NPTEL
NPTEL ONLINE COURSE
Discrete Mathematics
Relations
Number of Antisymmetric relations
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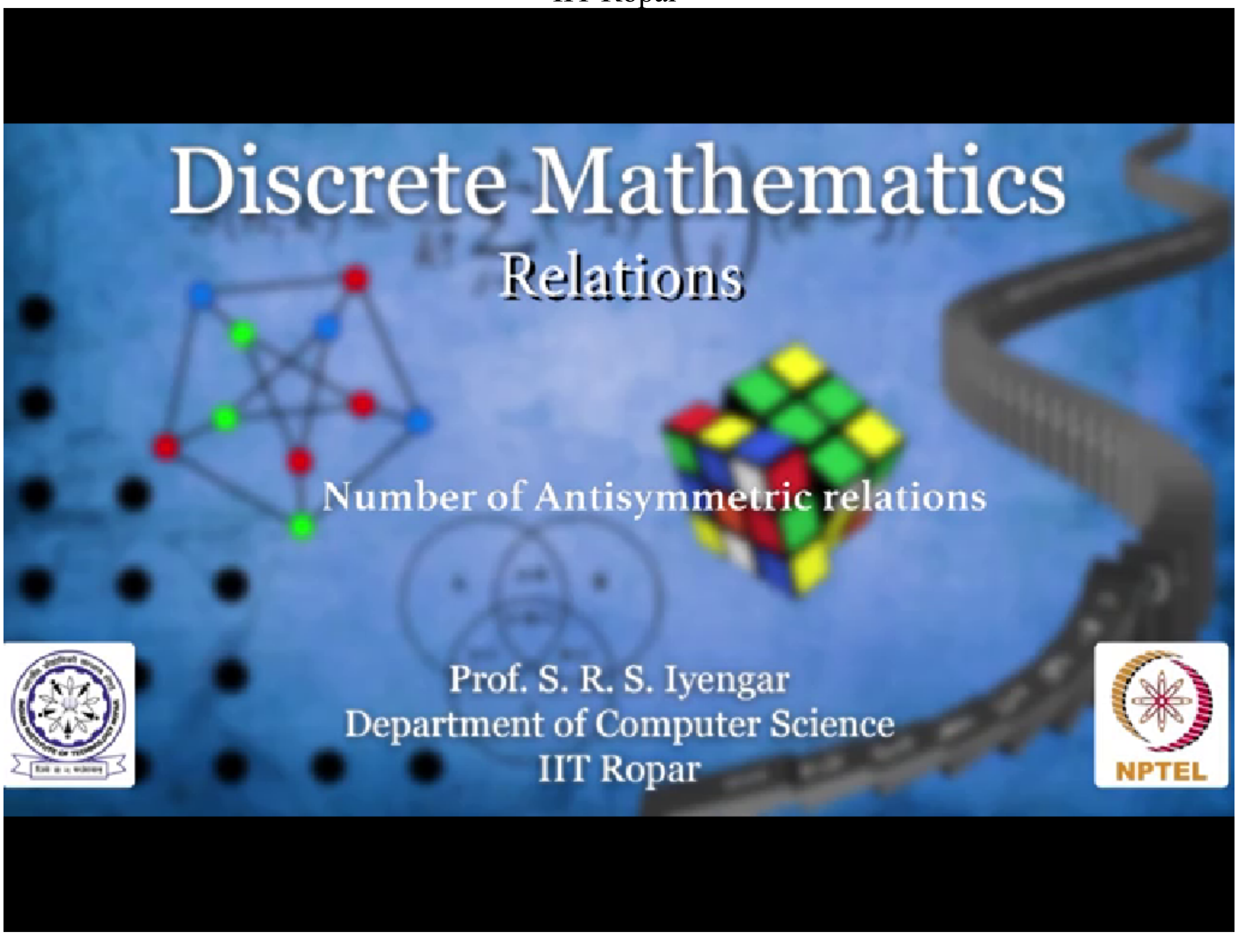
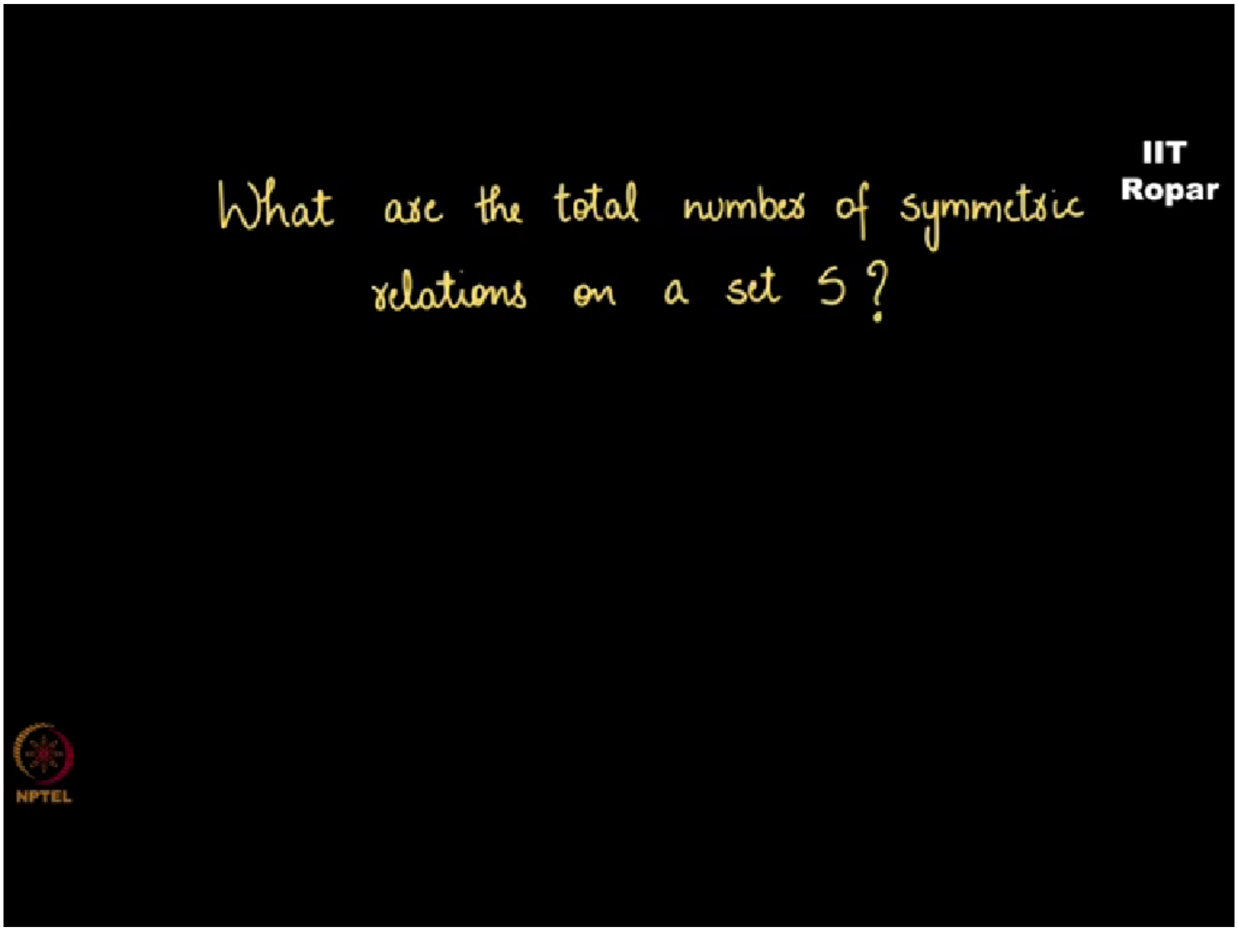
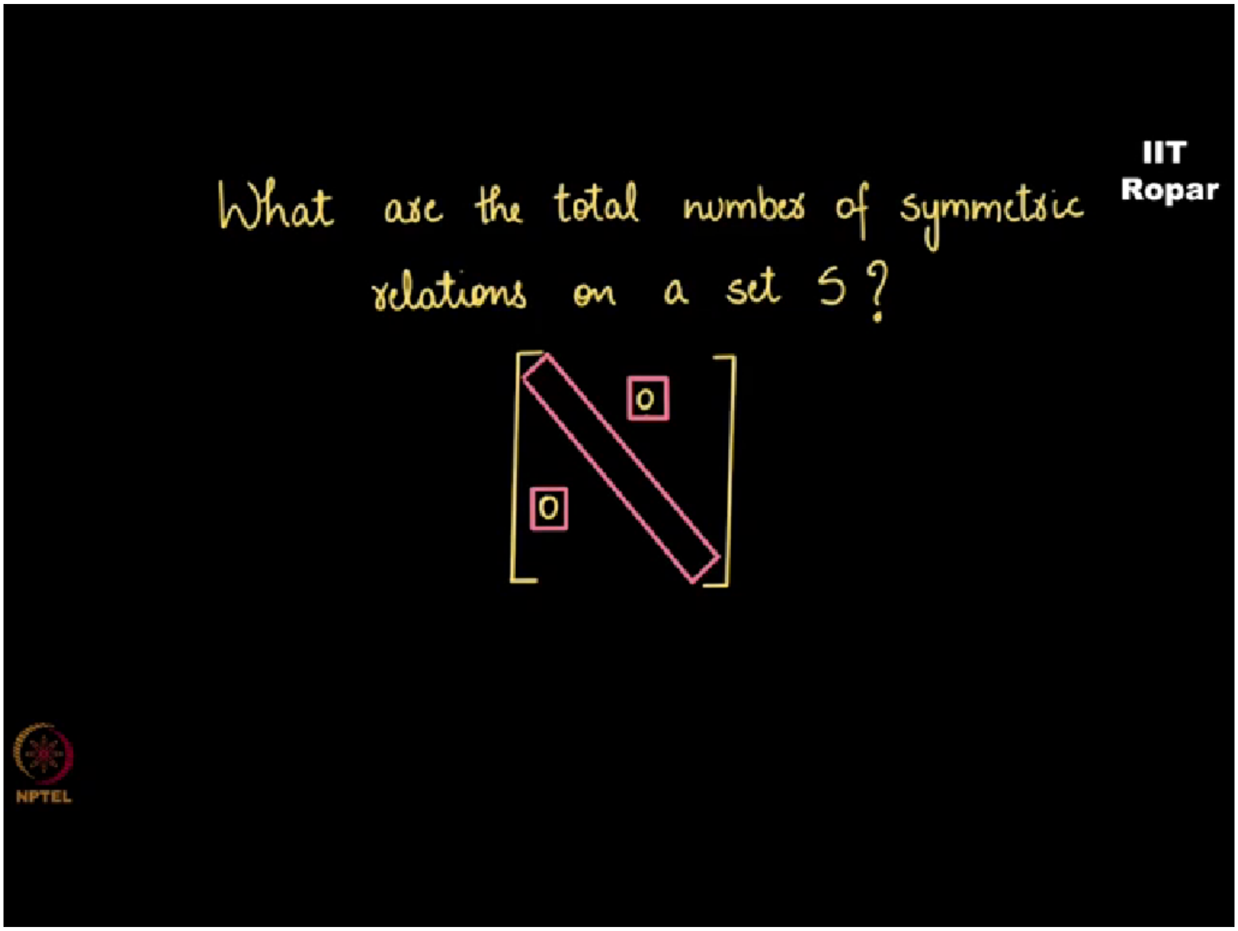


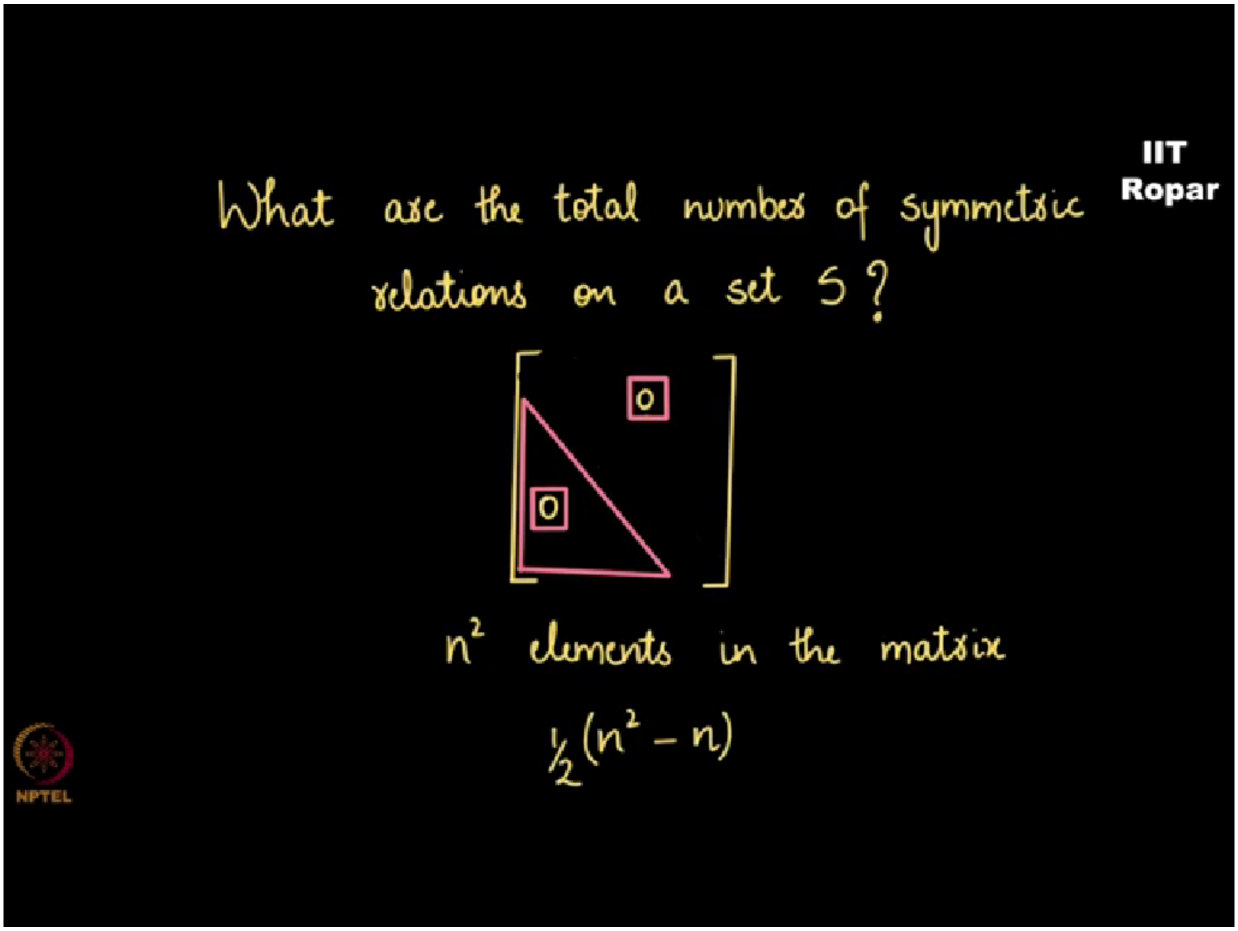
You remember right we asked this question, what are the total possible symmetric relations on a set S? Right?



We even derived an answer for it. I am going to derive an answer using a slightly different logic. You must be wondering why are we discussing symmetric relations while we were supposed to discuss antisymmetric relations. These two are related, so it is easy for us to see how we solve the symmetric relations question and counted the total number of them. With that help we are going to answer the question of how many antisymmetric relations are there on a set S? So how many symmetric relations are there on a set S?
Let's look at the corresponding matrix of the relation. We know if it is symmetric, diagonal can be anything, right, but then whatever you write below the diagonal, let's say I_J^{th} entry here should be the same as the J_I^{th} entry here. If it's a 1 here, it should be a 1 here. If it is a 0 here, it should be a 0 here while the diagonal can be anything.

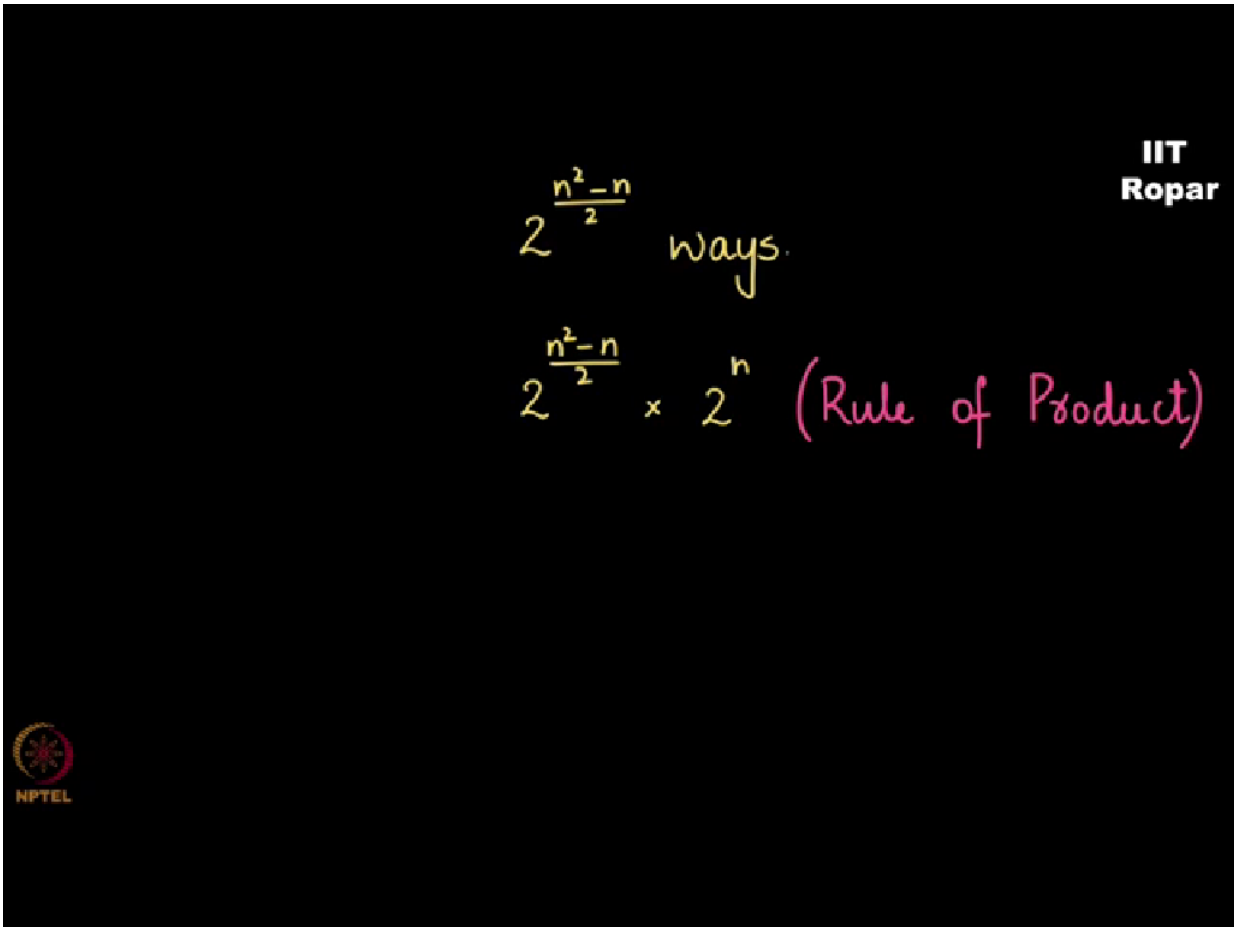


So now you have your freedom to fill up whatever you want here and things get fixed here based on what you fill here, right? So how many elements are there here? You have n^2 elements in total in the matrix, and then I remove the diagonal from the matrix, which is minus n elements, and then take only half of it, this part, half of it. So this will be the number of elements in this zone, right?

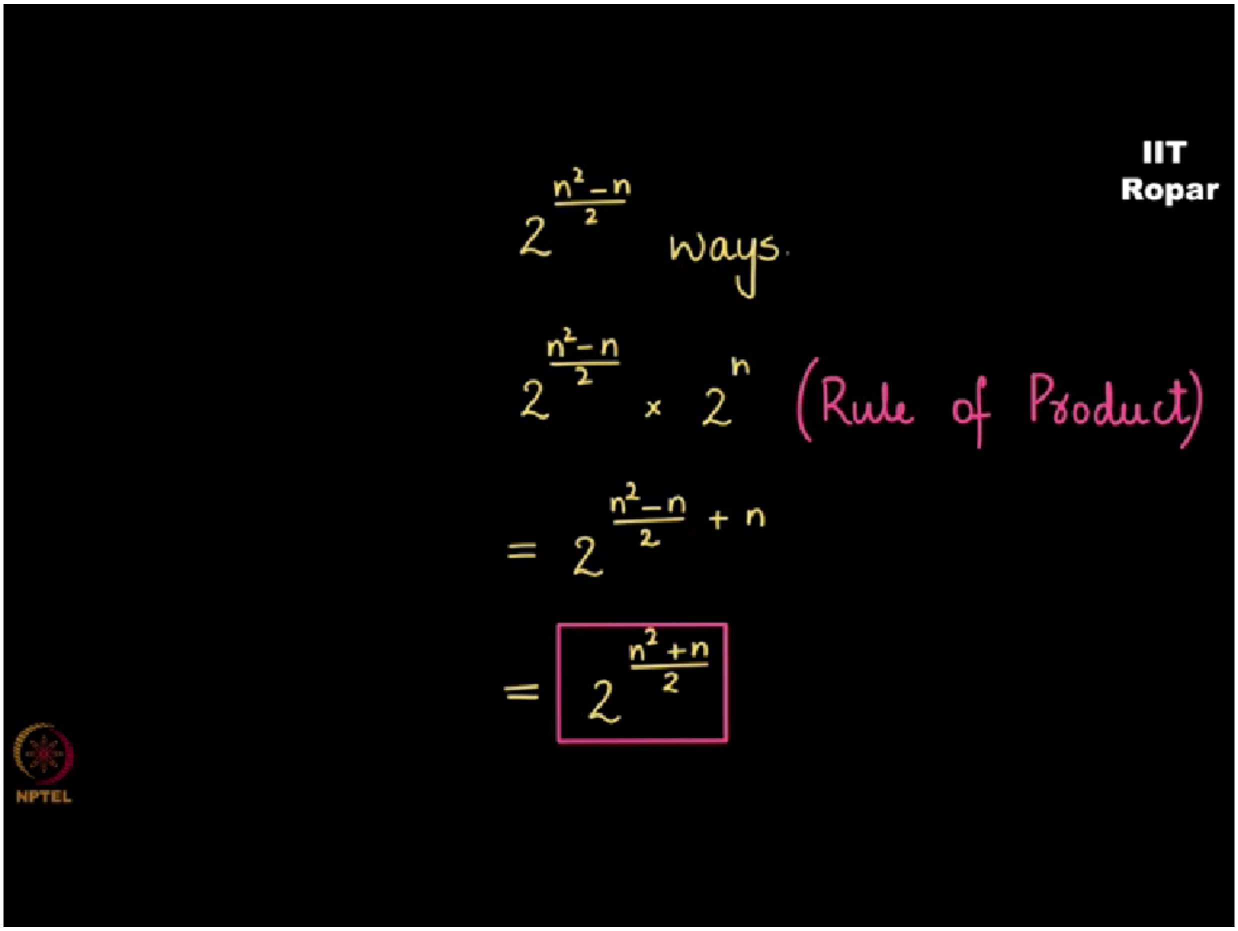


And I have my freedom to choose whatever I want here. Correct? So I can do it in 2 to the power of n^2 minus n by 2 ways. You all know why it is 2 to the power of n^2 minus n by 2 ways. Correct?

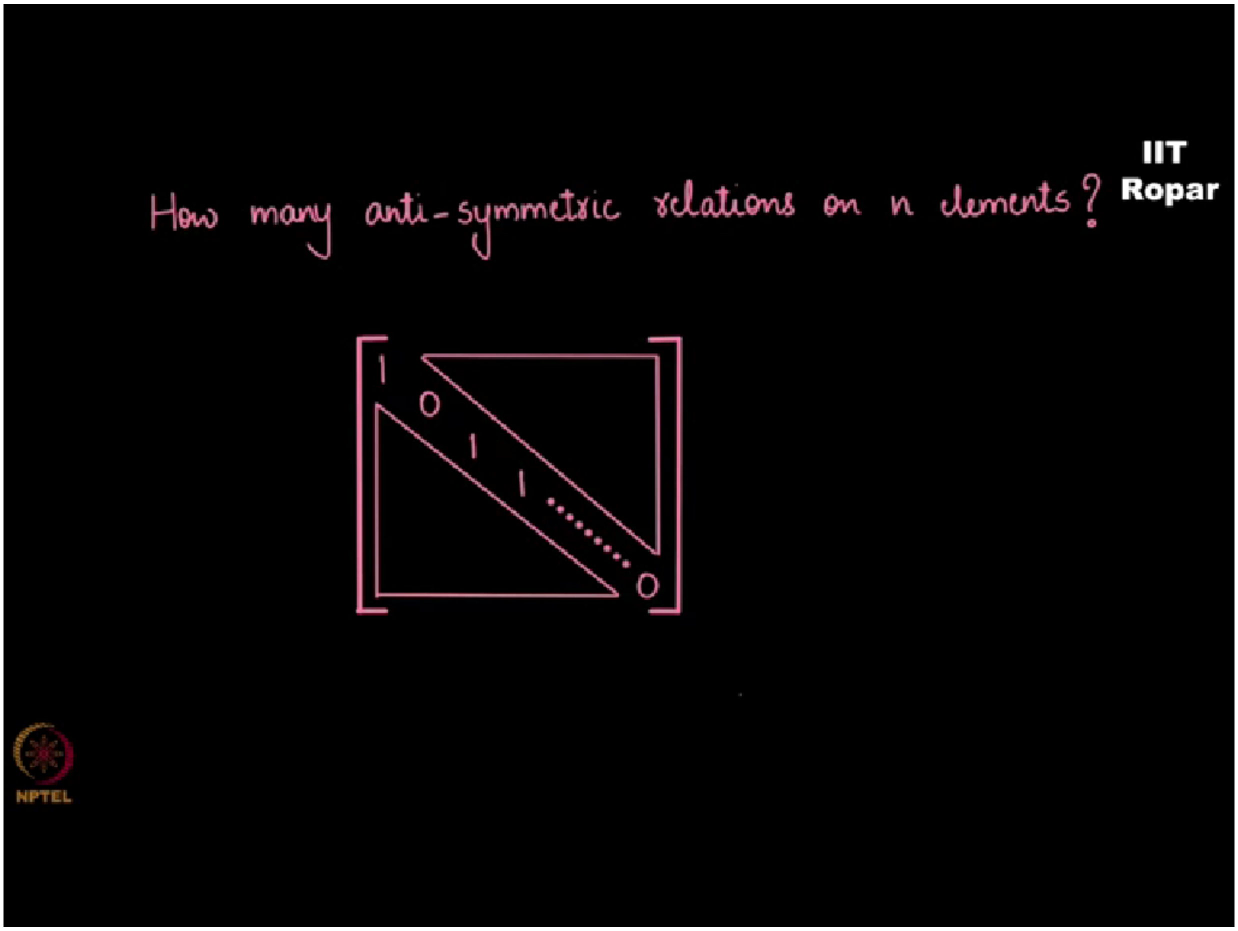
But then after this, I have my freedom to choose whatever I want on the diagonal. That has nothing to do with your symmetric relation condition, right, which means I am going to multiply this by 2 to the power of n. Why? That's because for every possibility here, you can write whatever you want on the diagonal. So this if you remember is the Rule of Product that we discussed earlier.



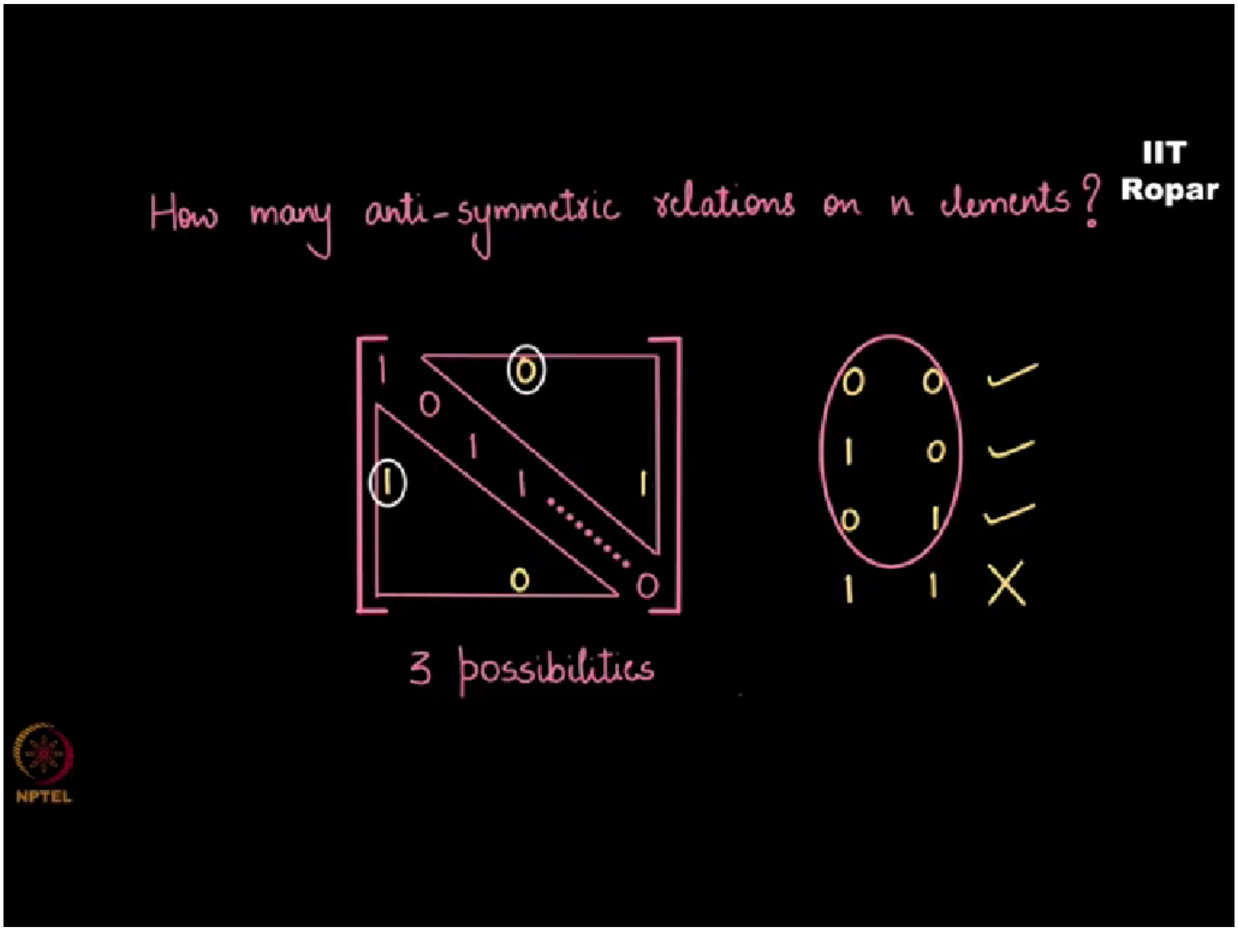
So, finally, this product will be equal to 2 to the power of n^2 minus n by 2 plus n, which happens to be 2 to the power of n^2 plus n by 2 and that's the answer that we saw already.



The logic was slightly different. This logic is slightly different given that we are multiplying it by 2 to the n, right?
Now with this as the logic, let us try to extend this logic and try to answer the question as to how many antisymmetric relations one can think of on a set S containing n elements?
Let us ask the question, how many antisymmetric relations can one think of on n elements? Correct? All right. Let’s see what is the definition of antisymmetric? As and always for counting it helps to see the matrix corresponding to the relation, right? Look at the diagonal and then look at the -- this is actually called the lower triangle. This one is called the upper triangle. I have not been using these jargons, but let's use it from now onwards.

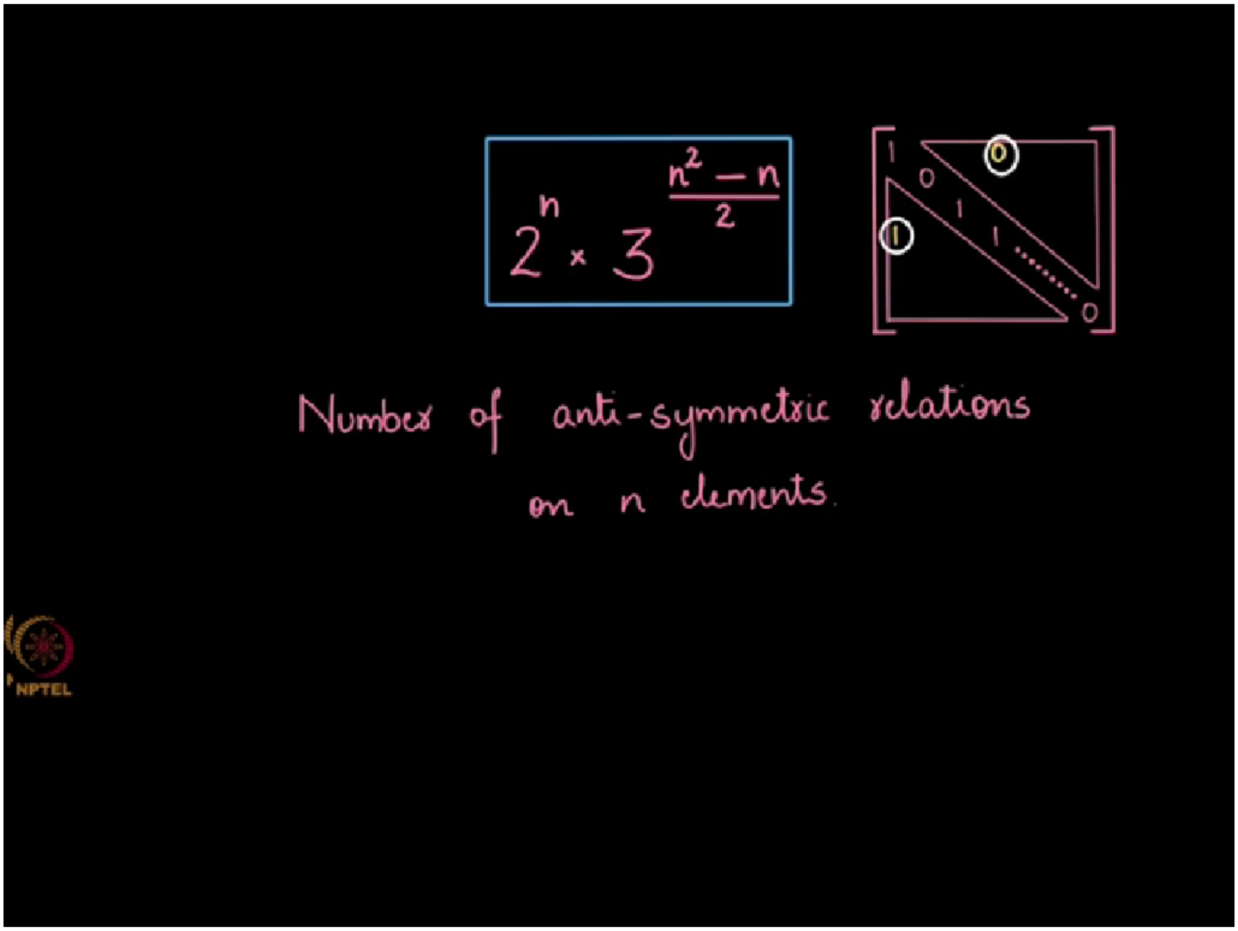


Okay. Here is a lower triangle. Here is an upper triangle. If a relation is antisymmetric, whenever you have 1 in the I_J^{th} entry, you will have a 0 in the J_I^{th} entry. You cannot have a 1 here unlike your symmetric relation where if you had a 1 here, you should have a 1 here. In antisymmetric, if you had a 1 here, for sure you will have a 0 here. But if you had let’s say 0 here, you may or may not have 0 here, right? That's what you mean by antisymmetric. By antisymmetric you mean 0 0 is possible, 1 0 is possible. 0 1 is possible, but 1 1 is impossible. So 0 0 is possible. 0 1 is possible. 1 0 is possible, but 1 1 is not possible.



So for every I_J^{th} entry and the J_I^{th} entry, you can fill them up with 0 0 or 0 1 or 1 0. You have three ways of filling this pair, so which means you can put whatever you want in this lower, upper triangle combination, but you must ensure that it has to be a 0 0, 0 1 and 1 0 only, which means you have three possibilities for so many pairs.

How many pairs? N^2 minus n by 2 pairs, right? How many such pairs are there? If you start counting this element and this element as a pair, I_J^{th} entry and the J_I^{th} entry as a pair, you have N^2 minus n by 2 such pairs. Correct? For each of these pairs, you can fill them with three possibilities: 0 0, 0 1 and 1 0. So these are the total ways in which you can do it, but wait a minute. Your diagonal can be anything. Correct? You can put whatever you want here, any combinations of zeros, ones or whatever. You can do that in 2 to the n ways.



So you got to multiply this huge possibility of 3 to the n^2 minus n by 2 with 2 to the n, thus making the answer to be 2 to the n times 3 to the n^2 minus n by 2. And what is this? This is the answer for our question, how many antisymmetric relations can one think of on a set with n elements? The answer happens to be this.