Probability & Computing
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Module - 01
Introduction to Probability
Lecture - 02
Segment 2: Axiomatic Approach to Probability Theory

So, we are now going to talk about a more formal or what is called axiomatic approach to Probability Theory. So, what is the plan? So, what we you may recall that we have already established a need for formalism and need for a quantifiable way to understand nature.

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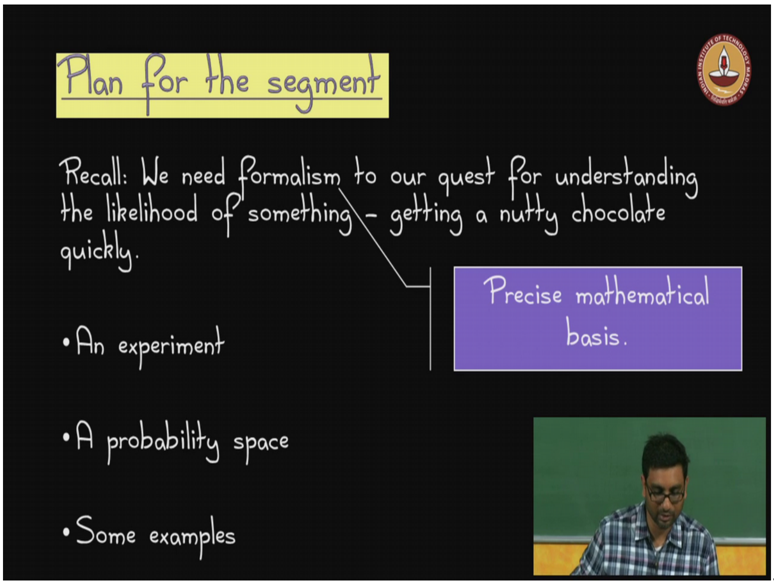

If you will in particular one of the examples that we talked about was getting a nutty chocolate in last seminar ok.

And what we want is a precise mathematical basis and so, we are going to provide just that. And I want to impress again I repeat myself a little bit, but impress again that it took about 300 and 400 years to get to this level of formalism. And it is easy to assume that this level of formalism and clarity comes for free, because someone has given it to you, but if you are interested in creating theory and new ideas that are going to last that long it is important for you to learn how these things come about ok?

So, take some time to appreciate the clarity with which people have formalized this notion. So, basically probability theory is just used to understand some natural phenomenon and we want to capture that by the notion called an experiment. And then on top of that experiment the notion called experiment, we are going to lay down some probabilistic a notion called the probability space. And on this basis we will conduct the rest of our understanding of probability theory in this course and pretty much all of modern probability theory right, you will see some examples in today is segments.

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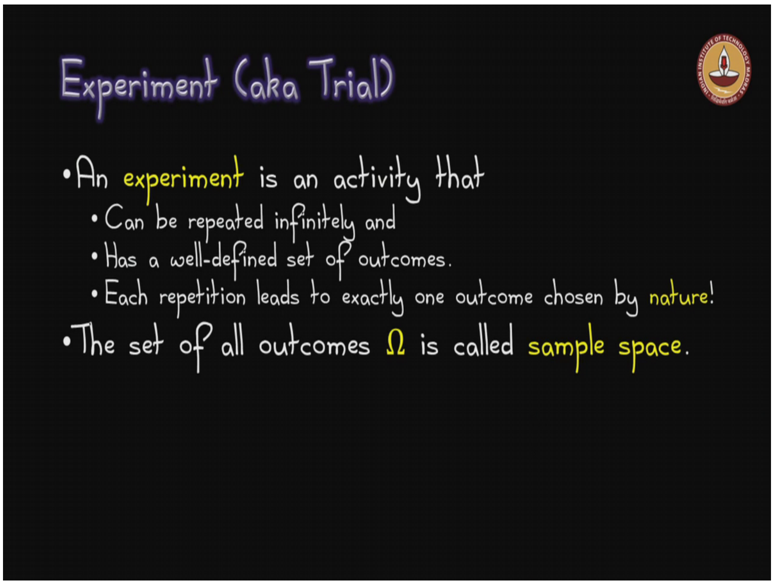

So, let us start with this basic notion called an experiment, if this is the very basis of probability theory it is often not explicitly mentioned, but you have to realize that this is the very basis it also goes by the term trial. What is an experiment it is an activity typically a natural phenomenon and natural activity, but sometimes it can also be artificially done. So, for example, a computer can also perform something in a in a random fashion and that also can be treated as an experiment ok.

It is something that can be repeated as many times as needed at least mentally speaking has a well-defined set of outcomes. So, in this experiment should have a clear set of outcomes? And each time the activity is repeated you can think of it as a nature as choosing one of the outcomes, through some means; we do not know we are not going to get into how nature behaves in specific terms?

But we are just going to assume that nature is going to pick one outcome and the set of all possible outcomes is called the sample space ok. So, given an experiment you will have to have a sample space non-empty sample space. And if the sample space is exactly one then the activity is called a deterministic active, but if the sample space is more than one then it is a randomized probabilistic at experiment ok.

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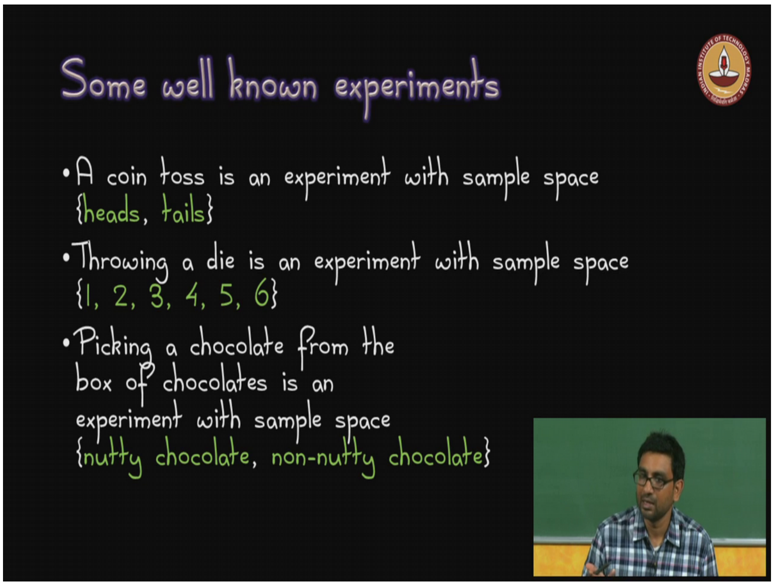

So, let us look at some examples and this is going to be a repetition of ideas that you have already seen before a coin toss is probably the most fundamental experiment that you can think of and of course, it is sample space is heads and tails these are the 2 elements in the sample space throwing a die is another experiment with sample space ranging from one through 6 ok.

There are all things that you have seen and connecting with the example that we discussed yesterday you know our sample space is one of 2 things either naughty chocolate or a non-naughty chocolate here is a more complex experiment ok.

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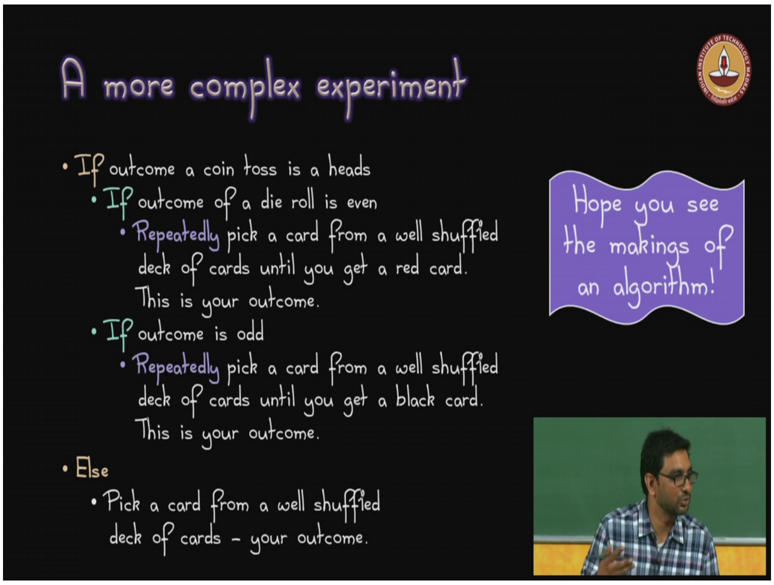

So, we are just going to put together a few experiments and create a new experiment you can do that.

So, now what we are going to do is we are going to toss a coin first ok. If the outcome is a heads we will do something otherwise we will do something else. If the outcome is a heads then we will roll a die. So, after we roll a die we repeatedly pick a car from a well shuffled deck of cards until you get a red card. So, that is if the die roll comes up even, but if the die roll comes up odd, we repeatedly pick a car from the well shuffled deck of cards until you get a black card or if the original coin toss was a heads well you pick a card from a well shuffled deck of cards that is your outcome.

If you did follow all the details do not worry, because the point was not for you to follow all the details, but the point was to illustrate that you can put together smaller experiments, in some interesting way and create larger experiments. Where what is the outcome of this experiment this complex experiment that is there on this slide you have 52 cards at your sample space.

And at the end you are drawing a card from this sample space right. All the inner workings are what nature does if you will ok? What matters is that the sample space of this experiment is the 52 cards and you are going to draw one from that ok. And importantly hope you see that the makings of an algorithm. So, look the way we have expressed this experiment should descend remind you of an algorithms course pseudo code and that is exactly the point.

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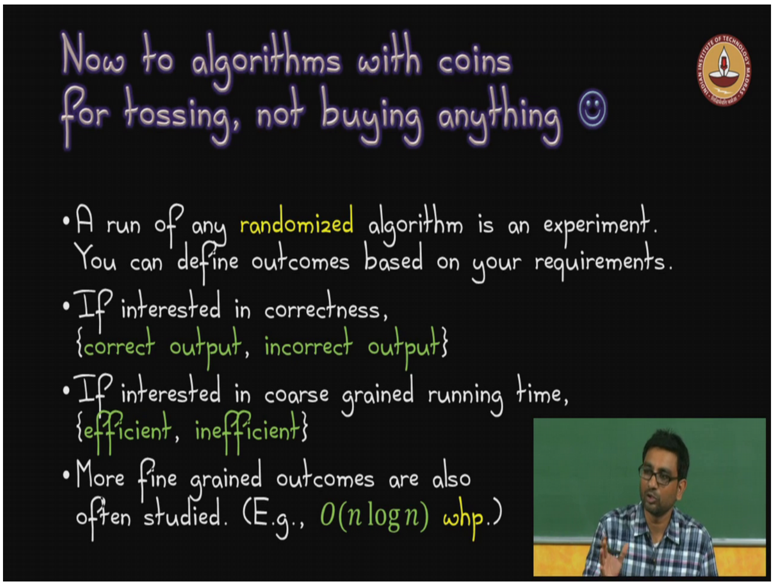

So, any algorithm that we design with the capability of tossing coins is essentially an experiment. So, now, the algorithm that we are going to talk about has at it is disposal coins that it can toss not for buying anything, but tossing and based on the outcome of those coin tosses it can make decisions right. So, basically any randomized algorithm therefore, is basically an experiment ok.

And you can define the if it is an experiment you should be able to define some outcomes, without looking at specific algorithms yet let us look at what are some interesting ways you can come up with outcomes. The most basic way of looking at outcomes is whether your algorithm is as arrived with the correct output or the wrong output. Now this algorithm is tossing coins along the way making lots of decisions going in some random way, what if it leads to a wrong output that is a serious concern that you should have ok.

So, you want to have it if that is your concern your outcomes that you care about will be correct output versus incorrect output ok. Let us assume you have established that your algorithm is correct or at least mostly correct. If you might be interested in the running time ok.

So, easier algorithm going to be and let us say at a high level you just want it to be able to execute in reasonable time, then you can still define the outcomes to be either efficient or inefficient ok.

And if you want to be even more nuanced about it you can look for some fine grained outcomes ok. So, an example context that we will be seeing in this course is that after you run an algorithm, you want to ask the running time been small as in within of n log n or not. And you want to then argue and here I am going to use terms that I have not yet defined, but you might be interested in things like this is running time o of n log n that happens most of the time. So, the technical term is with high probability that is what I have as w h p?

So, basically any algorithm that you run with the ability to pass coins you have to think about the outcomes and you have to reason about the outcomes. And so, you see how these outcomes naturally fit with our understanding of algorithms, if you had a basic course in algorithms all the questions that you talked about there can be phrased in this randomized or probabilistic way ok. That is the important insight that you should have now.

We defined an experiment.

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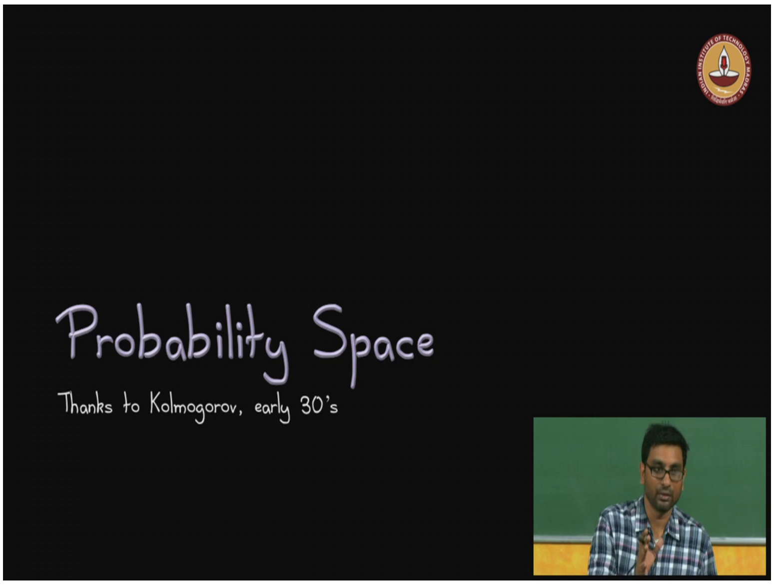

It is basically something that has outcomes we want to now provide some quantification to these outcomes ok. That is where the probability space comes up now you start with a sample space.

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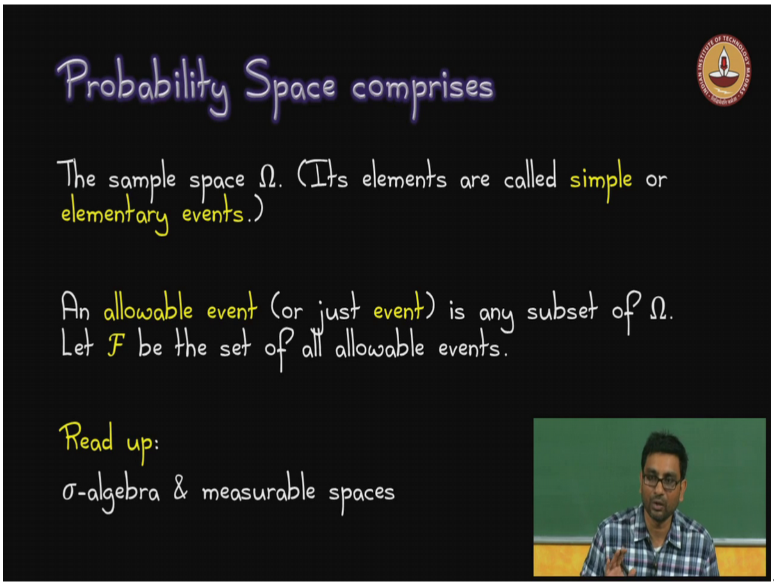

And of course, if there is a sample space; that means, there is an experiment from which the sample space is coming ok. The outcomes listed in the sample space are typically called simple or elementary events, but you can also define some more interesting events allowable events or just an event is basically any subset of this sample space give you a quick example before we move on.

So, now let us say you roll a die the sample space is 1, 2, 3, 4, 5, 6, but you can also define an event which is an even number and that is like 2, 4, 6 that is a subset of the sample space and that is an event odd even umm you know all any subset of the 1 through 6 is going to be a an allowable event. And the set of all allowable events we call it F, we are going to stop at this level for now basically you have a sample space and a set of subsets of the sample space and that is what we care about and called F?

If you are interested in exploring a little bit more what I would ask you to do is Google up these 2 terms; sigma algebra and measurable spaces ok. And you learn a little bit more about the formalism that goes behind this by the way I am going to be posting these slides as we go along. So, just to let you know. So, you may use that as a reminder to

Google these 2 terms to get a little bit more formal understanding of what we mean by these things ok.

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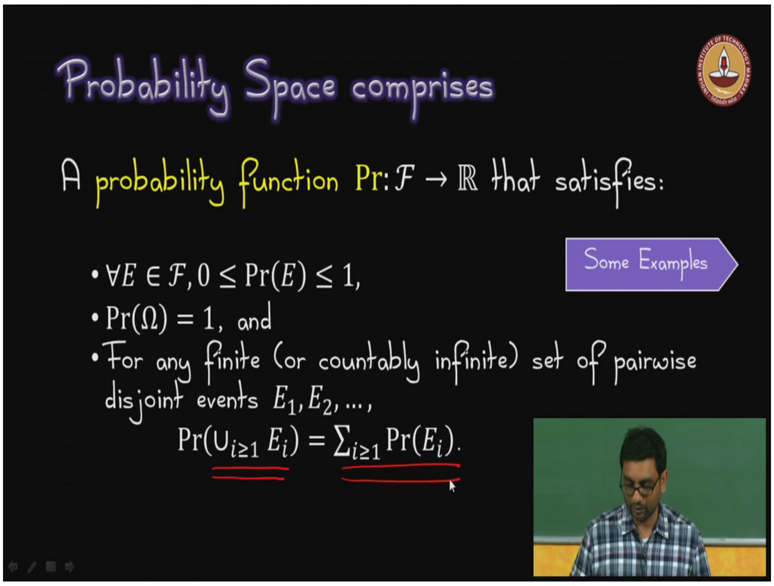

Now, we can quantify them. So, now, we define what is called a property function, basically it is a function that takes as input one of the events basic since that is the subsets in F and maps them to real values this is a quantification part.

So, any event that you can think of has a real value associated with it and that real value tells you how likely nature is to choose that particular event. And now we come to the axioms. There are 3 axioms that this probability function has to obey and that defines a probability space the first one is basically a normalization axiom basically quantification.

So, this probability that we talk about is at least 0. Also the probability of the entire sample space because the entire sample space itself sub is an event if you think about it and that the probability of that event must be a 1. And the third axiom is now if you look at any set of pair wise disjoint event.

So, now these are member events are sets. So, you can talk about pair wise disjoint sets. If you look at pair wise disjoint sets and you consider the event obtained by taking the union of those events. So, this is the new event obtained by taking the union of all these other events the these events e 1 through e 1 and so, on even need to answer.

Now, that has to be equal to the sum of the individual probabilities. So, these 3 are the basic axioms they are very simple and very intuitive help you helps capture the notion of probability. And the entire field of probability theory is resting on these axioms. So, it is as simple as it is it is also very important so, very clear about that ok.

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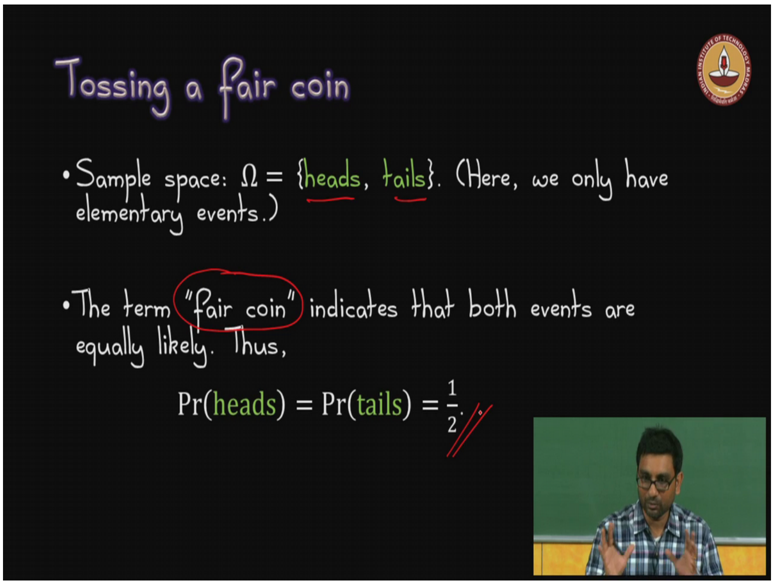

Now, let us look at some examples. So, what do we mean by so, let us go you know let us go back to this event this experiment of tossing a coin. Now what we were going to do to understand the notion of probability space we are going to impose quantification and we are going to do that by implicitly calling it a fair coin when we call it a fair coin.

What we are doing is for each of these outcomes, we are going to assign a value half. And because it is fair they both have the same value, which means they are both equally likely. This is of course, I am sure an experiment that you have performed you know very well, but I just want to take these basic concepts that you probably already know, make sure they are you know well set well within the theory of probability, you can go to the next level what if you toss the coin twice.

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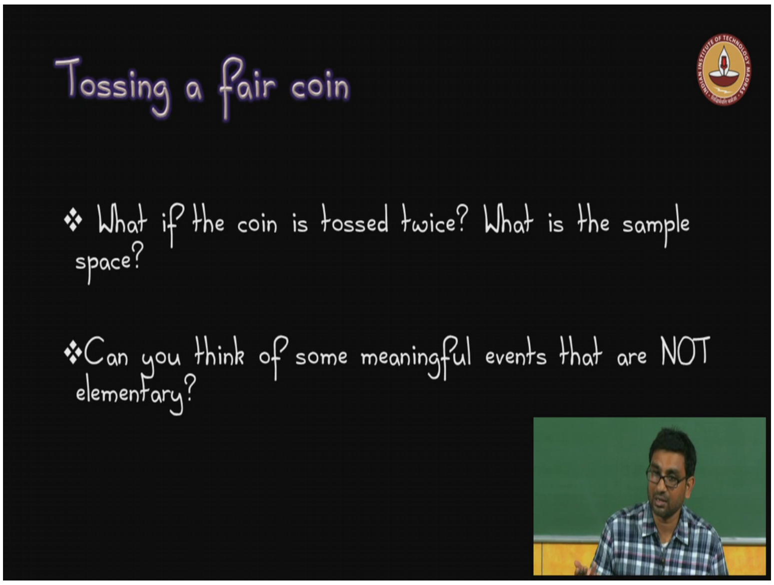

So, what is the sample space? So, at least what is the cardinality of the sample space when you toss it twice 4 ok? And then one more question for you. So, can you think of some meaningful events that are not elementary?

For example, if you think of this tossing of a coin twice you can think of a non-elementary outcome for example, both coin output toss outcomes have to be the same. So, that would be an event that corresponds to either H or T both heads or both tails ok.

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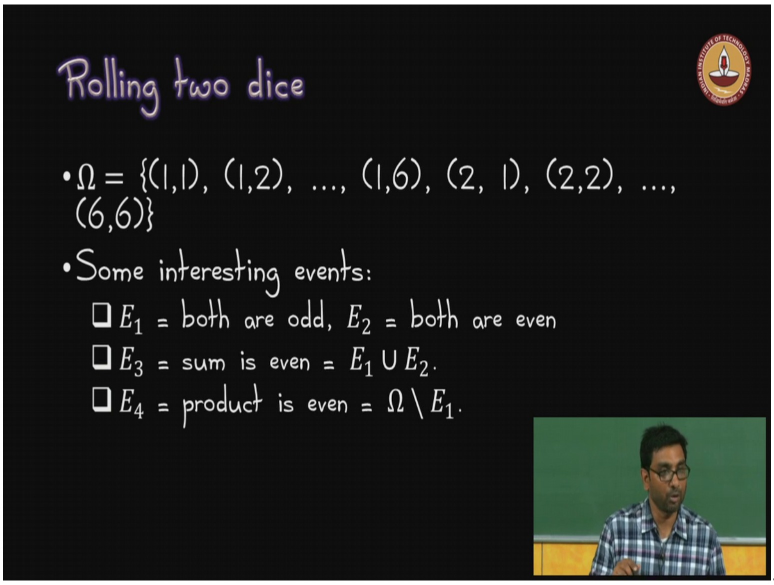

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So, in rolling 2 dice. So, now, what is the cardinality of this sample space? It is 36. So, now, we can think of some interesting events. So, E 1 so, this is the sample space, but even so, you can define it based on your requirement right. So, let us think of a few events E 1 is the event that both are odd E 2 is event that both are even, E 3 is let us say the event that the sum is even.

And that is by the way the same as the event E 1 equal union E 2, because if the sum has to be even either both are have to be odd or both are to be even E 4 is the event that the product is even ok. So, the only way you cannot have the product being even if is both are odd. So, basically it is a sample space minus E 1.

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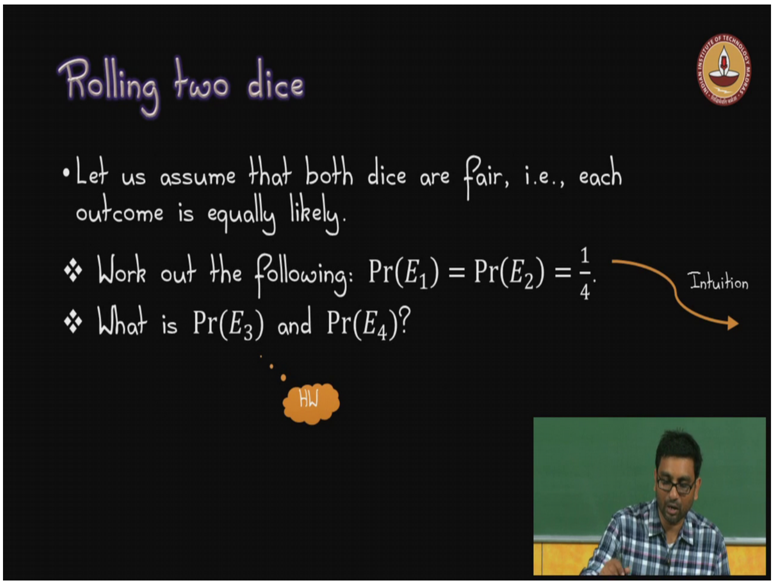

So, what we are going to a do now is we are going to argue that the probability of E 1 and probability of E 2 are both 1 4th intuitively, it should already make sense to you if you think about it, but we will just work through it carefully and you can similarly work out E 3 and E 4.

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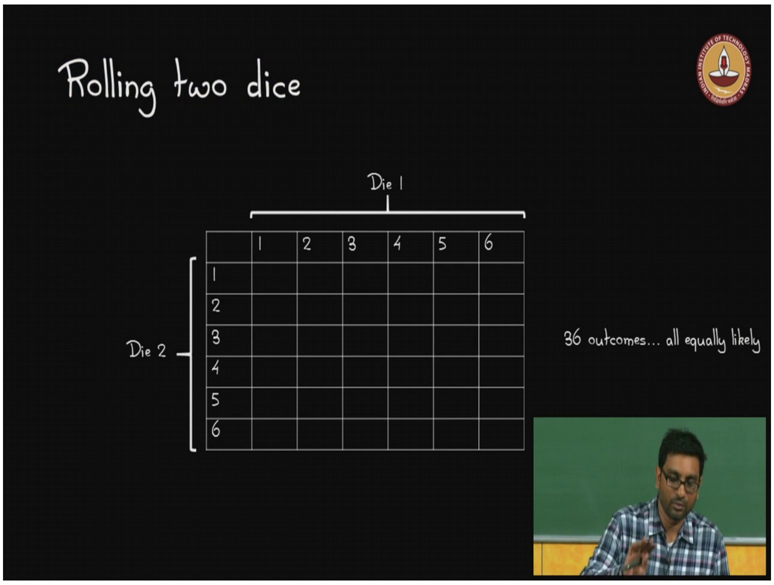

So, here is how we are going to work this out we have 2 dice, die 1 and die 2 are going to be rolled right. We are going to and so, there are a total of 36 outcomes.

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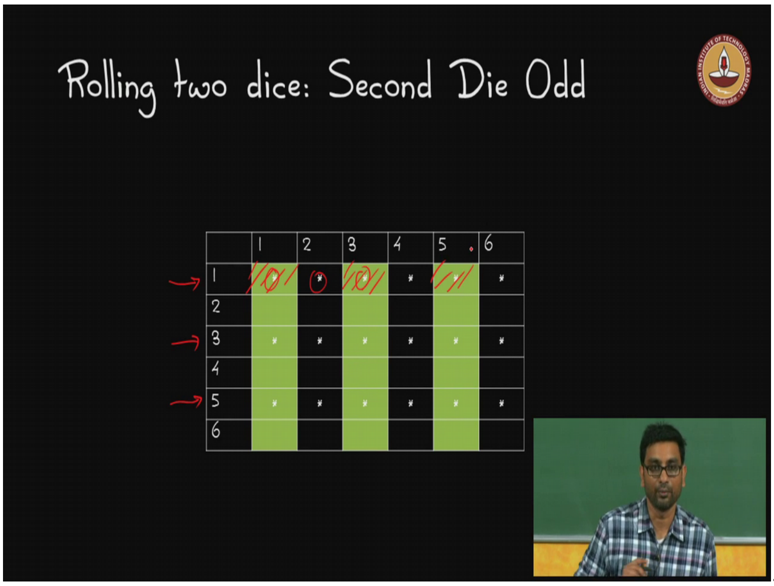

The let us look at the first die being odd. So, now, this so, you basically the events are marked in green. So, the outcomes which fall into this event that the first die is odd are marked as green.

So, now there is another event this is that the second die comes out odd ok. So, here if you think about it I am showing that as these stars. So, you have these stars basically corresponding to these rows.

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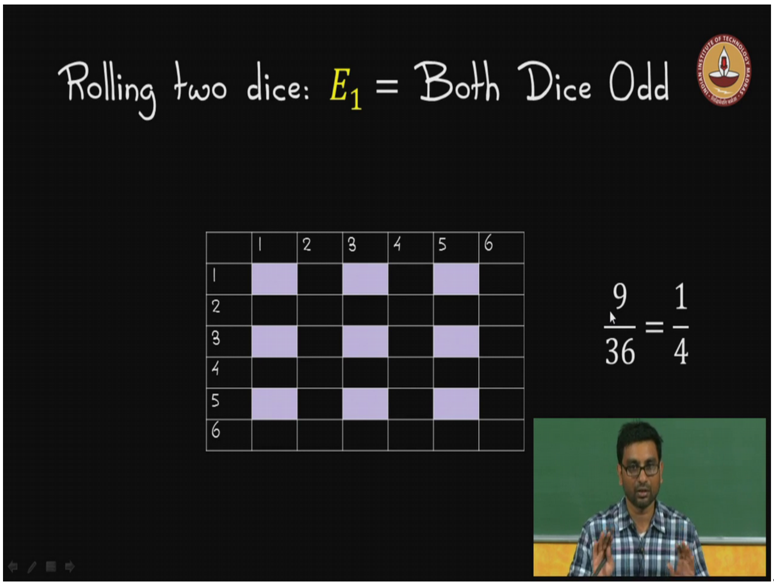

So, now, we are ready to understand what the probability of E 1 is E 1 is the event that both dice are odd ok. And this means that you only consider those events, those outcomes in which it is both green and has the star in it and so, on ok. And that is what you I have marked here in this purple colour?

And now if you and implicit in this is the assumption that all the outcomes are equally likely. So, now, you can use the third axiom to just add up the probabilities of the individual outcomes that are marked purple and you will get 9 over 36, which works out to 1 4th should make perfect intuitive sense, but it is good to see this in the context of the well-defined notion of probability space.

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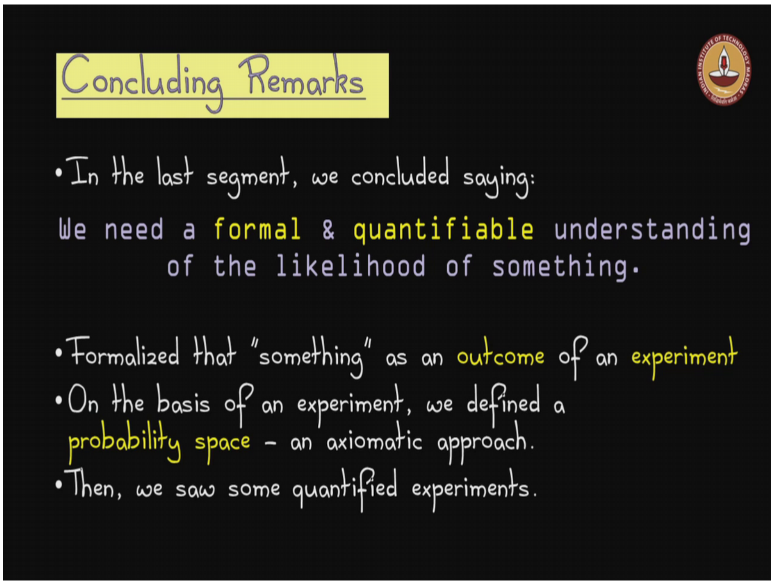

So, with that we are ready to conclude this segment remember the last segment we left off saying we need a clear formal and quantifiable understanding of the likelihood of some natural phenomenon and that is exactly what we have done in this segment. We formalized that natural something as an outcome of an experiment ok.

And on that basis we have defined what is called a probability space and it is basically an axiomatic approach in which we have defined the notion of a probability function giving 3 clear axioms that it has to follow. And then we saw some examples and quantified them and you know we have some understanding how these things work now?

So, in the next segment what we are going to see is an algorithmic example and along the way we will also learn a few more concepts in probability theory so.

Thank you.