**NPTEL**
**NPTEL ONLINE COURSE**

**Discrete Mathematics**
**Mathematical Induction and pigeonhole principle**

**Checker board and triominoes - Solution**

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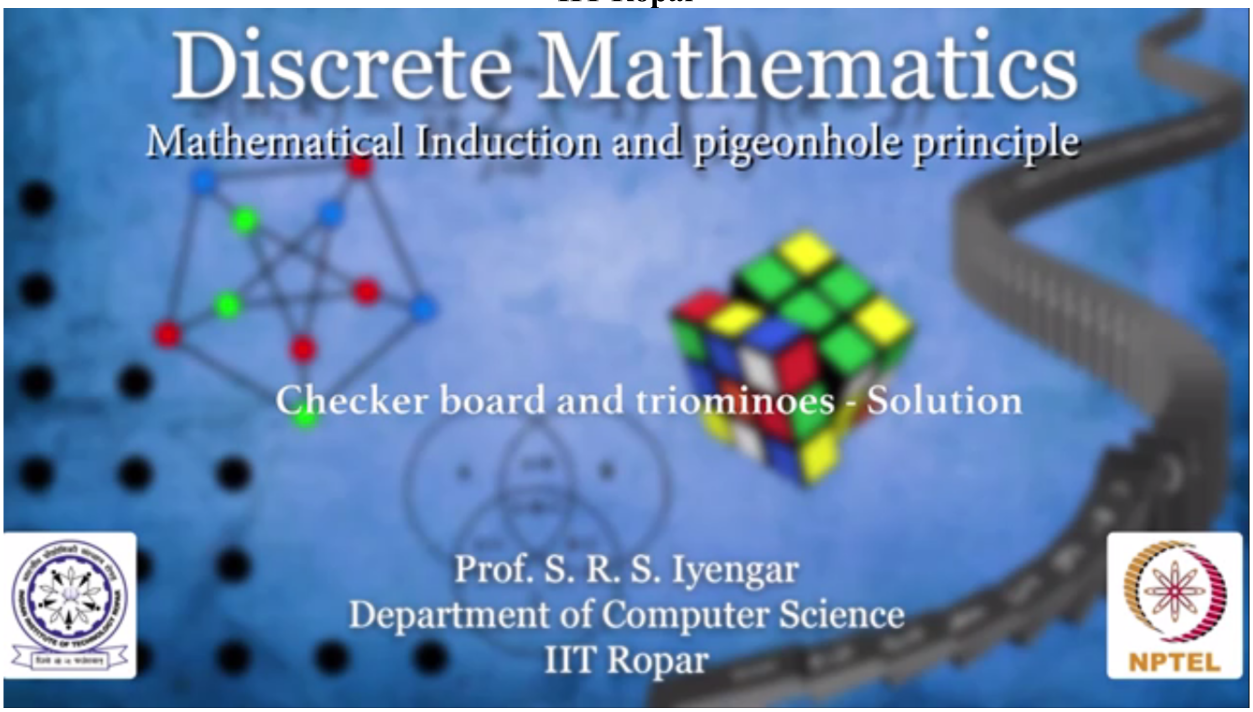
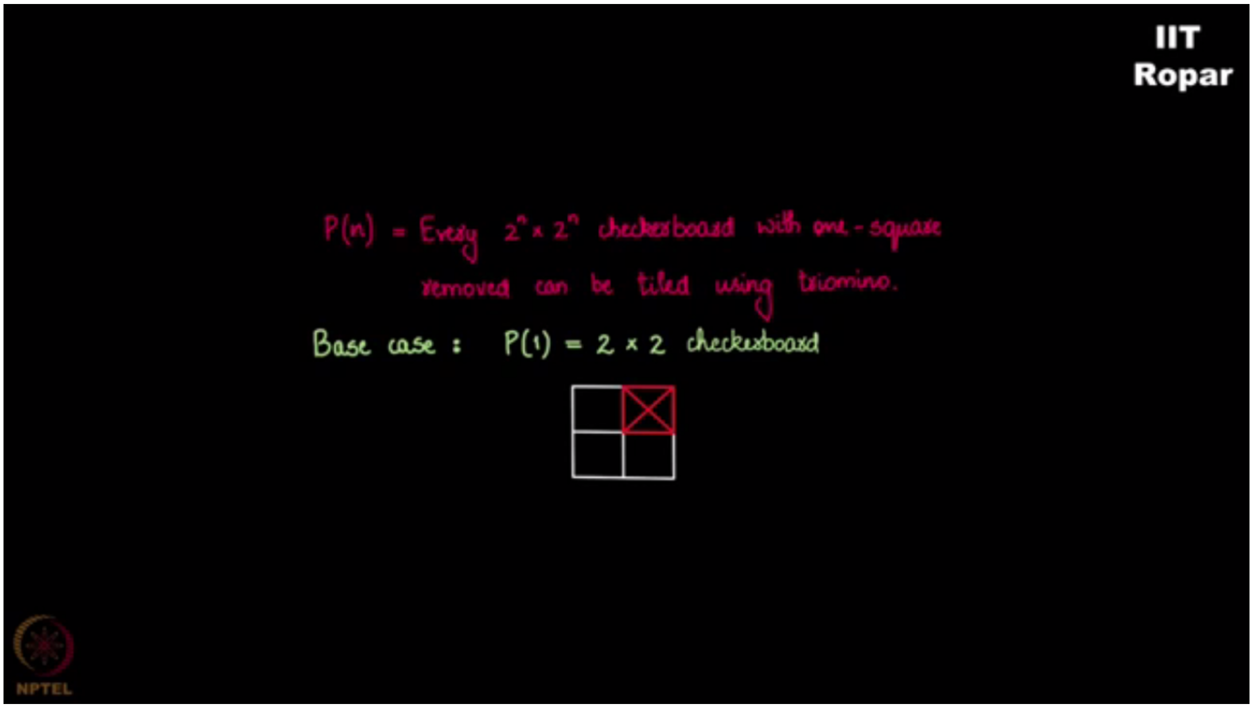
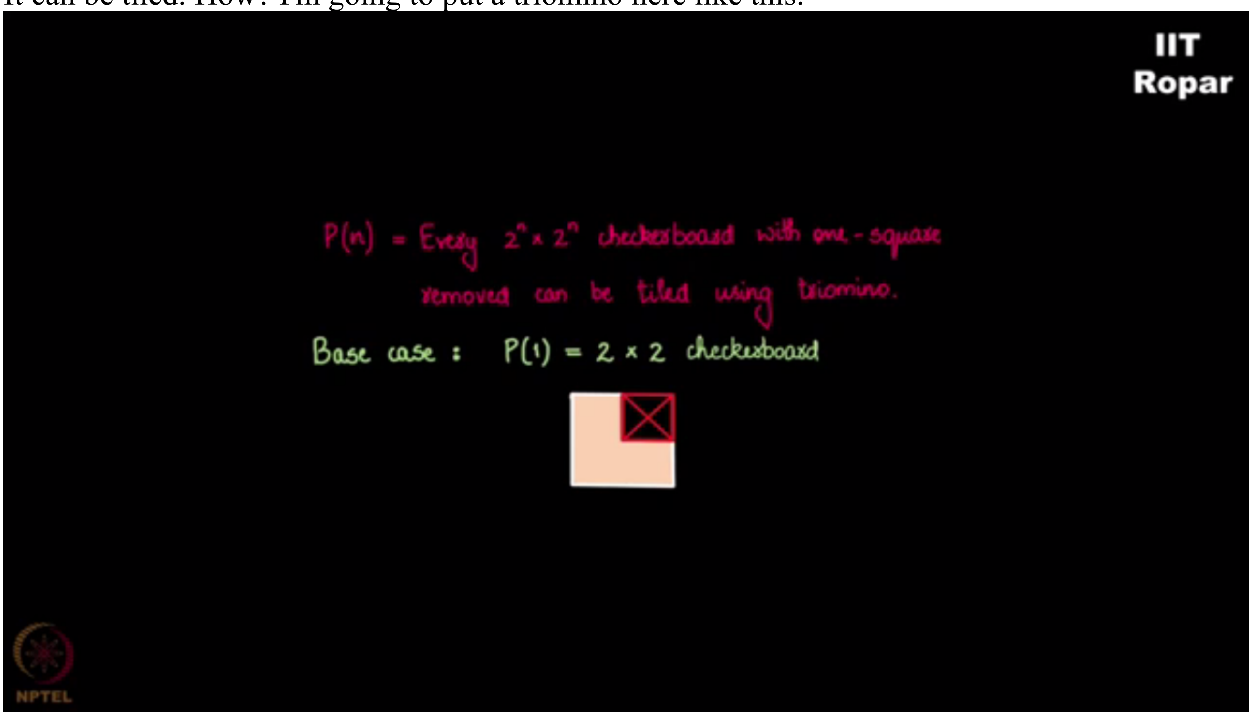



I hope all of you were wondering how to do it. Well, induction has come to our rescue. Let us see how we can prove it using induction.

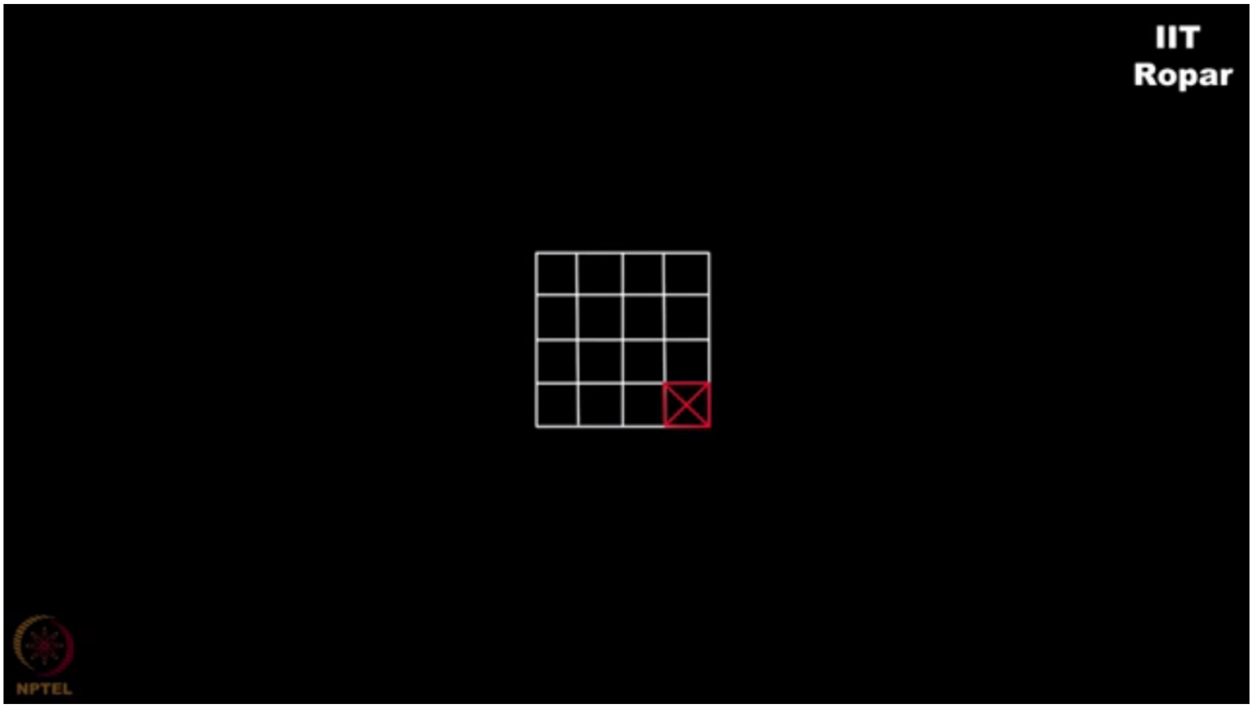
So what am I going to do first? I will consider the statement P(n) is every $2^n$ x $2^n$ checkerboard with one square removed can be tiled using triomino. What will be the base case? It is P(1) equals $2^1$ x $2^1$, which is a 2 x 2 checkerboard with one square removed. It looks like this.



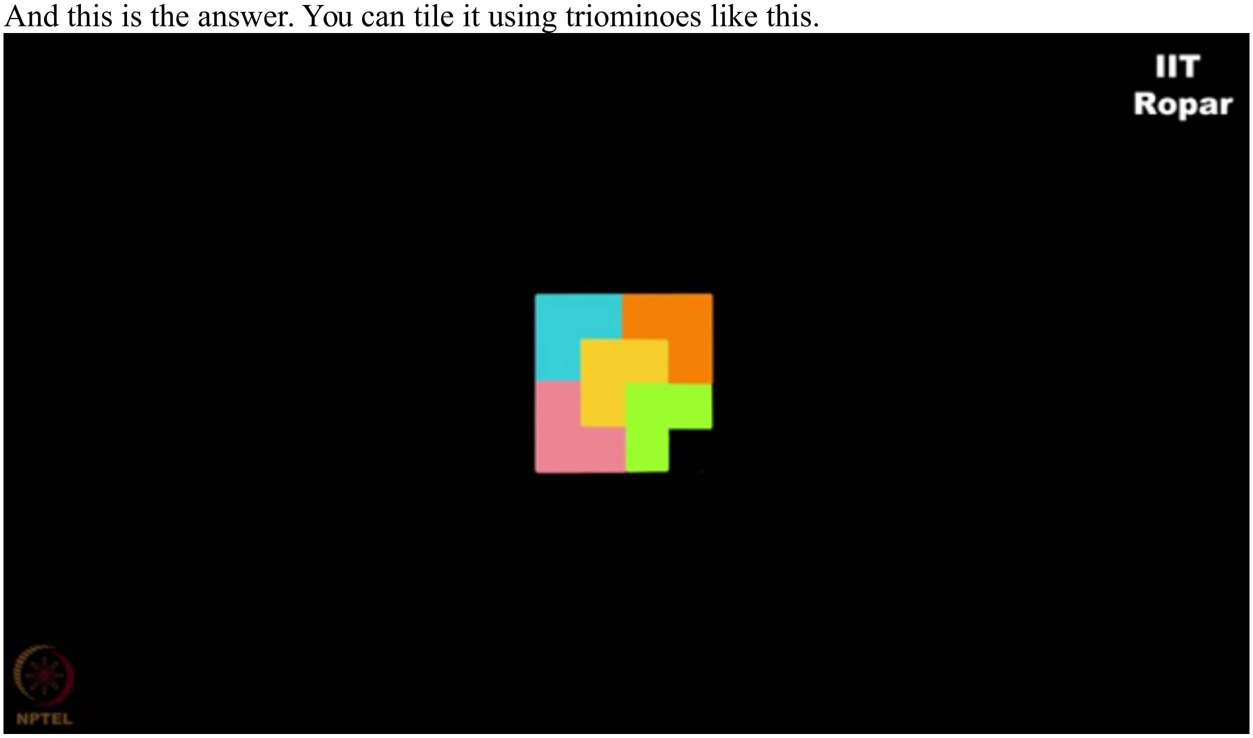
It can be tiled. How? I'm going to put a triomino here like this.


You see it is true.

So for better understanding, let me show you the answer for 4 x 4, that is P(2). This is the checkerboard. I am going to remove this square.




And this is the answer. You can tile it using triominoes like this.

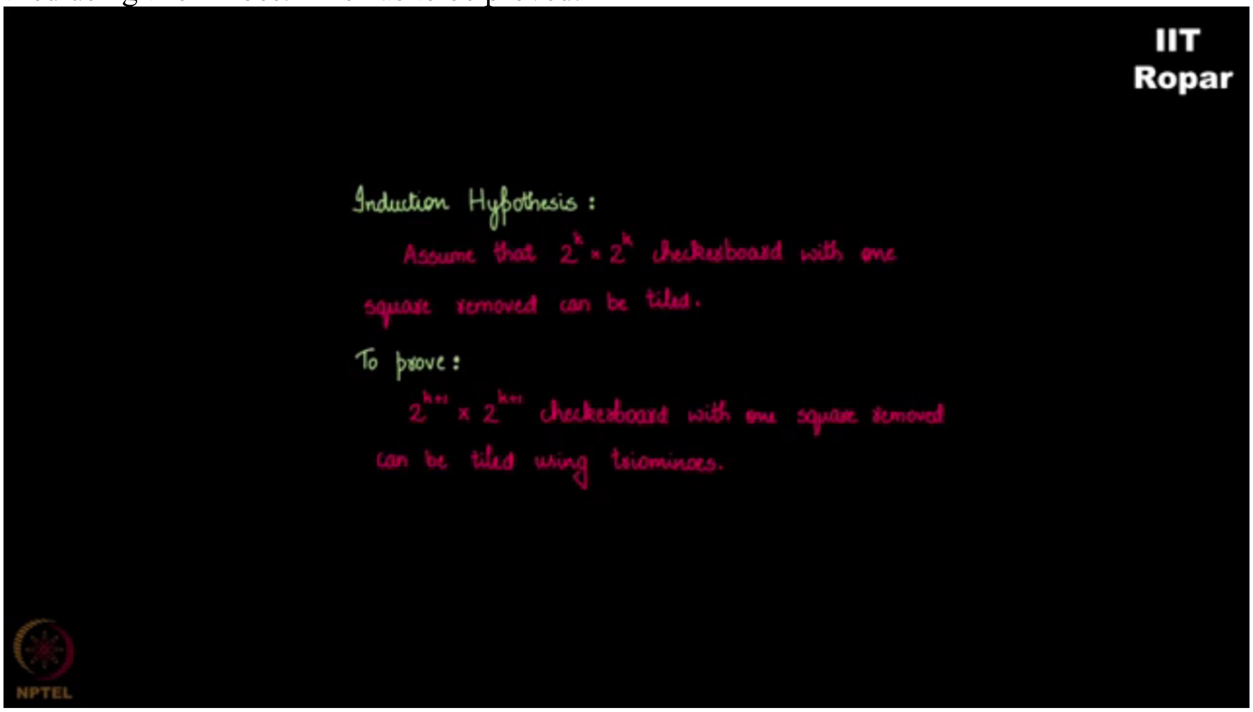



Now what is the Induction Hypothesis? It is assume that a $2^k$ x $2^k$ checkerboard with one square removed can be tiled.

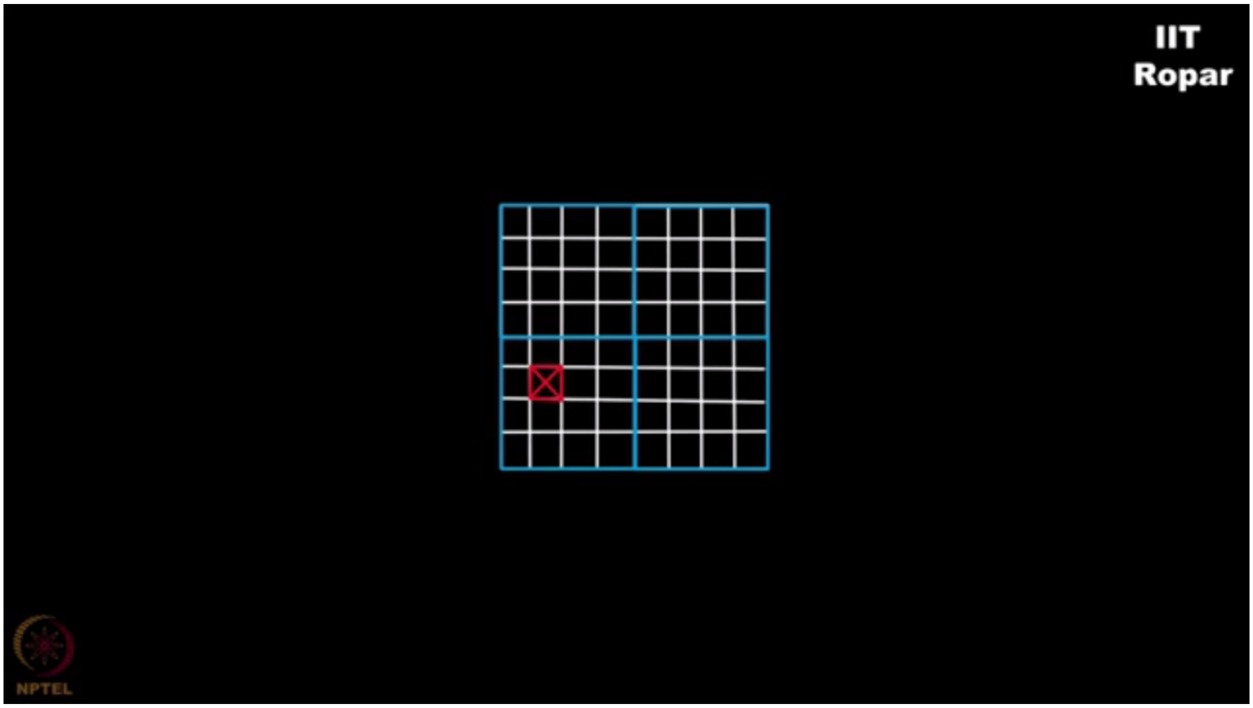
Now from now onwards I am not going to mention that one square is removed. It is assumed to be true. Whenever I am telling a checkerboard here, it means that one square is removed.

So the Induction Hypothesis is that it is assumed that a $2^k$ x $2^k$ checkerboard can be tiled using such triominoes.

Now what do we have to prove? A $2^{k+1}$ x $2^{k+1}$ checkerboard where one square is removed can be tiled using triominoes. This has to be proved.

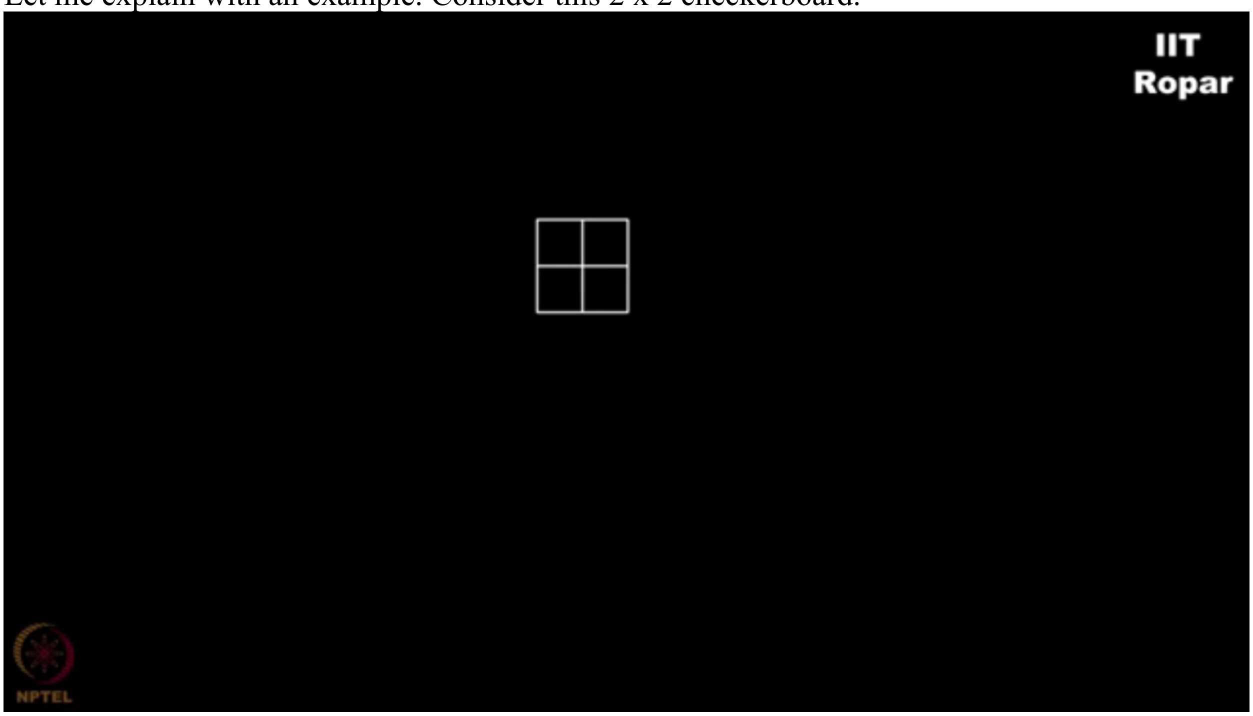



So assume this is a $2^{k+1}$ x $2^{k+1}$ checkerboard, this large one and I remove one square here. Without loss of generality, I remove this square.

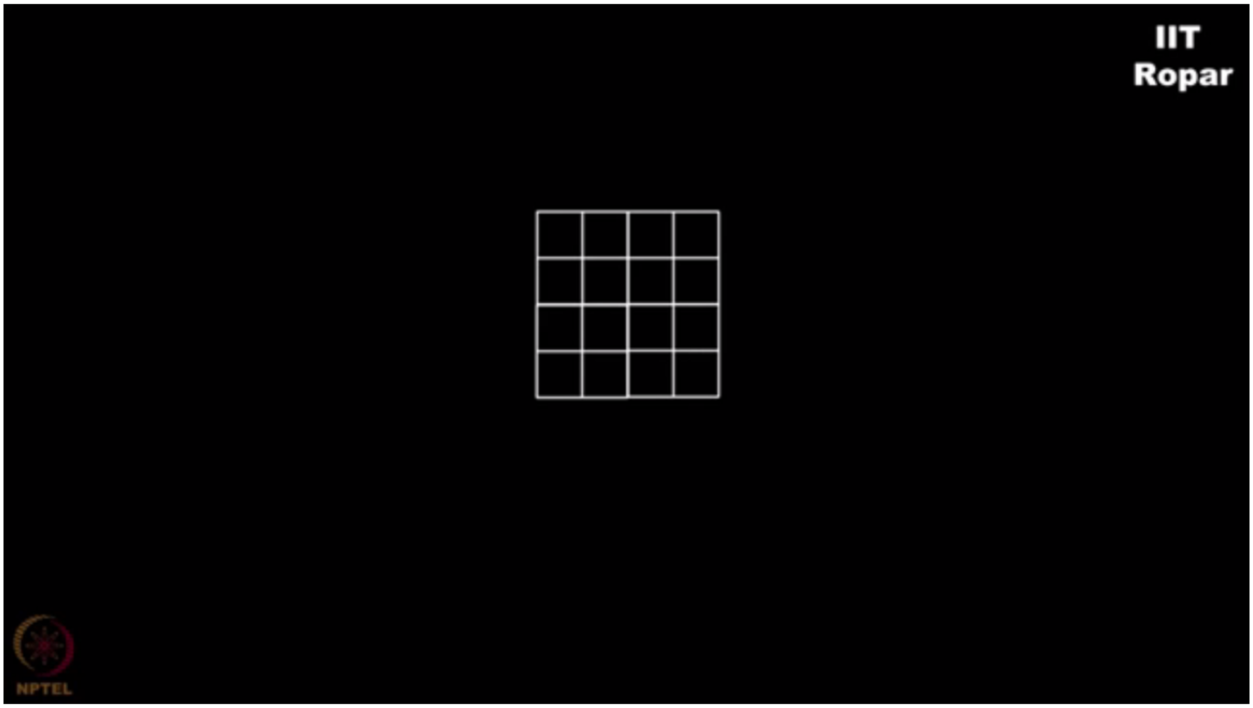



Now what we have to do is next we split this entire checkerboard into four $2^k$ x $2^k$ boards.

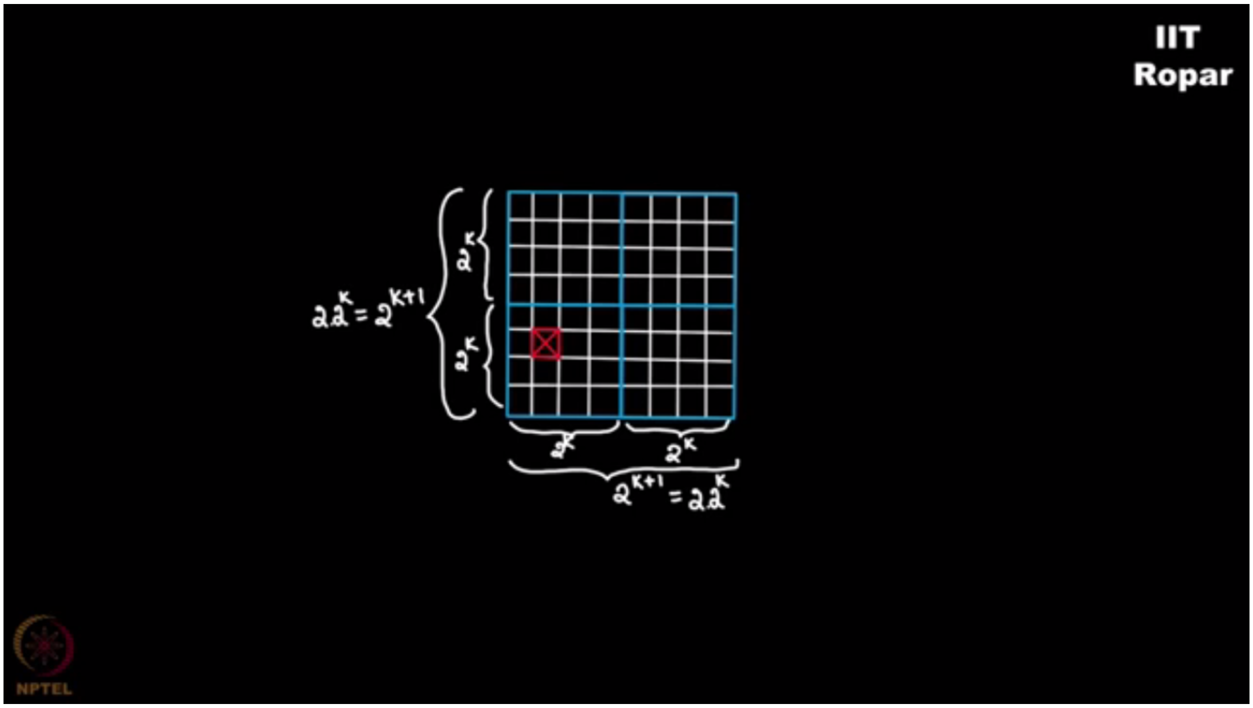
Let me explain with an example. Consider this 2 x 2 checkerboard.




Now if I expand this or if I double it, I have to double it this side as well as this side. So do you see four 2 x 2 checkerboards here?

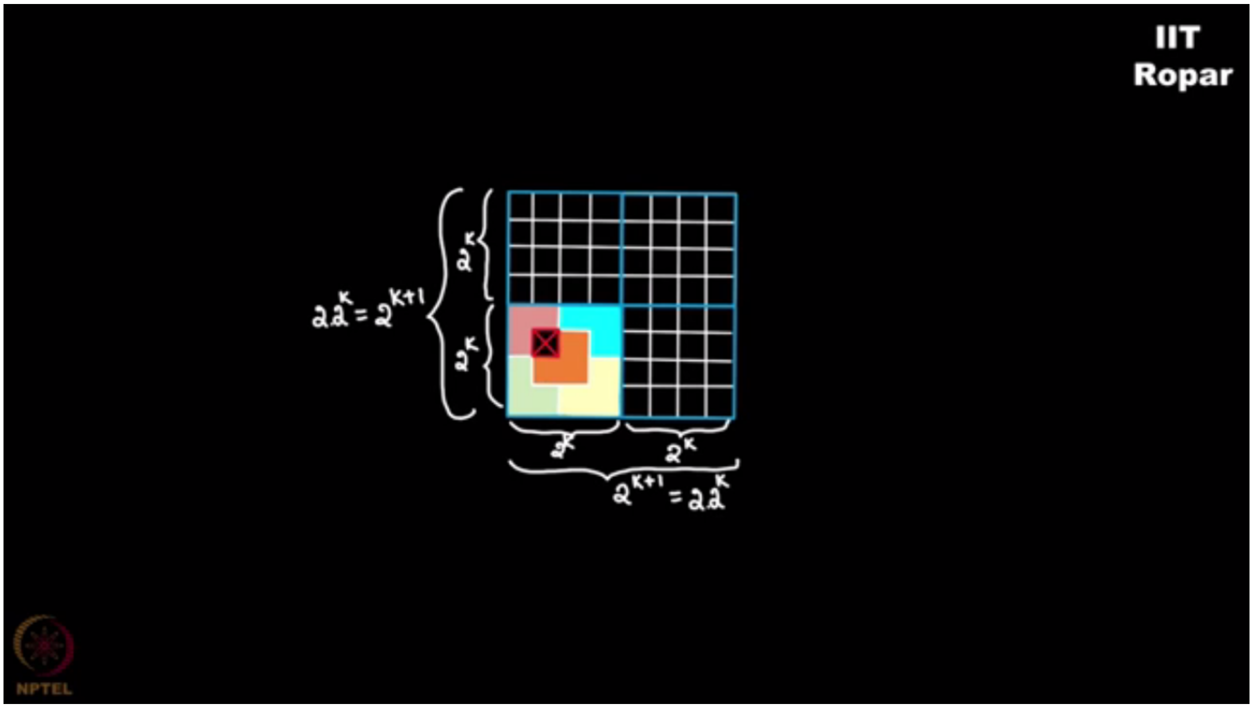



The same way here we have this entire $2^{k+1} \times 2^{k+1}$ checkerboard, which can be divided into four $2^k \times 2^k$ boards. Pause for a minute and think how it is done.



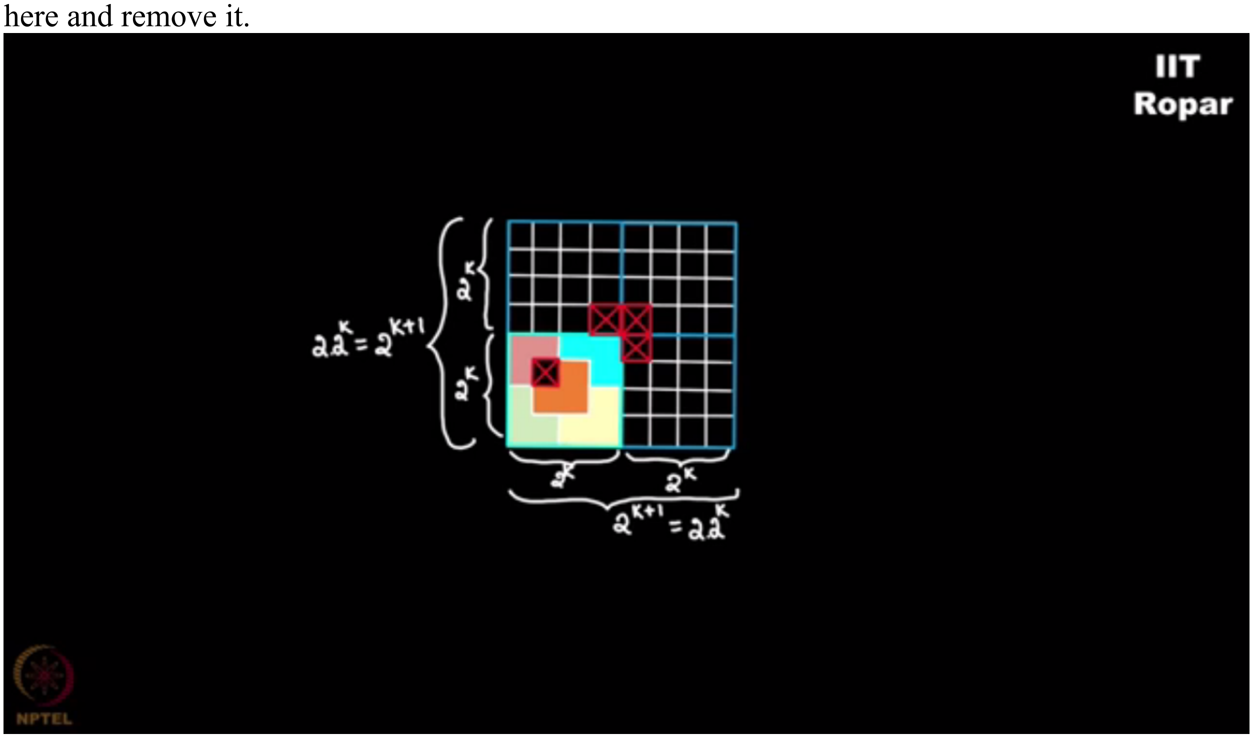

So it is divided like this. So we have four boards here and in one piece, one square is removed. I hope it is clear till now.

Now what does the Induction Hypothesis say? It says that $2^k \times 2^k$ checkerboard can be tiled. Correct? So this entire $2^k \times 2^k$ checkerboard can be tiled. Done. One is done.

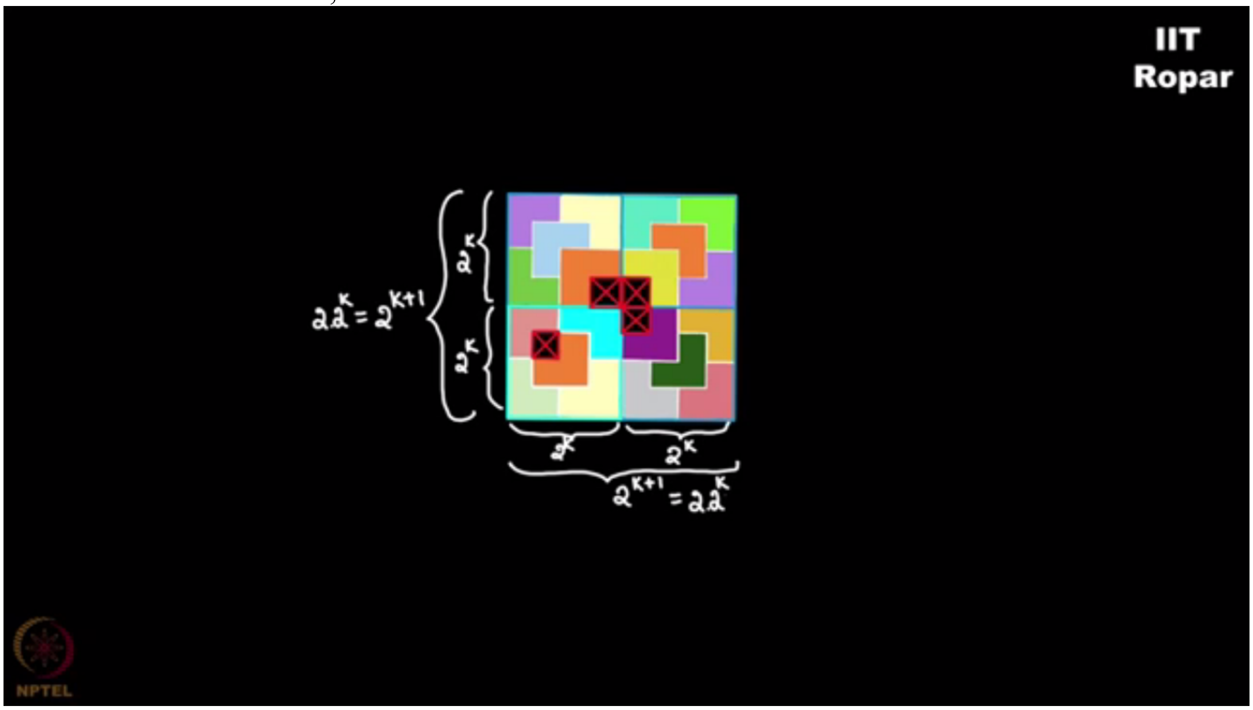



Now three more pieces remaining here. We have to tile all the three to say that the entire $2^{k+1}$ x $2^{k+1}$ checkerboard can be tiled. What do we do?

Look at the ingenuity of the proof. What I am going to do here is consider the center of the checkerboard. This corner, this corner and this corner. I am going to consider the three corners here and remove it.

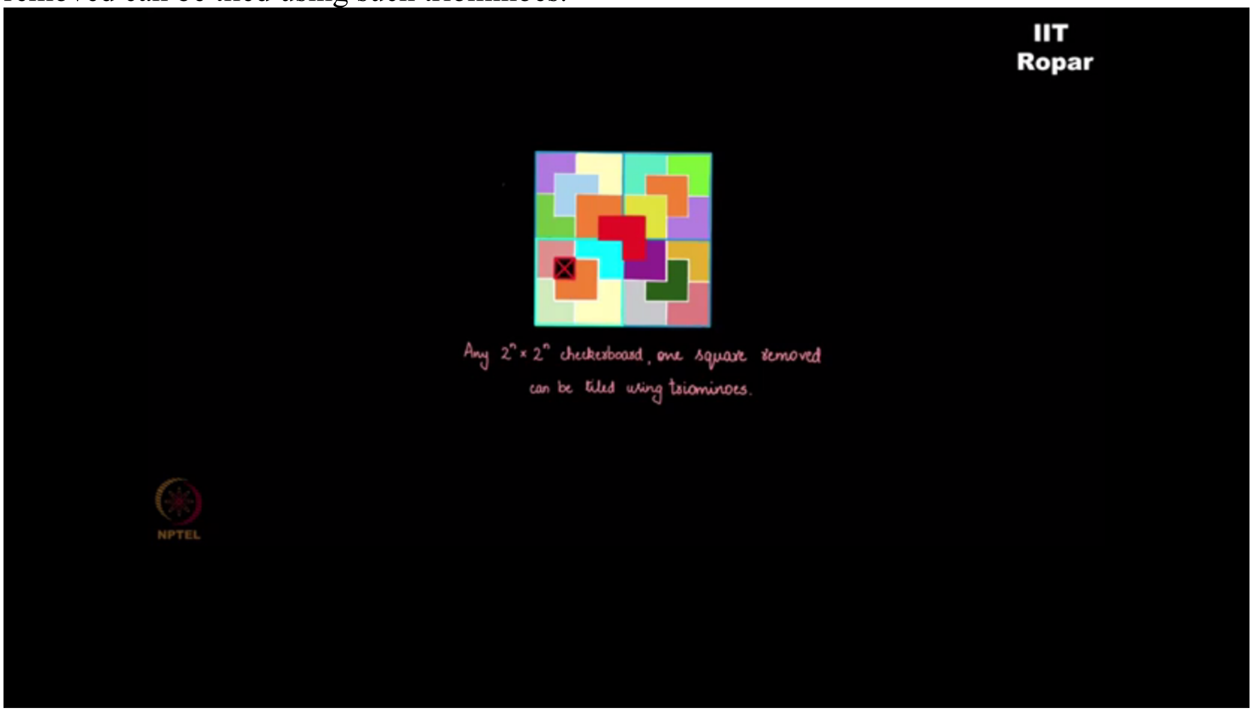


What do we have now? We have three $2^k$ x $2^k$ boards with one square each removed. Correct? Now it is very obvious from the hypothesis again that all these can be tiled. One piece was earlier tiled. Now the remaining three pieces can also be tiled by the induction hypothesis. Right? So it can be tiled like this, this and this.




Now do you observe something? What does this red part indicate? It is also another triomino you see. So what do we get? The entire $2^{k+1}$ x $2^{k+1}$ checkerboard has been tiled using triominoes by removing this triomino.

If you have keenly observed, it wasn't actually removed. This was yet another triomino now. One square was removed and I kept on tiling the other pieces too. Hence, you have tiled your entire $2^{k+1}$ x $2^{k+1}$ checkerboard. Hence, we see that any $2^n$ x $2^n$ checkerboard where one square is removed can be tiled using such triominoes.

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