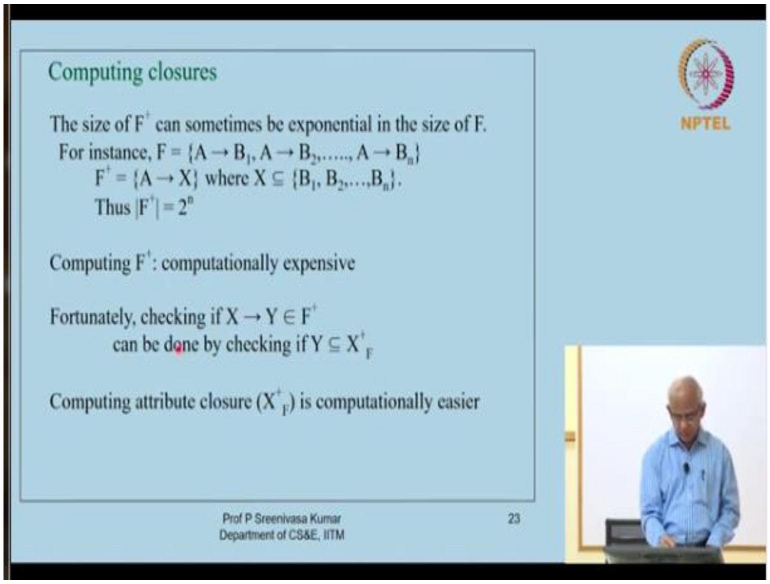

Now let us spend a little bit of time on computing this closures, how exactly this F + right. So here is I am giving an example of an F which some kind of a pathological case is the worst case. So if have, so F has these functional dependencies, the left hand side is same for everybody A, attribute A determines B 1, A determines B 2, A determines B n number of them.

Now you can see that A determines X, where X is a subset of B 1 through B n is always a new functional dependency that logically follows from these given functional dependency right. For any subset of because you can use the addition rule union rule, you can use union rule and then get back any set any subset of B 1 through B n on the right hand side, so it is a new functional dependence.

So now, you know that there are 2 power and subsets of this B 1 through B n. So for each of these 2 power n subsets, we can put it on the right hand side and then get a new functional dependency A determines that subset. So if you look at the size of F + number of elements in F + it can be exponential in terms of the number of n number of V functional dependencies that are given.

So theoretically speaking, F + is exponential in size compared to the size of F. So size of F + is 2 power n, whereas size of F is just n number of FDs in F ok. So obviously, the lower bound on any algorithm that computes F + given F is 2 power n (()) (42:5) at least because there are to output all of them you need 2 power n right**,** so that is the lower bound. So, any algorithm that produces F + has to take exponential term.

So that is why computing F + is computationally expensive, whereas fortunately because of all this setup we have made especially this attribute closure. So remember the claim 1 X determines Y belongs to F + is same as Y is a subset of attributes closure of X. That is the claim 1 first claim will make today morning that X determines Y belongs to F + ok, if and only if Y is a subset of this one and this also is consequence of the completeness of those.

Now, so though computing F + is expensive checking the membership of some functional dependency inside that F + can be done by checking Y is a subset of this one provided we can compute this we can compute this easily. If we compute this, then we can check whether Y is a subset of that X + and then if that is the case, we can conclude that X determines Y is indeed (()) (44:25). So computing this attribute closure is fortunately easier, I will just show that and then close.

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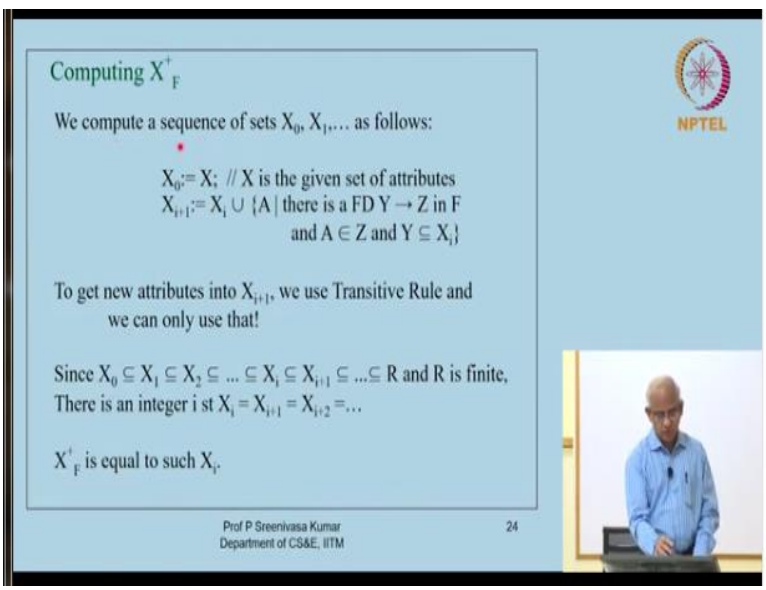

So we can indeed compute a, so let us focus on how to compute this attribute closure with respect to some F. So the algorithm for doing that is simple actually, it is very interesting also what we will do is starting from X we will compute a sequence of sets like this. So 0 is identically equal to X is the given set of attributes, then X i + 1will be derived like this ok.

X i union the set of all new attributes A such that now there is a FD Y determines Z in the given set of functional dependencies. Such that the Y is a subset of X i that means Y is already in the existing set of attributes and A is there on the right hand side somewhere. So we can now add that, basically we can use transitive rule to augment this X i with new attributes ok, X i is already there in our hand ok.

Now if some Y is a subset of X i, then we can apply this Y determines Z and then given X determines something inside X i ok and then Y determines Z can be now applied and then we can add new attributes. So like this we can keep on adding new attributes and then grow this sets successive sets, you can notice that X i is a subset of X i + 1 it kind of grows. So X 0 is a subset of X 1, X 1 is a subset of X 2 and so on.

And but then this sequence of things since r is finite there must be some i such that X i = X i + 1 because it cannot infinitely keep going like that it has to get stop somewhere r is fine, r has only n number of attribute. So in utmost n number of steps we would be actually able to compute X i + X i which is actually then equal to this ok. And so you can actually show that computing X + given F which has n any number of attributes does not really matter.

Because r is the one that matters because in some number of steps that is proportional to the size of the r, we will be able to compute this one and so this is a polynomial. So this we can compute and once we compute this, we can use this rule saying that X determines Y belongs to F + if and only if Y is a subset of X + ok and then this one.

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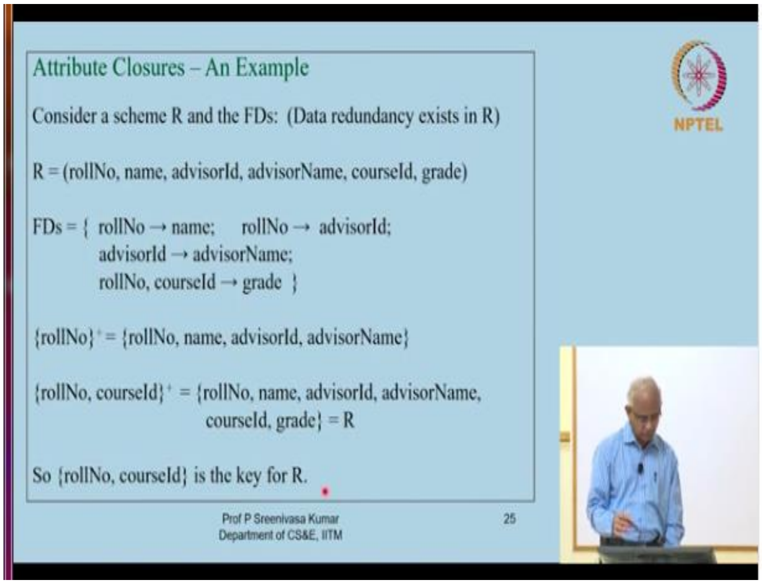

So let me conclude with this small example of attribute closures let us look it an example. Let us look at roll number, name, advisor Id, advisor name, course Id and grade ok; grade set up attribute and the FDs here R there is a lot of redundancy in the data here. So do not worry about too much right now, we will see how to address that later on. So the FDs are roll number determines name, roll number determines advisor Id right.

And advisor Id determines advisor name and roll number, course ID together determine the grade these are the FDs that hold. So now we can actually so if you start with roll number, so include roll number first and then look at keep on looking at all the FDs, what is the new attributes that you can bring into this thing. So we can bring in name, advisor Id, advisor name like that.

And then we can also; if you take these 2 you can derive that you will get the entire R into it anyway. And so actually this will be a key part R; this is another way of establishing that some set of attributes is a key for this key. Shows that are attribute closure is equal to R ok good. So I will probably repeat this example in the next lecture we will also close it now, thank you.