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Discrete Mathematics

Let Us Count

Examples of Catalan numbers

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We have discussed of what are Catalan numbers. Let us now see some examples of Catalan numbers. The first one parenthesis example. I hope you all know what is parenthesis. These are called as parenthesis. Now we will be given n pair of parenthesis and we would like to form a valid grouping of them. What do I mean by valid? Valid grouping means for every open parenthesis there must be a closed parenthesis. Now this is a valid parenthesis grouping. This is also valid but this is not valid because here is the open parenthesis. Here is another one. Here is a closed one for this so this is valid but again we have an open one and hence there is no closed parenthesis for this. Therefore, it is not valid.

Now the question is how many groupings are there for each value of n? Let us check. if n is 0 that is we do not have any parenthesis in how many ways can we have this? Just one way. This might seem to be a little confusing to you I say that there are no parenthesis and then I say one way but just bear with me at this place. When it is 0 we say that it's one way of having no parenthesis. When n is 1 what do I mean by one? We have one open and one closed parenthesis. So the only way to do this is this one and hence we have one way. When n is 2 we have two closed and two open. So the two possibilities are this is one and this is another one. So we have two ways of doing this.

When n is equal to three that is we have three closed and three open parenthesis. So all the possibilities goes like this. This is one. This is one. This is another one. This is another one. And this is the last one. So we have five ways of grouping three pairs of parenthesis. When n is equal to four, well this is a large number. You have 14 ways of doing this. You might want to stop here and do it all by yourself.

So we see that the sequence is 1, 1, 2, 5, 14 and so on. So here we see that Catalan numbers hold true in the case of grouping of parenthesis. The next one. This is called polygon triangulation. By triangulation we mean constructing triangles inside the polygon in a particular way. Let me tell you how. If this is a polygon I mean constructing triangles this way. We do not want to construct triangles like this. We don't want two lines to cross over each other. So here I'll directly start from n is equal to 3, n is equal to 3 we get a triangle itself. This is just one way of getting a triangle. When n is equal to 4 what do I mean by this I have a square and how do I triangulate it? I have to construct triangles so there are two possibilities of doing it and two ways. Now when n is equal to 5 that is I have a five sided polygon. This one. Now how can I triangulate this? This is one. This is another one. This is another one. Like this and this is the last one. So I have exhausted all the possibilities. So we have five ways of triangulating a five sided polygon. It can be done in 14 ways for n is equal to 6. So we get the sequence 1, 1, 2, 5, 14 and so on. So this gives the Catalan numbers.

The next one. Handshakes across the table. This is interesting if 2n people are seated around a round table please observe I am going to take only even number of people, in how many ways can they all simultaneously shake hands with another person at the table such that nobody crosses arms across each other. So we don't want to cross handshake. So when n is equal to 1 what do I mean by this there is one handshake which means there are two people here and there is one. So this can be done only in one way. When n is equal to 2 I have 2n that is 4 people seated like this and they can shake their hands either like or like this. So this can be done in two ways. Now when n is 3 it means there are six people. They are seated like this and now they can handshake this way, or this way, or this way, this way, or this way. So there are five ways in which six people can shake their hands without crossing. Now when n is equal to four there are eight people it becomes slightly complicated and there are 14 ways of doing this and so on as you keep increasing you see that the number increases. So we see that 1, 1, 2, 5, 14 so this again here when n is equal to 0 we have 0 people and 0 people can shake their hands only in one way and hence we see the Catalan numbers here.

The next example is binary trees. Do not get confused as I say binary trees. By tree I mean that consider this to be a dot and there are some lines coming out of it like this. I can draw some lines connected to some other dots. So this I will call it as a tree for now. This is a root and it is branching out into two branches, or let me say two leaves. So when n is equal to 0 please note here we are going to consider those dots which have two branches. So when n is 0 that means that there are no branches. There is only one node and this can be done in one way. When n is 1 that is there is only one node which is having two branches and this also can be done in one way. When n is 2 what does it mean that there are two nodes having two branches each. So I can draw it like this and here you see these are the two nodes and another way is like this and these are the two nodes which have two branches each. So I can when n is 2 it can be done in two ways. Now when n is 3 what does it mean that there are 3 roots which have 2 branches each. So this can be drawn in all these ways. So there are 5 ways of drawing these binary trees. Please note why did we use the word binary because we are here concerned only about two branches coming out. So as we increase n we see that the sequence becomes 1, 1, 2, 5, 14, 42 etc. You might want to stop the video here and watch the binary trees again.

So these were some of the examples of Catalan numbers. Well there are several many. I hope it was interesting. Please post all your questions and queries, doubts on the discussion forum.

We have now come to the end of the first chapter. We learned several elementary techniques on counting. These are going to be the builder blocks for more advanced techniques on counting. We will now continue with the next chapter which is set theory.

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