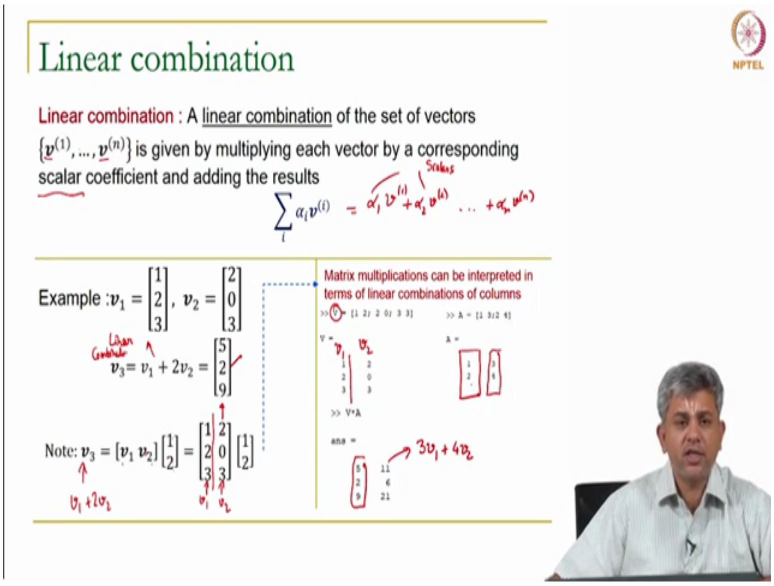
Machine Learning for Engineering and Science Applications
Professor Dr. Balaji Srinivasan
Department of Mechanical Engineering
Indian Institute of Technology, Madras
Linear Combinations Span Linear Independence

(Refer Slide Time: 0:14)

In this video we will be looking at three fundamental ideas in linear algebra, the idea of linear combinations, span and linear independence. The idea of linear combinations we will use multiple times through the rest of the course you can even think of this as simple definitions, but they are very very powerful ideas when you do a full linear algebra course or if you have done a full linear algebra course. In fact a lot of power of linear algebra comes from these three ideas, okay.

(Refer Slide Time: 0:45)



So the idea of linear combination is simple, so let us say you have a set of vectors v 1, v 2, v n. So remember I have shown this in bold which means each of this is a vector. So suppose v 1 through v n is a set of vectors, what you can get by combining each one of them through a linear combination is simply some coefficient multiplying this plus some other coefficient multiplying this.

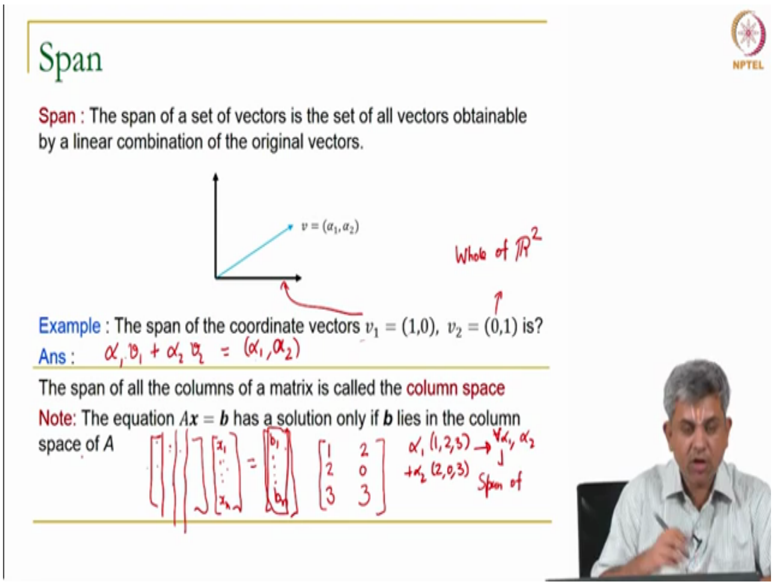
So some scalar coefficient multiplying each of these vectors and adding them is called a linear combination it is a very intuitive kind of definition. So mathematically you would write it as alpha 1 v 1 vector plus alpha 2 v 2 vector so on and so forth up till alpha n v n vector, where alpha 1, alpha 2, etc are different scalars. So let us take a simple example, so let us say v 1 is the vector 1 2 3, v 2 is the vector 2 0 3 and v 1 plus 2 v 2 for example is a linear combination in this case we get 5 2 9, okay.

Now interesting way of thinking about this linear combination v 3 remember was v 1 plus 2 v 2 is to write it as a product, okay. So we can write it as v 1 v 2 multiplying 1 2 so that you get v 1 plus 2 v 2. So in matrix notation you can write it as this vector which was v 1, this vector which was v 2. Now if you multiply this vector by this you will notice that the first element is 1 plus 2 times 2 5 2 plus 0 times 2 2 3 plus 3 times 2 which is 9, okay.

So essentially v 1 plus 2 v 2 can be thought of a linear combination of two columns v 1 as first column, v 2 as the second column so this is a tremendously useful way of thinking of things, okay so matrix multiplication when you take one matrix and multiply it by the other matrix it can actually be thought of as a linear combination of columns, okay. So for example if you have this matrix once again 1 2 3 2 0 3 and I multiply it by a matrix 1 3 2 4, okay so this is basically the first column of A, this is second column of A, okay.

So now what you can think of? You will notice that 5 2 9 is exactly what we had here, why? Because it is a linear combination of v 1 to v 2, now what will be this column? This column essentially is 3 v 1 plus 4 v 2. So the result of matrix multiplication each of the columns of matrix multiplication can actually be thought of as some particular linear combination of the columns of the matrix v that we are multiplying here, okay. So we will utilize this idea when we come to the idea of invertibility, etc.

(Refer Slide Time: 4:14)



The second idea that we want to discuss in this video is the idea of span, it is a natural outgrowth of the idea of linear combination. The span of a vector or a set of vectors is whatever you will get by every possible linear combination. So remember in the (last case last video sorry) last slide we had a simple linear combination v 1 plus 2 v 2 but suppose you write alpha 1 and alpha 2 free and you try and find out every possible linear combination of that, that is called a span.

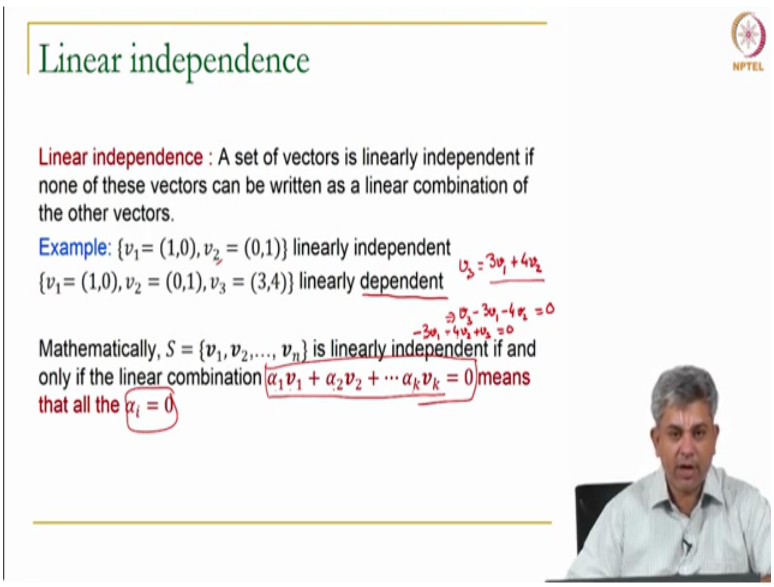
This should become a little bit clear if we look at an example, okay. So let us say we are looking at two vectors v 1 is 1, 0 and v 2 is 0, 1, okay. What would be the span of this thing? So mathematically what is going to be the case let us just look at it geometrically and then we can see quickly mathematically what happens. So v 1 remember or notices simply the unit vector in the X direction, v 2 is the unit vector in the Y direction.

Any vector that you have, so let us say that you have a vector (v 1) alpha 1, alpha 2 all alpha 1, alpha 2 is alpha 1 v 1 plus alpha 2 v 2, okay because alpha 1 times 1, 0 is going to be alpha 1, 0 and alpha 2 times v 2 is going to be 0, alpha 2 so you can add these two and get any vector, what it means is the span of the coordinate vectors is the whole of 2 dimensional space a whole of R 2, any vector that I choose can always be written as a linear combination of these two vectors, okay. So the span of these two vectors is going to be the whole of the coordinate space R 2.

Similarly you can think of you know multiple things for 3D if you define (0 1) 0 0 1, 0 1 0 and 1 0 0 if you define these three vectors their span will be the whole of 3D space, okay. The span of all columns of a matrix is called the column space. So if I have some matrix once again I will take the same example as last time all the vectors that you will get of the form alpha 1 times 1, 2, 3 plus alpha 2 times 2, 0, 3 for all alpha 1 and alpha 2 this will be the span of these two vectors, okay.

Now notice that if I have an equation Ax equal to b what it means is I have some matrix A and some vector x and I am obtaining some other vector b. So suppose I give you an A and I give you a b and I ask you to find out x, if this equation has a solution it automatically means that b has to be in the column space of A, why is that we just saw this in the previous slide, this vector is simply the linear combination of first vector multiplied by x 1, second vector multiplied by x 2, third vector multiplied by x 3 and nth vector multiplied by x n (all it) it automatically means that b is in the column space of A.

(Refer Slide Time: 8:04)



The final idea in this video is that of linear independence. A set of vectors is defined to be linearly independent if none of these vectors can be written as a linear combination of the other vectors, these three ideas of linear independence, linear combination, span are actually very very deeply related, unfortunately we will not have the time to go through all the interrelations between them, you can treat them as three sort of related ideas, okay. I think some of you might be automatically see the correlations between these ideas.

So in this all we are looking at is if I have a set of vectors and let us say I take these two vectors 1, 0 and 0, 1 you cannot write v 2 as a linear combination or as any linear multiple of v 1, okay. Now suppose I take these three vectors 1, 0, 0, 1 and 3, 4 now these are not linearly independent I would call it linearly dependent, why is that? Because v 3 is 3 v 1 plus 4 v 2 so since v 3 can be written as a linear combination of the other two vectors these three vectors are not linearly independent.

So mathematically we say that any set v 1 through v n is linearly independent if and only if the linear combination alpha 1 v 1 plus alpha 2 v 2 up till alpha k v k for any k is 0 automatically implies that there is only one possibility, notice that if I said alpha 1, alpha 2, alpha k equal to 0, obviously I am going to get 0, but that should be the only solution to this system of equations, why is that? Notice here, if v 3 is equal to (3 4) 3 v 1 plus 4 v 2 it automatically means v 3 minus 3 v 1 minus 4 v 2 equal to 0.

In this form I can write it as minus 3 v 1 minus 4 v 2 plus v 3 equal to 0. So I will check v 1, v 2, v 3 I find out some linear combination alpha 1 is minus 3, alpha 2 is minus 4, alpha 3 is 1 which gives me this equation equal to 0 without all the alpha i being 0, if that is the case then the set of equations is linearly dependent and if the only solution to this system of equations is that alpha 1 through alpha k is fully 0 then that means that a set of vectors is linearly independent, thank you.