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Discrete Mathematics
Graph Theory – 2

Proof for even degree implies graph is Eulerian

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Okay, let us now show the general theorem that, look at the statement, any graph with all degrees even, any graph with all degrees even will have this property that it will have an Eulerian circuit in it,
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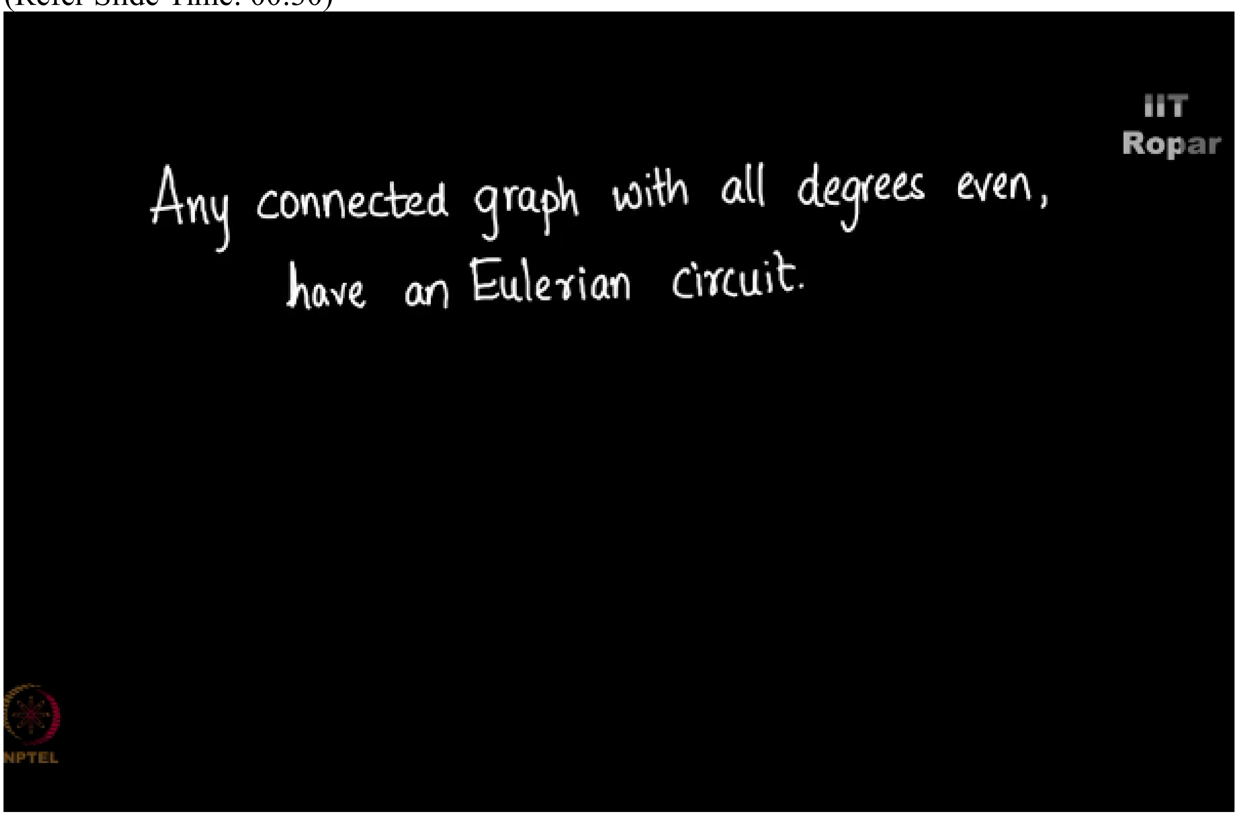

okay, any graph with Eulerian circuit is called simply Eulerian graph, right, so I'm saying is any graph with all the vertices having even degree will have this property that there will be an Eulerian circuit.

Now what should we show? You give me a graph and you ensure that degree of every node is even, then you need not show me explicitly an Eulerian circuit there, it is understood there is 1 there,

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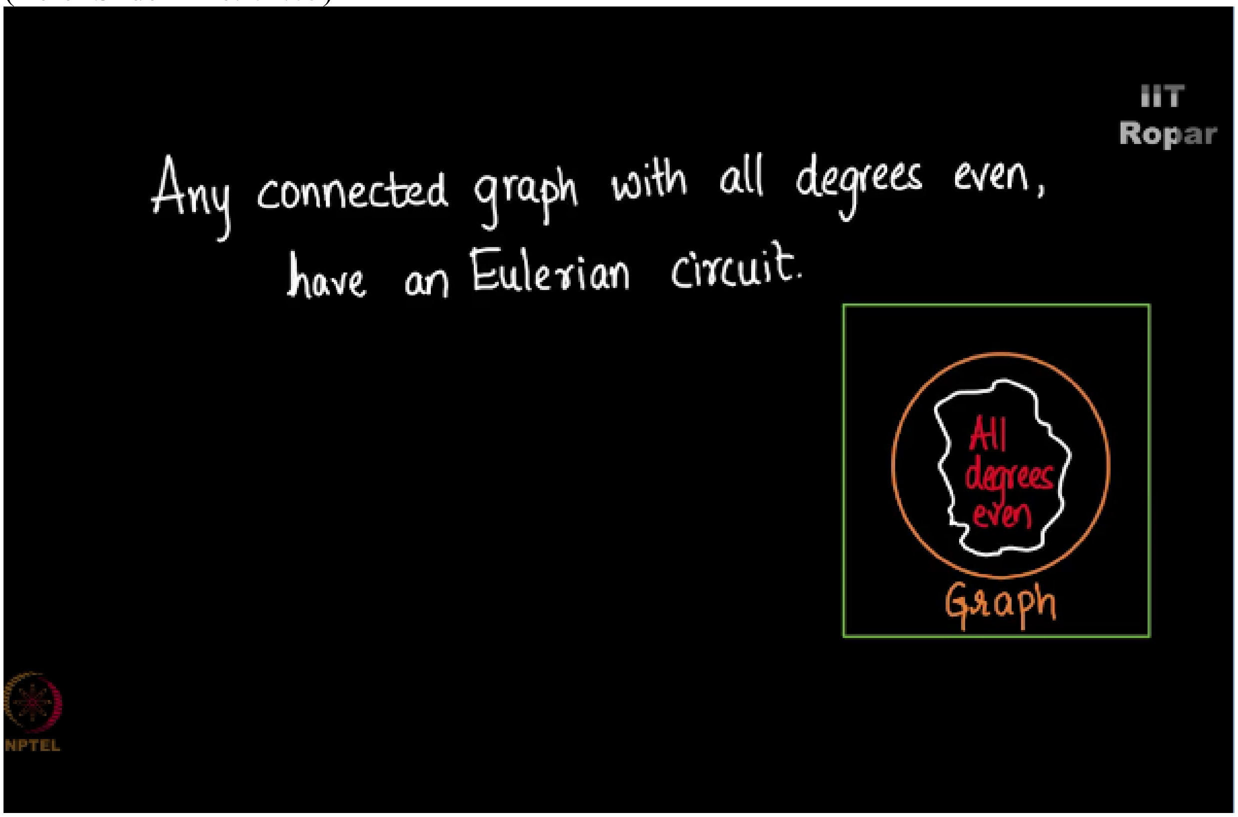

okay, let's see how this is true? Let us go slowly, now the proof involves a very, very, very settle strategy that requires some explanation with some real life and energy, okay, I'm going to tell you all a story to begin with, most of you would have studied this in your school days if you are from India, so there is this story of Tenali Rama who was a minister in the court of this king called Krishnadevaraya,

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he was both humorous as well as intelligent, and Krishnadevaraya would ask Tenali Rama to solve his problems, almost all the time, and Tenali Rama would come out with very ingenious and very comical ways of solving Krishnadevaraya's problems.

There was one fine morning where Krishnadevaraya sees two woman enter his court with a small kid,

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fighting that this kid is mine, the kid is so small and in fact that Krishnadevaraya could not ask the kid who is your mom, as the kid couldn't speak, given that it is an infant, Krishnadevaraya as perplexed, he doesn't know whose kid this, and who is the mom of the kid, and so as an always he says Tenali Raman please solve this problem, and Tenali Rama as an always comes

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out with a very neat way of finding out who is the mom, he goes sweetly towards the kid in the pretest of cuddling the kid, and then pinches the thighs of the kid very hard, so you all know what happens when you pinch an infant, the infant starts crying incessantly, and the moment the infant starts crying one of the two moms starts crying as well and comes and hugs the kids first, and the second mom has absolutely no reaction,
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and from this Tenali Rama concludes that,
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here goes the answer Krishnadevaraya you all now know who is the real mom.

Now let's analyze the story, Tenali Rama brings in the noise in this scene where he goes an unnecessarily, seemingly unnecessarily pinches the kid,

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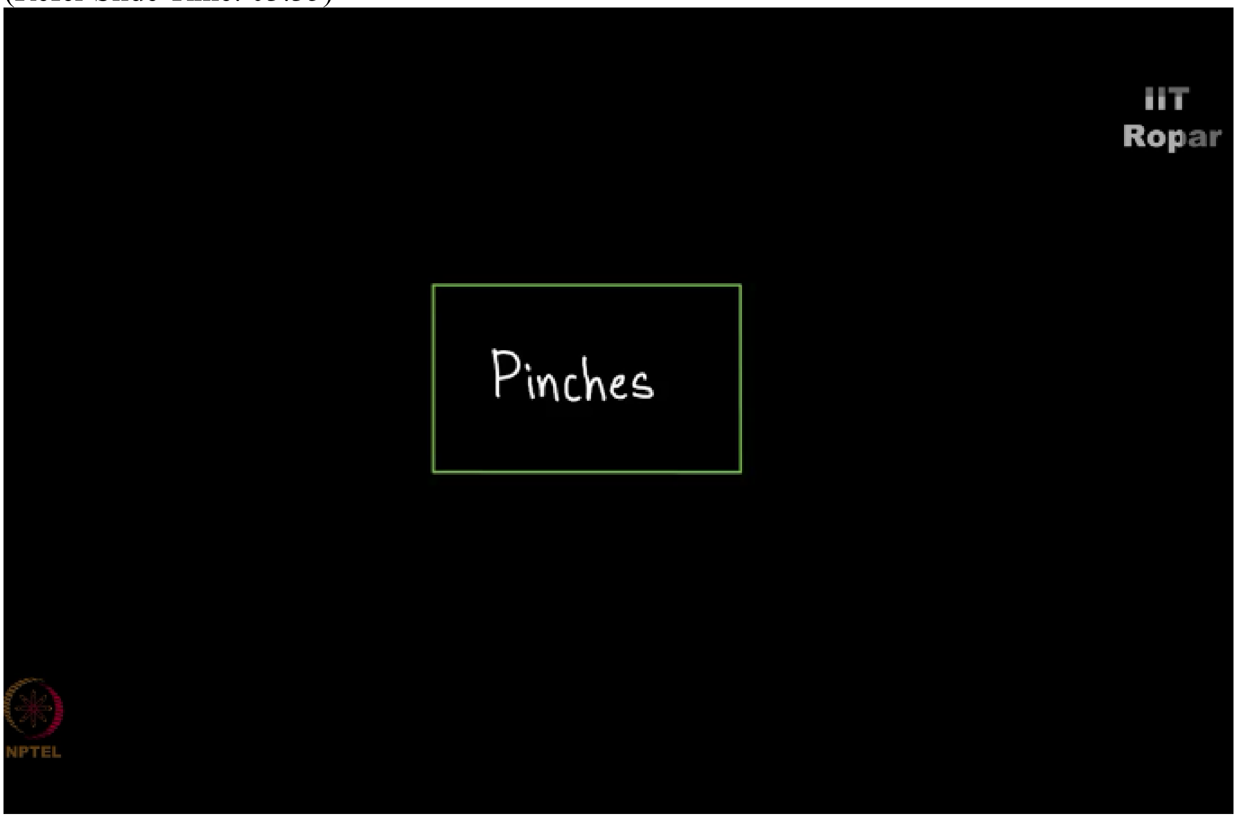

just to ensure that this becomes a Litmus test to understand who is the real mom, and then after pinching he understands that it's a damaged to, it's being violent to the kid and then he even massages the kid or _3:53_ the kid and ensures that the kid feels better, right so he brought in a factor and then removed that factor, correct, to understand what is what.

Similarly in our proof, I'm sorry for having told you a very long story, but I could not find a good analogy because this proof involves a small technique which is very confusing, and the technique when it comes very close to this story strategy where you adding a new noise which is seemingly irrelevant,

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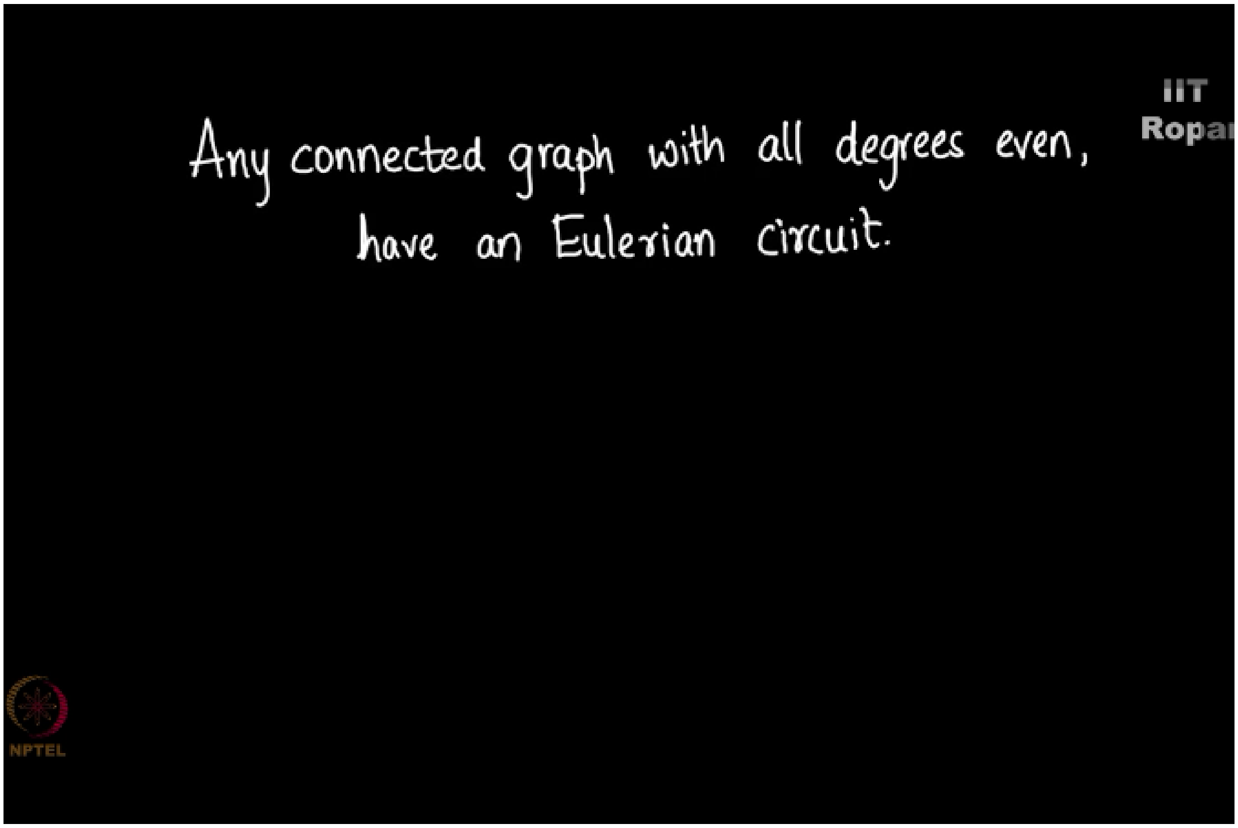

and then remove that noise back again and there you are, boom, you get answer in your hand, so what is this? What is the parallel between this Krishnadevaraya, Tenali Raman story and the proof in our theorem you will get to know when I go there, the right place I'll tell you how it's analogous to the story of Tenali Raman.

Now look at this, you are given a graph with all the vertices being even degree, let us take one node, some node namely X,
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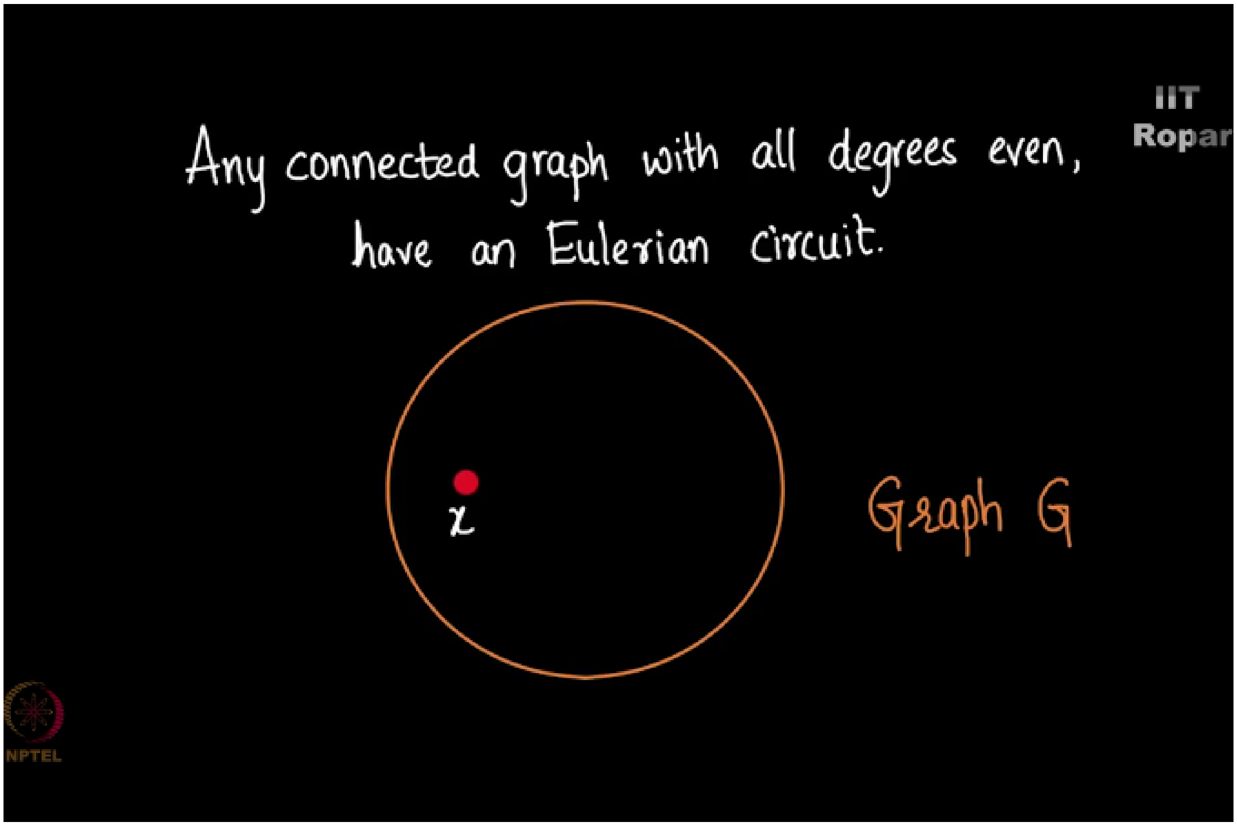

you know for sure that it has at least two edges going from it, why?

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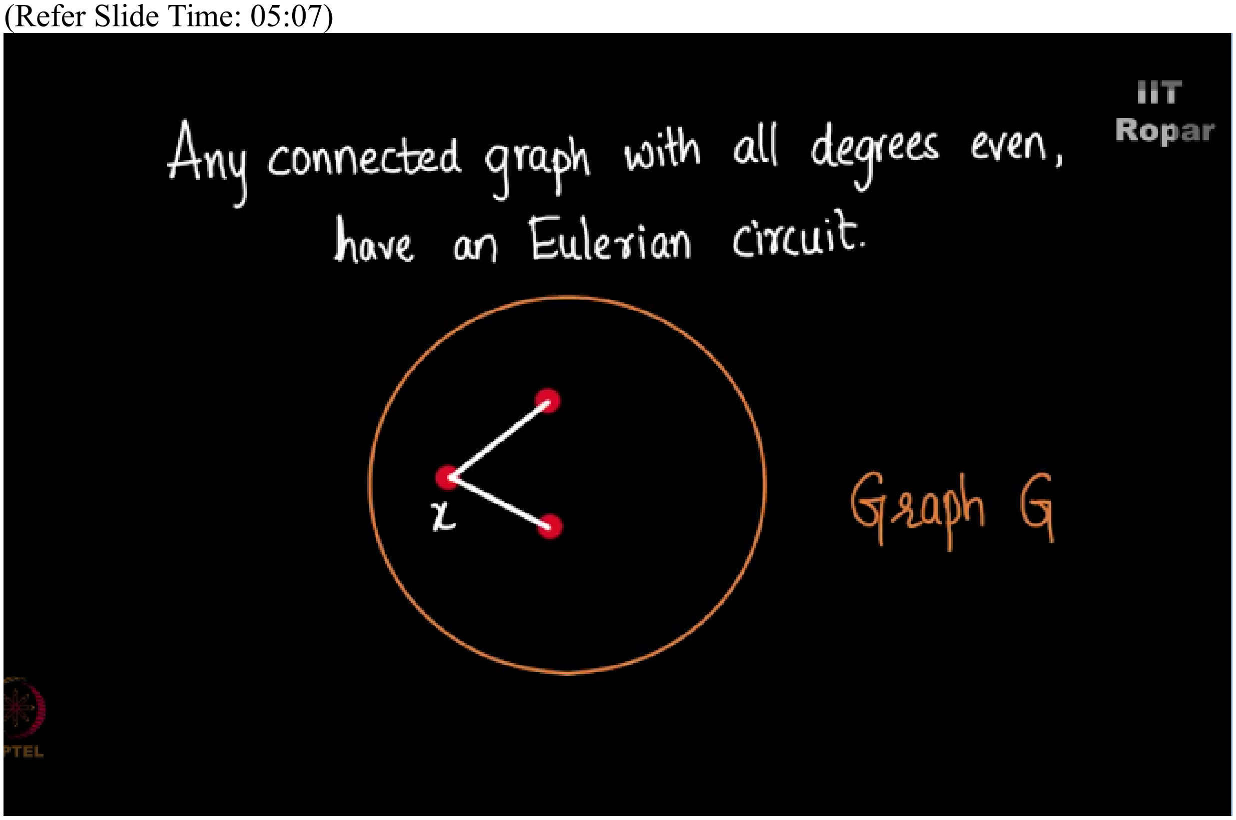

Because it’s degree is even, it’s degree is even means what? There should be at least 2 edges going, and please note these graphs are connected, any connected graph is what we wrote, any connected graph with all the vertices having even degree will have an Eulerian circuit, let me take one such vertex X, and I can guarantee you that it’s adjacent to W1 and W2, right.
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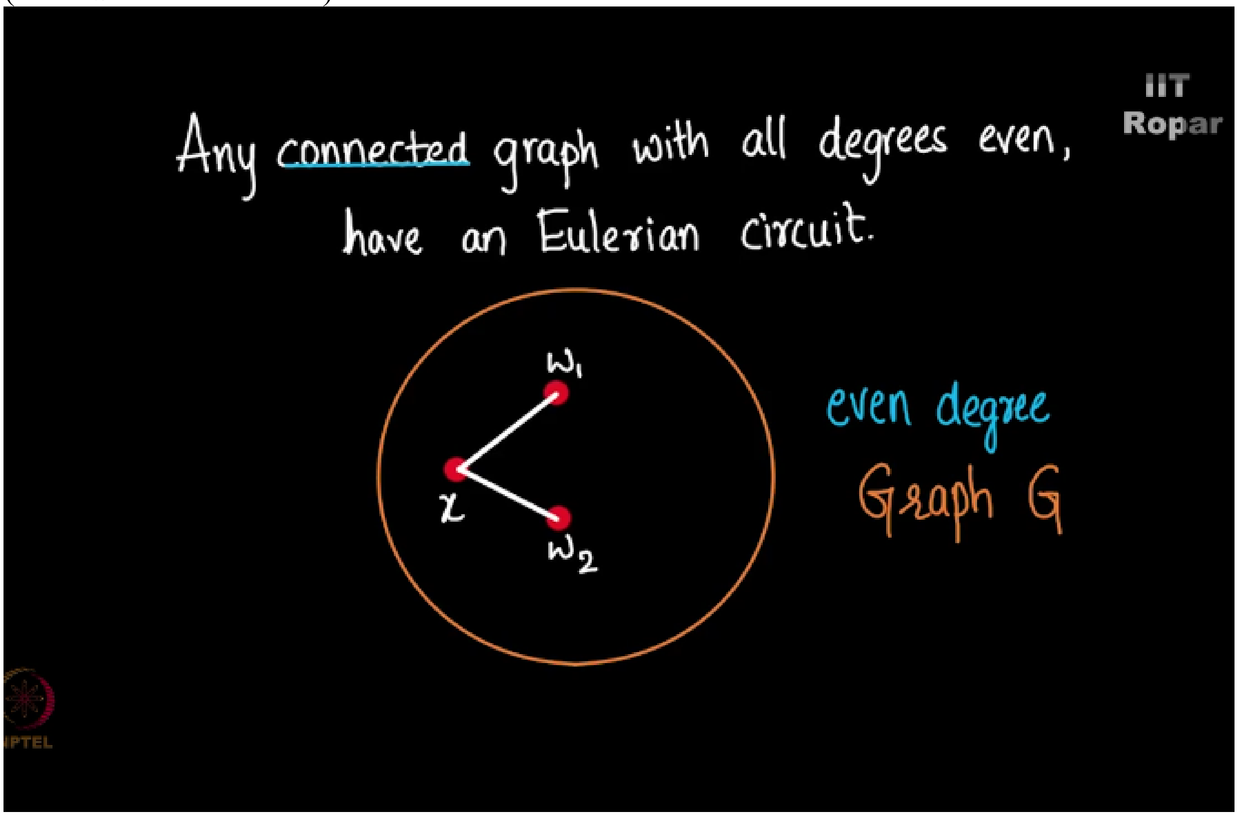

Now what I do is, I connect W1 and W2 and then remove this edge X, W1 and X, W2,
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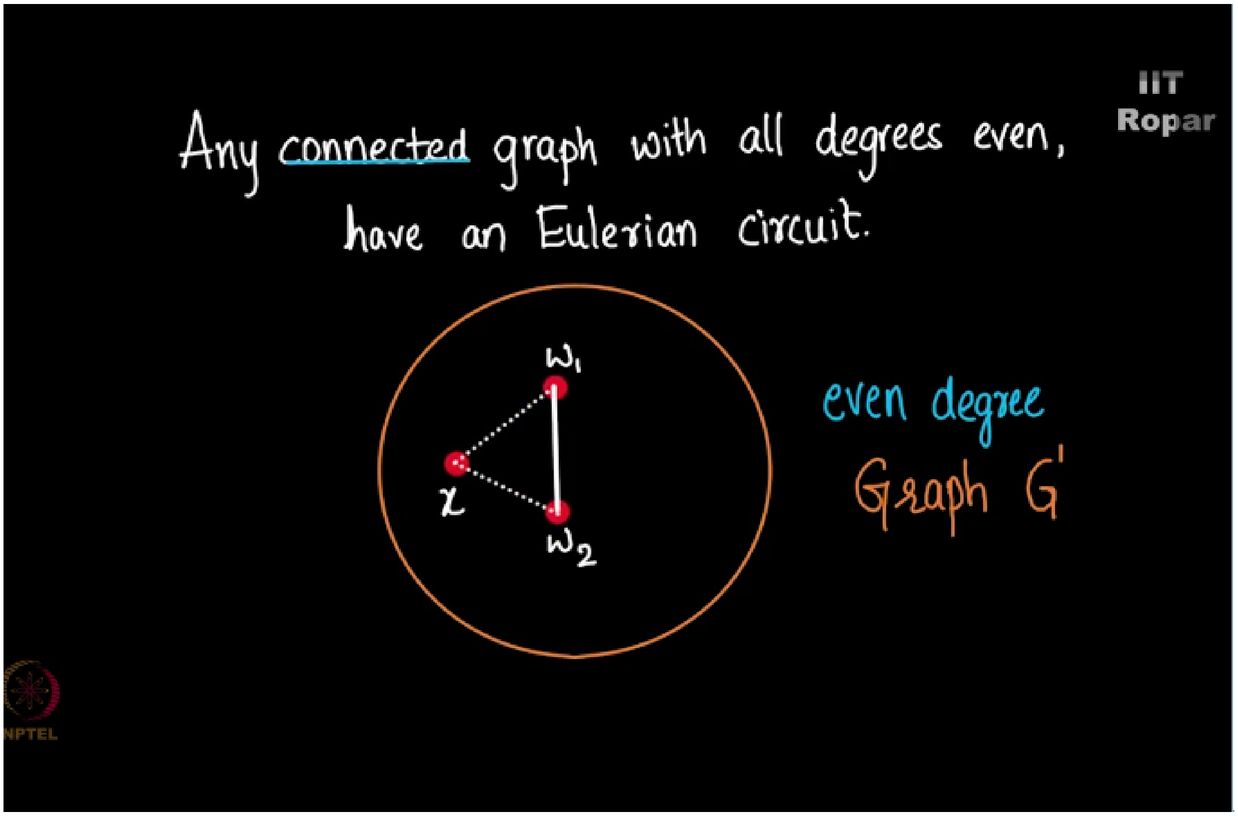

I simply remove it, so when I remove it you get a new graph all together, this is not the graph that was given to me, now my graph G has become G dash, how did it become G dash? You removed a couple of edges and introduced a new edge, let us observe this carefully, what happen to degree of W1 remains the same, why? Pause and think,

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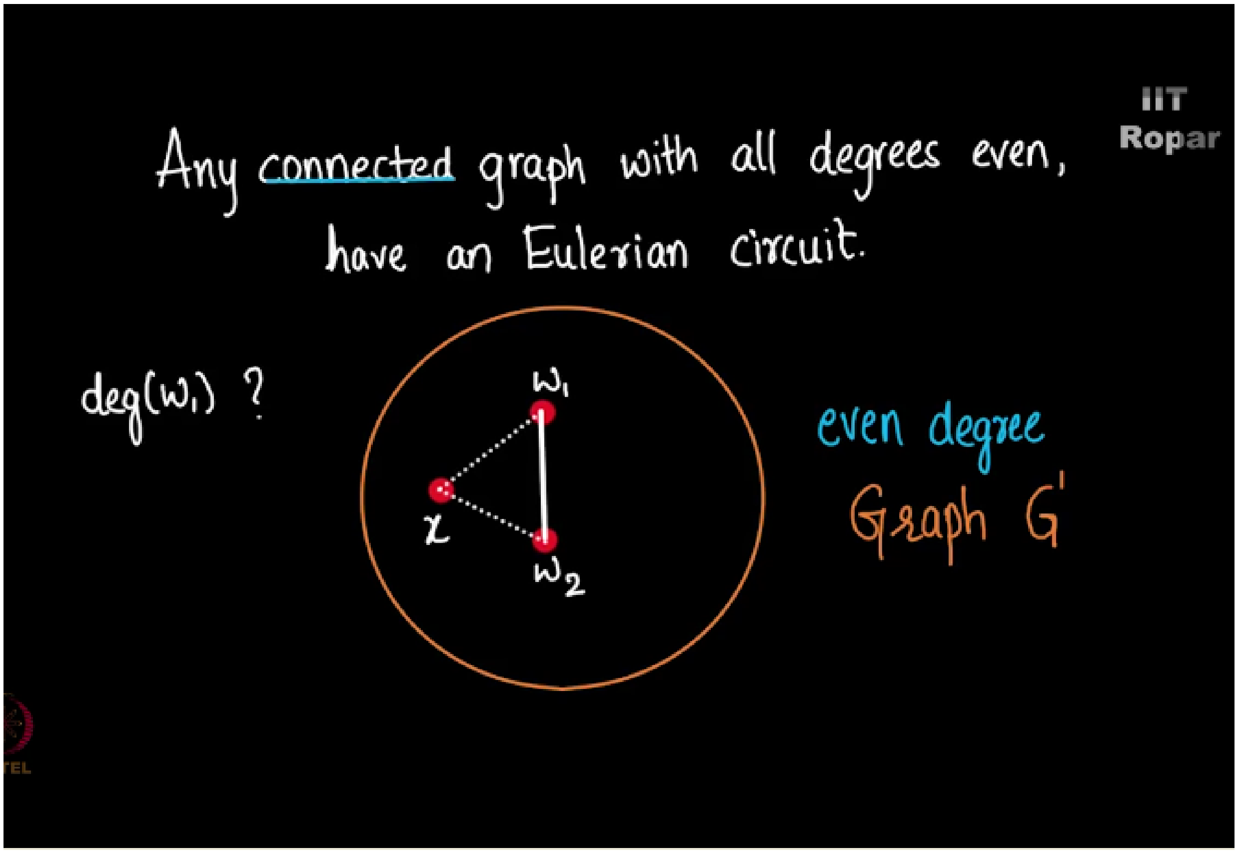

what happens to the degree of W2? Remains the same, so W1 was even and continuous to be even in the new graph G dash, what was the degree of X? It was even and you remove you some two edges it might have more edges too, right, you simply remove 2 edges,

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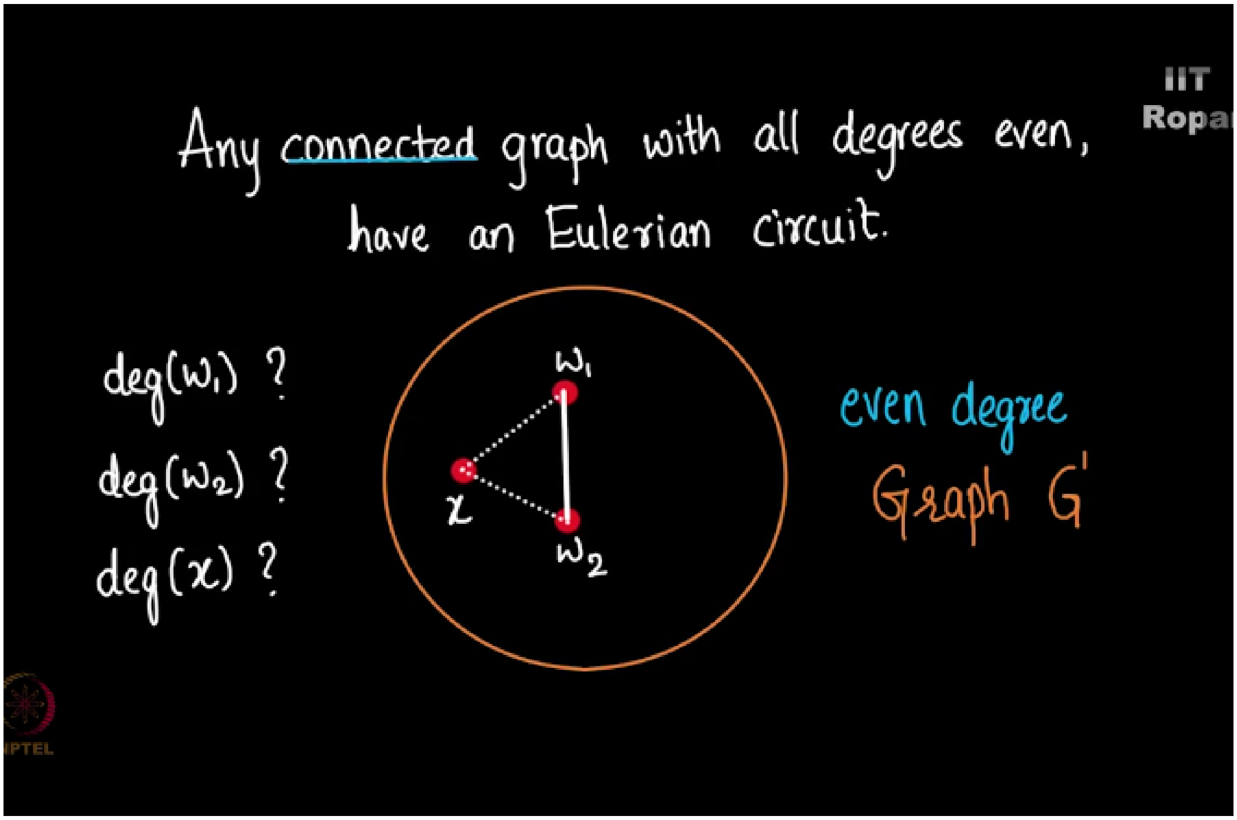

when you remove 2 edges from a node with even degree it continues to have even degree, okay.

If it had only degree 2, then it becomes an isolated vertex, it is not part of the graph at all, right, or if its adjacent to other vertices it will be part of the graph, now what I do is I know very well this G dash has as many vertices as G but one edge less, why?
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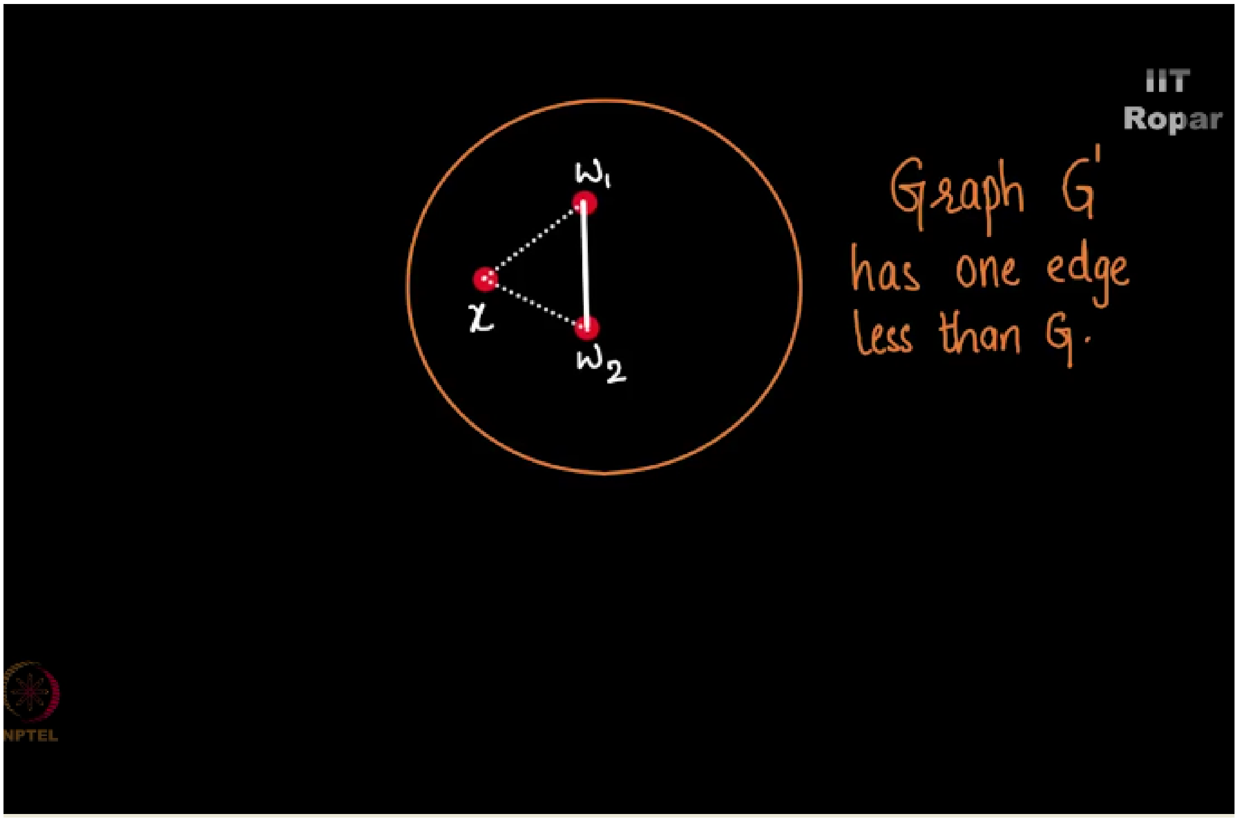

Because you removed two edges and added a new edge, so from G you went to G dash, correct, so you have one edge less, now, now, now think about this, what is induction? Induction is all about if you can show that something is true for step K and if you can show that its true for K+1 then it is true in general.

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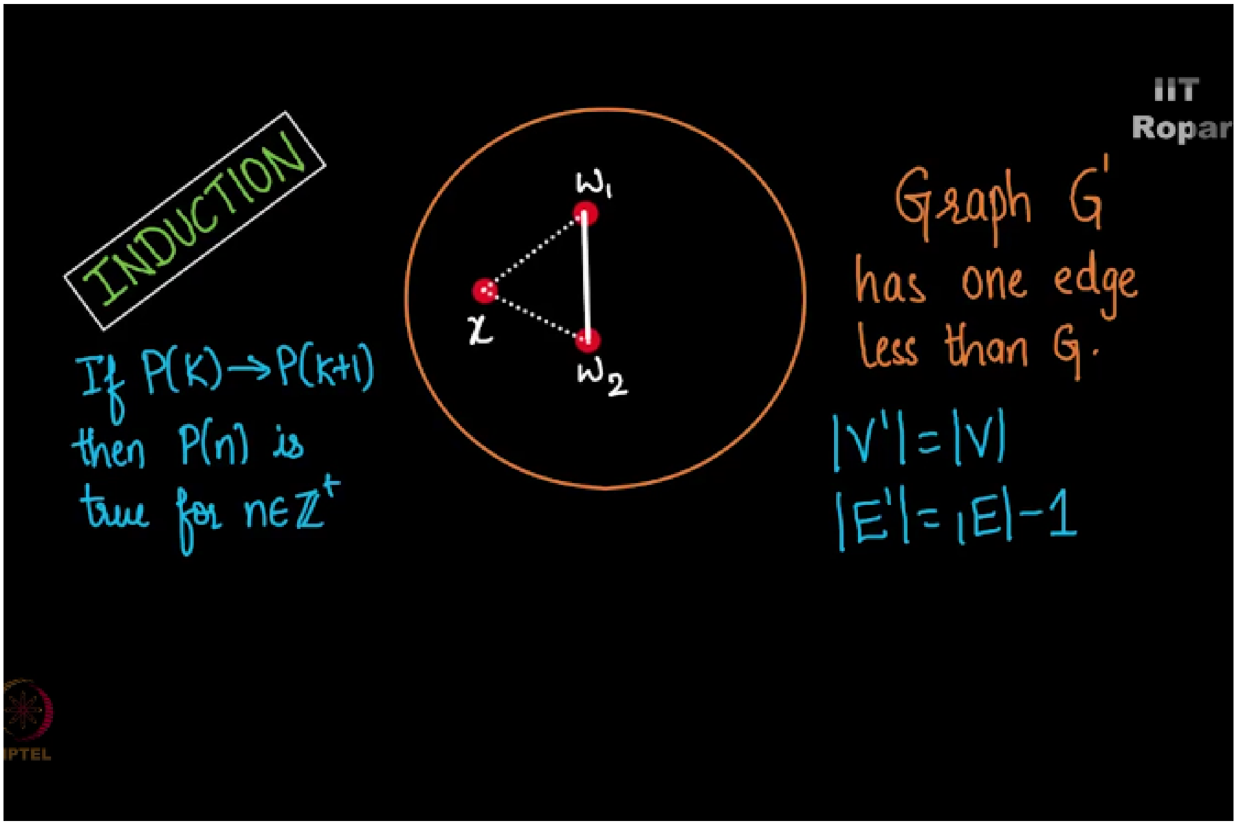

Now I took a graph with some M edges let say, now it became M-1 edges, so let me apply induction here and say it is true for M-1 edges, what is true? A graph with M-1 edges with all degrees even has a Eulerian circuit,

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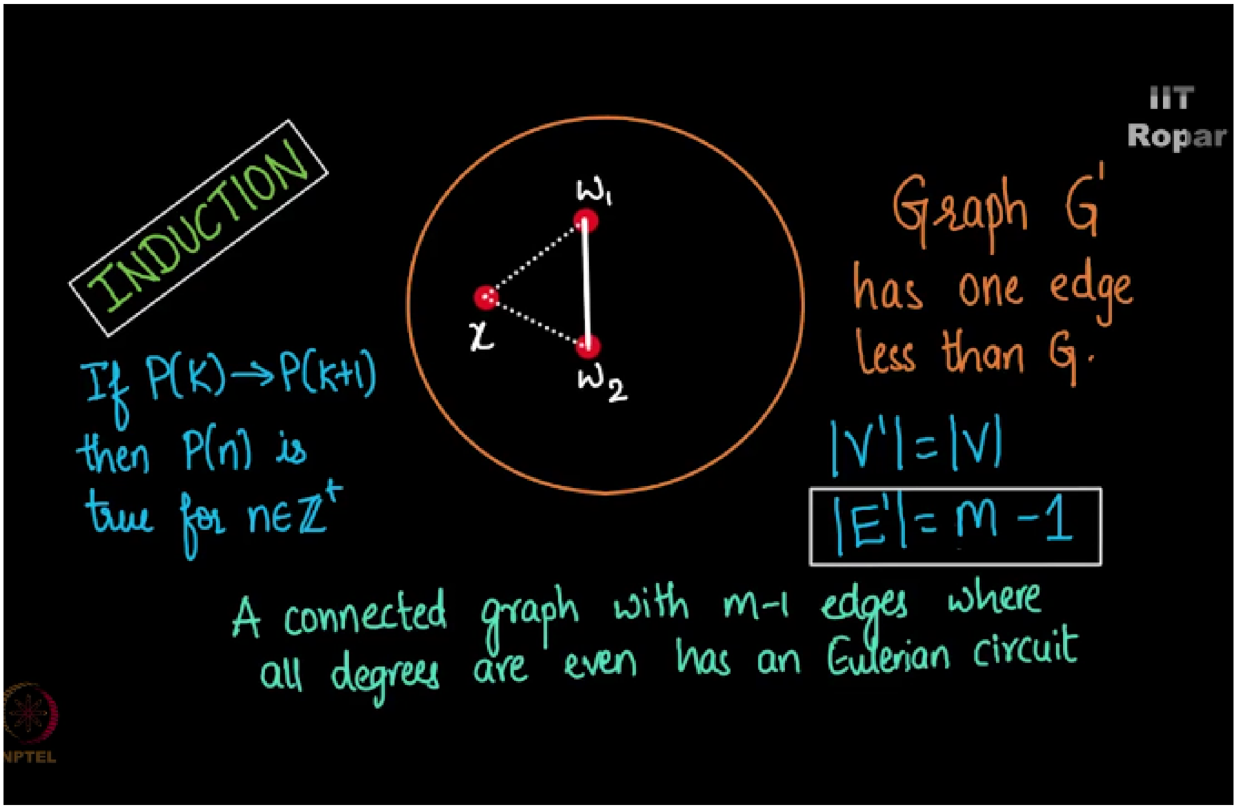

which means there is an Eulerian circuit on G dash, so what I will do is I will construct that Eulerian circuit, observe it carefully and I see that its starts from W1, goes around and ends in W1, and eventually it would cross W2, W1 also,

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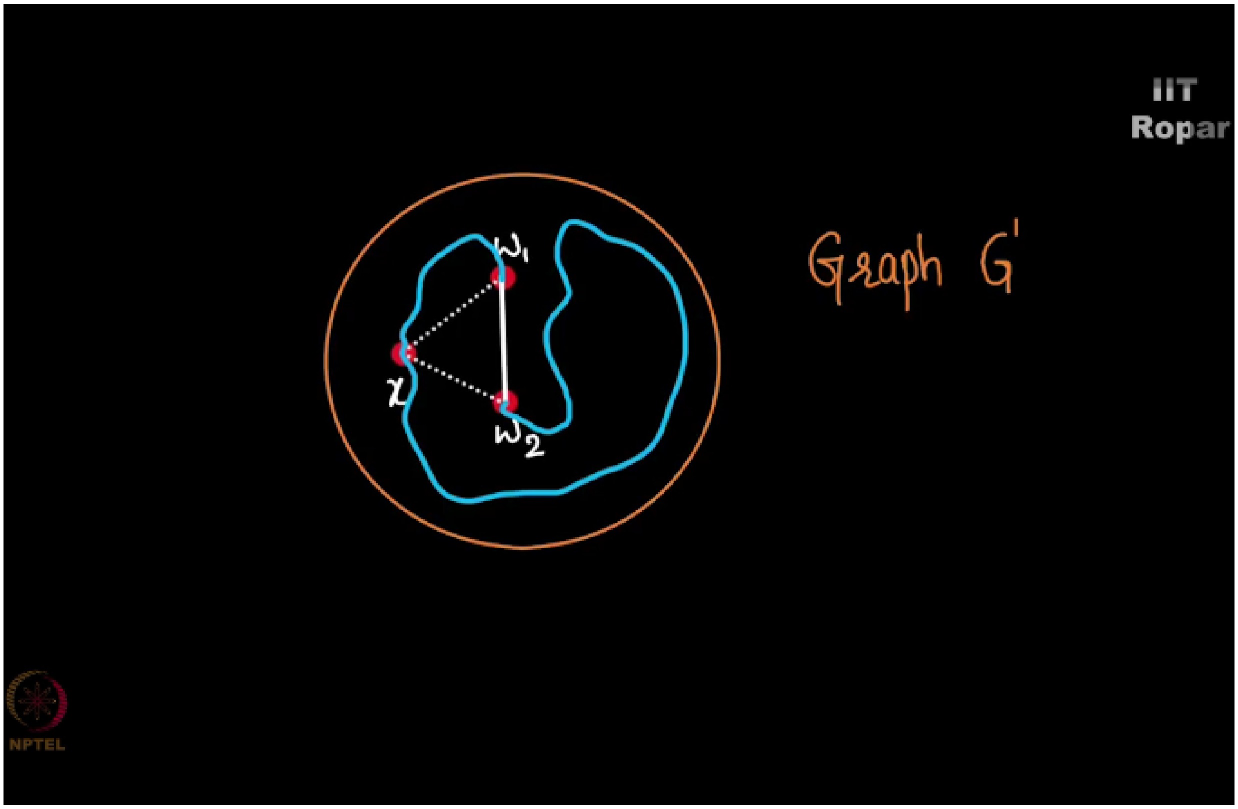

right, what I will do is when it starts from W1, and crosses W2, W1 somewhere this edge is basically, all I am saying is it will cross this particular edge and all the edges in fact, when you start from W1, correct.

Now whenever you start from W1 and come to W2, all I do is I will not consider this edge from W2 to W1, I will make these two edges reappear instead, and I'll go to X and then reach W1,
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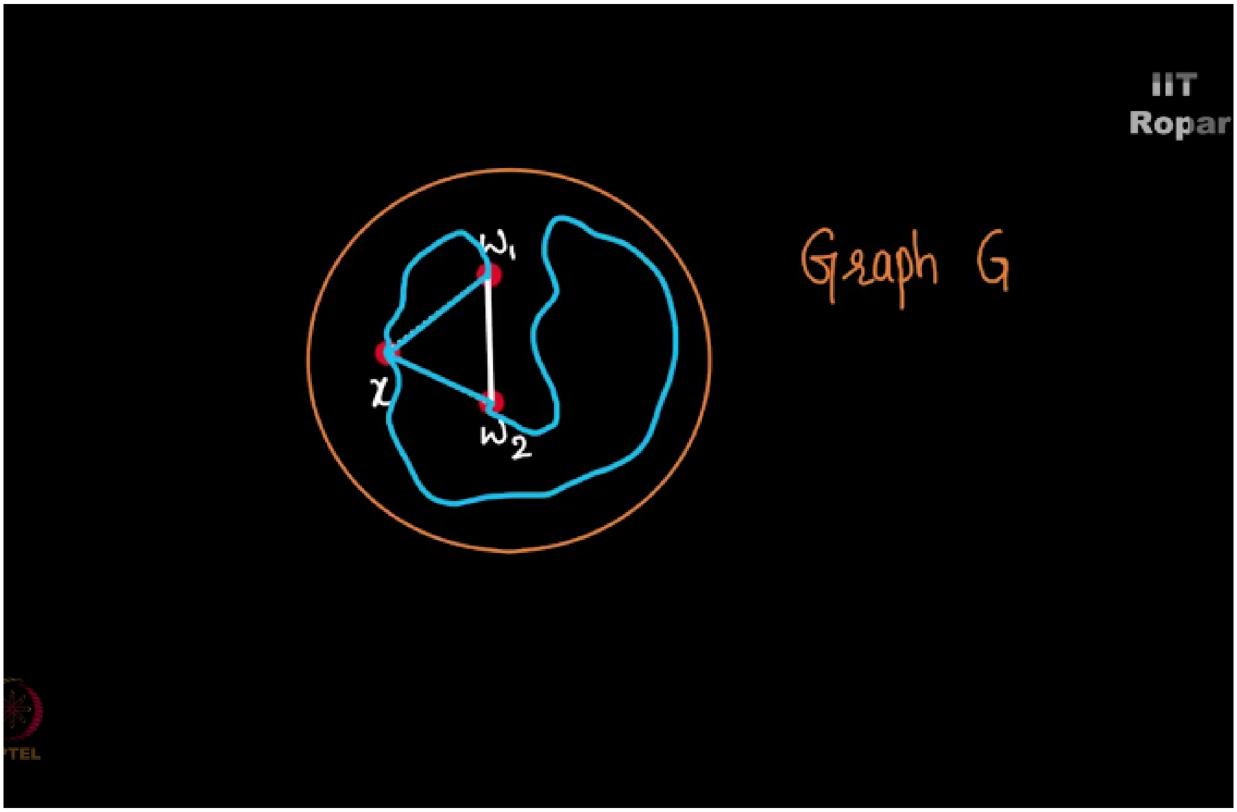

I repeat see there is a Eulerian circuit starting from W1 going around everywhere and then coming to W2 and then ending in W1, correct, when you start from W1 finish all the 8 edges, and come to W2 you don't take W2 to W1 you have a twist in the story, you go to X, and then come to W1, that way you show that you exhaust all the edges in the graph G, please note G dash simply had one edge included and two edges removed, in G dash there is a Eulerian circuit, so you use that Eulerian circuit to obtain an Eulerian circuit in G, correct,

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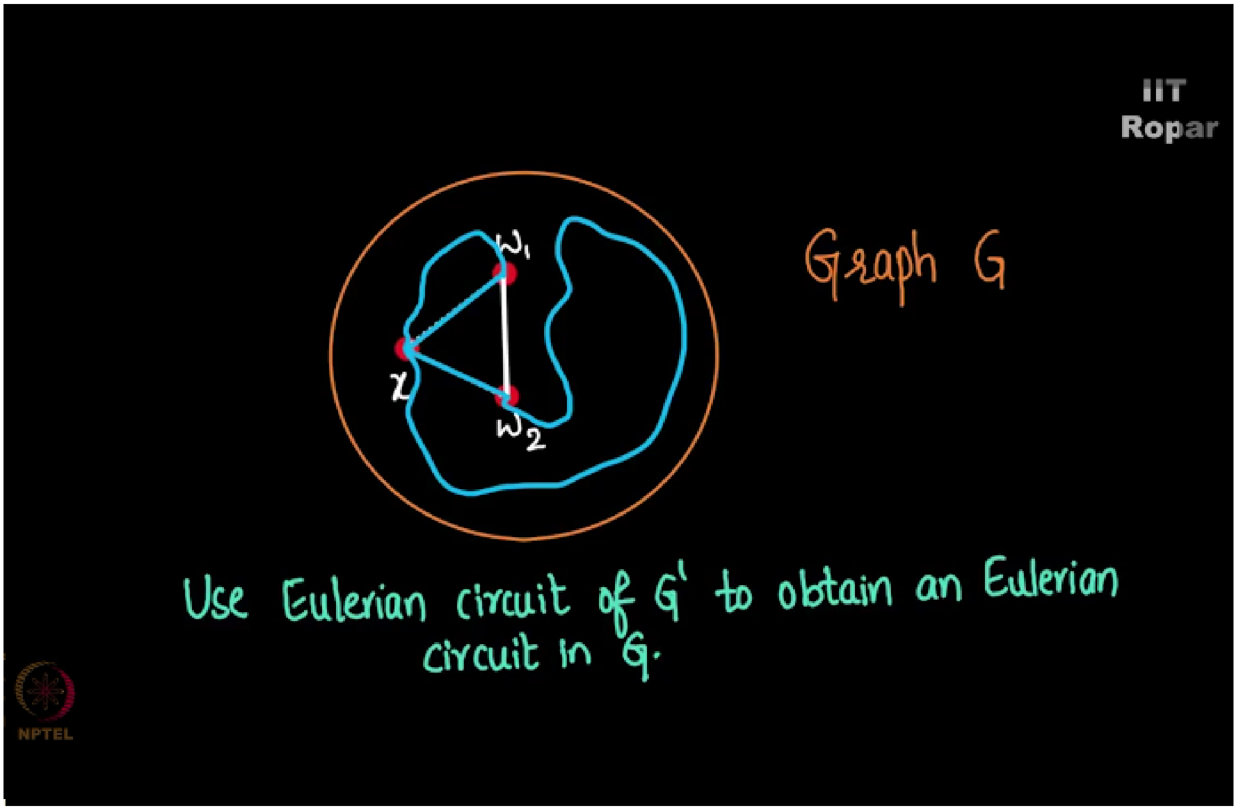

I repeat, the structure of the proof is very simple, given a graph G take a node X, it will have at least 2 vertices adjacent namely W1 and W2, that's because it has even degree, correct, and then what you do is you take XW1, XW2, and then remove these two edges and join W1, W2, right, when you join W1, W2,

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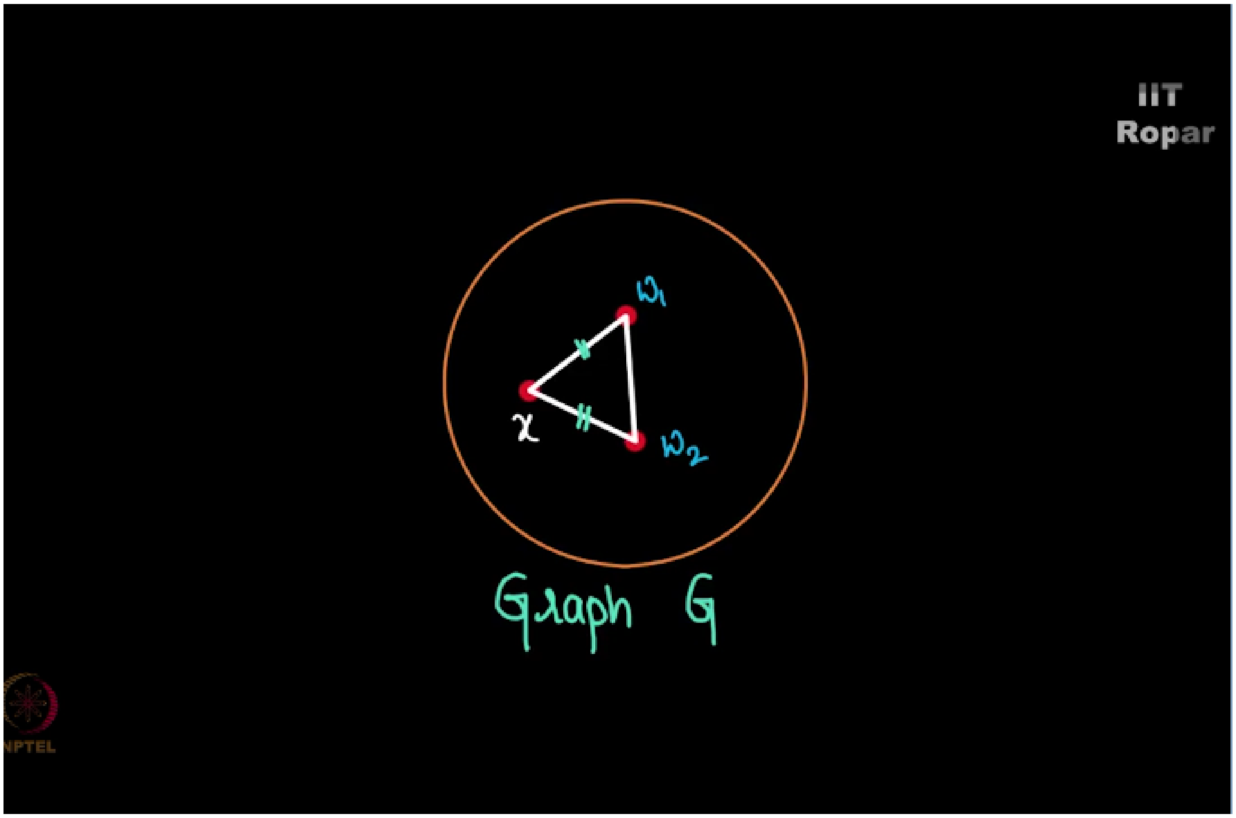

you observe that you went down by two edges, but you've included one edge which means the deficit is one edge, what do you lose? Just one edge, when you lose one edge from M edges of the original graph G, you come to a smaller graph with M-1 edges which is G dash.

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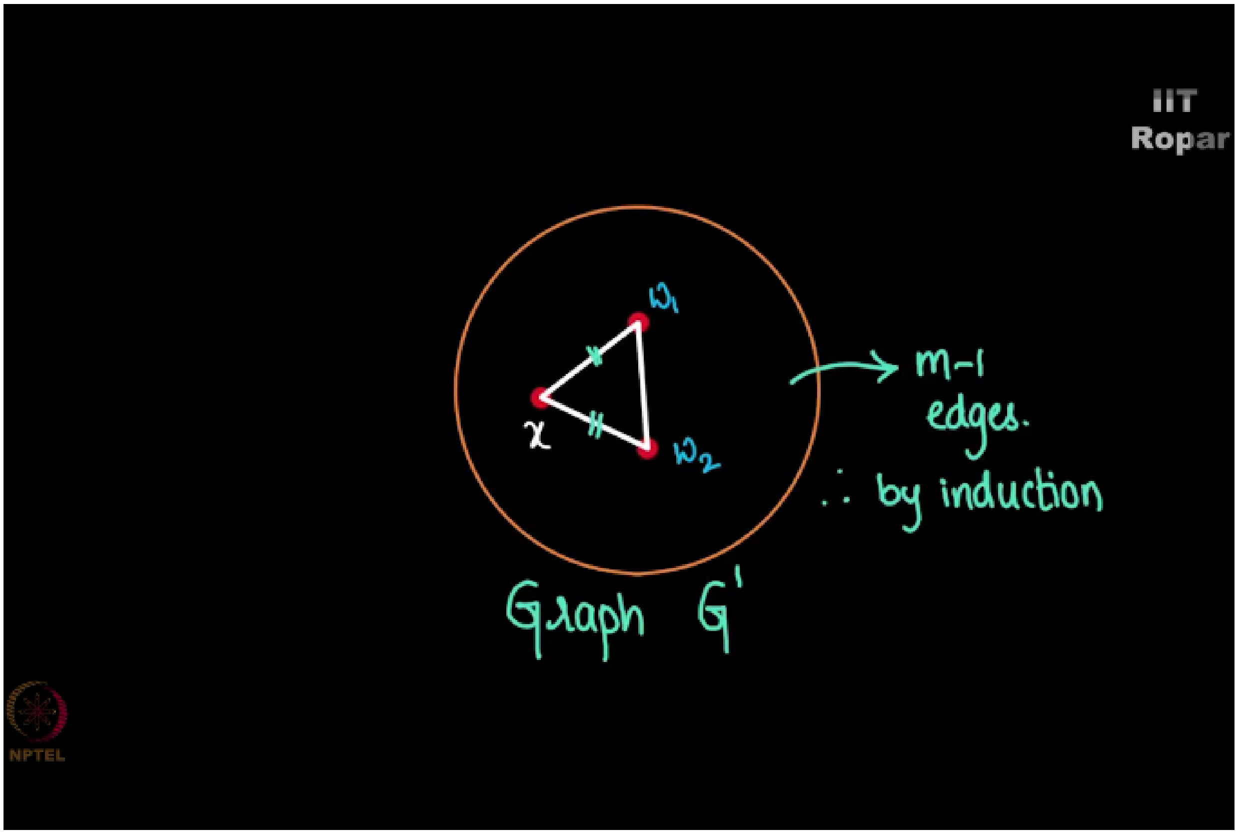

And in G dash given the lesser number of edges by induction there is a Eulerian circuit, right, there is this Eulerian circuit, correct, why is an Eulerian circuit? Because it has lesser number of edges and hence there is an Eulerian circuit, and after you see this Eulerian circuit what you do,
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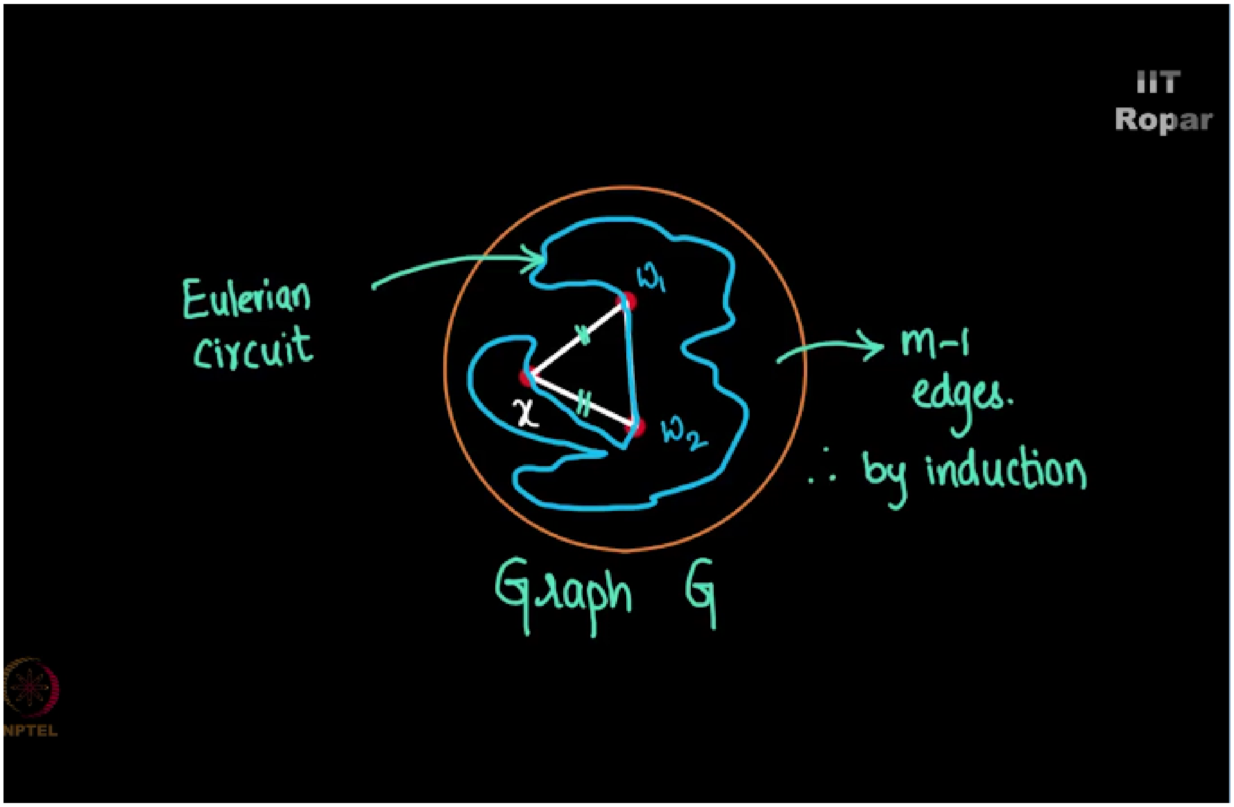

is you know that there is a way in which you start from W1 and end in W1, and somewhere W2 to W1 also comes in that Eulerian circuit, so what you do is whenever that edge comes from W2 to W1 instead of that edge you put W2 to X, and X to W1 and bingo there you are, in the original graph G you just found an Eulerian circuit.

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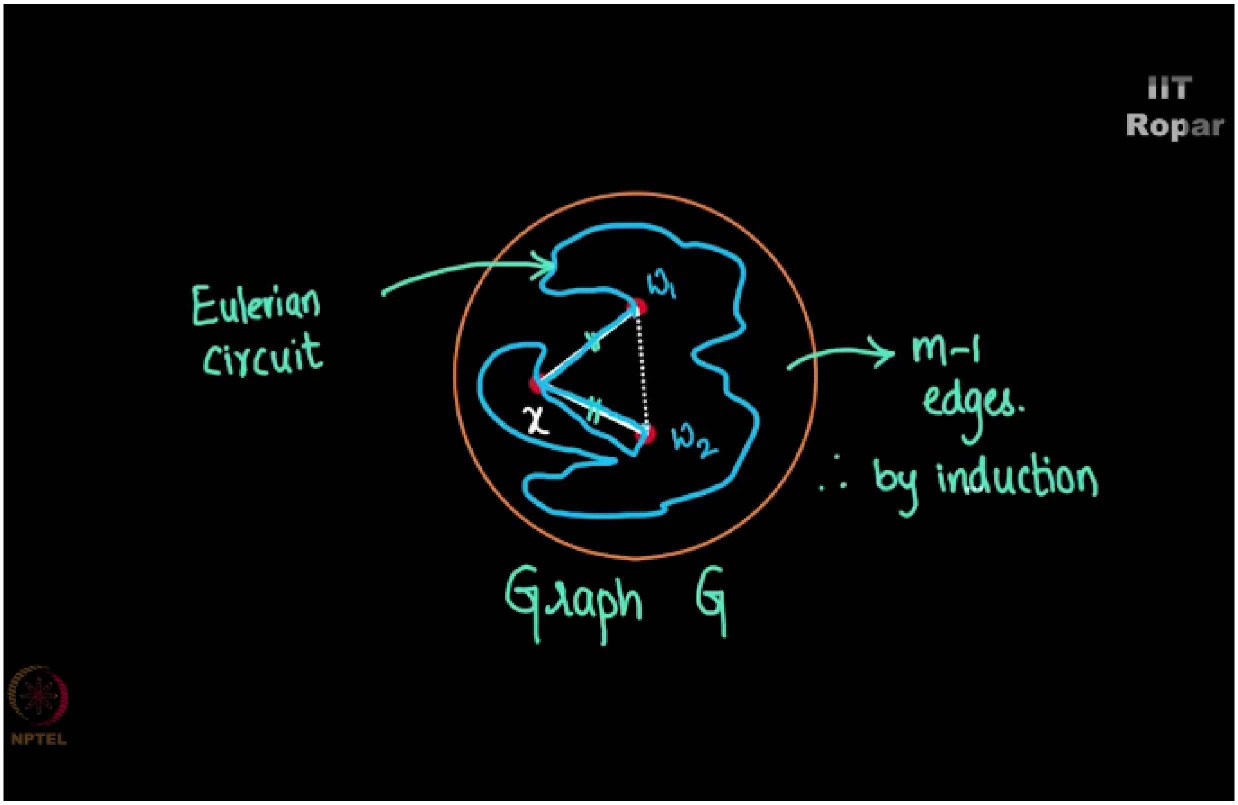

Now what was the analogy for, you see what I did, I removed a couple of edges in the given graph G, I brought in some new kind of noise, I included an edge W1, W2, why on earth did I do that, I'll be getting a different graph all together, if you are asked to solve, if you asked to solve a problem in your house, you should not go to your neighbor's house and then solve their problem, right, this sounds something like that, should have solving the problem in my graph G I go to some other graph G dash but then when you go to G dash and then solve the problem there you observe that boom, the problem get solved in G, why is that?

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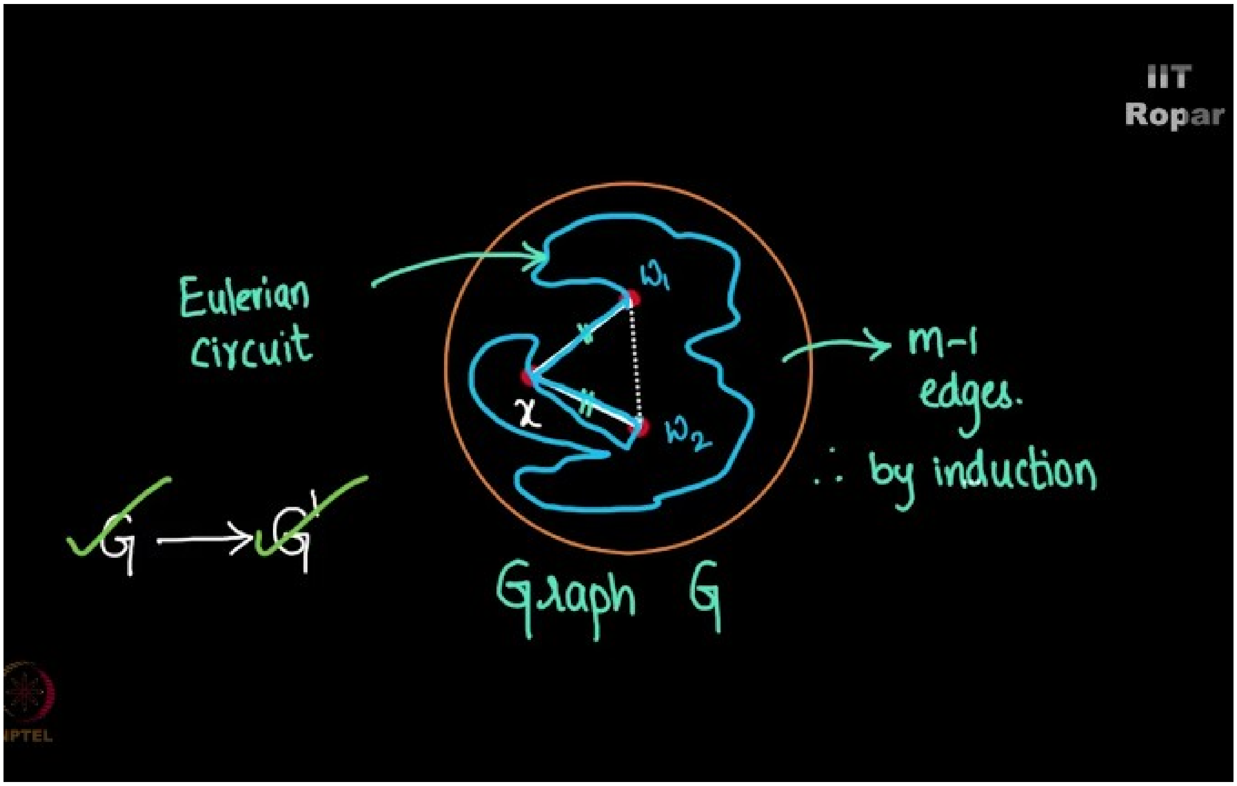

That's because G and G dash are just one edge less, and you are able to apply induction there, you see this is very, very analogous to your logarithms you see, when you apply logarithm, multiplication becomes addition, and you simply add there, when you add there you don't get the answer, you should take the antilog and you will get the answer, right, so this is just like that from the world of G, you go to the world of G dash you solve it there using induction and then come back to G and you see that you indeed have solved the problem. So Tenali Rama story is just to tell you people that adding some noise and then removing it sometimes helps us get to the solution.

Well, what we did not observe in our discussion so far? Had a long discussion you see for almost 10 to 15 minutes we discussed, how, when a given graph has all nodes even degree, there is always a circuit that exhaust all the edges, also called the Eulerian circuit, there was a small point in the proof where there were two cases,

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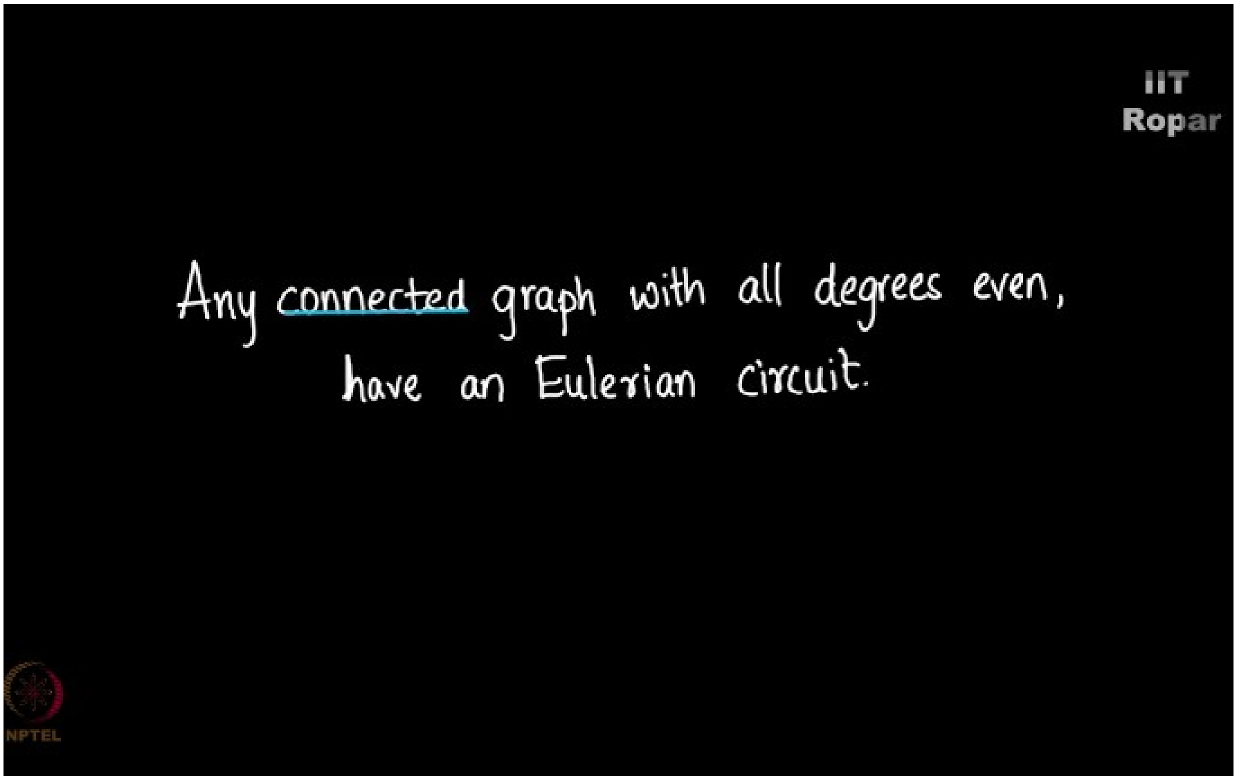

I didn't want to disturb your minds with considering all the cases there, then the cases become more important in the proof itself, right, so I kept it for later discussion.

Now when you remove these two edges XW1, and XW2, you might end up having a disconnected graph, right, so there could be a component with this X,

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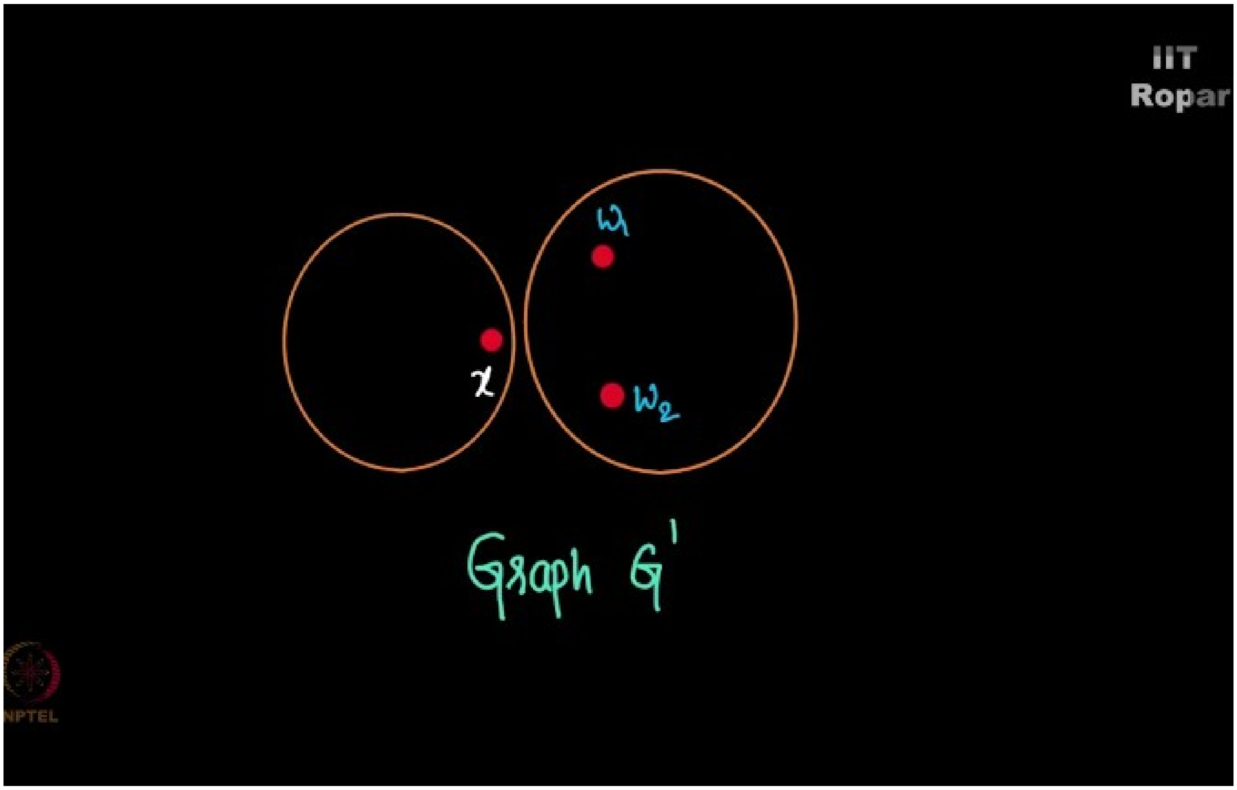

completely away from this side, in that case edges of course will go less by 1, but it will be a connected component with even fewer edges, and hence there will be a Eulerian circuit on this smaller component.

Now what I do, is I finish that Eulerian circuit and then again the same logic W2 to W1 and then I have finished an Eulerian circuit on W1, but instead of W1, W2 to W1 I remove that edge and I put back these two edges, so which means I go to X and then go to W1 and I end there, but then when I go from W2 to X,

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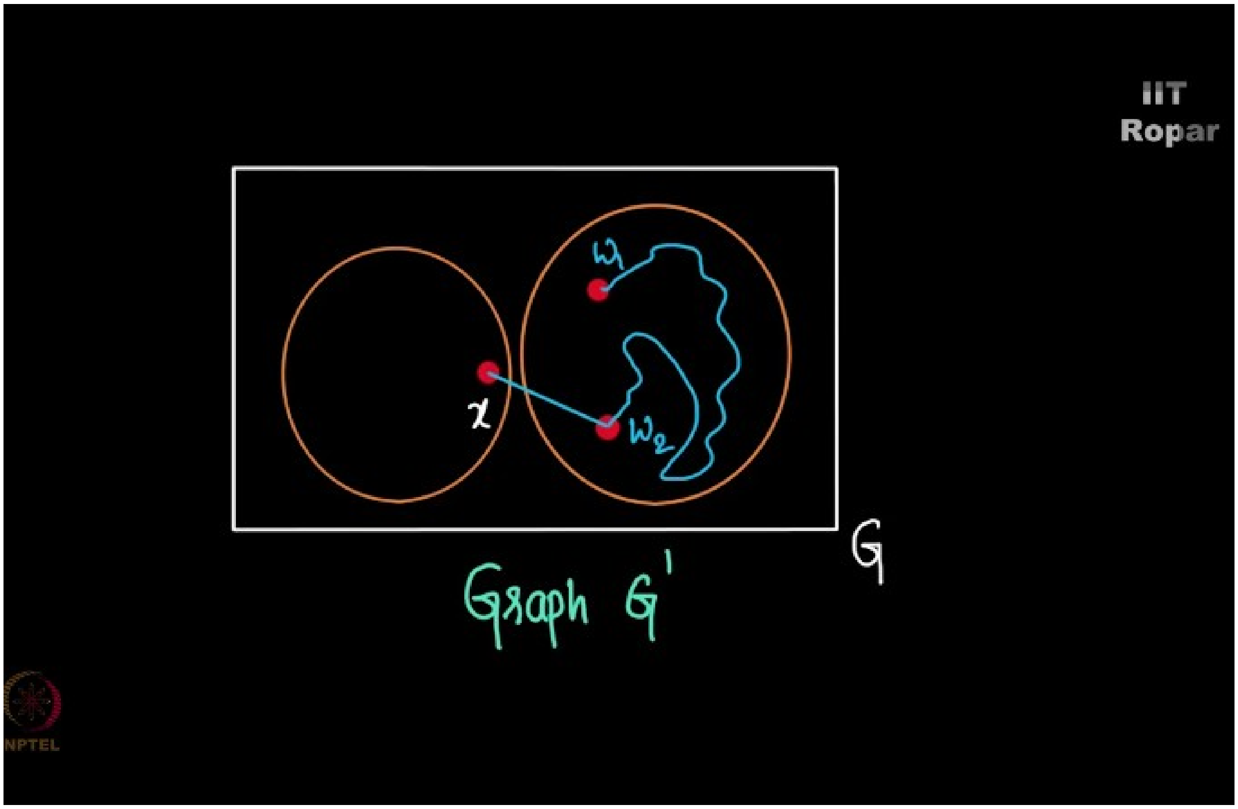

what I do is in this component also there is a graph with all nodes having even degree, you see that, correct, why do all of them have even degree? By hypothesis, the graph has all vertices with even degree, right, and just got disconnected, the smaller component by induction even this should have an Eulerian circuit, I go to X and boom, I travel this entire place

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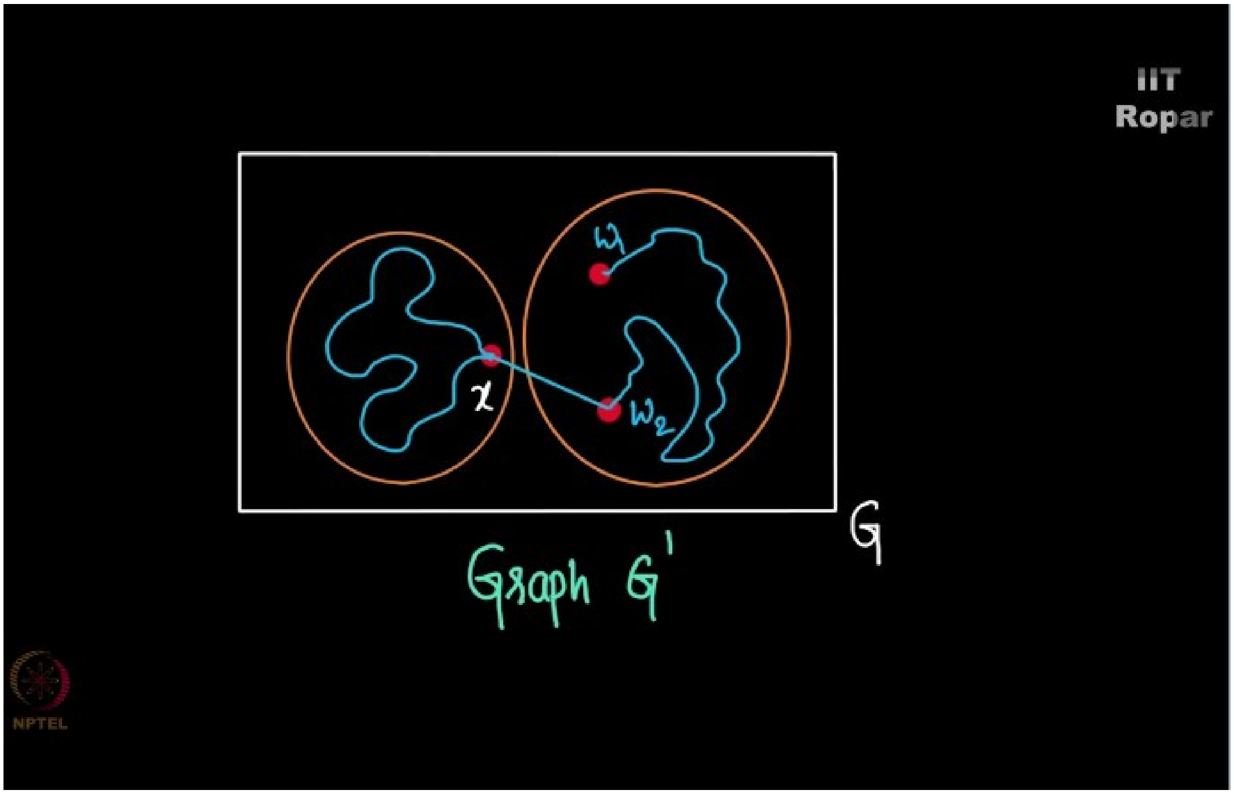

and then come back to X, that's possible, why? The graph is Eulerian, correct, so it's a smaller graph, so I come back to X and then go to W1, I repeat when you remove these two edges and put the edge W2, W1 this component can get disconnected, and then what you do is you find an Eulerian circuit in this world,

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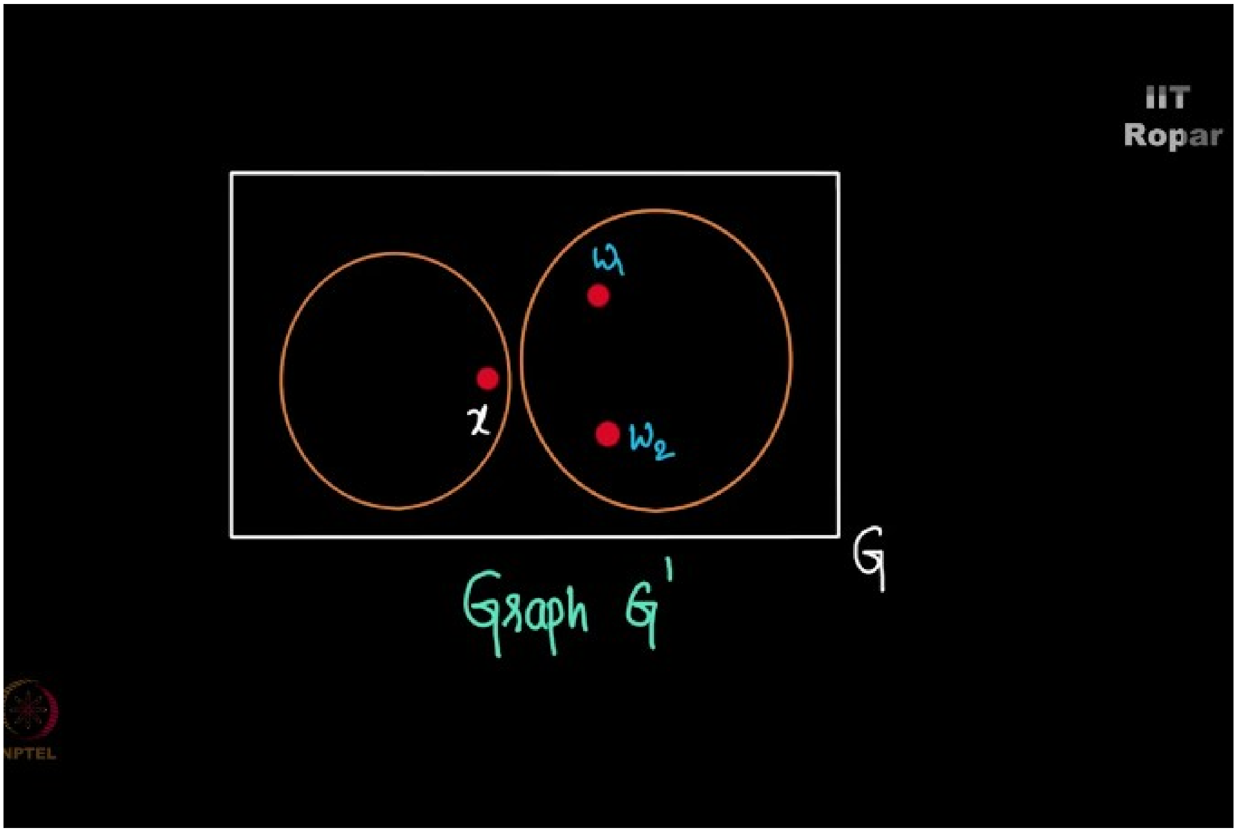

and when you find it out you will have an edge going from W2 to W1 you remove that and you put W2 to X, and then X to W1, and when you do that W1 to everything, boom you finish then come to W2 and then broom you go to X and then before going to W1 you exhaust the Eulerian circuit on this component and then go to W1,

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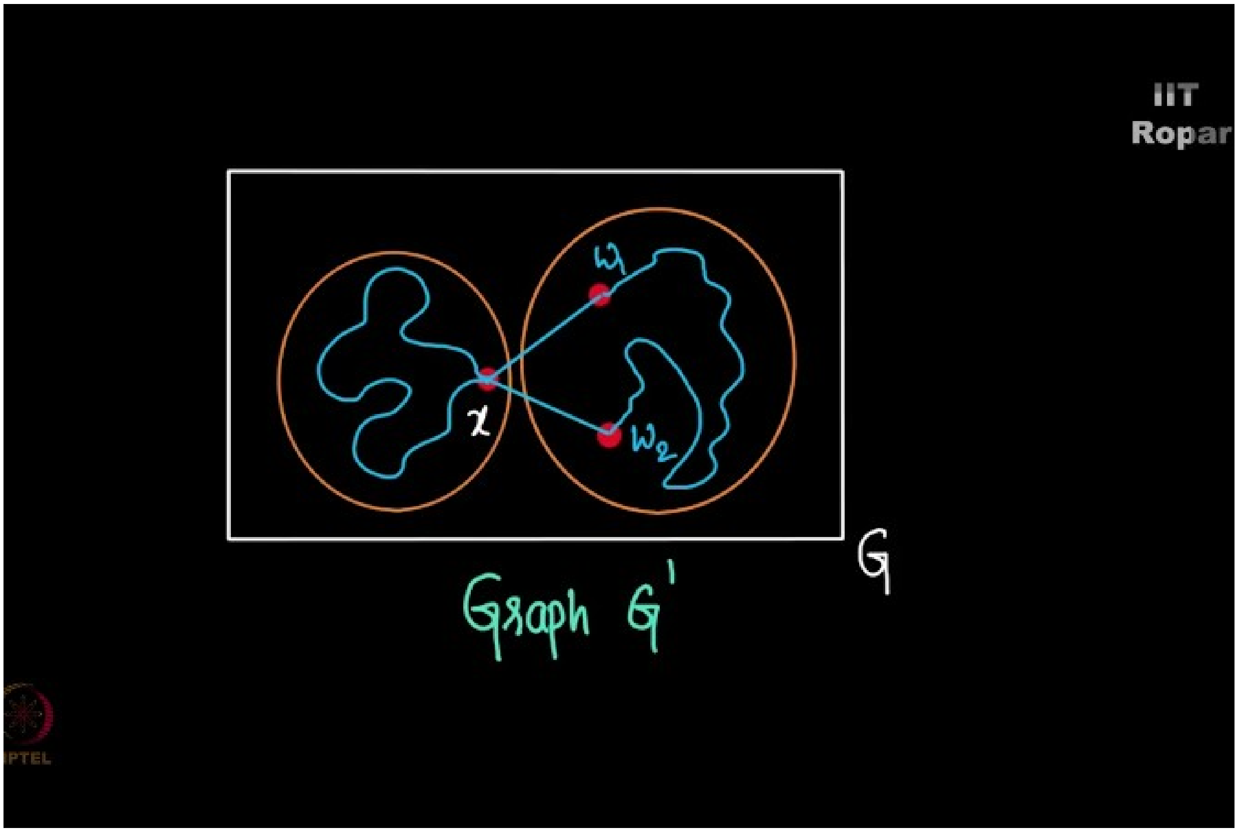

and hence the case when the graph gets disconnected is also taken care of.

Now if this is very clear to you then no, that you have actually miss some points somewhere because this is something that should be not at all clear to you in your first attempt of reading, given the lot of chunky facts here, there are lot of fact, 1 fact, 2 fact, 3 fact before we were using induction, we are using some that noise reduction, that Tenali Raman example, you must be wondering Sudarshan just lectures long video and calls it a proof, but yes bear with me, this is one of the upper bonds for hardness you see, no other result is as hard as this, that's because it's a very involved theorem, this change the world of mathematics, people started believing that something as silly, and as simple as Konigsberg bridge question can have some deep math associated with it, so you cannot solve Konigsberg bridge problem because of this theorem,
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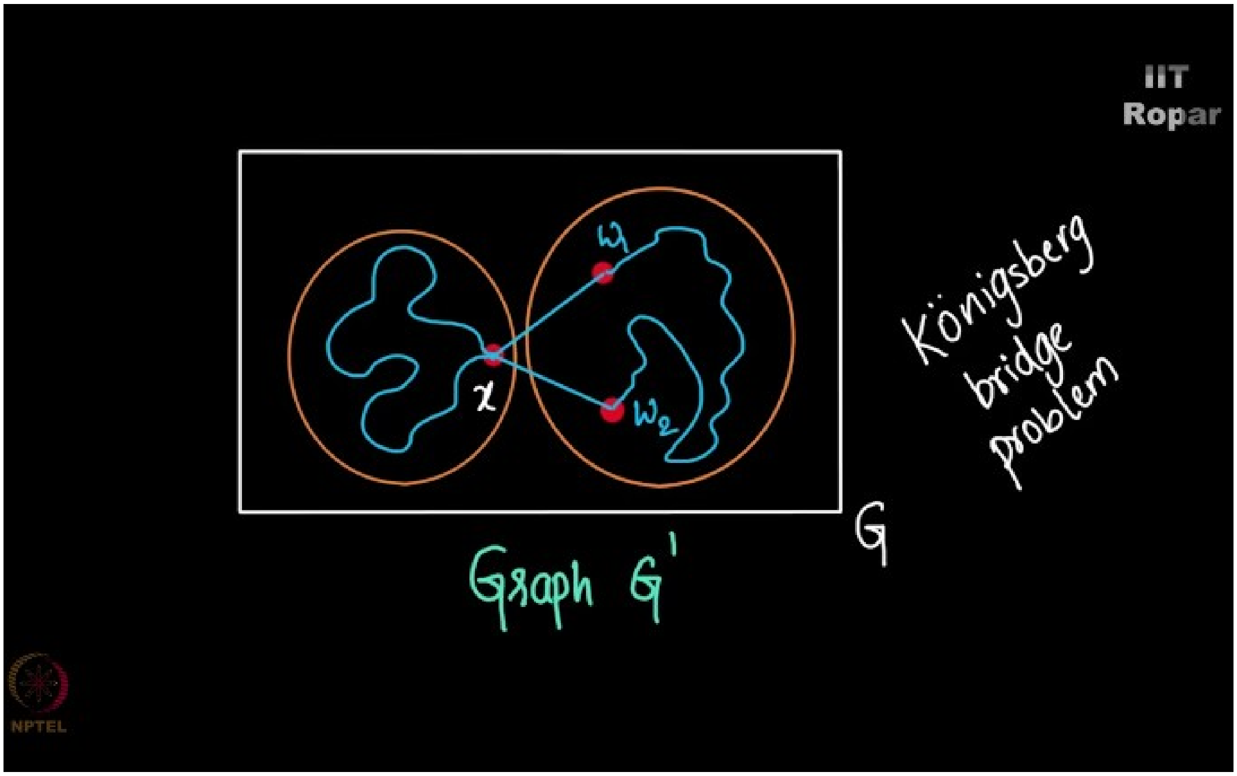

you see if you want to traverse a given graph it must and should have all nodes even, here is the case where, Konigsberg bridge is the case where not all nodes are of even degree, and hence you will not be able to traverse that, that is one way of seeing it, but then what is it that you can traverse completely? By exhausting all the edges such a graph should have the property that all nodes are even, only then you can traverse, right, so it takes some practice maybe you even want to watch the video again and again, please email us in case you have questions, if there is a better way to explain this I'll try to do it once more maybe, but here is my best at this given point of time, so it will probably help you if you can write some non-trivial graphs and let say 10 nodes ensuring that every node has even degree, and try to see if this theorems proof helps you in observing an Eulerian circuit there.

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