NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Discrete Mathematics
Graph Theory – 3 &
Generating Functions

Picking 7 balls - The creative way

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You saw that there were 36 ways of picking 7 balls from this big bunch of red, blue, green balls, right, there is no restriction here, you were not required to pick at least one red ball, at least 1 blue ball, like the previous question that we saw while before, right, you could pick all 7 red, or all 7 green, or all 7 blue, or 1 red, 2 blue, and 4 green and so on, right, you observe it was 9 choose 7, or 9 choose 2 ways which happens to be 36.
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Now I want to try solving this problem using some complicated mathematics, I might sound very crazy to invoke some very spooky mathematics to solve something as simple as what was explained to you right now by our friend Soujanya, but I as a professor we'll try to pretend to be a teacher who will complicate the stuff unnecessarily, you will realize once I proceed further

explaining things that sound very not so require, look at this, so let me look at the expansion of 1+X whole to the N, by the way our motive would be to solve the problem of picking 7 balls from R, G, B in a very different way, what is expansion of 1+X whole to the N, N choose 0 + N choose 1 and X+ N choose 2 times X square and so on up to N choose N times X to the N.
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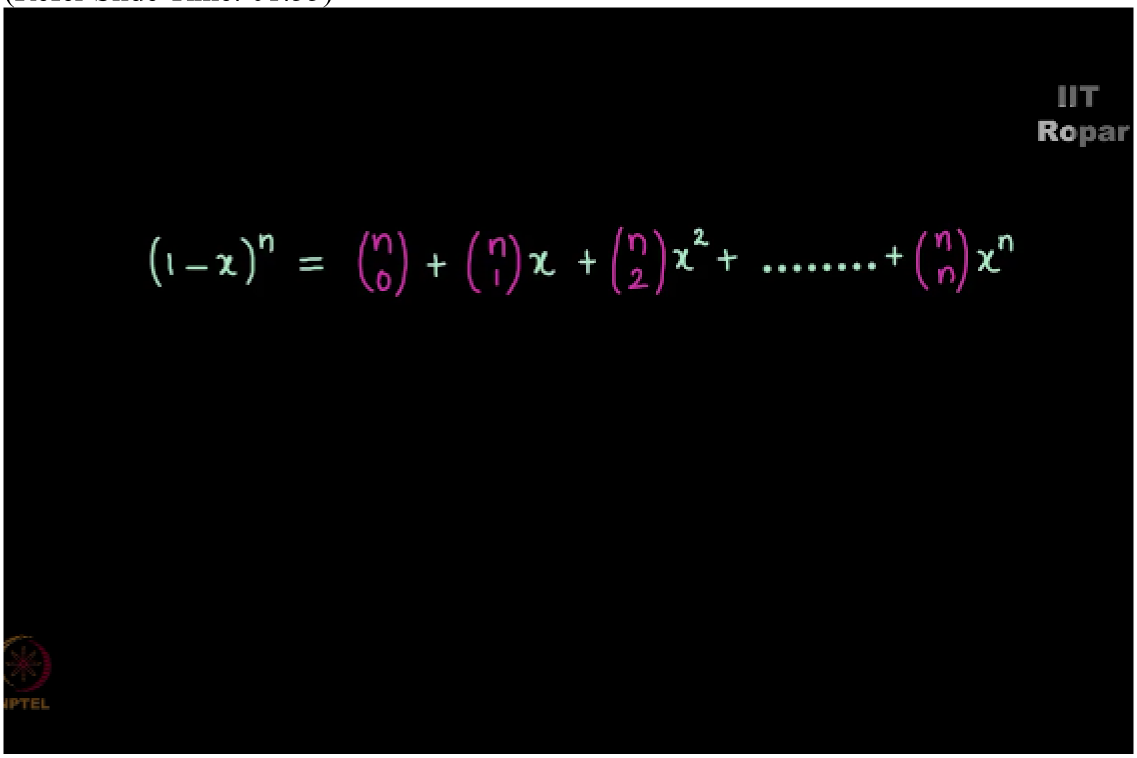

If I were to put –X in place of X we will get N choose 0 – N choose 1 times X + N choose 2 times X square and so on in general the rth term will be N choose R times X to the R, but note the minus symbol here, it will carry the parity of R, what do I mean by this? I'm saying something very simple that you see the sign before N choose 1 times X was minus, because it alternates between plus and minus you see,
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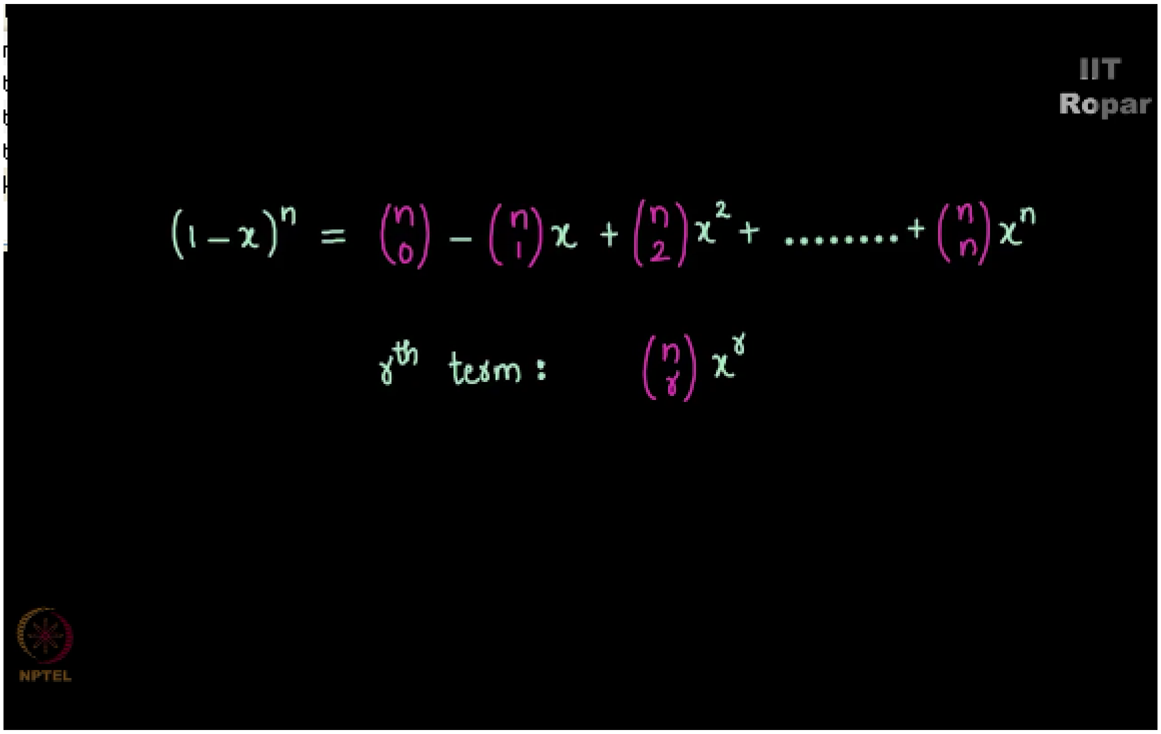

the reason is very simple, in place of X you are putting –X, right, if it's X square the minus goes away, if it's X cube the minus comes there, think about it, right, so it alternates and X to the R, X square, X square if you see 2 is even, so it is +X to the 1, 1 is odd so it is –X cube, 3 is odd so it is minus, so –N choose 3 times X cube and so on, and hence X to the R will have the coefficient N choose R the pattern, and then the sign will be -1 if R is odd, +1 if R is even, this is very smart way in which mathematician denote this, they simply say -1 whole to the power of R, R is odd this becomes 1, if R is odd it becomes -1 if R is even it becomes +1, so on up to N choose N into X to the M, and again the symbol will be -1 to the N here, it could end with - or + based on whether N is odd or even, perfect, so 1-X to the N happens to be this, the expansion is going to be this.

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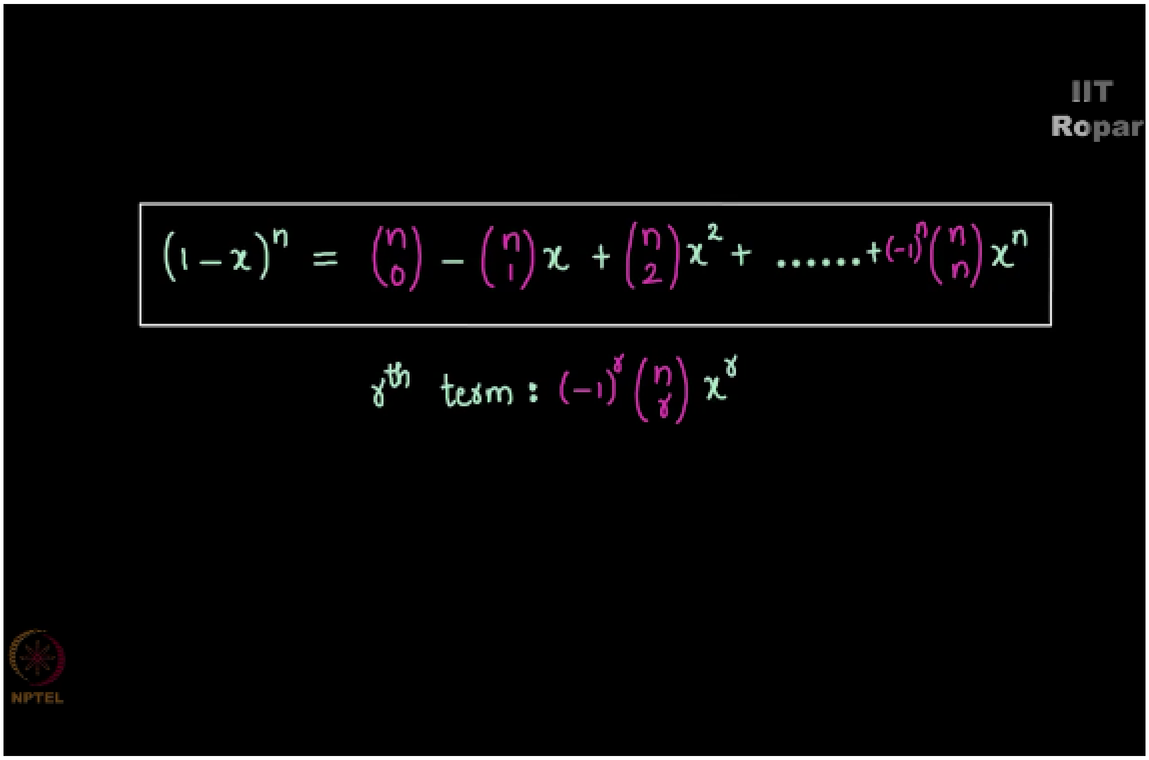

Let me now expand my N choose R's here, 1 – X whole to the N is 1 – N times X, N choose 1 becomes N you see, N choose 2 becomes N(N-1/2), so 1-NX + N (N-1/2) times X square + so on up to, so what is N choose R? N(N-1)(N-2) so on up to (N-R+1), why is that?

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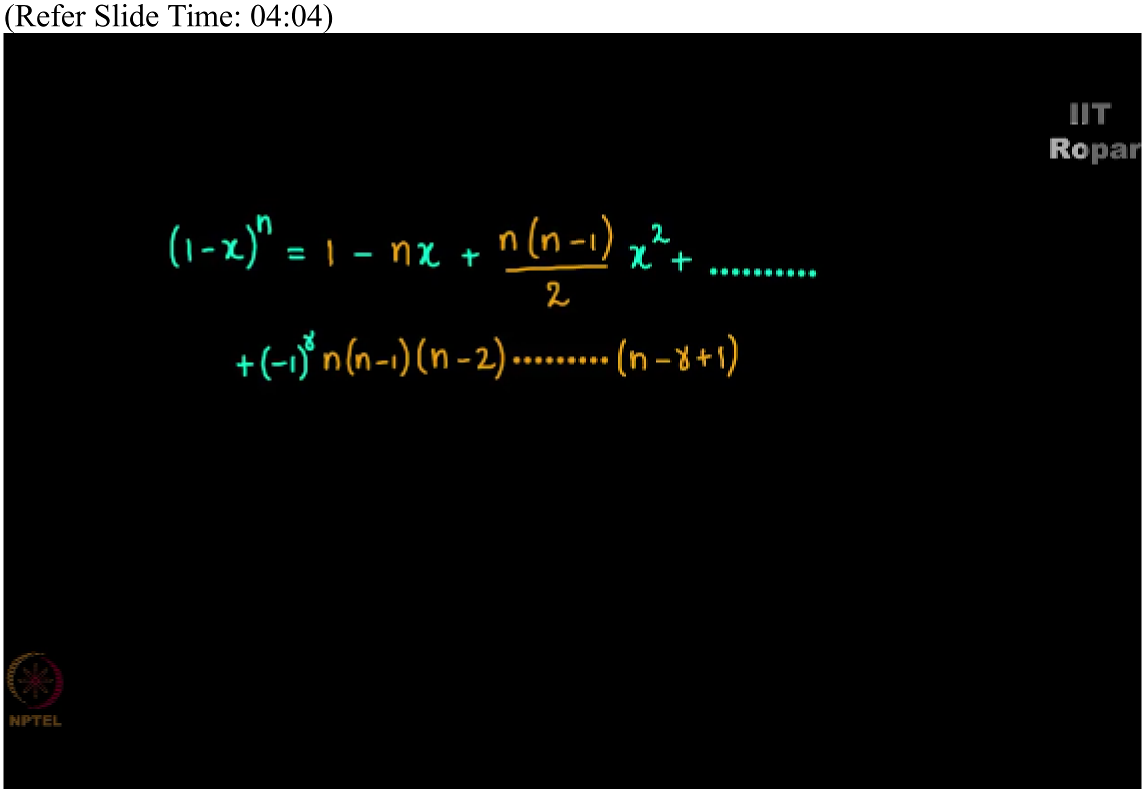

There are R-1 term, rather R terms here, so it will be N – (R-1) which makes it N-R+1, think about it patiently you will get it, let me not explain everything here, so that intuit is to spoon feeding, right, so you should be able to look through all by yourself, so N(N-1) up to (N-R+1)/R factorial, and that's the coefficient of X to the R, correct in general, and so on so on and so forth up to N choose N times X to the N, correct, which is one times X to the N here.
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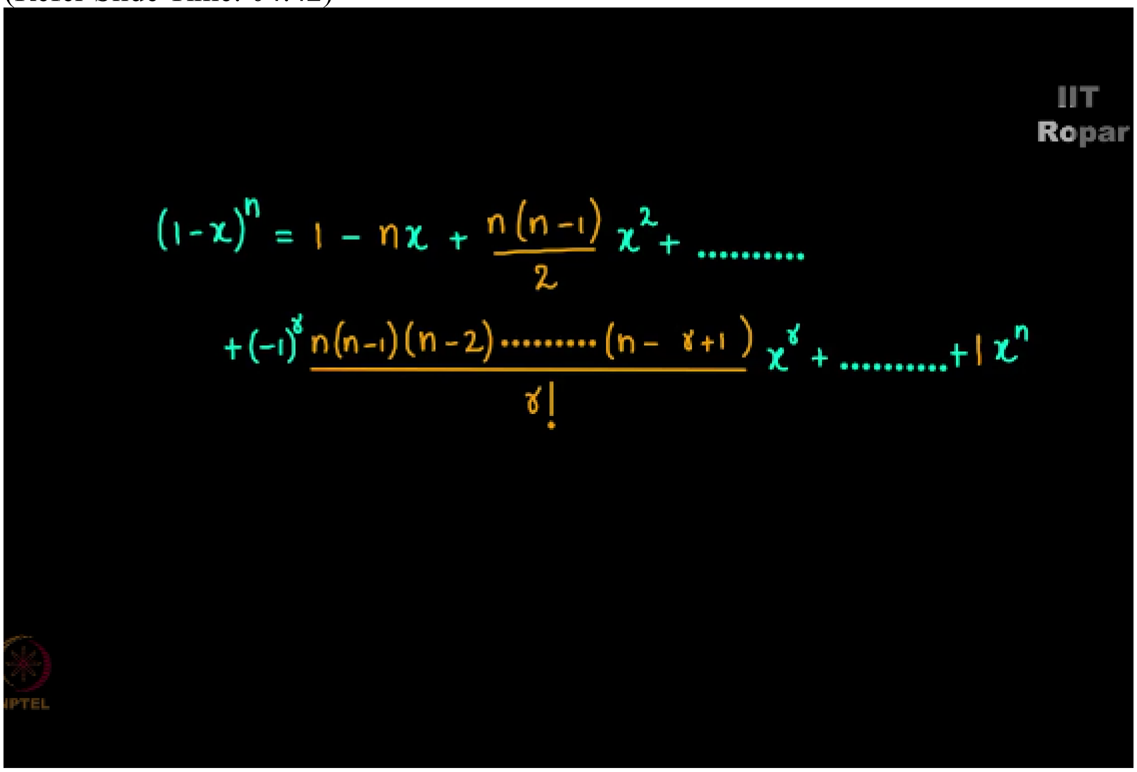

So what have I done in the second step? I have simply expanded my N choose R's, this is the expansion of 1-X whole to the power of N, let me now ask this question, what is the expansion of 1-X whole to the power of –N? This is the simple math, simply put –N in place of +N, please note I'm making a small conceptual error here if any mathematician sees this he will have an objection because I write N choose R's here, what do you mean by –N choose R, when you are talking about 1-X whole to the power of –N, but then I have expanded N choose R, whatever is N here I can try putting –N here, this is actually a very deep concept in mathematics, I wish not to take you people to an advance level in this course, so without any loss of abstraction or rigor we can go ahead and simply put N = -N and see what happens, you will be startle to see that this takes it to a brand new level and a brand new direction.

So N = -N you see 1 – of N becomes –N here times X + N becomes –N here, -N into -N -1/2 factorial into X square so on so forth up to N(N-1) (N-R) + 1/R factorial remember, it had a sign -1 to the R before it, correct, so -1 to the R, N becomes –N, N-1 becomes –N -1, N-2 becomes –N-2 so on, you see you can pluck out the minus sign there up to –N –R +1/R factorial into X to the R, if I remove -1 from everywhere what you get is -1 to the R and there are R terms here, observe carefully, N -N -1, -N –N-1, -N -2 up to –N, minus of R-1 observe that,
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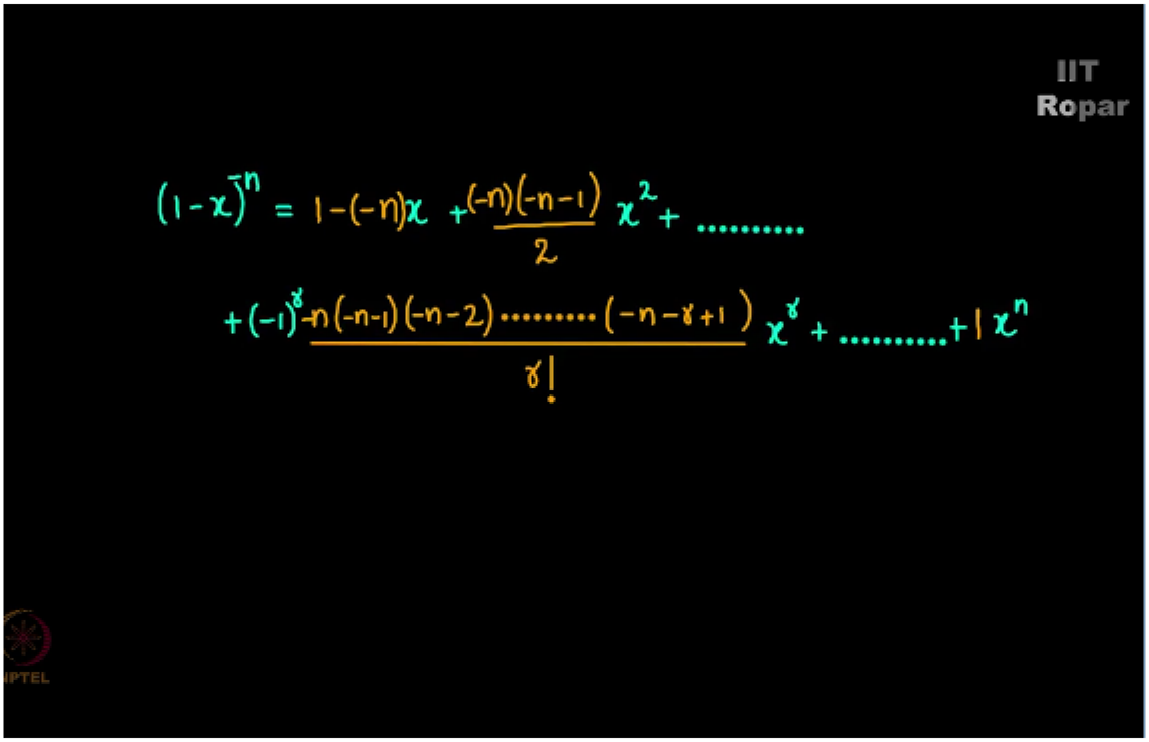

correct, so that becomes R+1, so you pluck out -1, -1, -1, all times that becomes -1 to the R again, you already have -1 to the R, that makes it -1 to the 2R basic observation,

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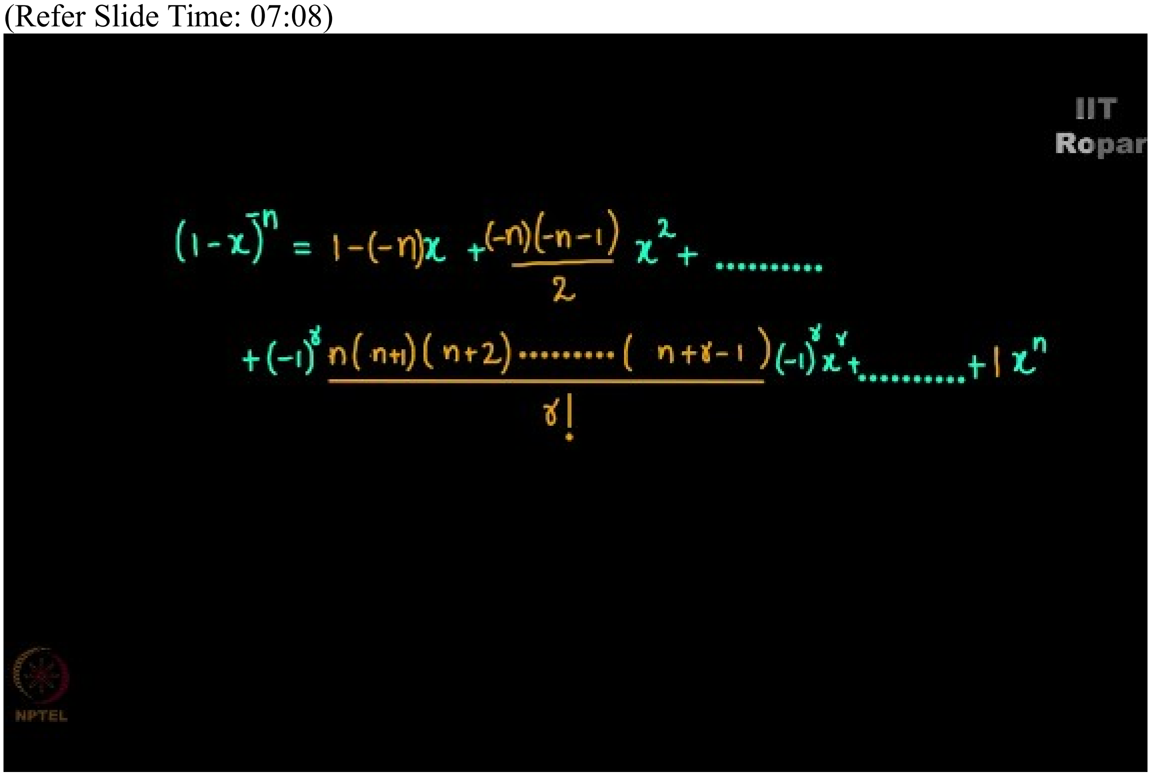

-1 to the power of 2R, 2R is even, so -1 to the even will be a +1 which means throughout we are going to have +1, +1 everywhere, because in general for R you are getting +1, so +1 times

N(N+1)(N+2)(N+3) up to N+ what is that? R-1, N+R -1, why you are plucked out the -1 there, right.

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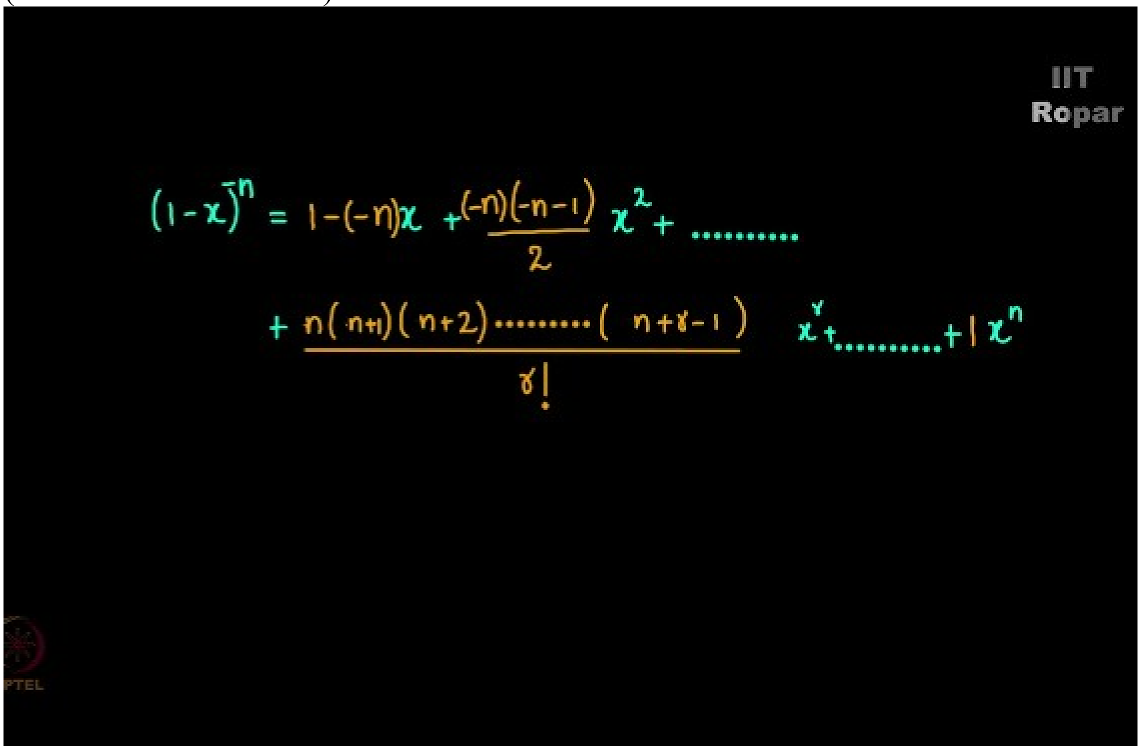

Now this is, observe carefully, initially we had you see N(N-1)(N-2) numbers were decreasing, now when you put N = -N, surprisingly now the numbers are increasing, you see N(N+1)(N+2)(N+3) up to N+R-1, right, observe this carefully again I'll not do any spoon feeding, you should be able to realize that this is nothing else, but N+R -1 choose R, isn't it? I repeat look at this term N(N+1)(N+2) up to N+R -1/R factorial, this is nothing else but N+R-1

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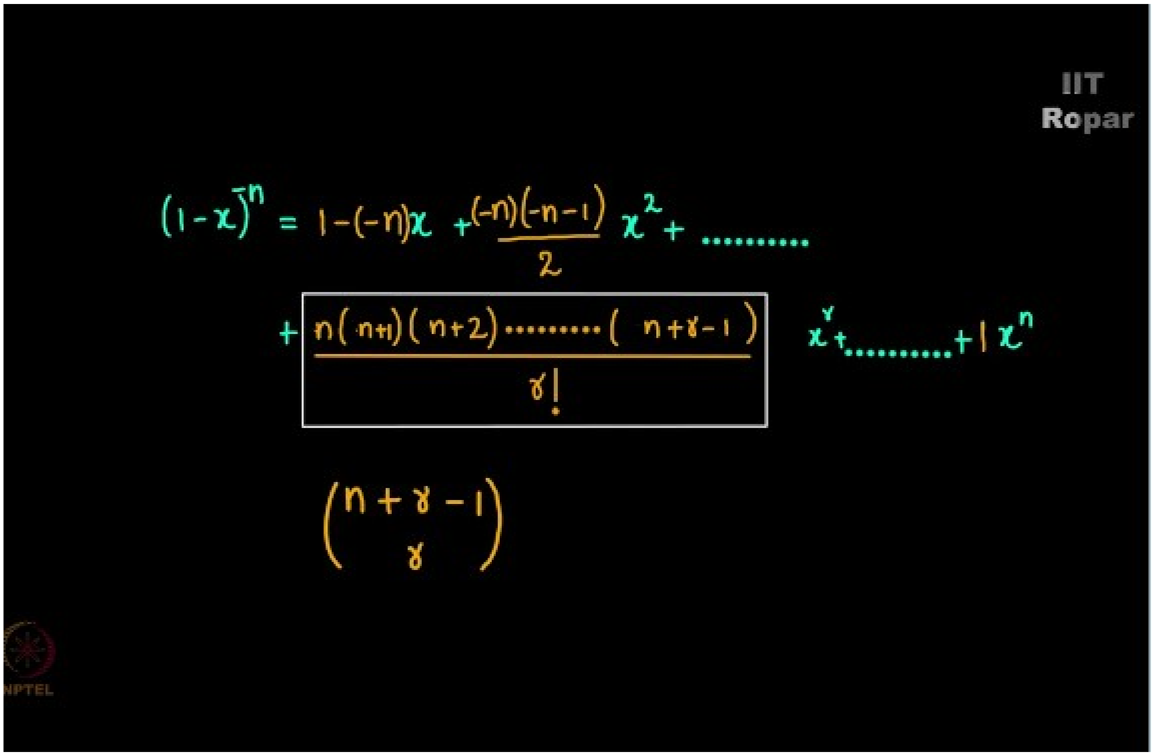

choose R, the coefficient of X to the R here is N(N+1)(N+2)(N+3) up to N+R-1/R factorial and so on up to X to the N of course, so what do I observe? When you put N = -N your N choose R becomes basically this, N(N+1) up to N+R -1/R factorial which is also as observed N+R-1 choose R which is the coefficient of X to the R bracket this, I mean box this, the coefficient of

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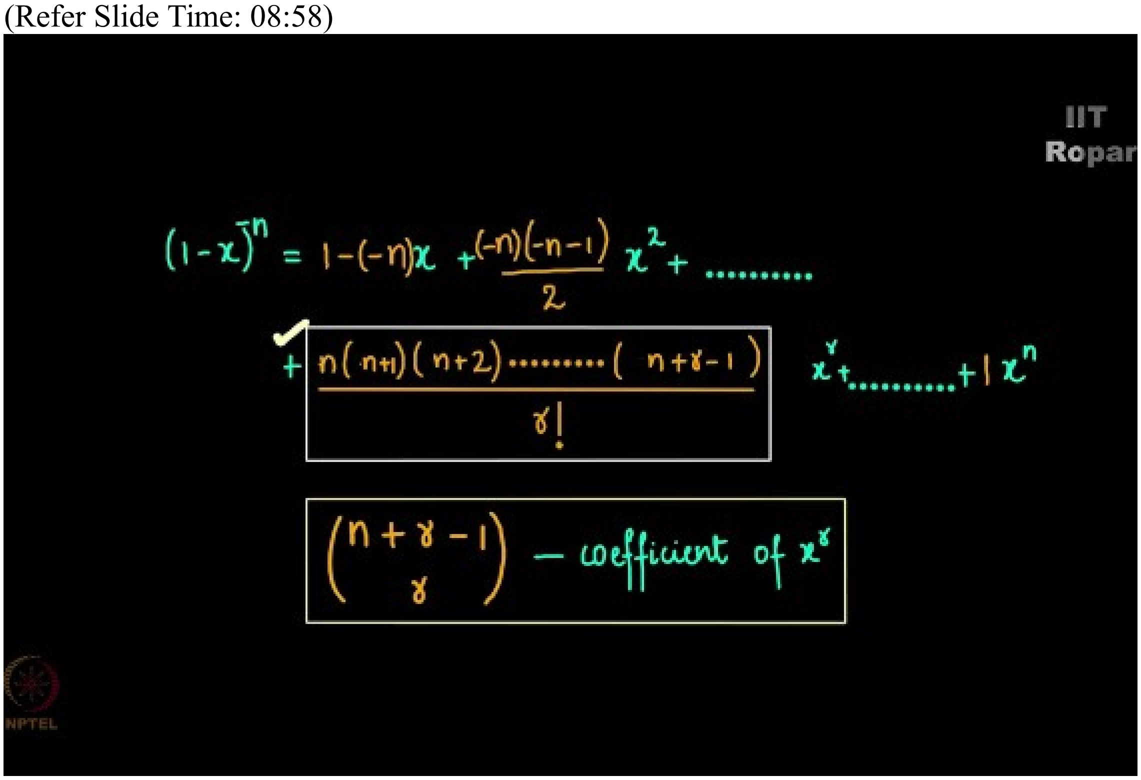

X to the R happens to be N+R-1 choose R, pause, stair at it, think, what have we arrived at right now, let us separate everything else and only look at this in the full screen, the coefficient of X to the R in the expansion of 1-X whole to the –N is N+R-1 choose R.
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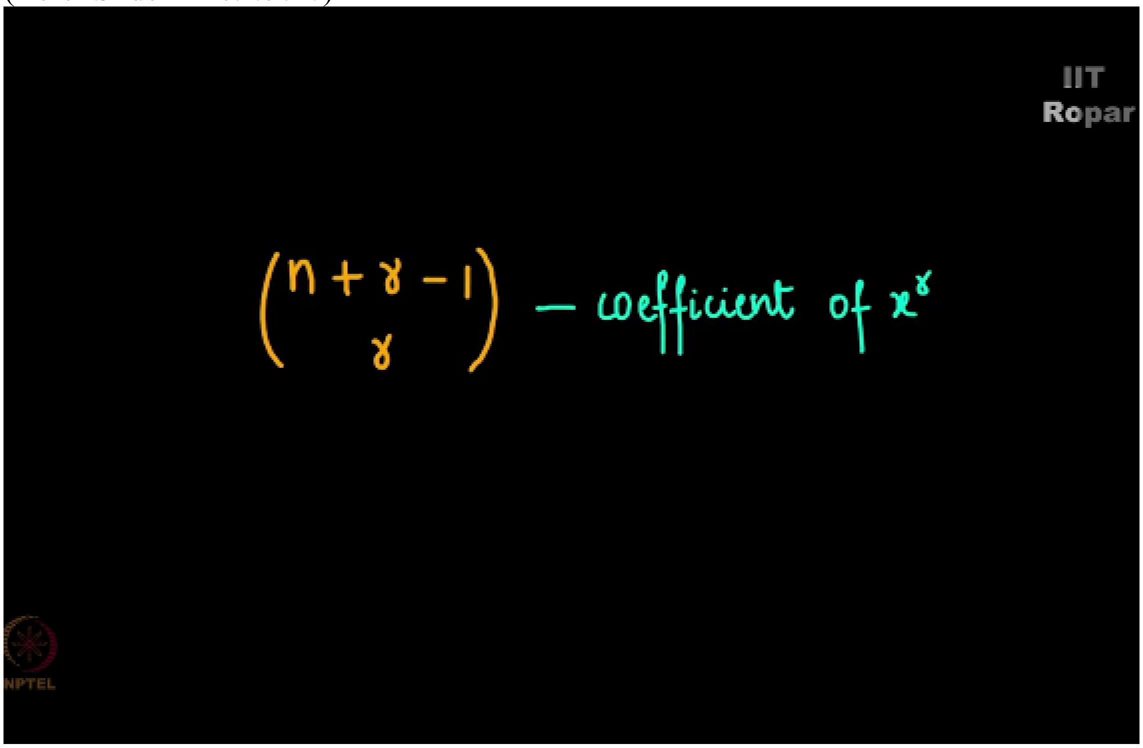

Now keep this aside, let us look at the red, blue, green question in a brand new light, I can take no red, or 1 red, or 2 reds or 3 reds, or 4, 5, 6, 7 reds, of course I cannot take 8 reds you see, there is no way in which I can pick 7 balls by picking 8 reds, you will be left with no choice then, right, so we should stop X to the 7, same is the case with all the colors red, blue, and green,
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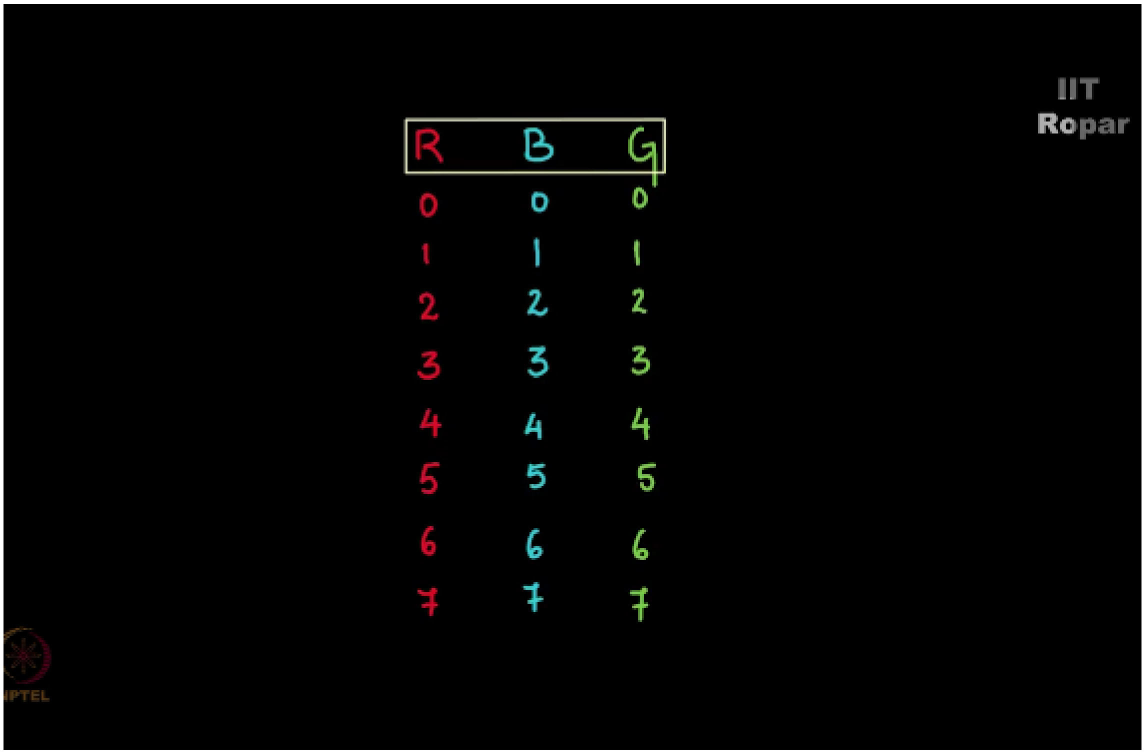

so what is the total possible ways in which you can pick 7 balls so that you pick all red, blue, green balls is simply the coefficient of X to the 7 in the expansion of 1 + X+ X square + X cube + X to the 4, X to the 5, X to the 6, X to the 7 meant for the red ball times 1 + X+ X square + X cube + X to the 4 + so on up to the X to the 7 for, first one was for red, second one is for blue, third one will be for green, and in this expansion what is a coefficient of X to the 7 is what we ask.

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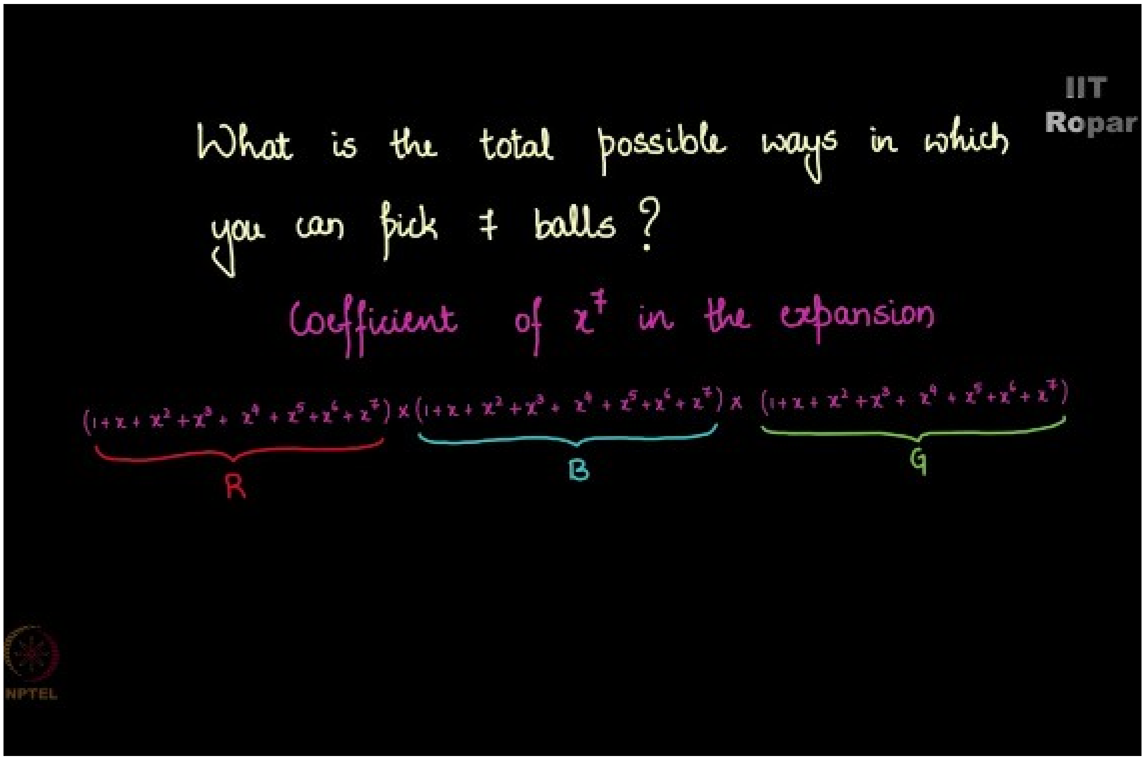

Observe a very small trick but a very powerful one, you see we are going to invoke some non-trivial math by doing a small trick here, I can in fact right X to the 8 in the reds polynomial here, right, why should I stop in X to the 7, or you may say when will we pick X to the 8, it's nonsensical, don't worry you are asking the question, what is the coefficient of X to the 7 in the expansion of these 3 things, X to the 8 will get multiplied by something else and give a bigger power of X which will never be used, so why care? So what I'll do is I'll safely include X to the 8, X to the 9 up to infinity, not just for red, but for blue and green as well,

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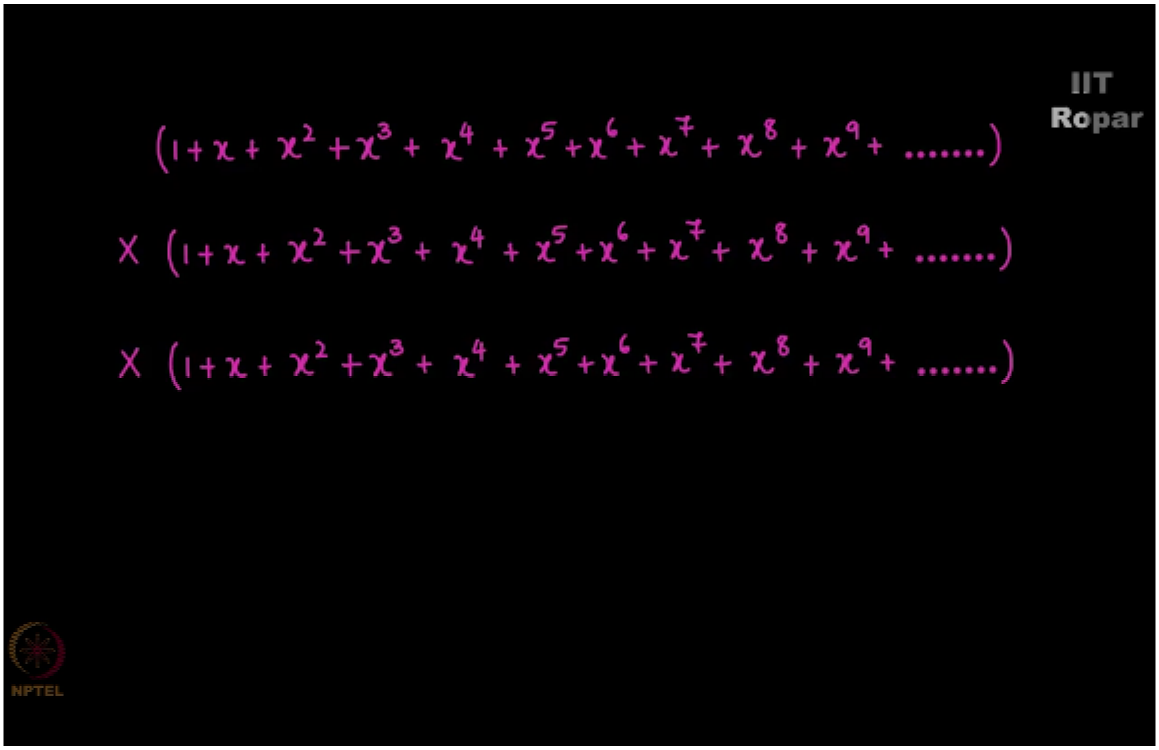

do you see what I have done just now, I've simply taken 1 + X + X square up to infinity, 1 + X + X square up to infinity again 1 + X + X square + X cube up to infinity and I'm multiplying these three things and I'm asking you what is the coefficient of X to the 7 here, pause and observe what just happened, now what is this? This is nothing else but 1 over 1-X, same is the case with this as well, and put together this is 1 over 1-X whole cube,

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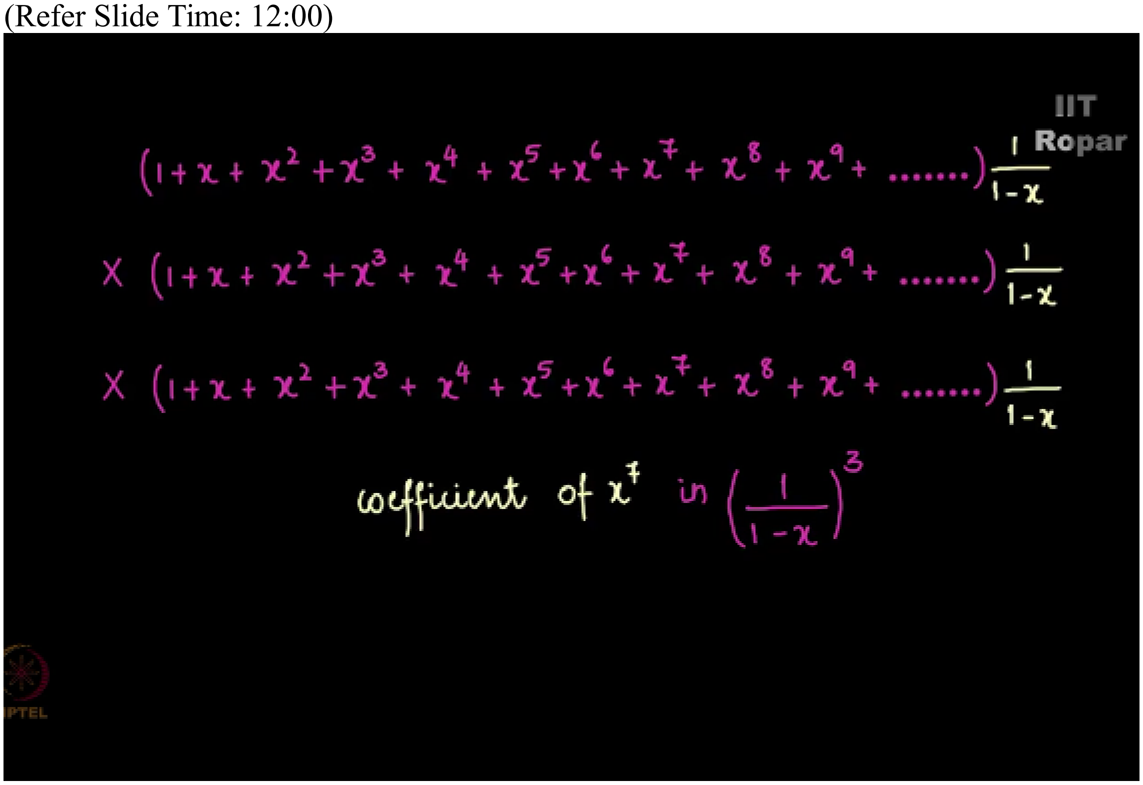

and they are asking what is the coefficient of X to the 7 in the expansion and that's the answer for our red, blue, green, question, correct, but, but, but, but, but we just now saw what this is, we saw what is the coefficient of X to the R in the expansion of 1-X whole to the N, 1-X whole to the –N, I'm sorry, right, so now i have 1-X whole to the power of -3, but I'm asking this question what is the coefficient of X to the 7 here, what will be the answer, let me think, the answer is nothing else but simply plug in this formula or N+R-1 choose R, what is N here? 3, what is R here? 7, think through, correct, so 3 + 7 – 1 which is 9, and choose R is again 7, 9 choose 7 which is 9 choose 2, which is 36,

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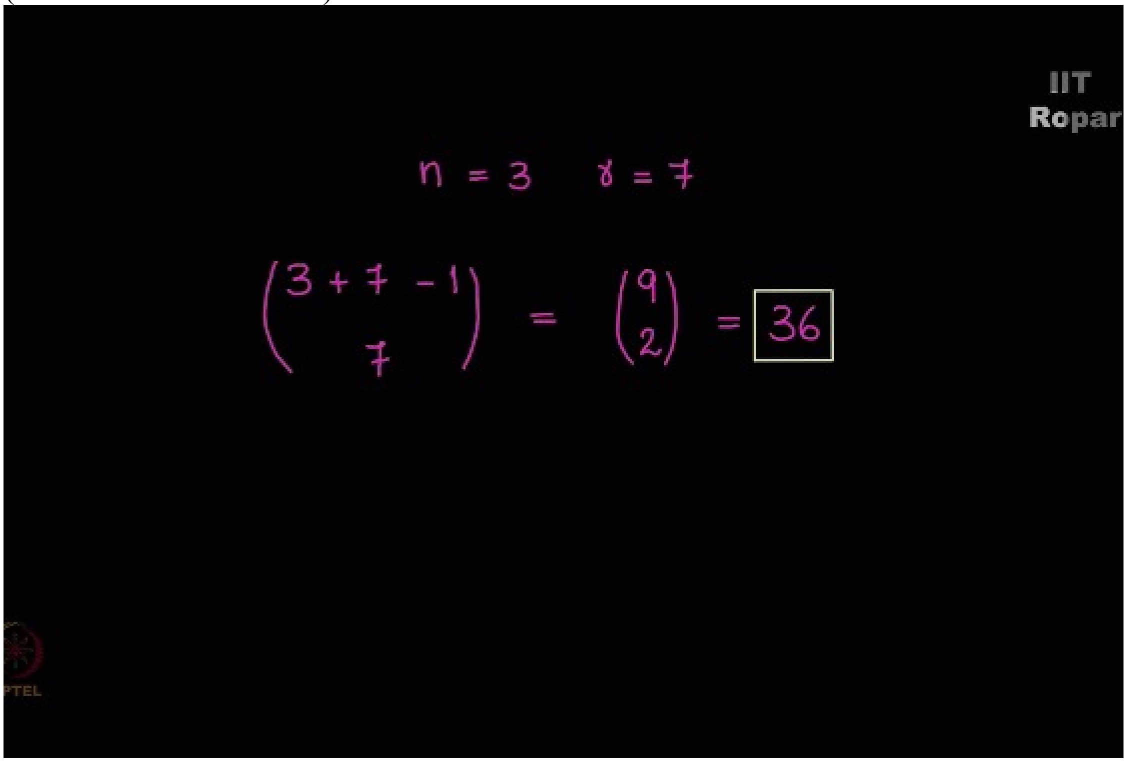

exactly the same answer that we saw just now, but the math was completely different, so do you observe how? A very simple solution, can we made complicated by using very complicated mathematics, so I'm yet to answer, why I invoke this kind of a spooky math to explain something that can be easily explain with sticks and cups, that our friend explain just now, there is a reason, let us discuss this over the next video.

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