NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Discrete Mathematics
Principle of Inclusion and Exclusion

Rook Polynomial

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Let us now ask a very non trivial question, and this question we have picked from the text book discrete and combinatorial mathematics by Ralph P Grimaldi,
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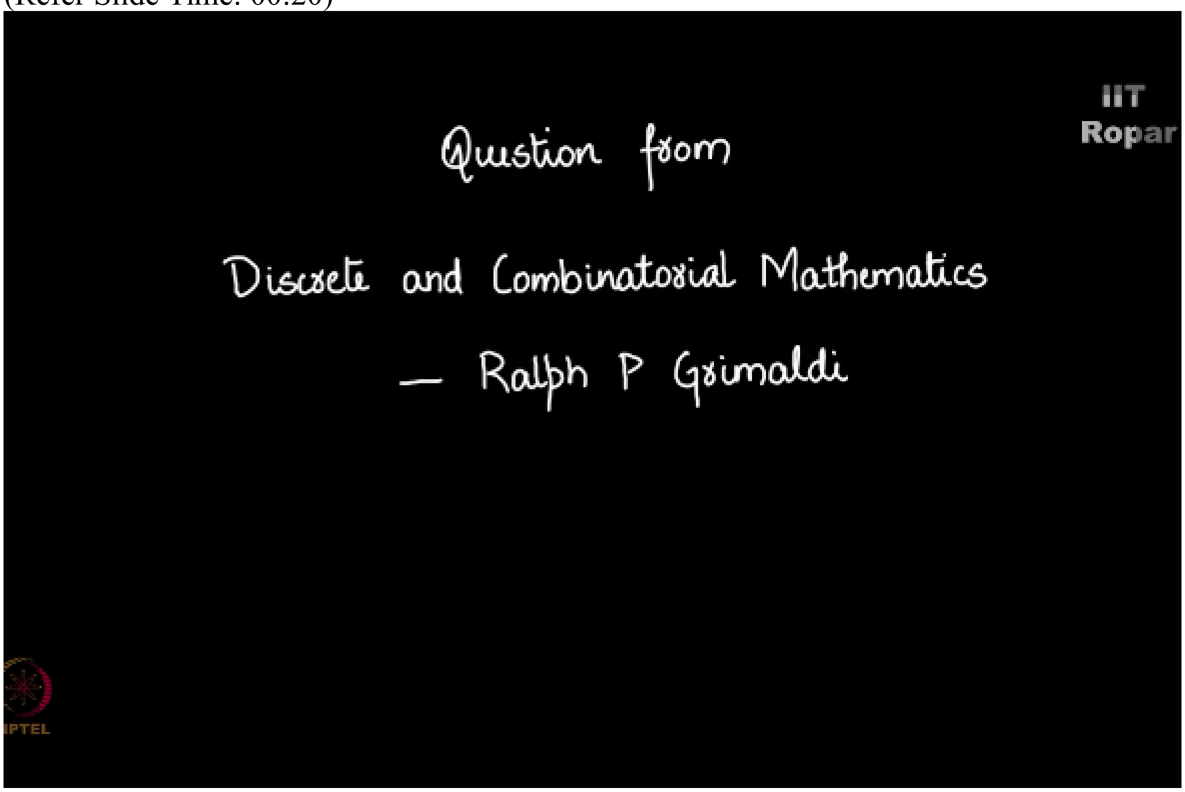

and the treatment of, this concept called rook polynomials in the book is very good and we would want you all to refer to the text book for all the notations and other, if you want to solve more problems, this book is the best possible book that we can recommend you as part of this course.

So look at this word example from the book, look at this chessboard here having 1, 2, 3, 4, 5, 6, 7 cells,
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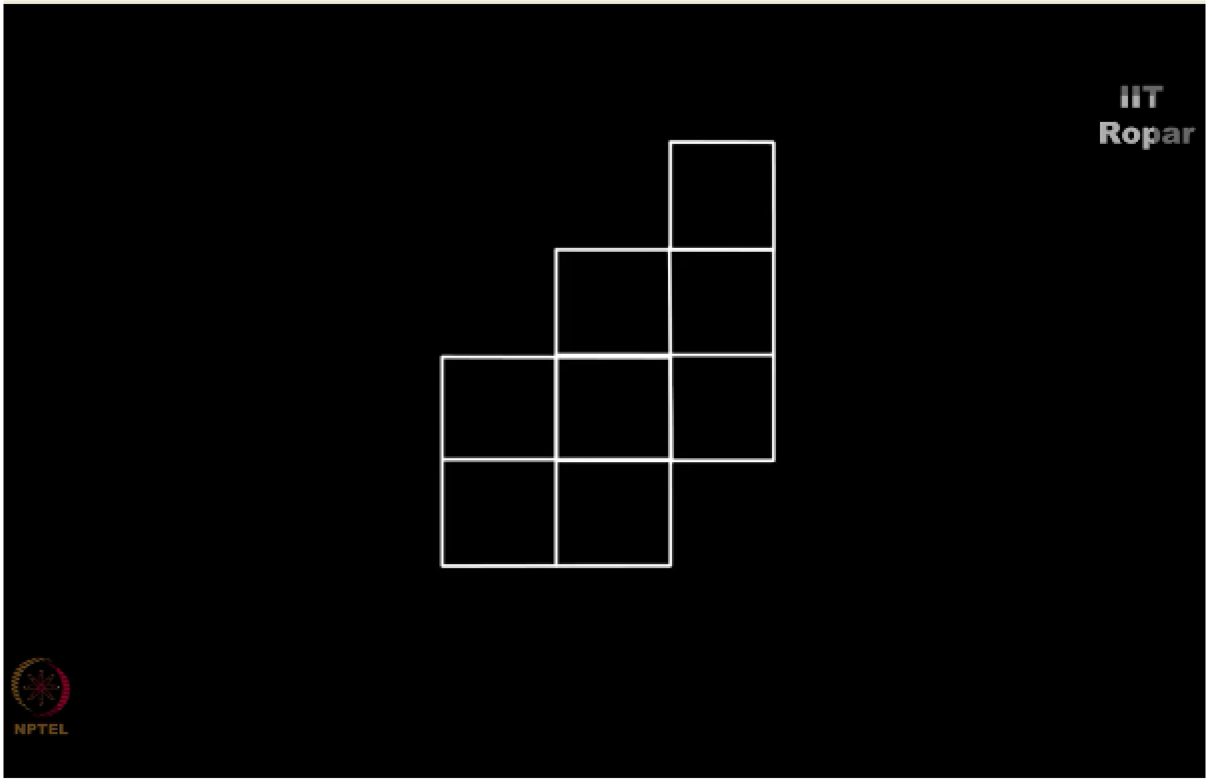

what is the rook polynomial of this particular chessboard, that sounds very difficult to compute, of course the coefficient of X is easier to compute that which is 1, 2, 3, 4, 5, 6, 7, 8 cells, so it will be 1+8X, but what is a coefficient of X square? By that we mean in how many ways can you place 2 non-taking rooks on this chessboard?

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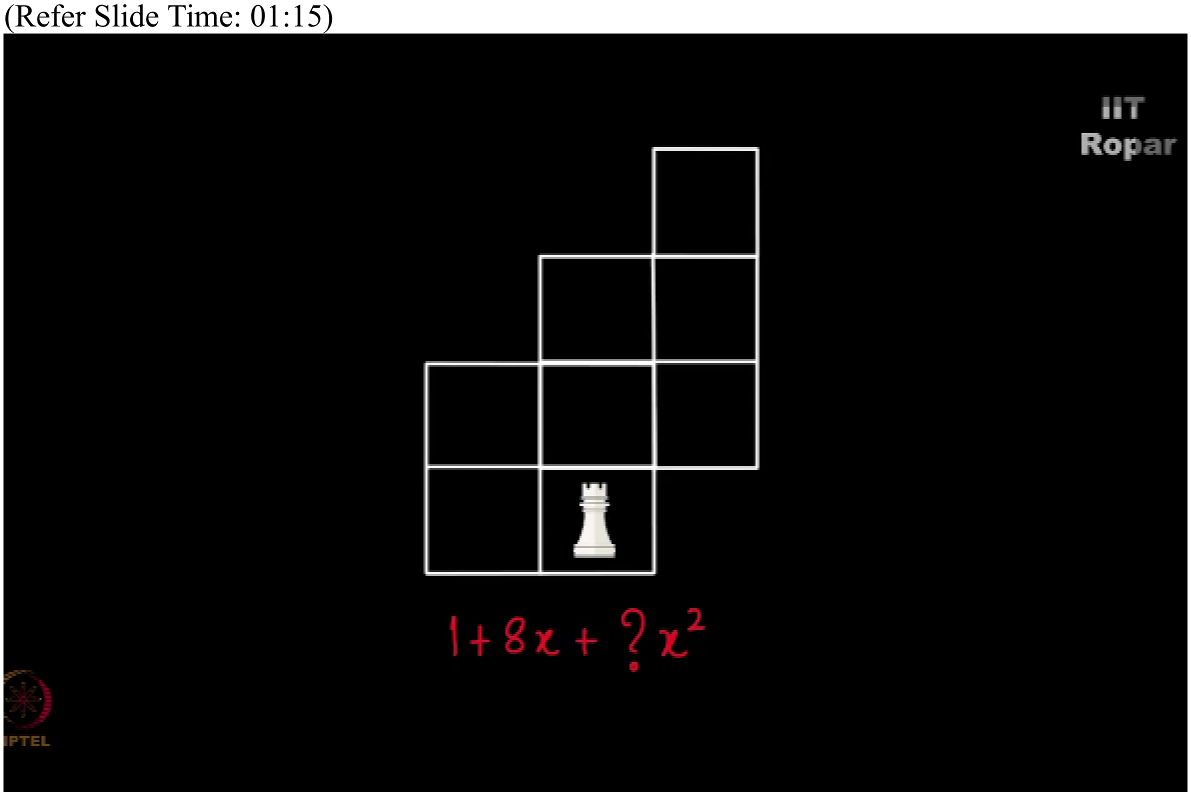

That’s very difficult but there is a surprisingly very elegant way of solving this problem by breaking this onto smaller chunks, you will see what I mean as we proceed slowly with this problem.

Now look at this chessboard with 4 + 4, 8 cells,
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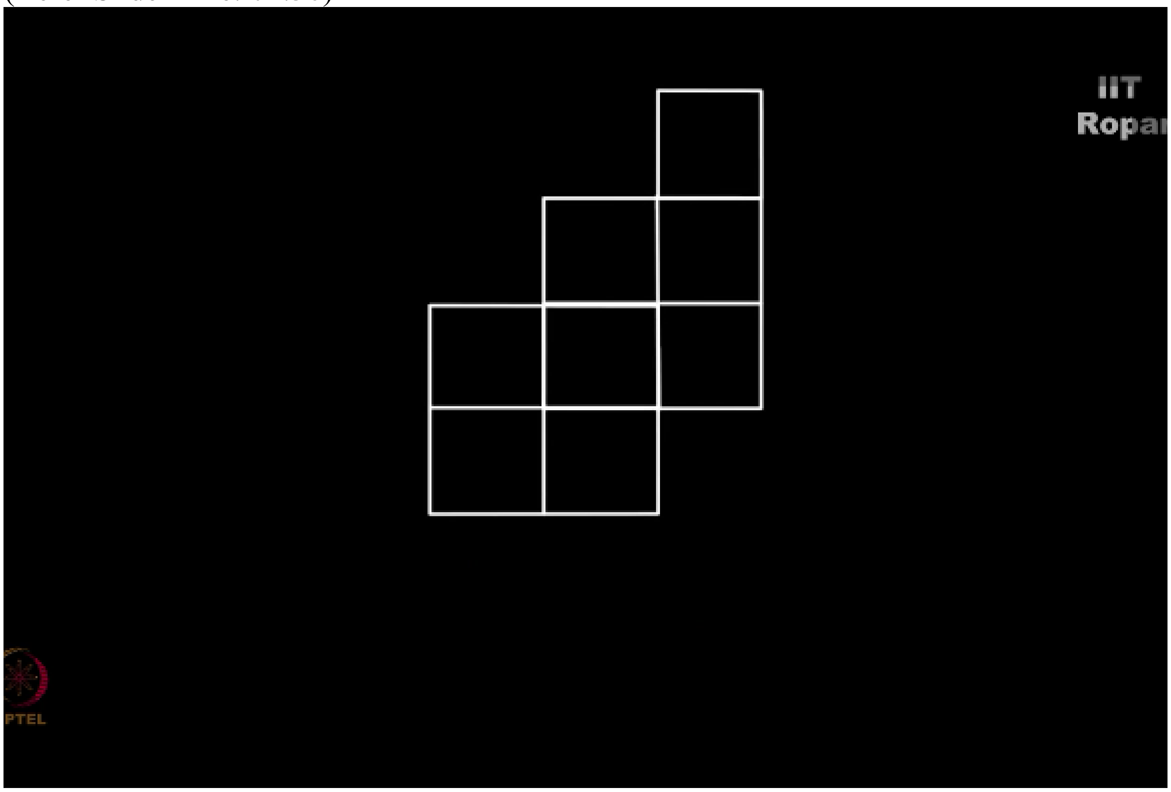

right, now I need to compute the rook polynomial of this chessboard, let me mark this one first cell, let me put a star there, and let me ask this question, the total number of ways in which you can place K rooks on this chessboard is the total number of ways in which you can place one rook in this particular cell, the star itself, 1 rook is definitely there, and there is K-1 will be in other places, this is case 1,
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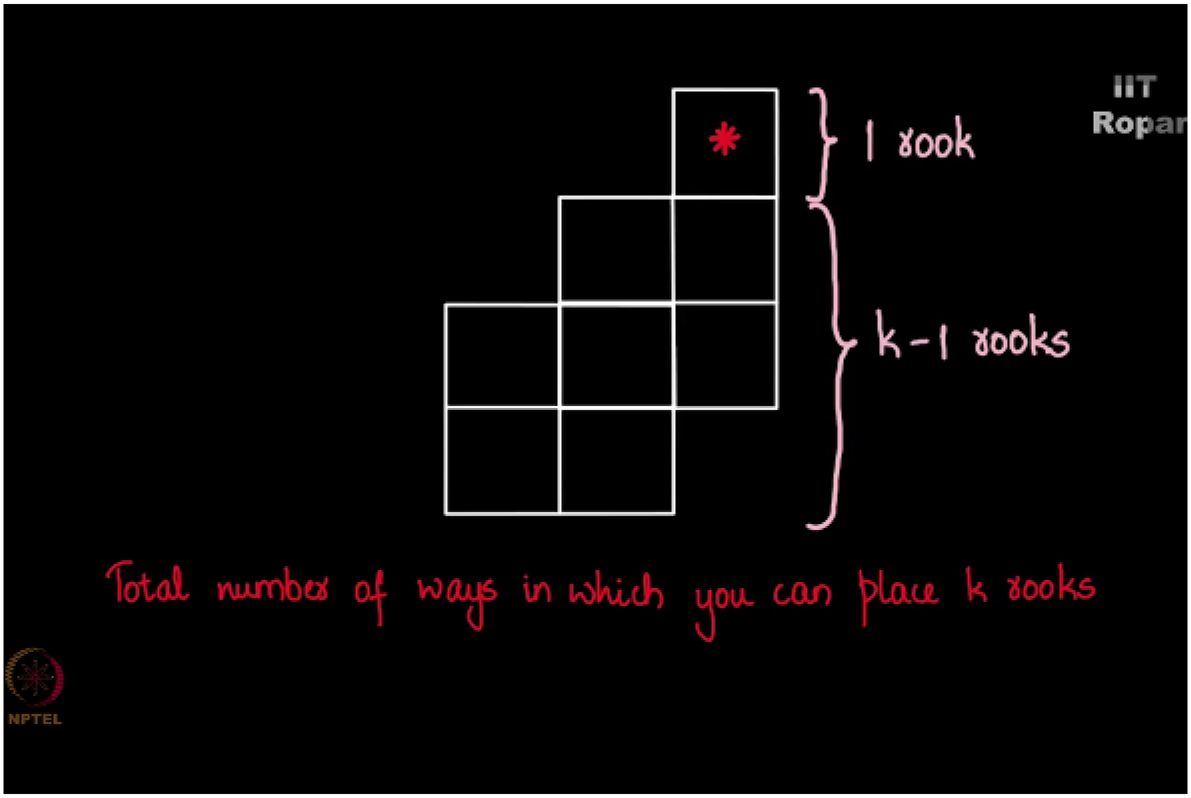

and case 2 is there is no one rook in this cell, all K rooks are kept in the other 7 cells,
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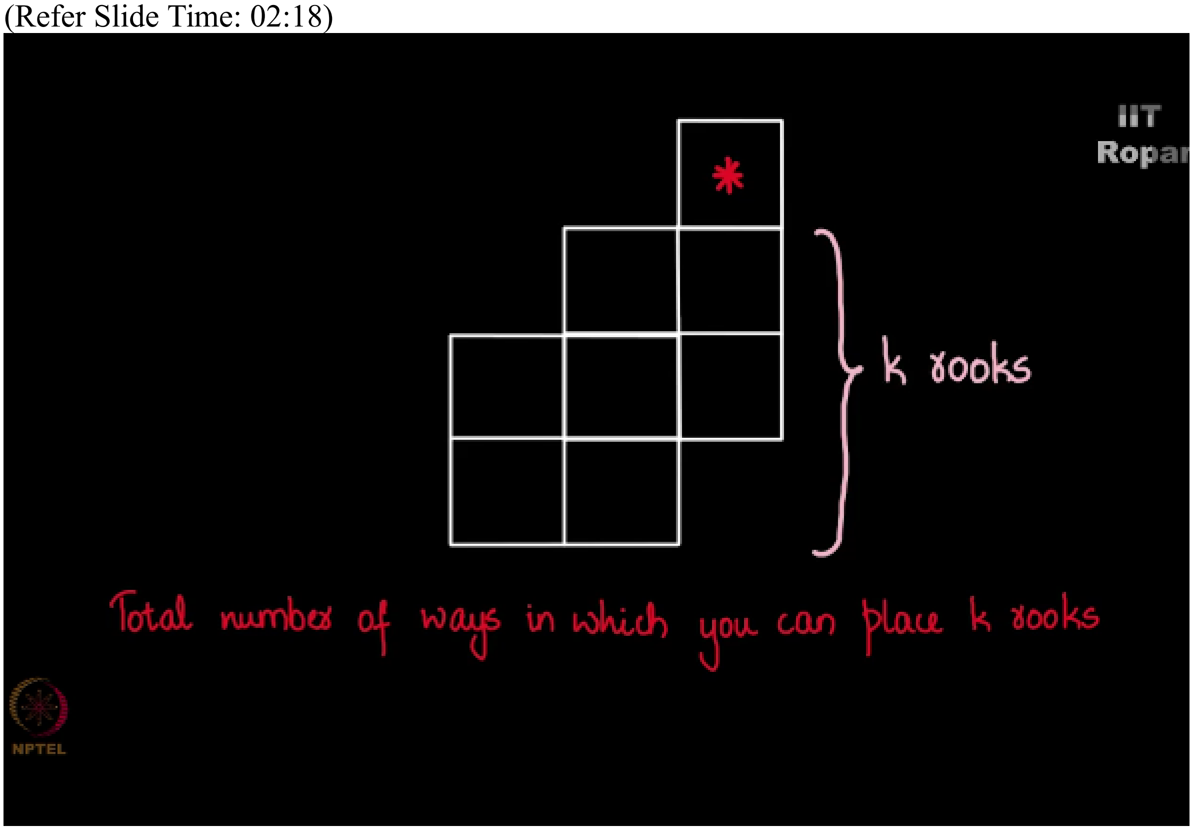

this are mutually exclusive events, so if you want to count the total possible ways in which you can place K rooks on this chessboard I can count these 2 cases separately, and then add them up with the rule of sum as you can see, okay.

So let's proceed, I must warn you people that concept is a little involved, you may want to go through it very slowly, okay, step by step and you want to pause and then think a lot before proceeding further, it's no story book where you can start from the beginning and then go until the end by just following the words, you may want to interpret and reinterpret some steps as I take you through the solution of this question, what's the question? What is the rook polynomial of this chessboard, that's the question.

Okay, so what I'll write is let me call this chessboard as C, I'm trying to find the Kth coefficient of the rook polynomial, let us represent this as RK(C) as I said the notations are directly taken from the text book,

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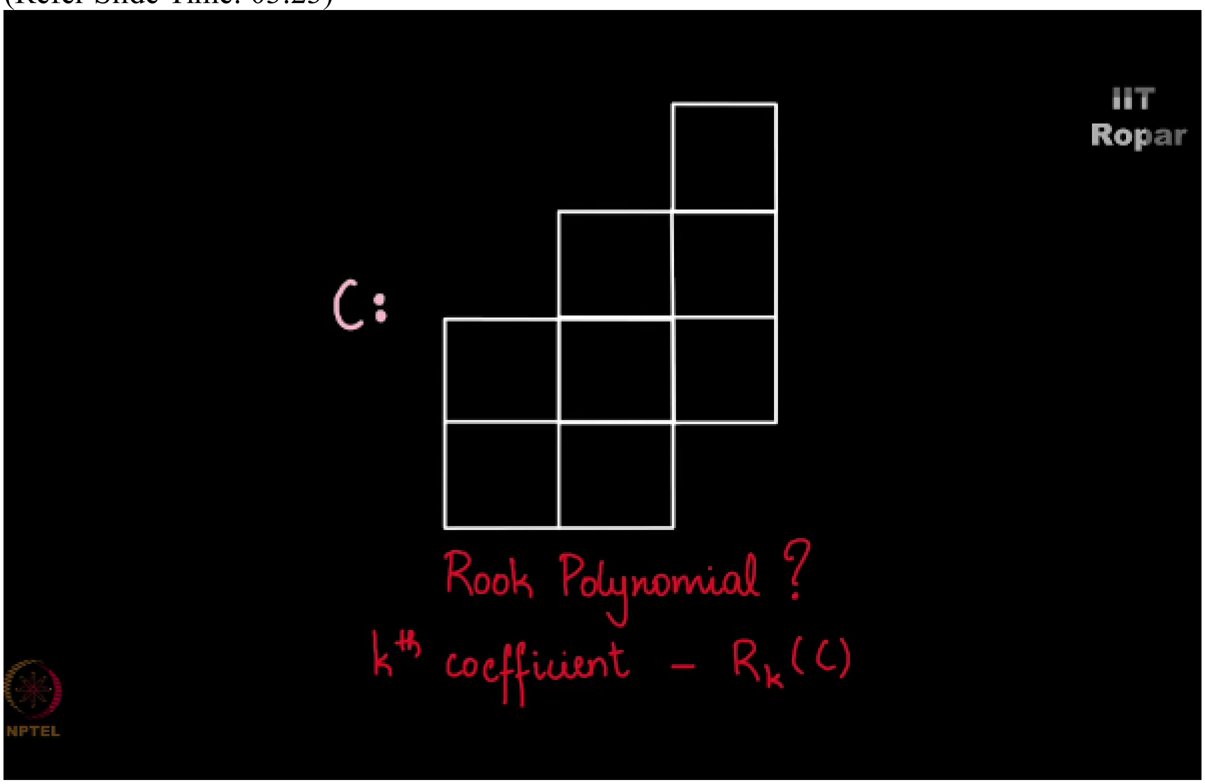

RK(C) will be equal to, now put 1 rook in the cell which means you have K-1 possible rooks that are to be placed in the smaller chessboard which appears like this, you see what I have done, I have removed the entire column here,

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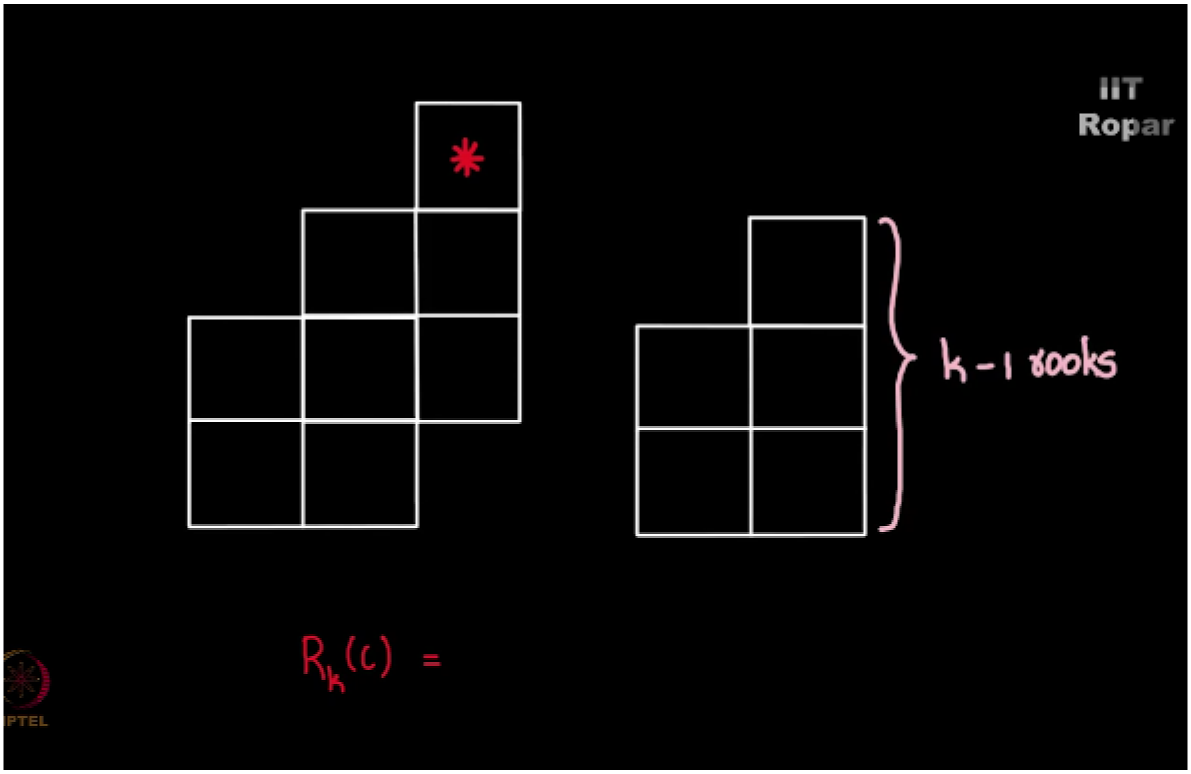

that's because one rook is kept there, that's the case I'm discussing right now, so this is as good as total number of ways in which you can place K-1 rooks in this smaller chessboard, let me call it CS,

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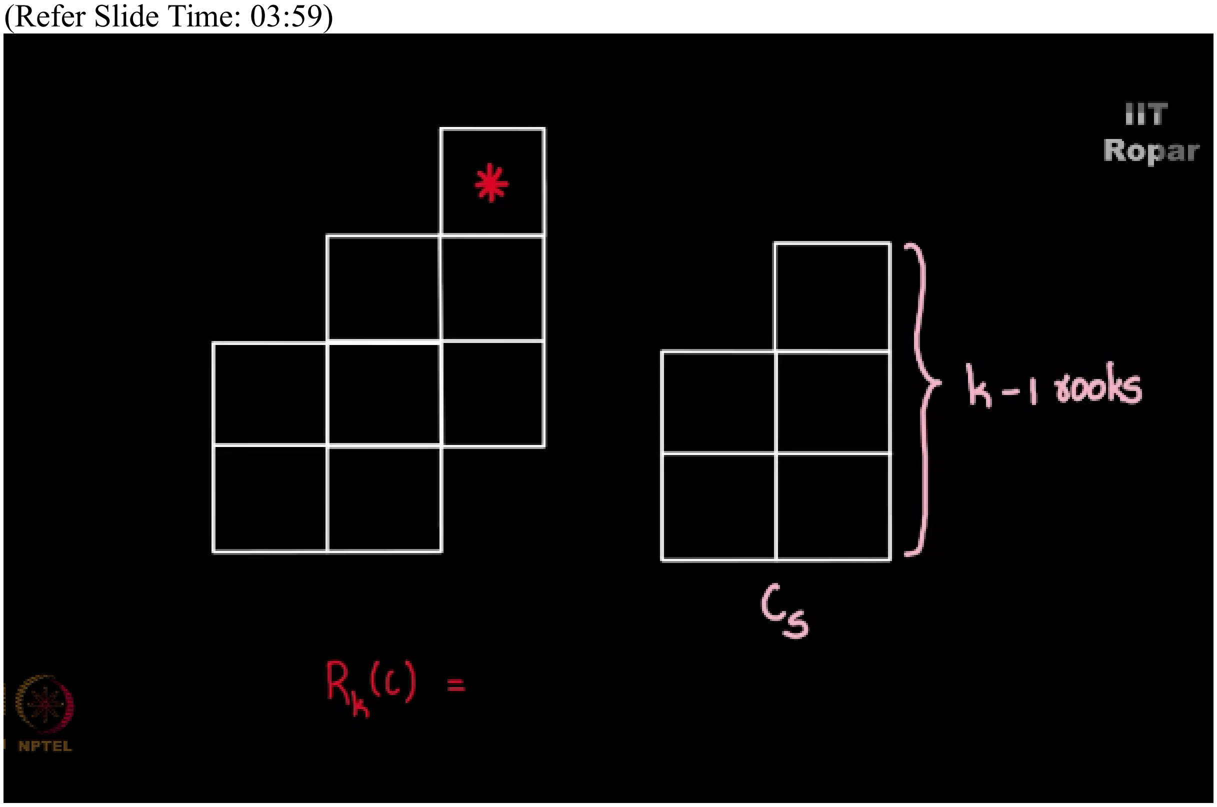

okay, S for smaller, RK-1(CS) + right this is the number K-1 rooks in this chessboard CS + the case where you do not put any rooks in the cell which means you have K options to be put on this chessboard, what is the chessboard? It's same as your original C chessboard but this one cell is removed, okay. Let me call this chessboard as CE,

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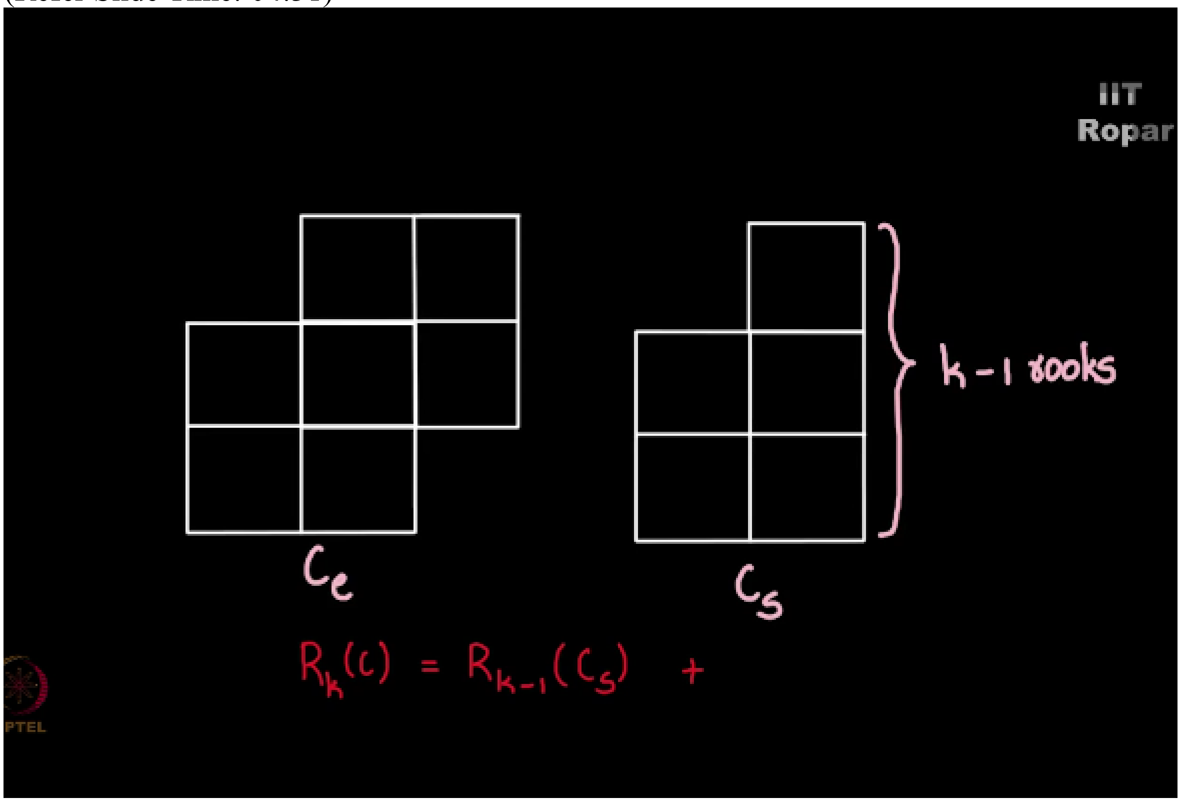

CE means let's say eliminated, one cell eliminated chessboard, okay.

And now this will be simply RK(CE), you see RK(C) this chessboard is RK – 1(CS) this chessboard + RK(CE), right, now what I'll do is I'll multiply X to the K throughout, why? You will soon realize why I am doing this little trick, so when I'm multiply X to the K throughout I simply get this RK(C) times X to the K which is equal to RK(-1) CS X to the K + RK(CE) X to the K, right, okay.

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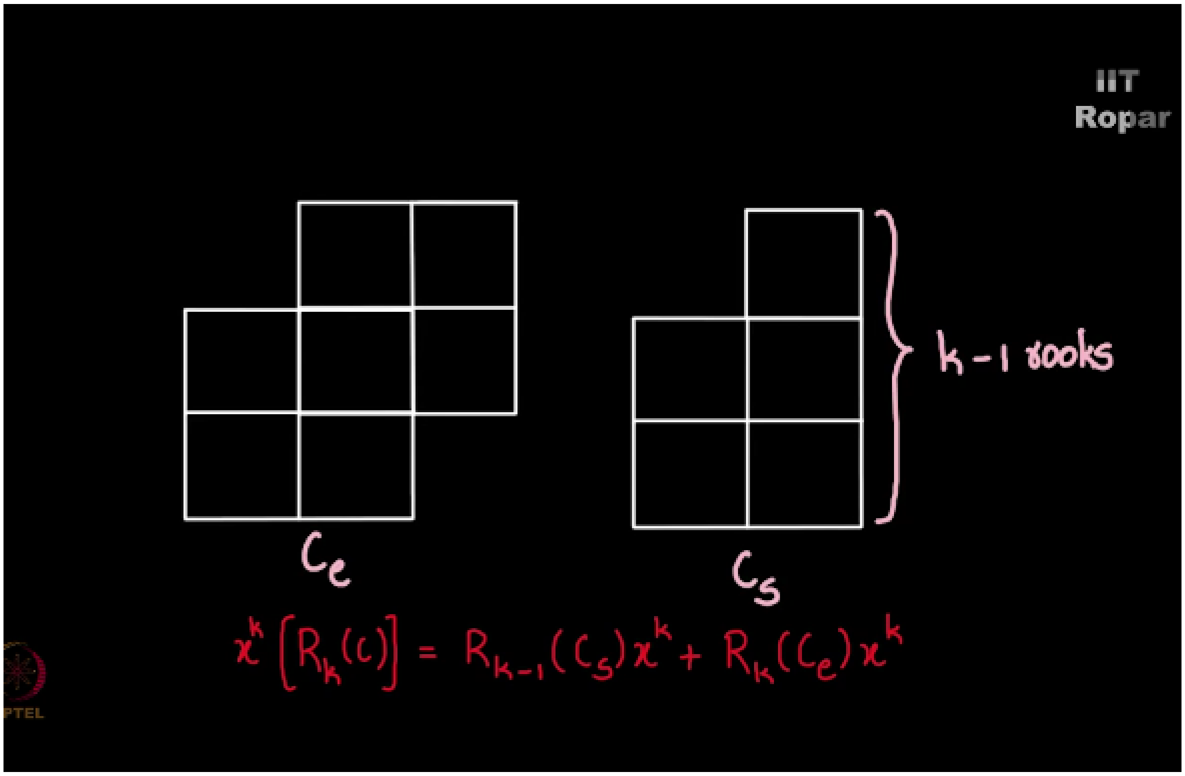

Now I'll take summation running through K=1 to N throughout, for K, this K is fixed K, but I'll run this K from 1, 2, 3 up to N,

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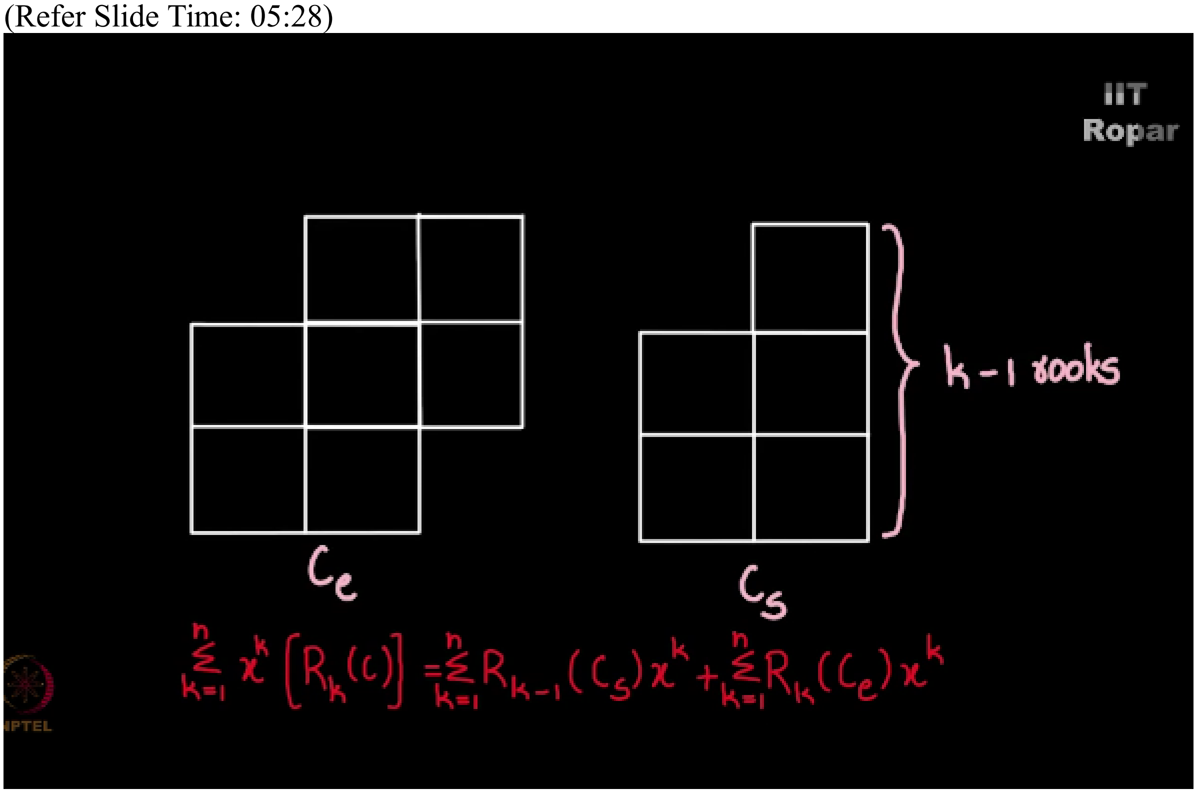

and that summation will be there throughout, correct, when I take RC basically it's like taking K = 1, K = 2, K =3 and adding all the equations, you will get summation K = 1 to N, RK(C)

(SK) = summation K = 1 to N, RK-1(CS) times X to the K + summation K = 1 to N, RK(CE) X to the K, okay, so far so good.

Now this particular equation can be rewritten as again I wanted to appear more like a polynomial, right, what is missing here? A constant 1 is missing, right, on the left hand side, so let's fixed that, now look at this,
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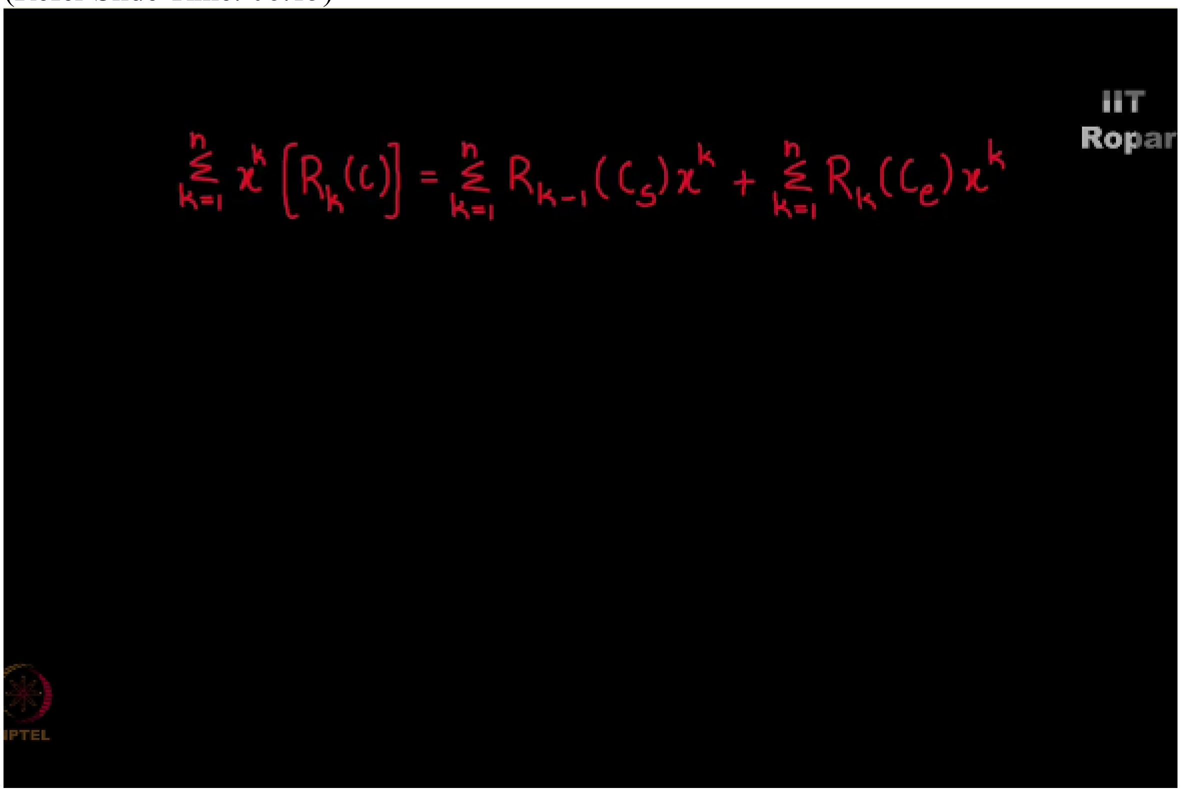

what I'll do is I'll simply write summation K = 1 to N, RKC(X to the K) I'll add 1 left side, on the ride side I'll again add +1, so basically I'm adding 1 to both left hand side and the right hand side, and then I'll take X outside and write this as X times summation RK-1 times CS times X to the K-1 + summation RK(CE) X to the K, right, so what I have done, I've just plucked out X so that the K-1, K-1,
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$$\sum_{k=1}^{n} x^k \left[R_k(C)\right] = \sum_{k=1}^{n} R_{k-1}(C_s)\, x^k + \sum_{k=1}^{n} R_k(C_e)\, x^k$$

$$\sum_{k=1}^{n} R_k(C)\, x^k + 1 = \sum_{k=1}^{n} R_{k-1}(C_s)\, x^k + \sum_{k=1}^{n} R_k(C_e)\, x^k + 1$$

$$= x\left[\sum_{k=1}^{n} R_{k-1}(C_s)\, x^{k-1}\right] + \sum_{k=1}^{n} R_k(C_e)\, x^k + 1$$

the RK-1 here and the polynomial X to the K-1 both of them are the same, so that I can write it more like a polynomial, right, and then I add +1 on both sides and what you get, you simply get the rook polynomial R(C,X) = X times R(CS,X) + R(CE,X), so rook polynomial of C = X times rook polynomial of CS + rook polynomial of CE.

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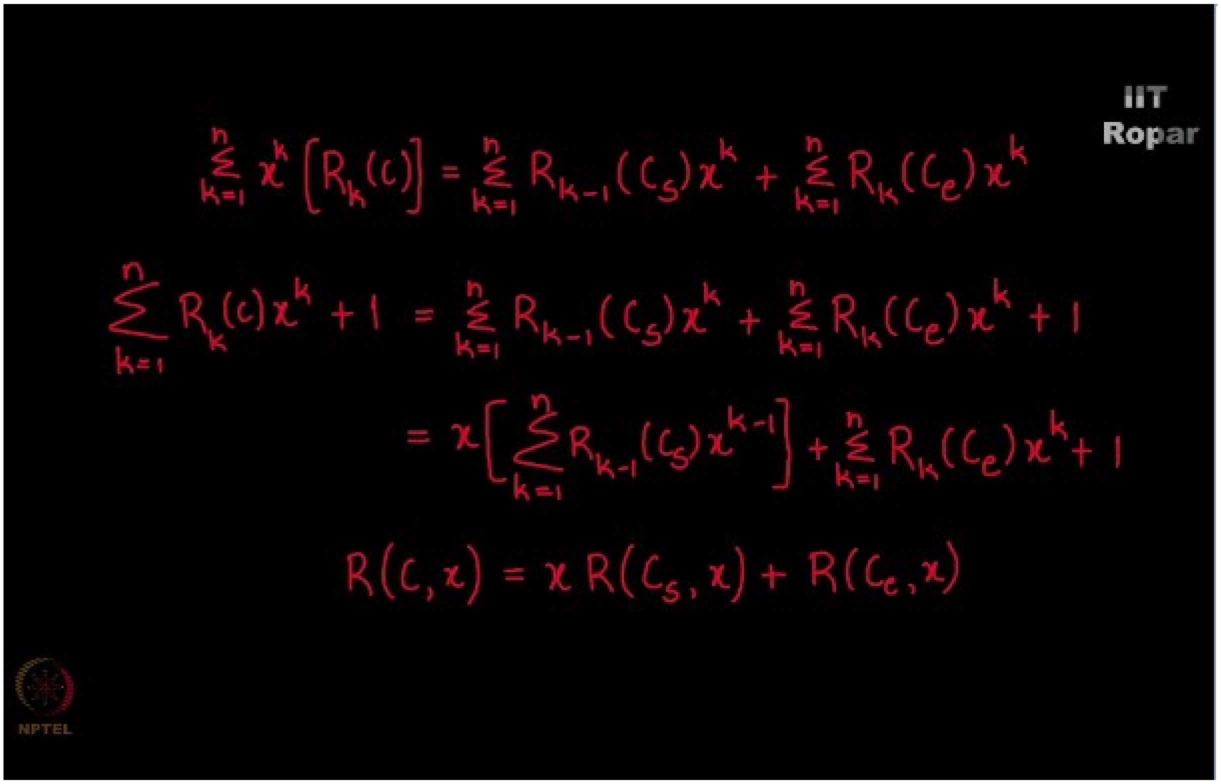

Now what did we just do? What we did is actually magical, you see we took a bigger chessboard and we ask this question, what is the rook polynomial of the bigger chessboard, and we got that the rook polynomials of the bigger chessboard is X times the rook polynomial of a slightly lesser chessboard + another smaller chessboard which means we can recursively keep applying this, and then make the question small and smaller and finally get the answer, right, we will precisely do that.

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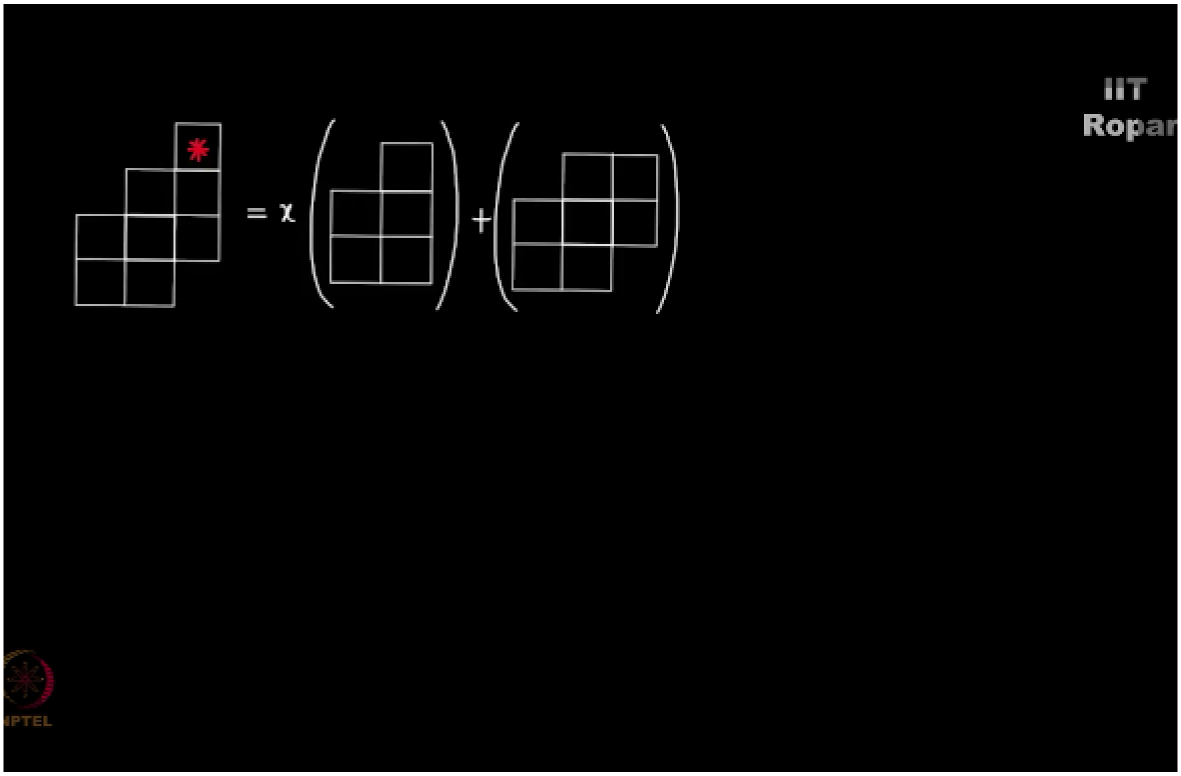

Now look at this, all I'm saying is rook polynomial of this chessboard, here is a chessboard itself is equal to X times the rook polynomial of this 5 cell chessboard + this 7 cell chessboard,

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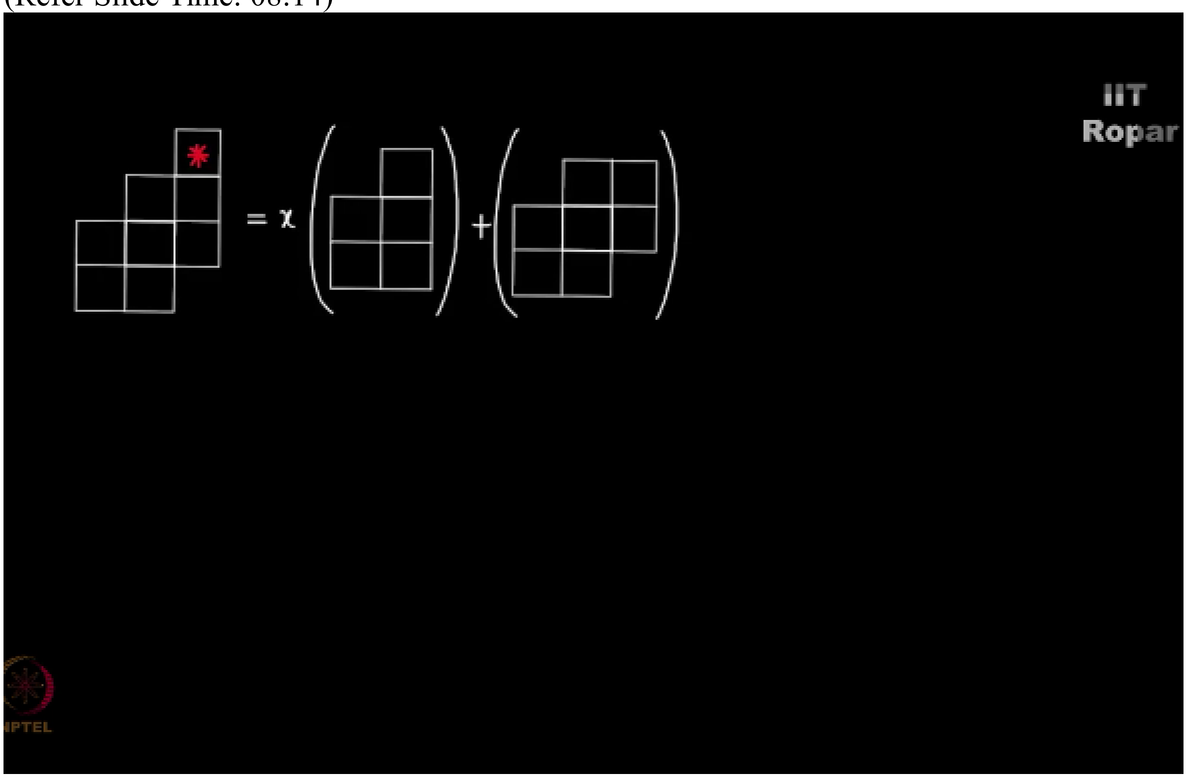

correct, and then this becomes again I'll divide this particular 5 cell chessboard and I'll apply the formula further on it by keeping a star on this cell, and what I'll get is my CS chessboard will be simply 2 squares like this, and my CE chessboard will be a 2 x 2 square.

Similarly when I put a star here for this chessboard I will get a similar breaking down, so ultimately you will get this X times, X times 2 cell chessboard + 4 cell chessboard + X times 2 x 2 chessboard + this 6 cell chessboard,

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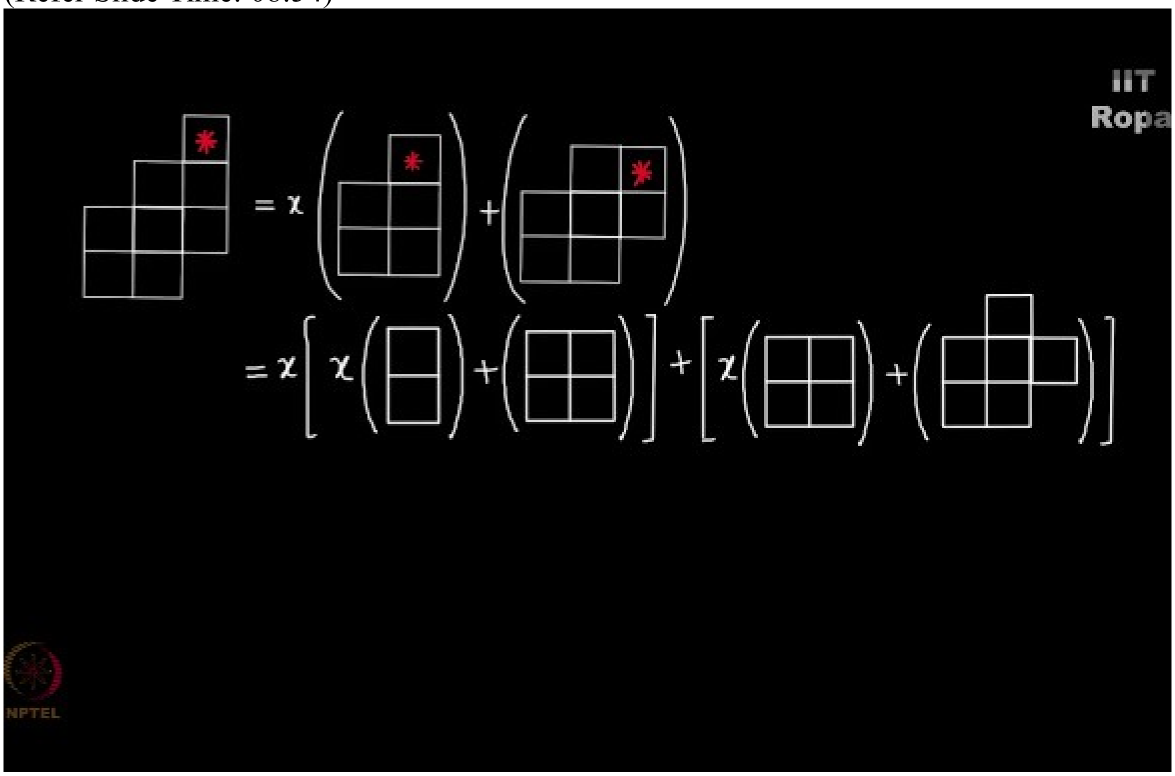

again further breaking this down in the next step you will observe that we will get a polynomial, so I'll just directly write the polynomial right now for this smaller chessboards, right, so I'll now write for 2 cell chessboard, the polynomial will simply be 1 +, in how many ways can you place 1 rook here? 2, so it is 1+2X, that's it, okay, X times 1+2X for this board, and for our 2 x 2 chessboard whatever is the, this is the rook polynomial I'll write that here, and for this chessboard it is again 2 x 2, so I'll write this polynomial here, and what I will do is for this chessboard I will again break it down by putting a star here, okay, it just again breaks down like this.

And ideally I will finally get a big polynomial which looks like this, as you can see the steps, intermediate steps,

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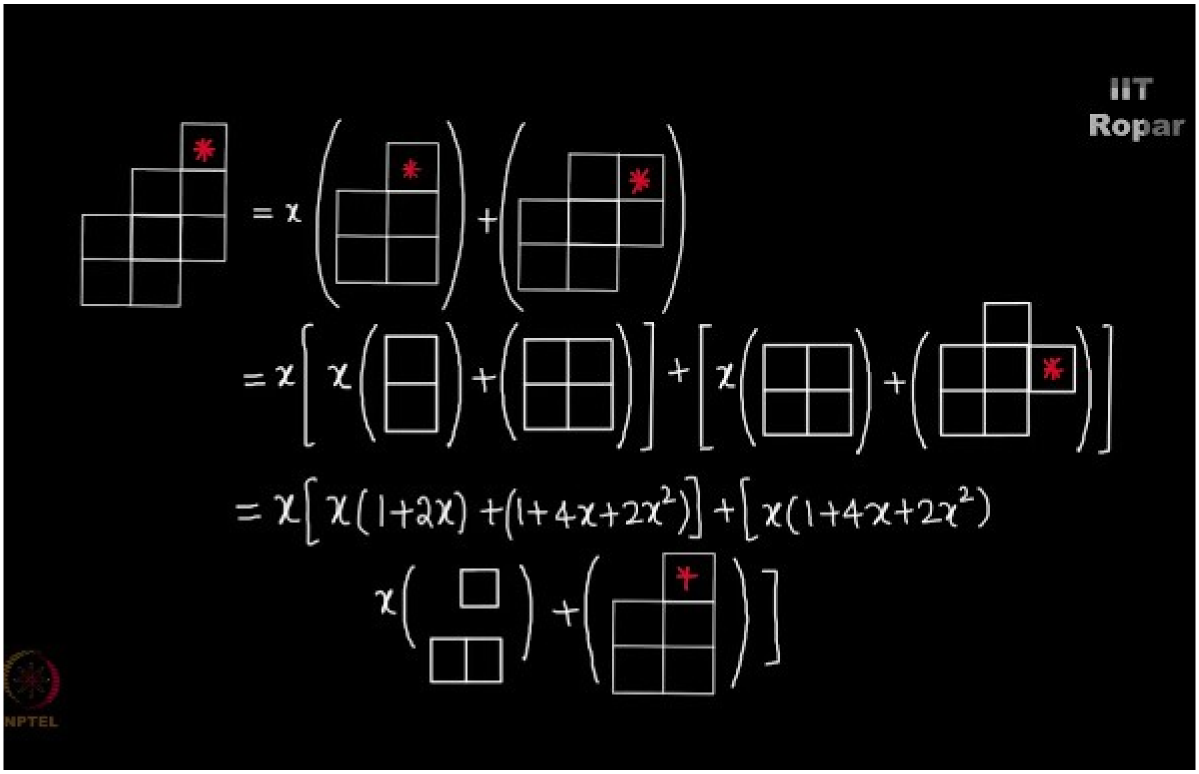

we are just displaying it you can verify them, they are very easy, very straightforward, the way, you know how you replace a polynomial, a chessboard with the polynomial, simply write down 1+ number of ways in which 1 rook can be kept times X + number of times in which 2 rooks can be kept, non-taking rooks of course times X square and so on and multiply and add all of them and finally you will get this as the answer,

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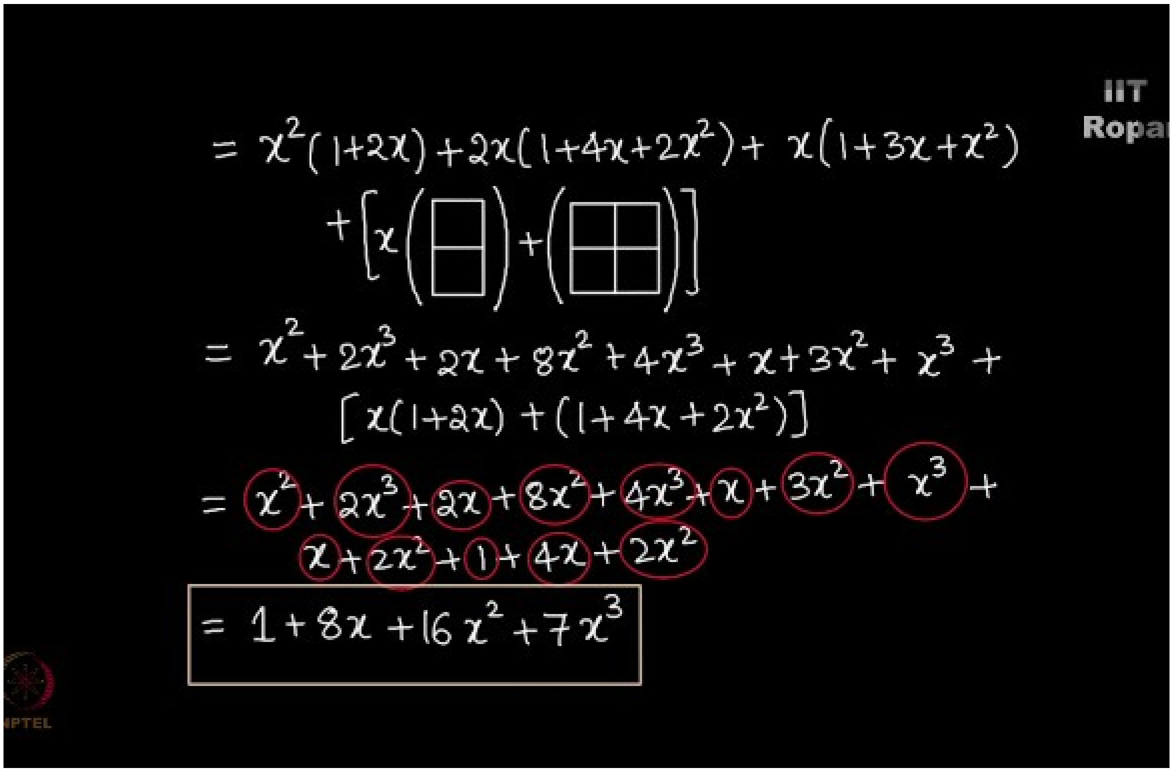

you can check the intermediate steps by staring through this slide,

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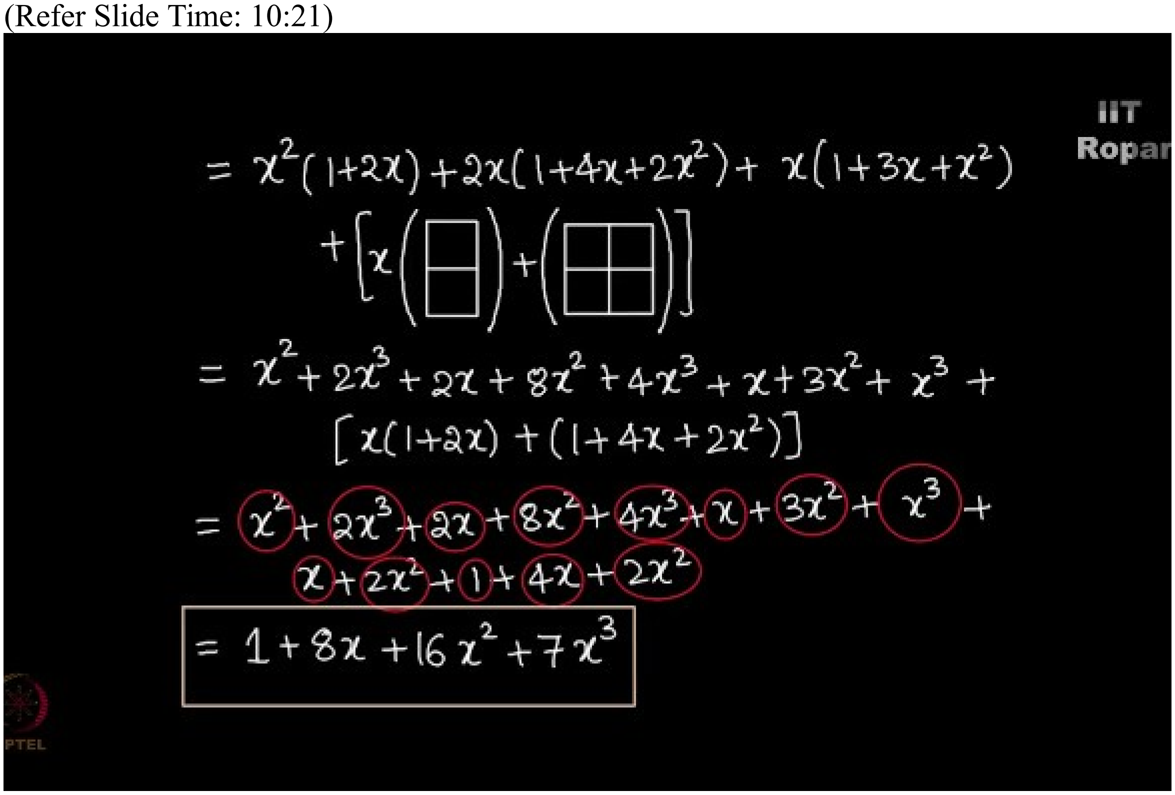

you will get 1+8X + 16X square + 7X cube which means in this chessboard you can place 1 rook in 8 ways, 2 rooks in 16 ways and 3 rooks in 7 ways.

Just observe what happens so far, you were given a chessboard and you were asked to compute a rook polynomial, it sounded very non-trivial task, but you came out with a fantastic formula that breaks your chessboard into smaller chessboards, right, and this smaller chessboards were in no way simple, there were just smaller, so what you do is you make this smaller, one step smaller even further, and even if that is complicated you make it even smaller by reapplying the formula recursively and finally the chessboard start appearing smaller and smaller, but there are lot of them, but you don't know, but you know very well how to handle them, so you replace it by the polynomial, corresponding polynomial and multiply and add, you will get the final answer, right, so a very powerful counting technique using, again simple polynomial arithmetic and you observed that something as not so obvious question as in how many ways can you place rooks on a very small chessboard, we will go further and try to see examples where this can be further applied,

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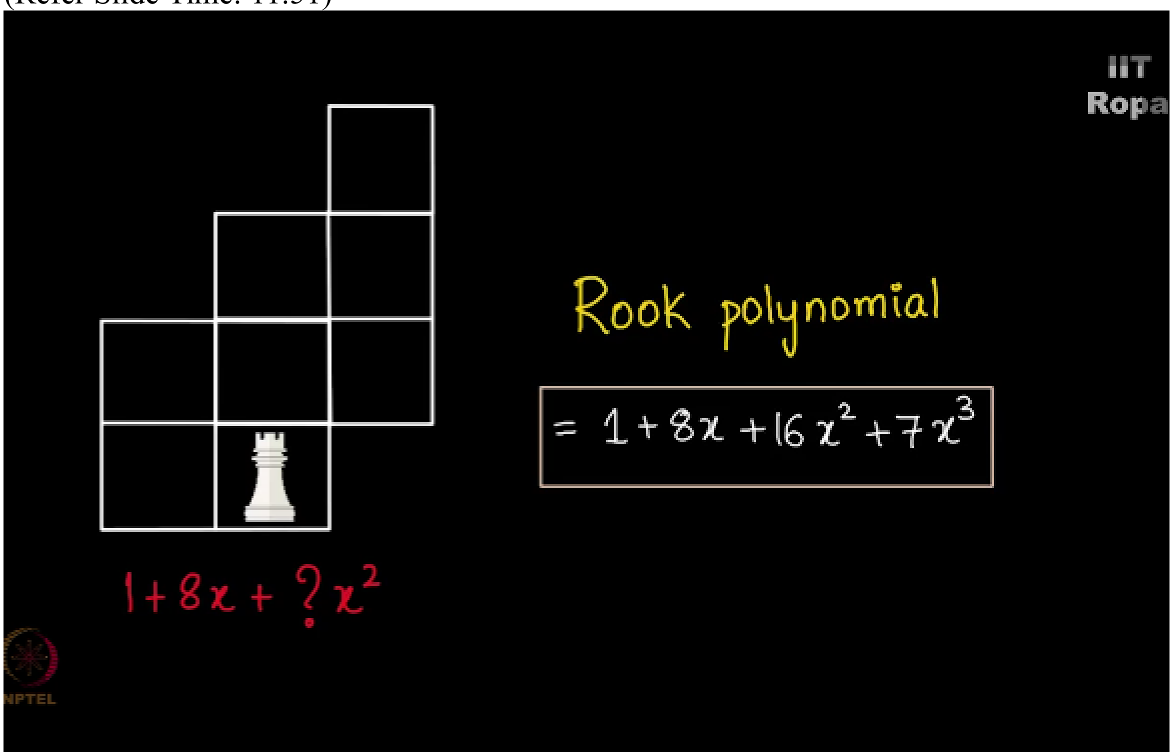

as an always please refer to this chapter called principle of inclusion and exclusion from this book by Ralph P Grimaldi is given here on this slide, you can even try solving more problems, we could not find the better book than this when it comes to good exposition on the topic of rook polynomials.

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