NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Discrete Mathematics
Principle of Inclusion and Exclusion

Inclusion-Exclusion Formula

By
Prof. S.R.S Iyengar
Department of Computer Science
IIT Ropar

Let us consider a small example,
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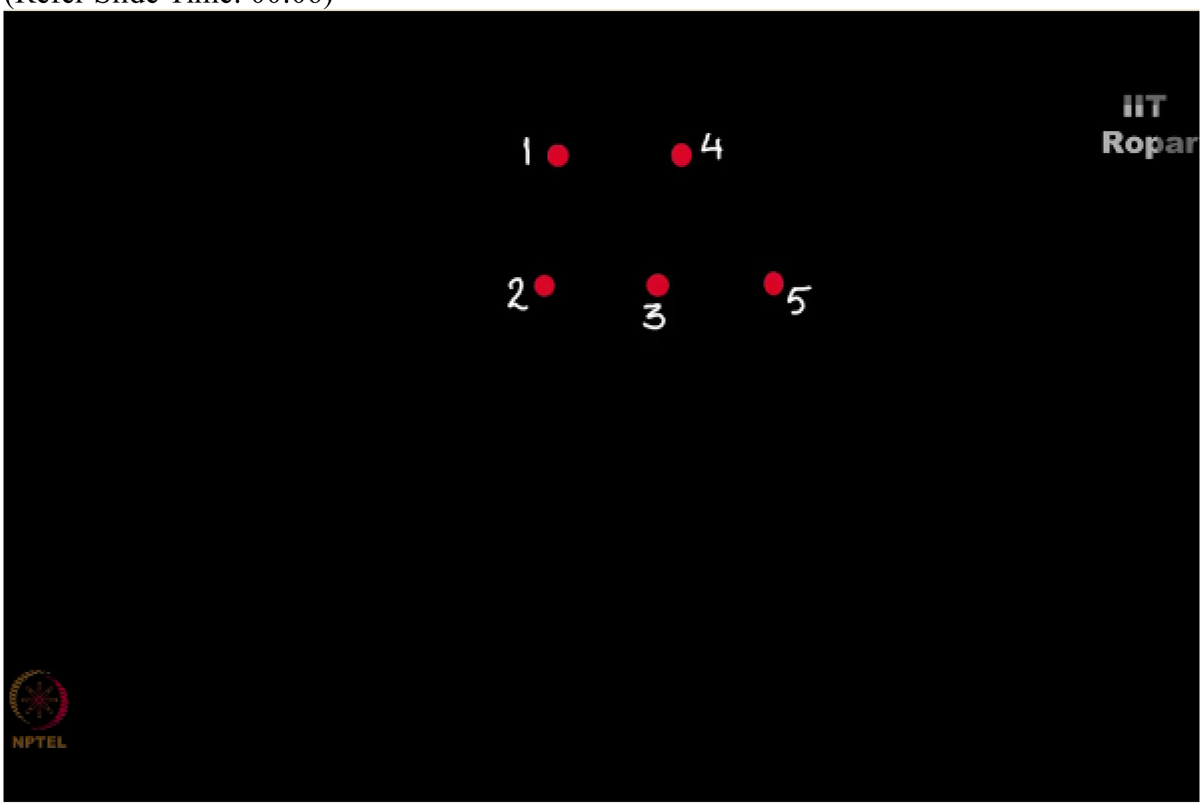

imagine these are the 5 cities in how many ways can you construct roads between these 5 cities so that no city is left out,
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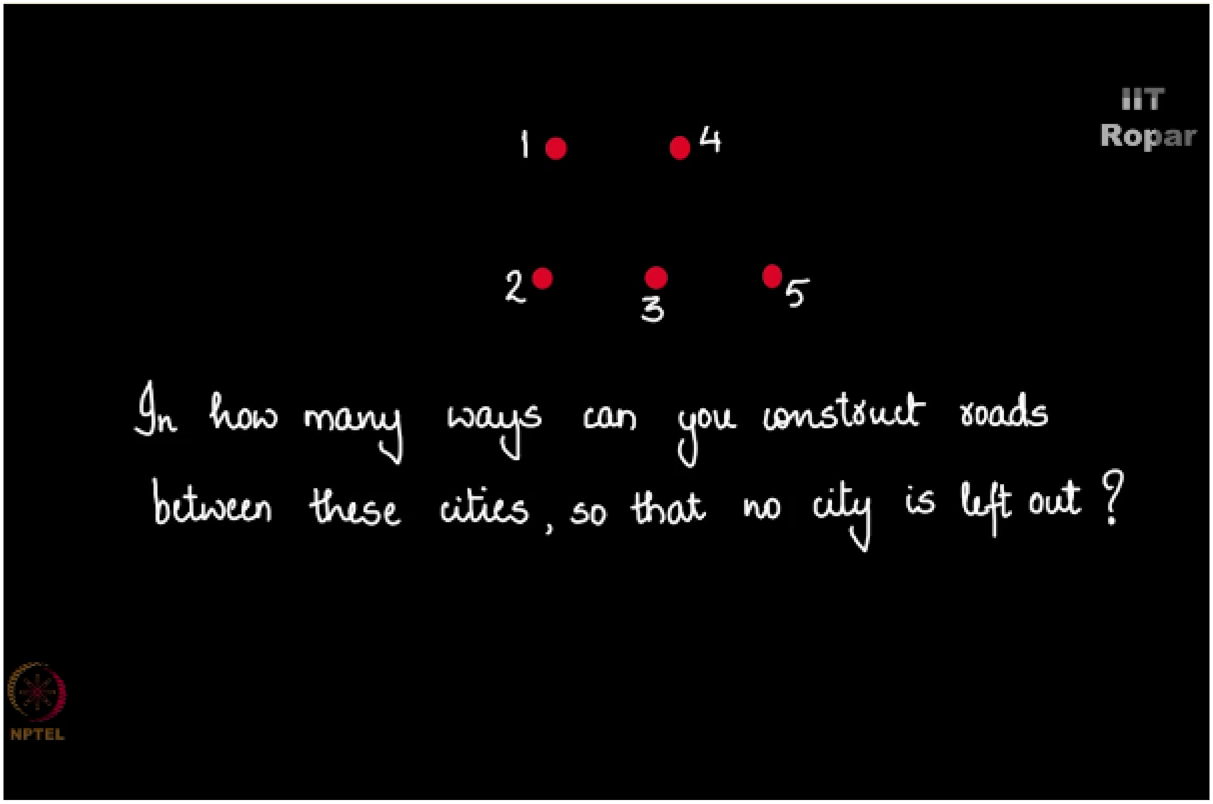

you see this is one possibility, see all though there are, there is no way in which you can go from node 1 to node 4 or 5, still every city has at least 1 road,

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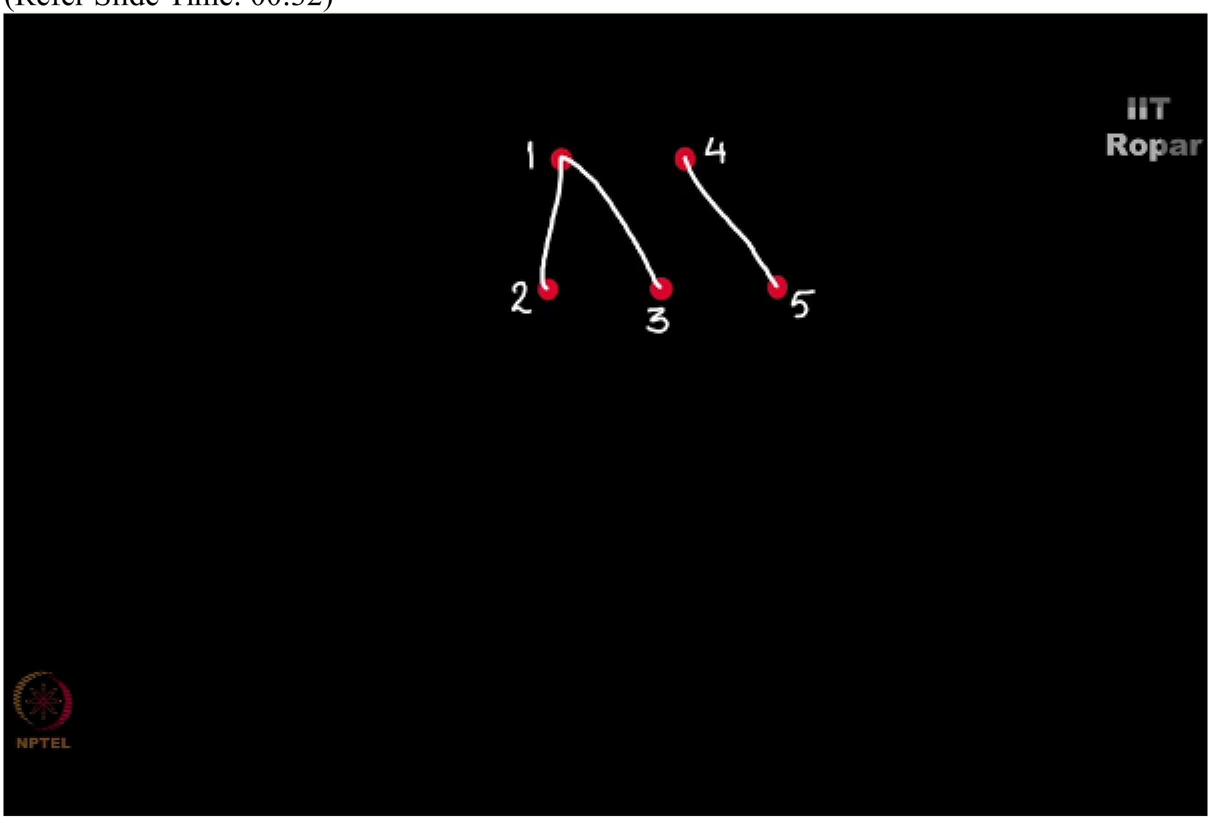

okay, in other words in the language of graph theory we are saying in how many ways can we have a graph so that there are no isolated vertices, isolated vertex means a vertex of degree 0,
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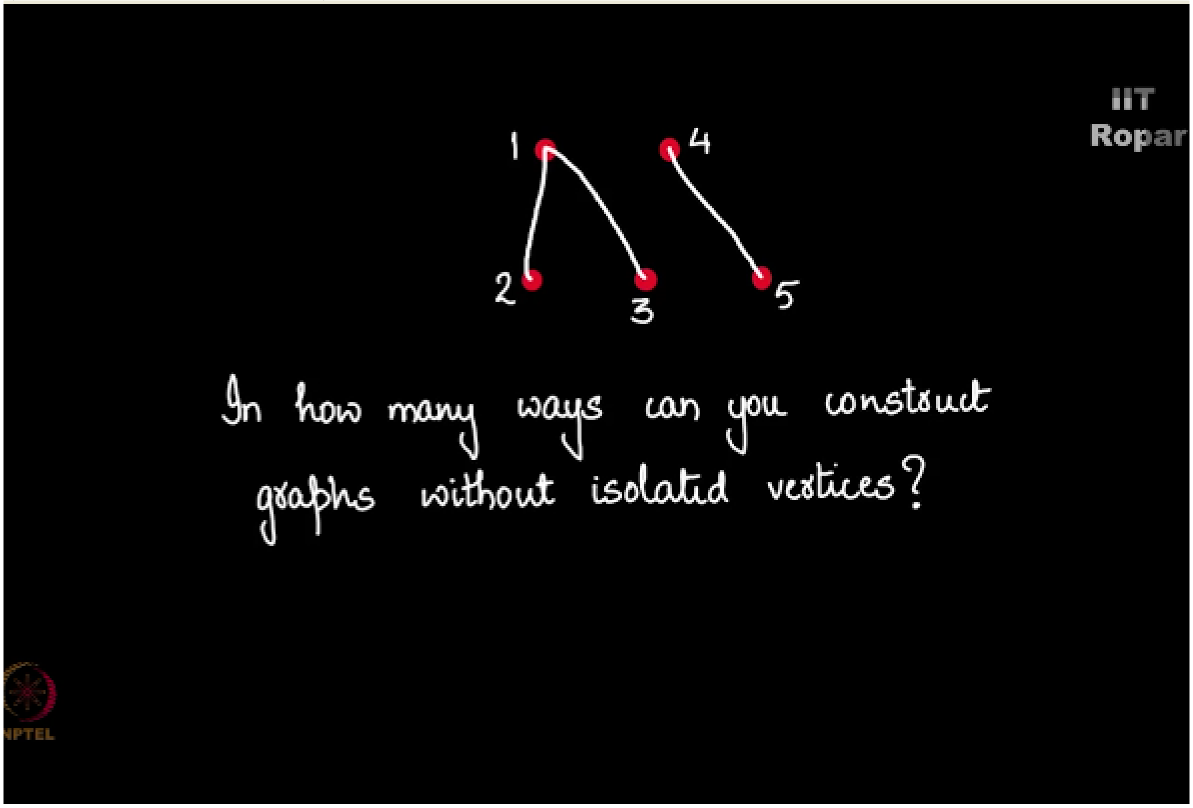

in how many ways can we construct a graph without isolated vertices? Now look at this, I'll now count in how many ways I can construct a graph with 1 being isolated, let me call that as condition 1, condition 1 is vertex 1 is isolated, and N(C1) is a total number of ways in which I can isolate 1,
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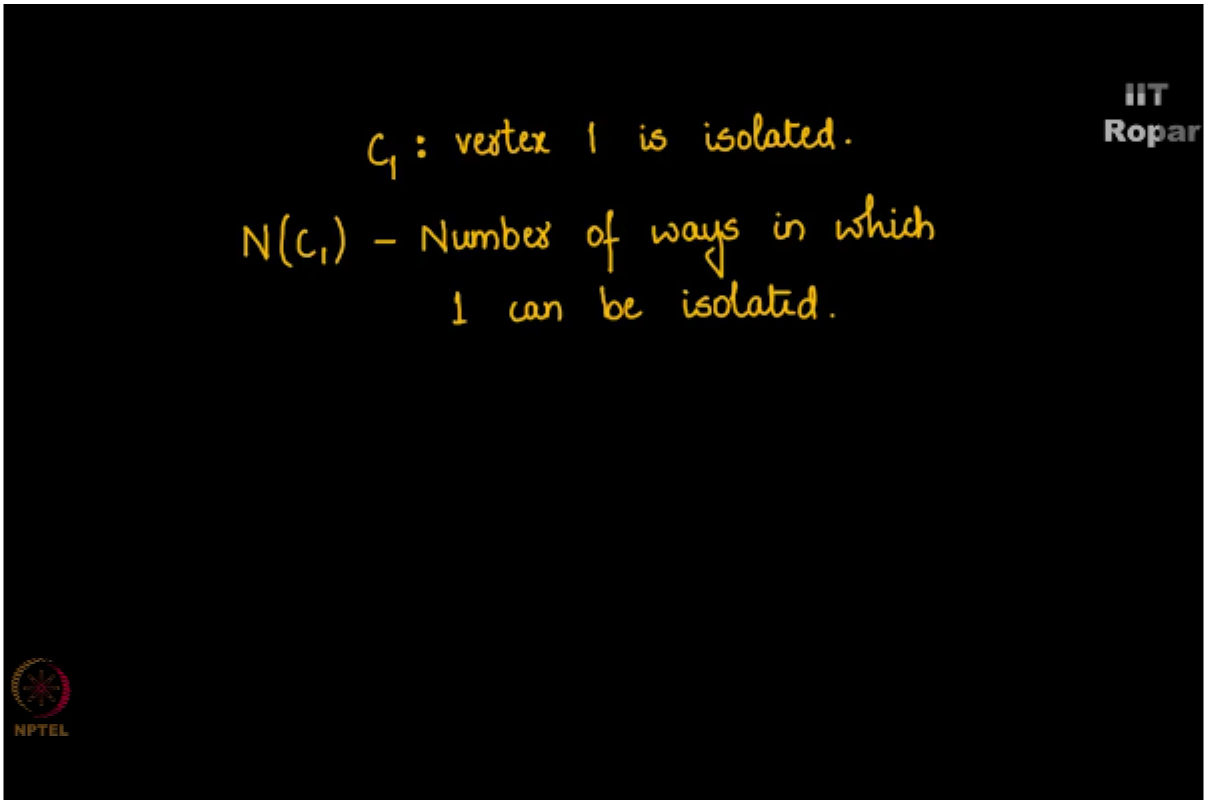

just a convention, I say I call it N(C1), okay, N(C1) is going to be, I'll keep one isolated and I'm left with how many vertices? 2, 3, 4, and 5, 4 vertices, in how many ways can I construct a graph on 4 vertices? We have seen this problem already, it's a straight forward question, it is 2 to the power of all possible edges which is 4 choose 2, which is 2 to the power of 6,

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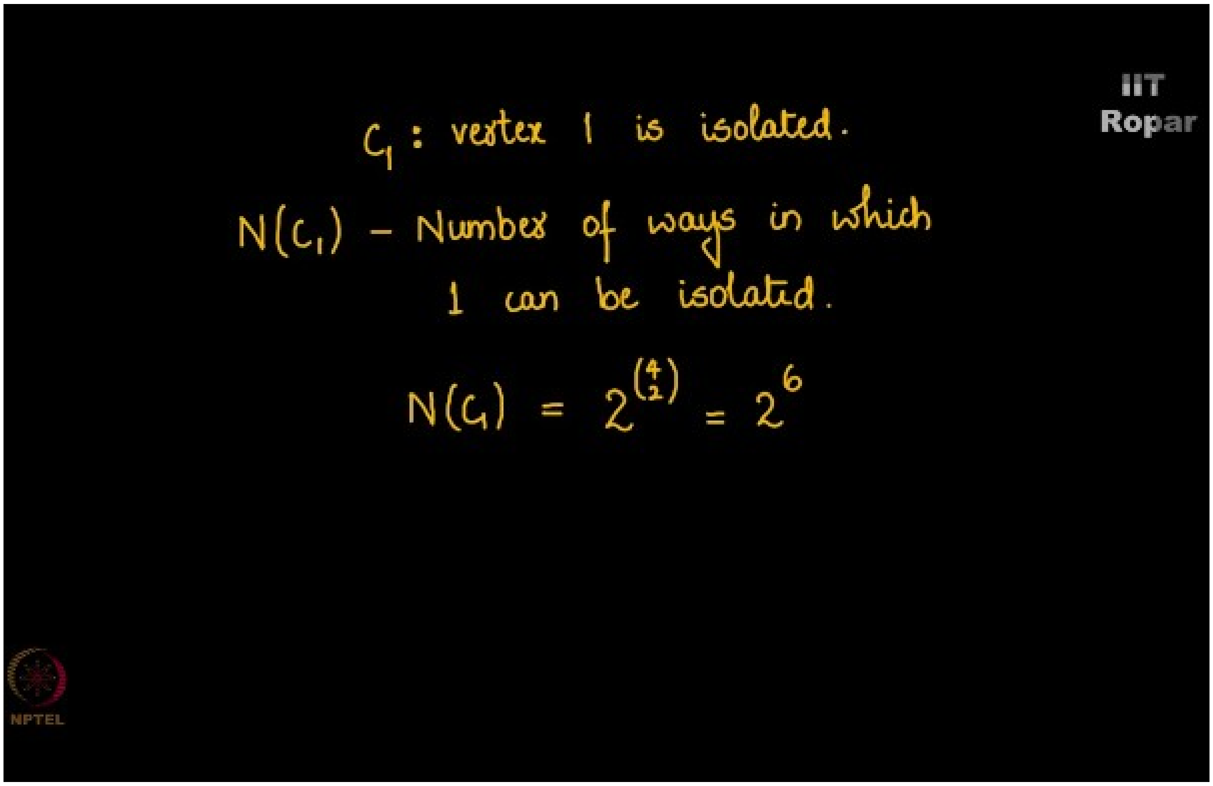

so N(C1) is 2 to the 6, pause and think if you have not figured out what is C1, what is N(C1) and how N(C1) is 2 to the power of 6.

Similarly what is C2? C2 will be condition 2 which is the vertex 2 is isolated, okay, so N(C2) is also 2 to the power of 6 as you can observe,
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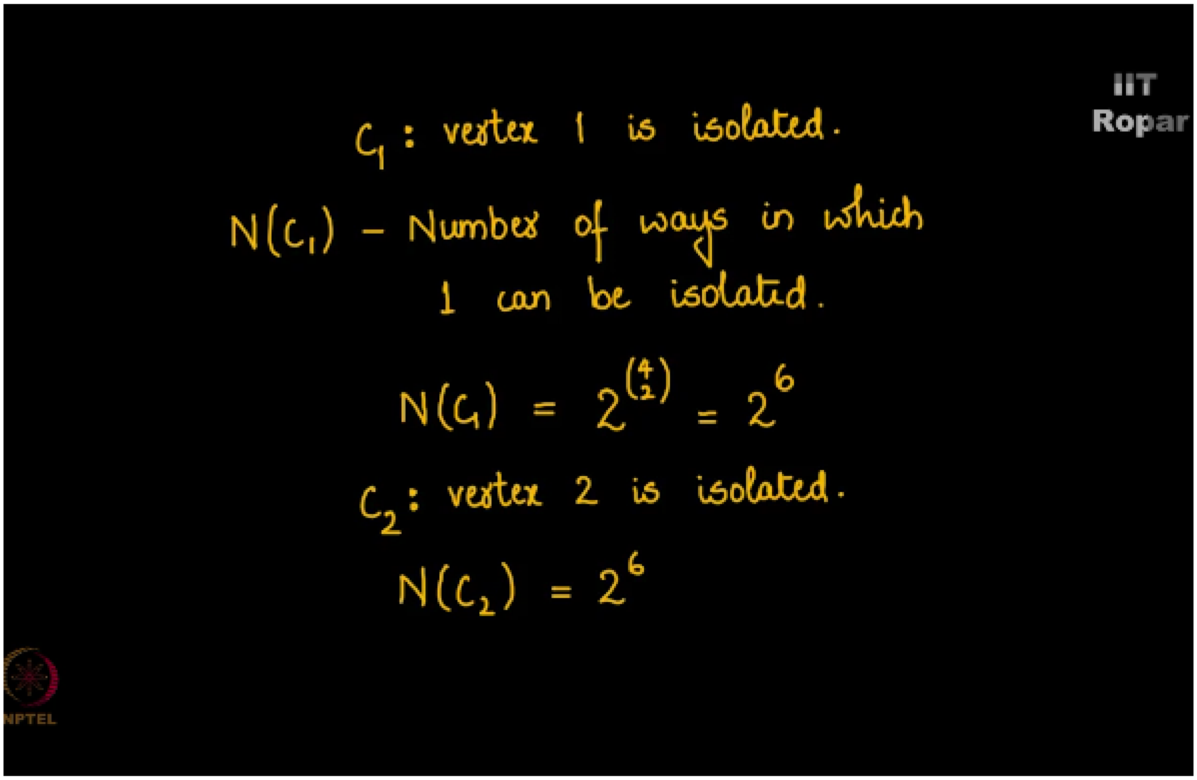

same way the way you said N(C1) is 2 to the 6, N(C2) is also 2 to the 6, so is N(C3) as you can observe and so is N(C4), all are 2 to the power of 6.

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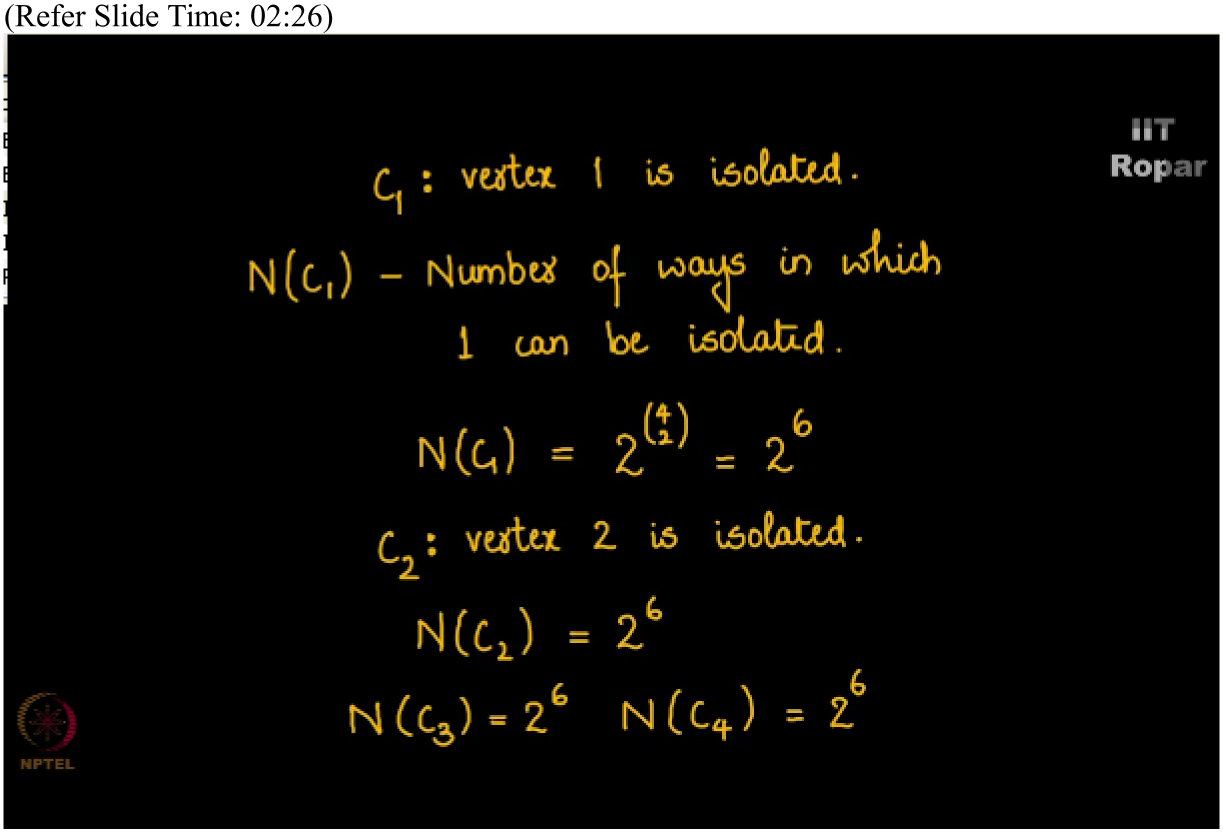

Now what is N(C1 times C2)? What do you even mean by this? By this we mean, what are the total number of ways in which condition 1 and condition 2 both are satisfied, what do we mean by that? By that we mean 1 is isolated, that's what C1 stands for and 2 is isolated, that's what C2 stands for, so N(C1, C2) denotes in how many ways can the vertex 1 and 2 both be isolated,
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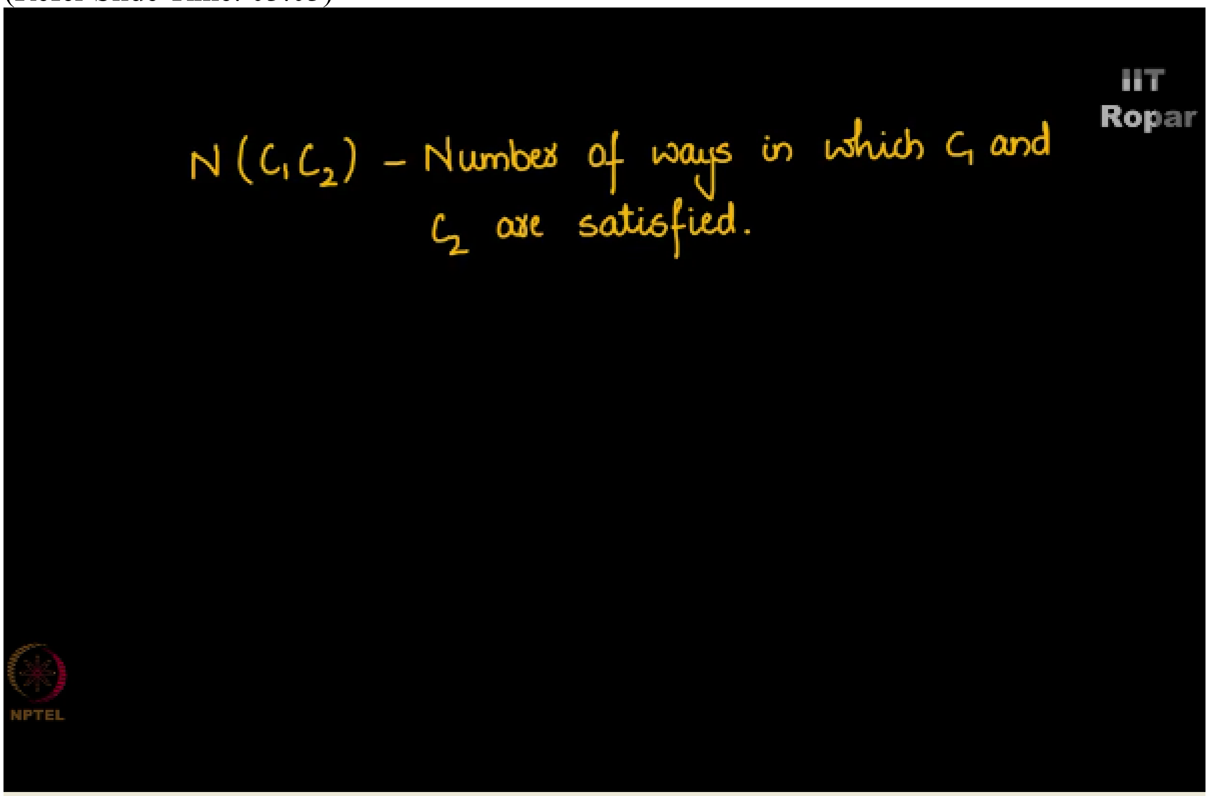

now again very simple 1 and 2 you don't have any edges, you ask this question in how many ways can you construct the graph on these 3 nodes, which is 2 to the power of 3 choose 2 which is 2 to the power of 3, so similarly N(C1, C3) is also 2 to the power of 3, think about it, it's all symmetric argument, N(C1,C4) is also 2 to the power of 3, take any 2 vertices, so N(CI,CJ) will be 2 to the power of 3, so (C1, C2) (C1, C3) (C1, C4) (C1, C5) (C2, C3) (C2, C4) (C2, C5) (C3, C4) (C3, C5) and (C4, C5), all these will be in fact 2 to the power of 3 only.
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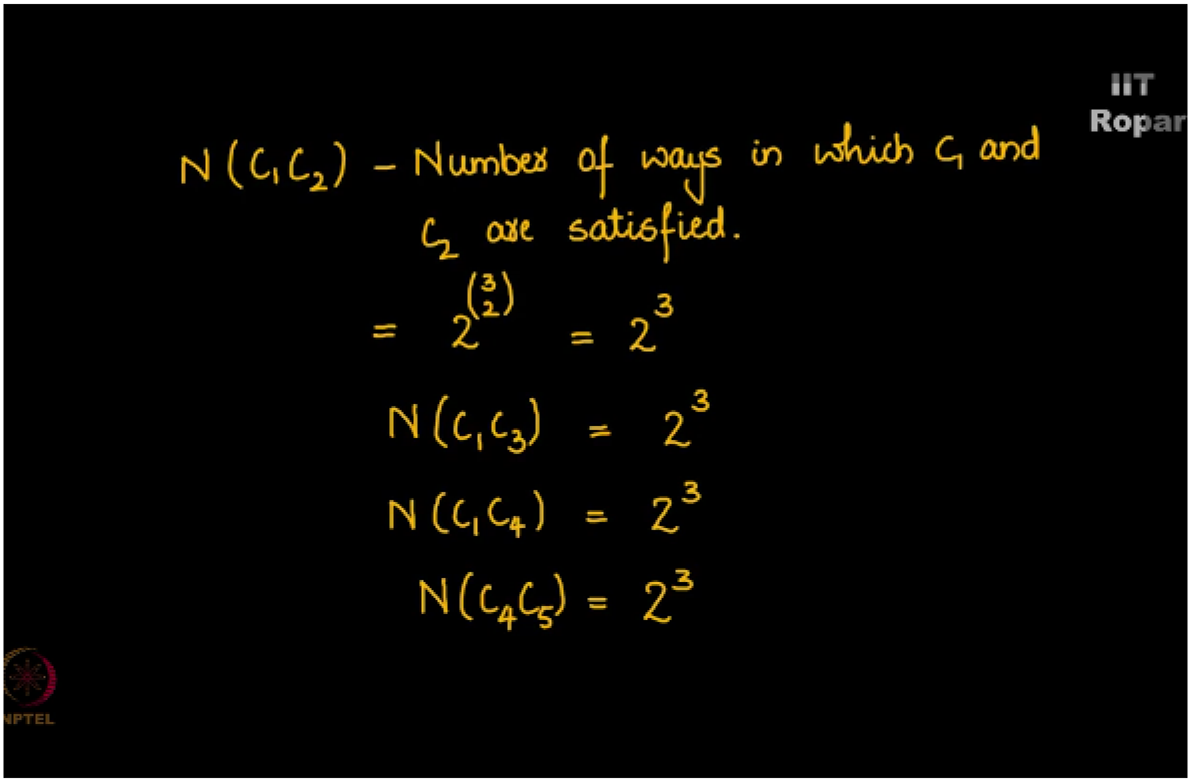

What is N(C1, C2, C3)? Where we even doing this, you will get to know very soon, C1, C2, C3 means 1 is isolated, 2 is isolated, 3 is isolated, in how many ways can this happen? 2 to the power of, how much is it? 1 ways, why? 1, 2, 3 are isolated means between 4 and 5 you have an edge or you don't have an edge, there are only 2 possible graphs, so it is 2 to the power of 1 and so N of any CI, CJ, CK will be 2 to the power of 1,

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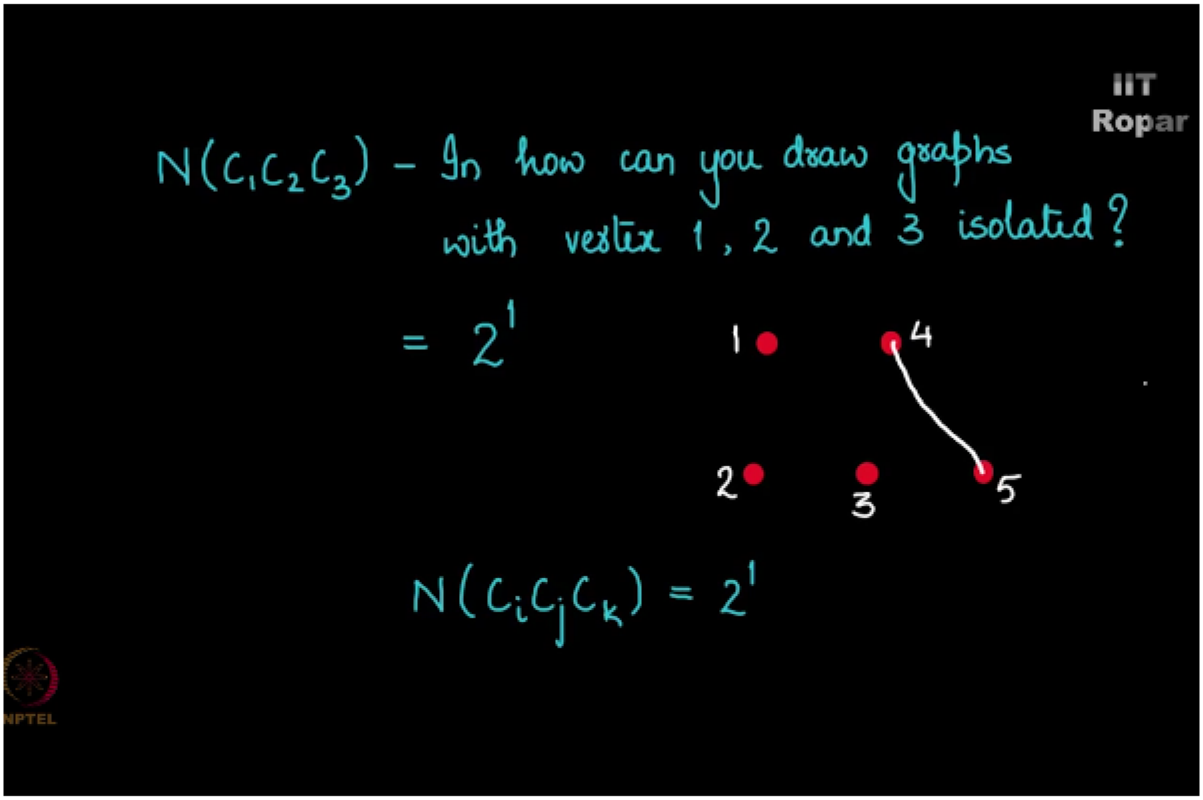

similarly N(C1, C2, C3, C4), in how many ways can you have all 4 vertices isolated is precisely one way, why? Think about it.

In how many ways can we have, what is the other way? 1, 2, 3, 4 is one way, 1, 2, 3, 5 is another way, 1, 3, 4, 5 is another way, so you have all possible 5 ways of having 4 vertices isolated,
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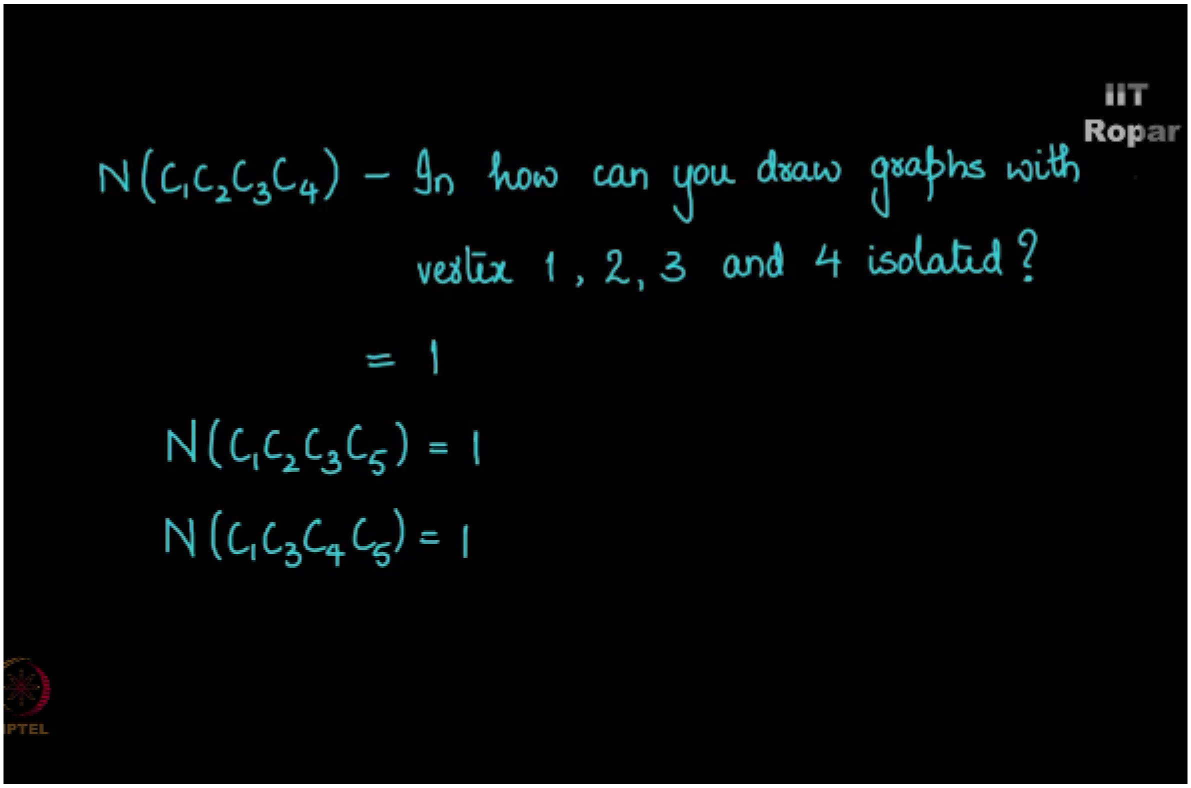

now please note we are counting the same graph again and again you see, now that is a point we noted, but let's go ahead, we'll simply write down for completeness sake, N(C1, C2, C3, C4) is 1, N(C1, C2, C3, C5) is 1, N(C2, C3, C4, C5) is also 1 and so on.

Now what is N(C1, C2, C3, C4, C5)? Sounds like a joke to you, right, because we seem to be counting the same graph again and again, in how many ways can we have 1, 2, 3, 4, 5? All the 5 nodes isolated, one way.

Now the question is if I write a big diagram, and write a C1 circle, C2 circle, you see just the way we wrote A, B, A union B, and found out the number of elements in A union B, right, if we were to write C1, C2, C3, C4, C5 my question would be, what are the elements outside these circles, which means an element here would denote that graph where no vertex is isolated, if vertex 1 is isolated that graph is considered in C1,
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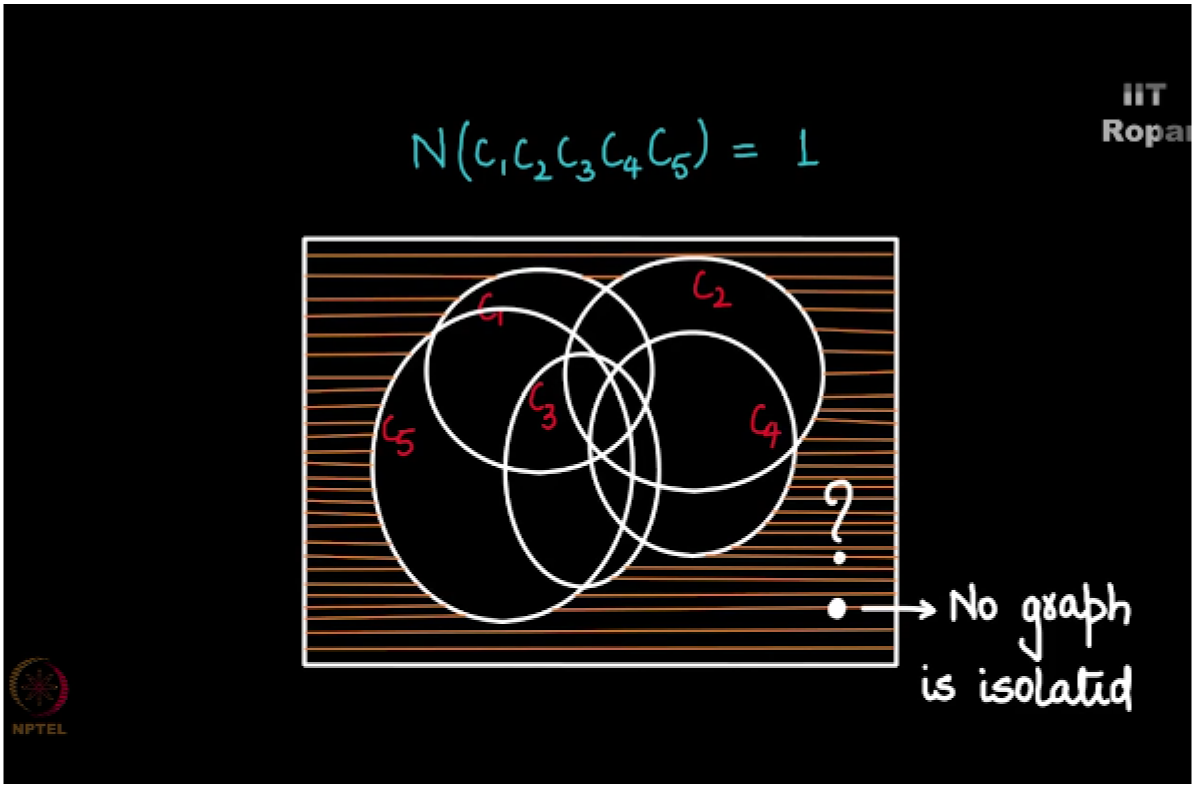

if vertex 2 is isolated that graph is considered in C2, think about it, I'm not explaining much, I'm trying to map the question to this kind of a mathematical model, right, so C4 stands for, that vowel stands for all those graphs with the vertex 4 isolated so on and so forth, right.

So now all I want to compute is what is outside these circles, so for that I should compute what is inside this circles, N(C1) + N(C2) etcetera up to N(C5) will be inside this circles,

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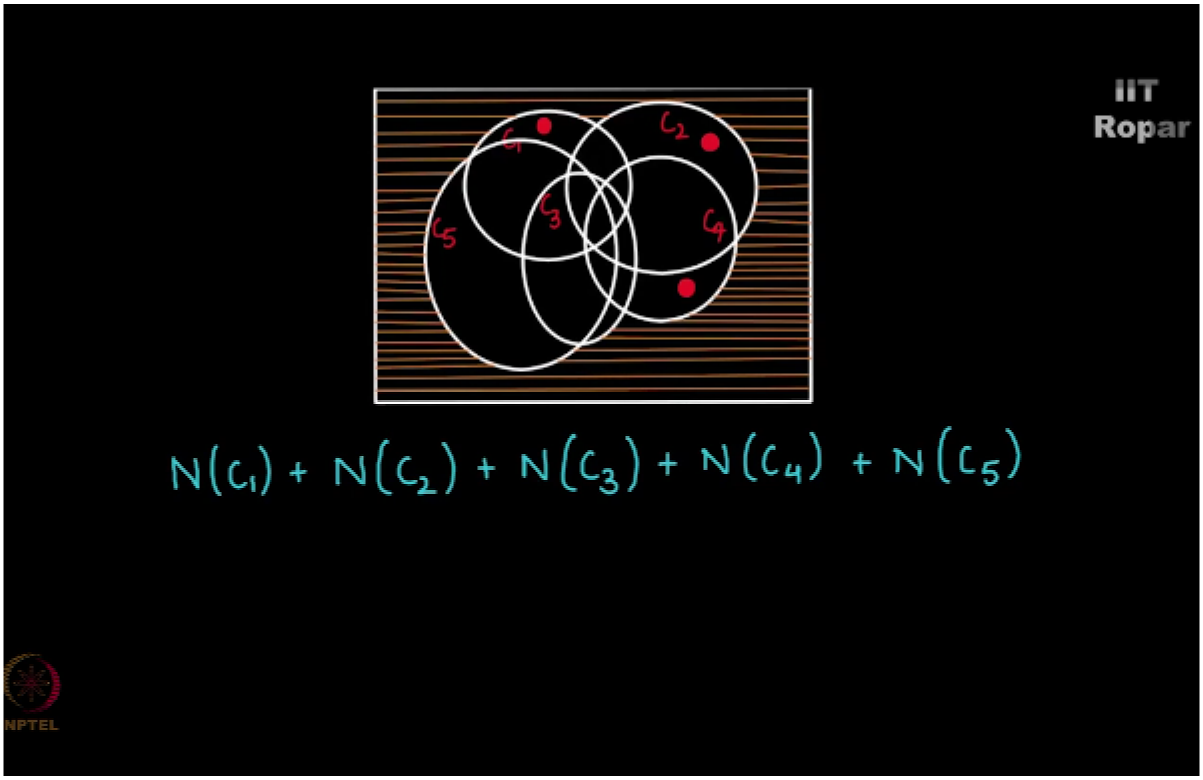

this is general formula for this as you can see for A union B you saw that it is cardinality of A + cardinality of B – cardinality of A intersection B, so for A union B union C you observed that cardinality of A + cardinality of B + cardinality of C – cardinality of A intersection B – cardinality of A intersection C – cardinality of B intersection C and all of these plus cardinality of A intersection B intersection C, you observe that this plus and minus signs keep toggling like this.

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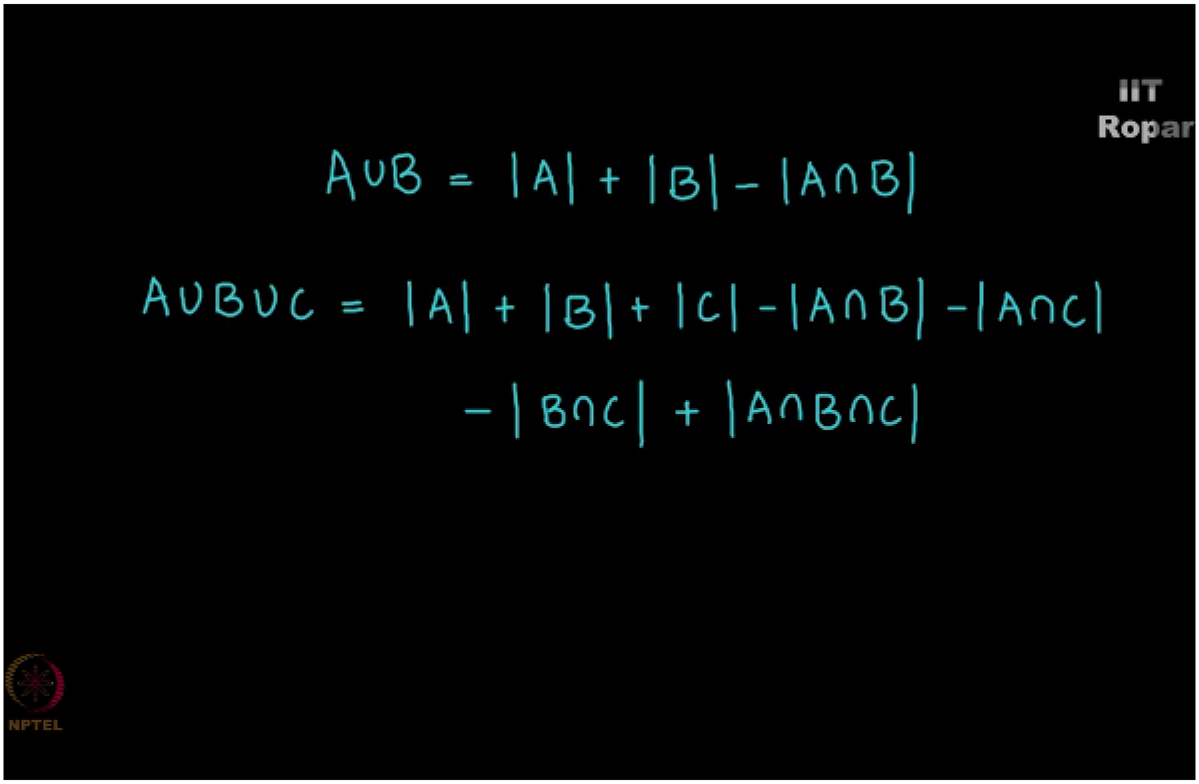

So what do we need to compute? We need a general formula to compute the elements outside all these 5 vowels, which is let us say all the elements inside this rectangle is N,

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okay, and I'm going to separate all possible entities inside this C1, C2, C3, C4, C5 from N and that will give me the number of elements outside these vowels, and I'm going to call that N bar, N bar are the elements here, elements inside this rectangle all of them put together is N in number, and now what I need to do is I need to remove all elements inside the vowels, it's not very easy to compute them in a straight forward way, you saw that formula for A union B union C, right, it was quite twisted and confusing in the sense that there was an alternating plus and minus sign there, if there are 5 vowels, 5 sets, it will be even more complicated, I'll just like that go ahead and write the formula as though and so familiar with it without any proof and once written we will try to prove that general case in some time, okay, the formula looks like this N bar elements outside the vowel is N which is all the elements inside the rectangle minus all the elements inside the circles, okay, the formula goes like this, -N(C1) number of elements in C1 + number of elements in C2 + number of elements in C4 + number of elements in C5, okay.

And then after minus comes your plus symbol, how, why you will get to know just trust, have trust in me and then assume that this is the way to do it, okay, plus N(C1, C2), N(C1, C3) + N(C1, C4), (C1, C5), (C2, C3), (C2, C4), (C2, C5), (C3, C4), (C3, C5), and (C4, C5) all possible, 5 choose 2 combinations I will write,

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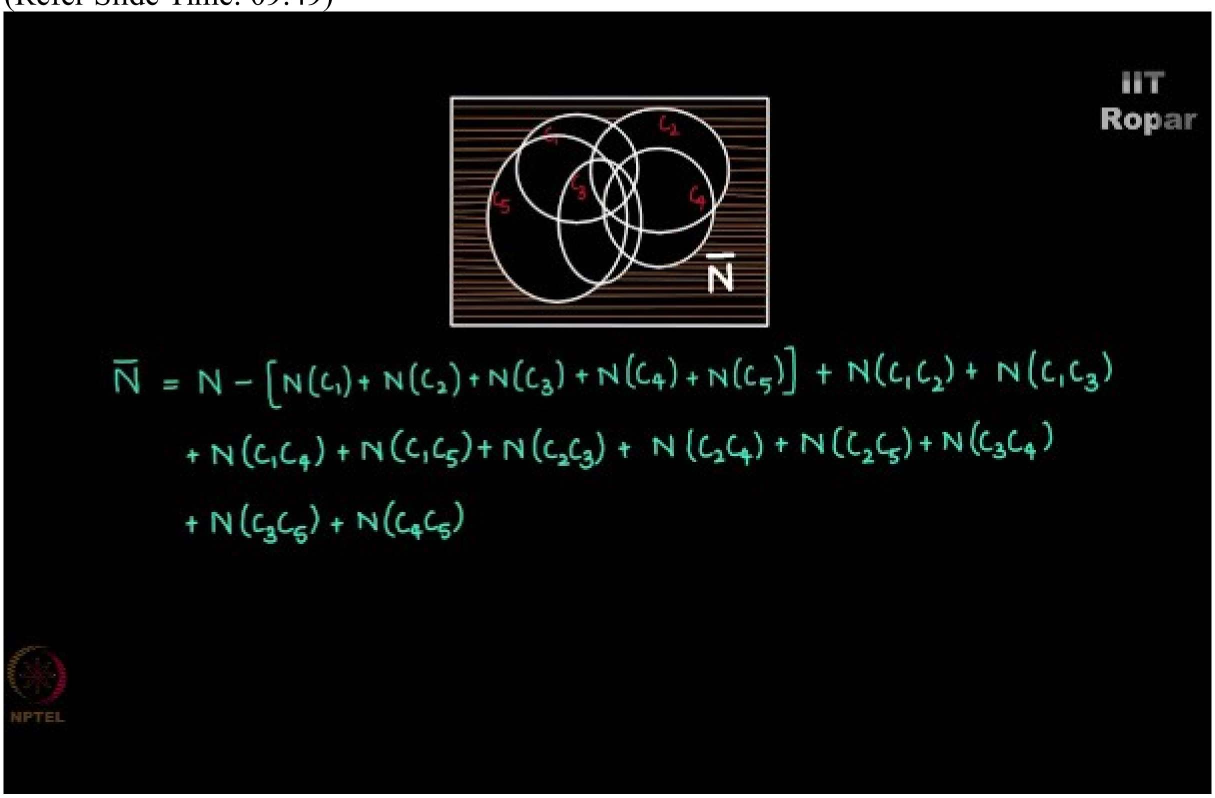

and here as you know N(C1, C2) stands for number of elements that are both in C1, C2, which means in the intersection, this is exactly like the formula of A union B union C's cardinality, just that the notation is different and using N of something, right, and then next will be a minus sign continuing N(C1,C2, C3) and so on up to N(C1, C2, C3), (C1, C2, C4), (C1, C2, C5) so on up to C3, C4, C5, and then again comes a plus sign N(C1, C2, C3, C4) + N of, you left out C5 here, then you'll leave out C4 here, which will be C1, C2, C3, C5 and then you leave out C3 which will be C1, C2, C4, C5 and so on, the last term would be N(C2, C3, C4, C5), right, and then again minus sign, and then the last term would be N(C1, C2, C3, C4 and C5), so this is the formula,

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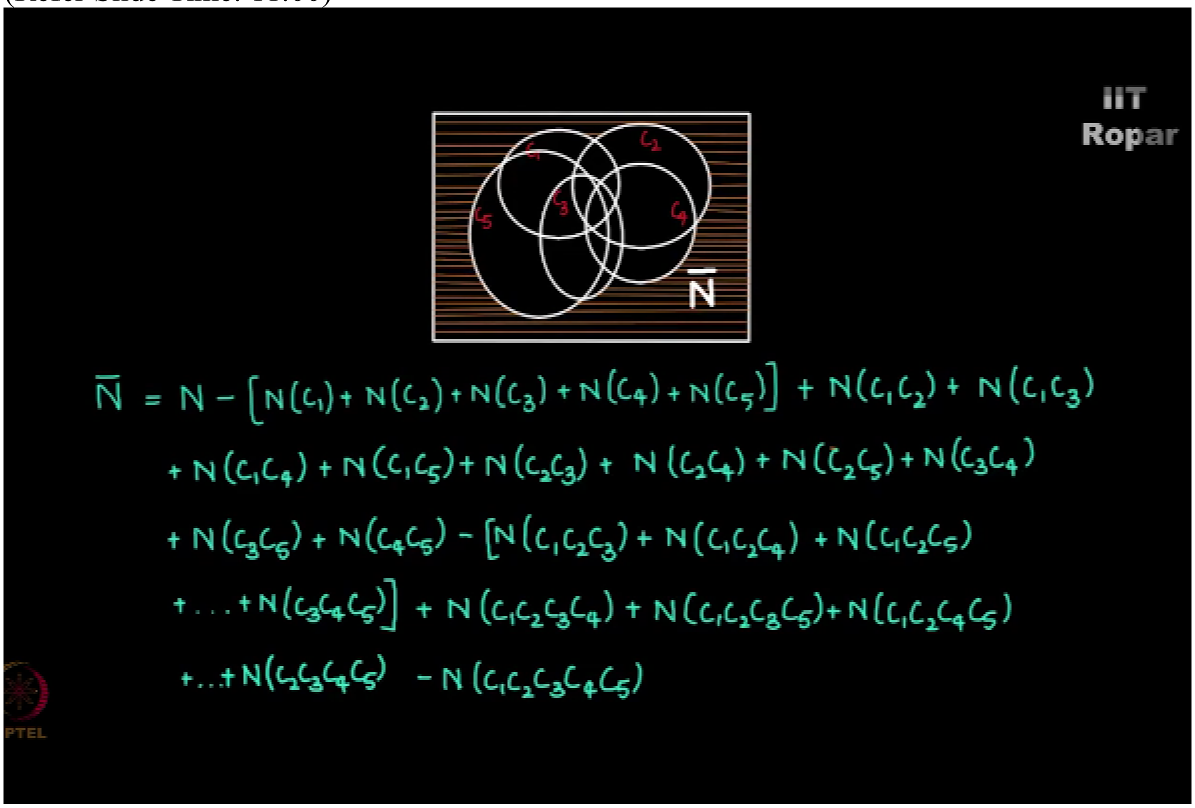

trust me which will help you compute the number of elements outside this vowels, you might be wondering why are we even computing entities outside the vowel, you see, you remember what was the question we started from, there are 5 towns and we are constructing all possible graphs on these 5 towns so that no vertex is left isolated, so all those graphs with vertex number 1 getting isolated is put into C1, all those graphs with vertex 2 getting isolated is put into C2, all those graphs with vertex C3 isolated is in C3, okay, N(C3) denotes the number of such graphs and a similarly N(C4), N(C5), okay.

N(C1,C2) means all those graphs where vertex 1 and 2 are isolated, okay, that's in the intersection, now this formula will actually give us the answer, pause and think, I'm computing the total possible graphs outside this vowels means what, if they're not in any of this vowels it means vertex 1 is not isolated, if it's not in C1, vertex 2 is not isolated if it's not in C2, vertex 3 is not isolated if it's not in C3 so on and so forth up to C5, and any entity outside this vowel means here is a graph where no vertex is isolated and that is precisely what we want to count, think, correct, so this formula will magically give us the answer.

Let us now try to compute the answer for this, so N bar as you see is N is all possible entities inside this rectangle, let us compute that, please note we are just being quite well organized, in the sense that we are trying to find what is N, N(C1), N(C2) and this is the formula that will work, I've promised you, you will get to know why in a few minutes I'll give a proof for this formula as well, but we are using this formula to simply solve the isolated vertex graph question, okay, so N will be the total possible graphs within the rectangle, and what is that? All possible graphs on 5 vertices, you all know what that is, that is 2 to the power of 5 choose 2 which is 1024 as you can see, it is 2 to the power of 10, 5 choose 2 is 5 x 4/2 which is 5 x 2 that is 10, 2 to the 10 is 1024, and then minus you have N(C1), N(C2), N(C3) up to 5, okay, so which is N(C1) is simply vertex 1 is isolated and all possible graphs in the rest of the 4 vertices, think about it, N(C1) what is it? Total number of graphs where vertex 1 is isolated, I'm not saying vertex 2 is not isolated, I'm only saying that vertex 1 is isolated which means you should compute all possible graphs which 2, 3, 4 and 5 in it, that is simply 2 to the power of 4 choose 2, right, which is 2 to the power of 6 basically, which is 2 to the power of 6, and N(C2) is also the same you see because N(C2) is vertex 2 is isolated, the rest can be anything , so 2 to the power of 4 choose 2 once again.

So on and so forth every single term here will be 2 to the power of 4 choose 2 which is actually 2 to the 6 which is 64, right, so at times 5 because there are 5 terms here N(C1), N(C2), N(C3), N(C4) and N(C5), so 2 to the 5 choose 2 will become 1024 – 5 times 64, right, plus look at the next thing, N(C1,C2) what is that? C1, C2 stands for vertex 1 and vertex 2 are isolated, which means of the 3 vertices you have 2 to the power of 3 choose 2 possible graphs that is precisely 8 graphs, and similarly with N(C1, C3) which is again 8, N(C1,C4) again 8, so on and so forth, there are 5 choose 2 terms here, so I'll write 5 choose 2 times 8, we know 5 choose 2 is 10, so 10 times 8-, this is an alternating symbol, -N(C1,C2,C3) which is a 5 choose 3, similarly times C1, C2, C3, three are isolated which means on 2 nodes you have only 2 possible graphs, rather 2 to the power of 1 possible graphs, so 5 choose 3 into 2, now think about it, correct, which will be 10 times 2, 5 choose 3 is 10, 10 times 2 and then plus N(C1,C2,C3,C4) like that you have 5 terms here, 5 terms into N(C1,C2, C3) in how many ways can all these 4 vertices be isolated? 1 is isolated, 2 is isolated, 3 is isolated, 4 is isolated, okay, this is only one way, even 5 is isolated there as you can see, so this is one way, and 1 1 1 1 there are several entities here, so it is simply 1, 5 times 1, you may note that all of them N(C1,C2, C3, C4), N(C1, C2, C3, C5), the next term and so on up to N(C2, C3,C4, C5) they all represent the same graph, you see as I told you this formula, the reason why we have minus plus, minus plus minus here as you can see in the formula of A union B union C we tend to over count and hence we remove some entities and then while removing we under count and then we add some entities, this stuff goes on and on, so the last term actually is about adding a few stuff and we over count and then we again subtract something from that, don't worry you will understand the formula very soon, so just go by the formula whatever it says, you will have a deeper understanding of it very soon.

Okay, 5 times 1 and the last term is N(C1,C2, C3, C4, C5) which is just one term and the number of graphs you can think of is again 1, so it is 1 into 1,
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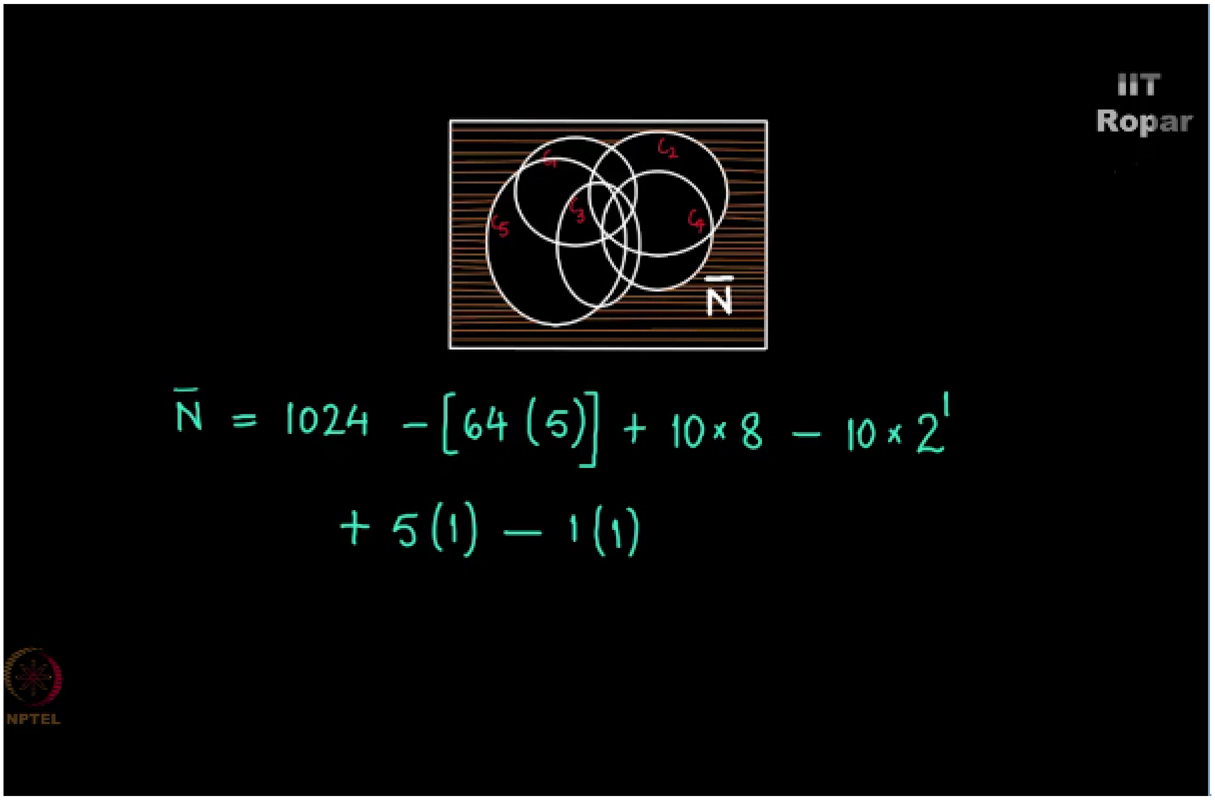

if you can see 1024 – 5 x 64 is 320 + 10 into 8 is 80, 10 times 2 is 20 + 5 – 1, this entire thing as you can see if you can add this up, it took me sometime, I got 768, some basic arithmetic,

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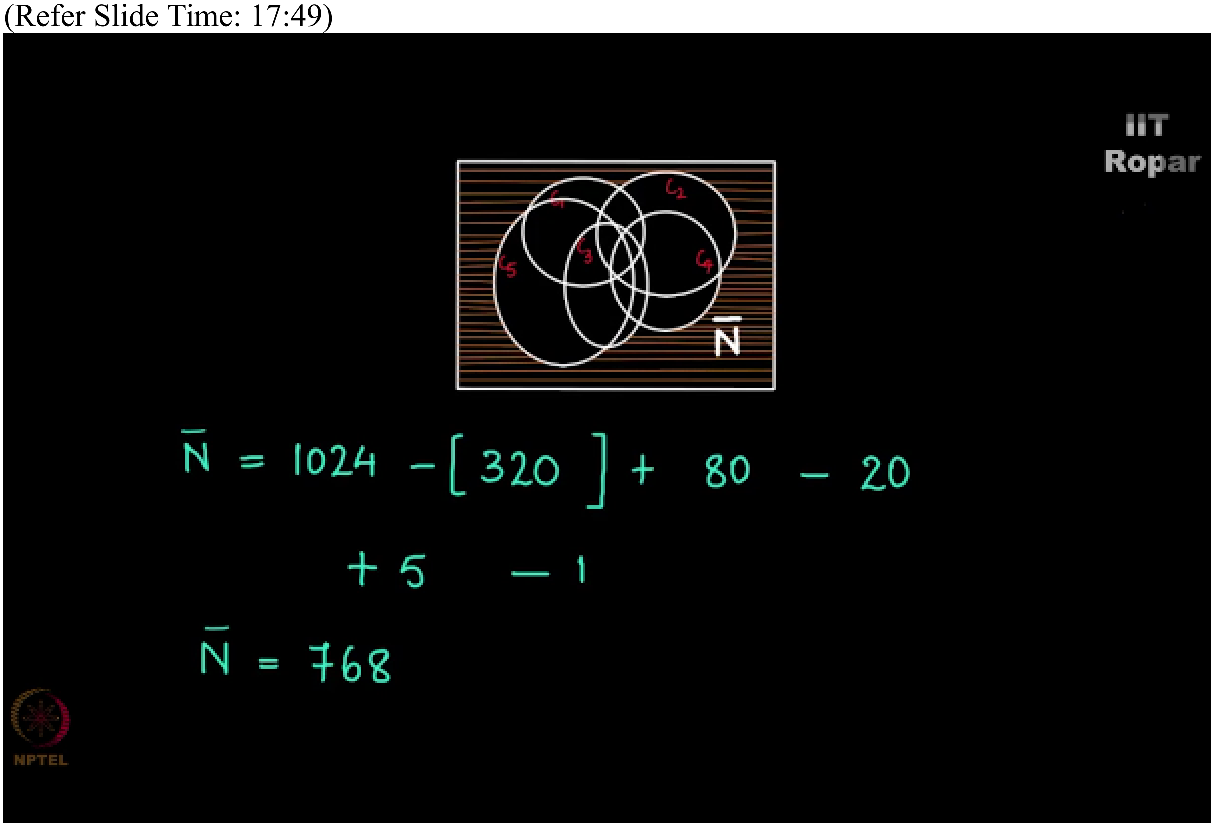

so what is this number? This is the total possible ways in which you can construct a graph on 5 nodes such that no vertex is left isolated,

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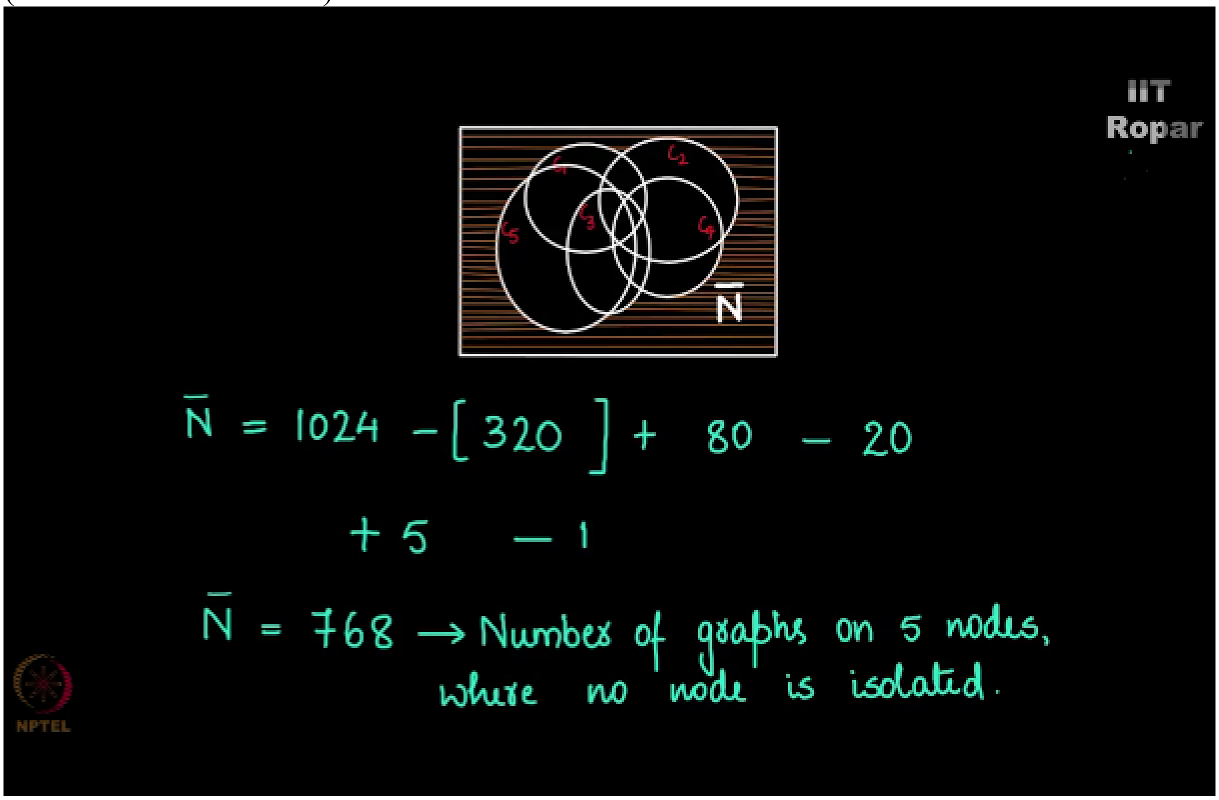

think, now what is left? The proof of this formula, magical formula that I used and that's coming next.

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