Algorithms for Big Data

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Lecture – 04
Karger's Mincut Algorithm

We now come to the second half of this lecture in which we move on from data structure to talk about a Randomized Algorithm. In particular we are going to look at a randomized algorithm for the Mincut problem, and this was proposed by a David Karger.

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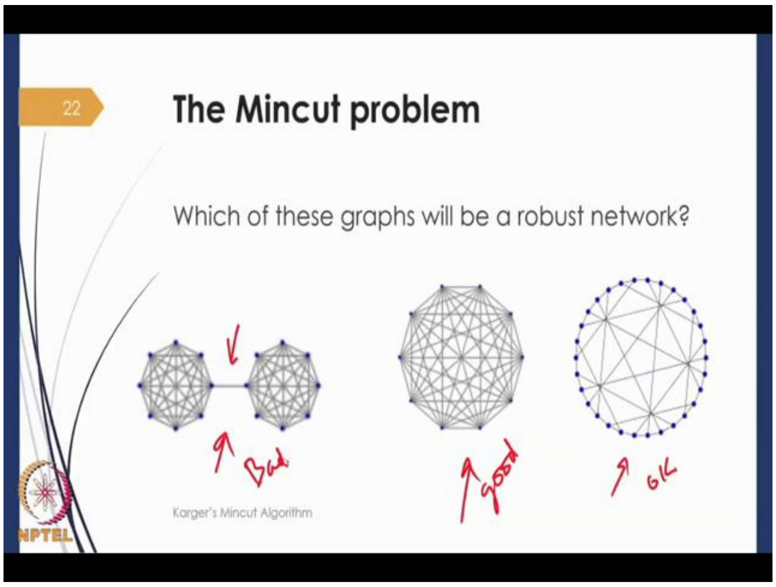
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The Mincut problem

Which of these graphs will be a robust network?

Karger's Mincut Algorithm

And for this very fundamental problem let me motivate this problem first. Here, in this slide we have three graphs, and if these were network graphs and you are concerned about the robustness of the network, how easy it gets to connected things like that. Quite obviously, you will notice that this graph is very bad, this graph is ok, and this graph is good. And what makes you spot this? Because you can see in this graph that there a single edge if it is deleted can disconnect the graph into two components. This is the nature of real world issues that motivate the mincut problems.

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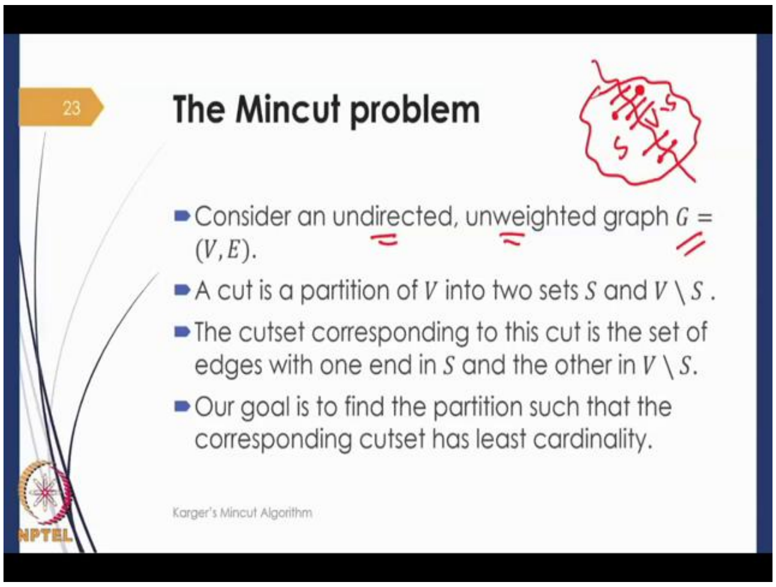
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The Mincut problem

- Consider an undirected, unweighted graph $G = (V, E)$.
- A cut is a partition of V into two sets S and $V \setminus S$.
- The cutset corresponding to this cut is the set of edges with one end in S and the other in $V \setminus S$.
- Our goal is to find the partition such that the corresponding cutset has least cardinality.

Karger's Mincut Algorithm

And chances are you have seen this problem in the context of flow based solution, but in today's lecture we are going to talk about a very simple elegant randomized algorithm for this problem. Let me define the problem. we have input as graph g, it is an undirected graph and is unweighted graph. A cut is a simply partition of the vertex set into two sets. If this is your set of vertices a partition into set S and V minus S is a cut.

The cut set corresponding to this cut will be the set of edges with one end in S and the other end in across the cut N V minus S. So, all these edges that go across the cut comprise to the cut set. Now goal is to find the partition, the cut such that the corresponding cut set has least cardinality. And so naturally this is called the minimum cut or mincut problem.

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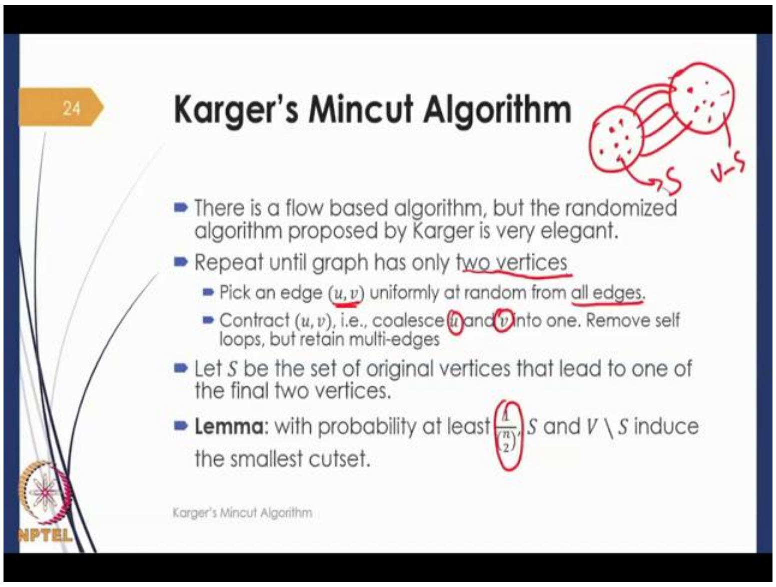

Karger's Mincut Algorithm

- There is a flow based algorithm, but the randomized algorithm proposed by Karger is very elegant.
- Repeat until graph has only two vertices
  - Pick an edge (u, v) uniformly at random from all edges.
  - Contract (u, v), i.e., coalesce u and v into one. Remove self loops, but retain multi-edges
- Let S be the set of original vertices that lead to one of the final two vertices.
- **Lemma**: with probability at least $\frac{1}{\binom{n}{2}}$, S and $V \setminus S$ induce the smallest cutset.

Karger's Mincut Algorithm

As I mentioned earlier you may have seen a flow based algorithm, but in today's lecture we are talking about a randomized algorithm and here is the algorithm course. We start with the full graph and we repeat these next two steps until finally we left with just two vertices. So each iteration we pick an edge u, v, uniformly at random from all the edges that are currently in the graph. And once we pick an edge we contract that edge. And would mean by that, we coalesce the two vertices the edge u v is connecting vertices u and vertices v, we coalesce them into a single vertex. Any edge between u and v now becomes a self loop.

So we simply remove those self loops, but then there could be multi edges. We will see how such multi edges could form from an example that we will look at shortly. We will be retained multi edges loop. The self loops we will discard, but the multi edges we retained. As we call as we keep track of this super notes and the original notes that were actually comprise. So, each super note was formed by smaller super notes or smaller original notes, we keep track of all of that.

Finally, we will be left with the two vertices after n minus 2 contractions. And we will be left with two super nodes if you will, with several possible several multi edges going across them. But each of these super nodes actually is made up of several original notes.

So, let us pick one of these super nodes and the set of nodes that they actually are contained in this super node is simply S. Which means this would be V minus S, and this is our candidate cut.

The claim now is that with a good nonzero probability in fact 1 by polynomial n probability, S and V minus S actually induce smallest cut set. This seems quite magical, let us look at an example that might give us some intuition has to why this is true and in subsequent segment we will actually analyze it carefully.

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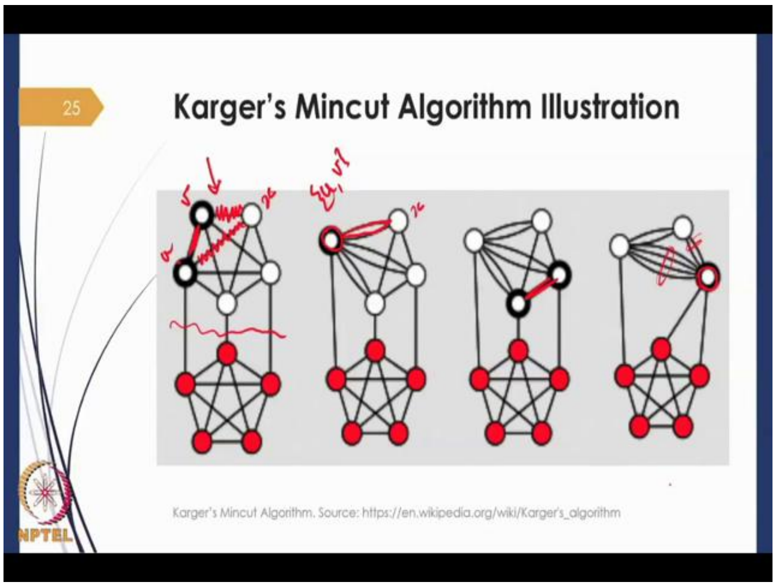
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Karger's Mincut Algorithm Illustration

Karger's Mincut Algorithm. Source: https://en.wikipedia.org/wiki/Karger's_algorithm

Let us look at how the algorithm operates, and this is our original graph. As you can probably see by this eyeballing it, this is going to be their minimum cut and this execution that we are going to see now illustrated here actually is going to discovered that particular a mincut. So, let us see how that works. There are several edges; the algorithm picks one of these edges uniformly at random, so here the edges pick this one. And it is connecting to verities (Refer Time: 07:08) v, we call it even v. Now, this edge has to be contracted and we get a super note and this super note includes both u and v.

Of course, the self loop is thrown away, but notice that there are multi edges that are created. For example, there is an edge between v and let us say call this as x, there is also an edge between u and x. Now between the super note that includes u and v we have to include both of those edges, and then we proceed.

Similarly, all other multi edges also notice are being retained. So, we now again randomly choose one of the edges and the notices that these multi edges you cannot discard them so each one of them multi edges is also candidate for being picked. In this case let say this edge is picked. We contract those two vertices, we get this super note. And notice that we get four multi edges over here and two multi edges over here, we retain all of them and we proceed with this in this manner.

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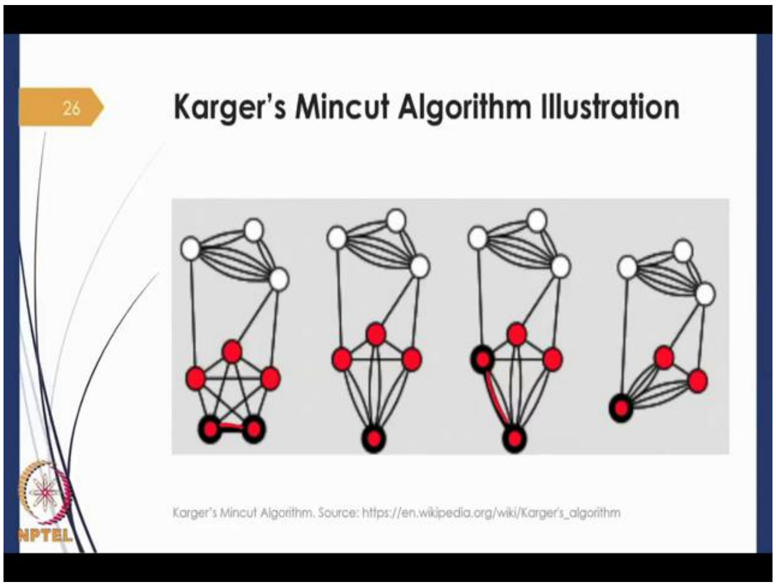

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Karger's Mincut Algorithm Illustration

Karger's Mincut Algorithm. Source: https://en.wikipedia.org/wiki/Karger's_algorithm

Again, this edge uniformly at random contracted we get this new structure. And look here one of the multi edges here is being picked, again contraction takes place, self loops are thrown away and this procedure is continued; now here we are retaining all the multi edges.

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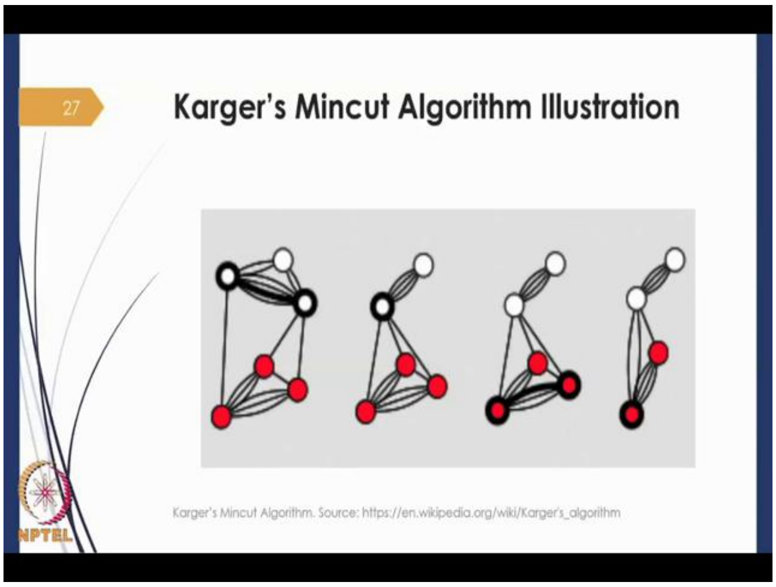

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Karger's Mincut Algorithm Illustration

Karger's Mincut Algorithm. Source: https://en.wikipedia.org/wiki/Karger's_algorithm

We continue on and finally we will be left with a scenario like this, where you have two vertices and a few edges going across them.

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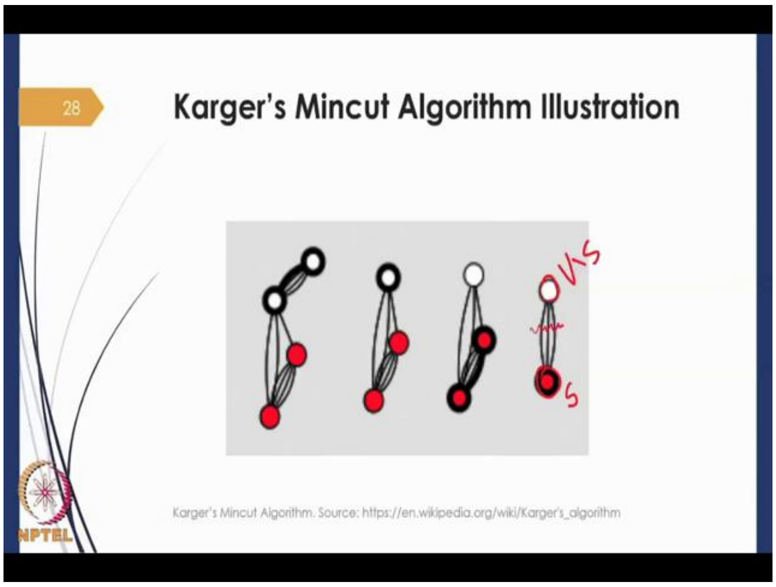

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Karger's Mincut Algorithm Illustration

Karger's Mincut Algorithm. Source: https://en.wikipedia.org/wiki/Karger's_algorithm

And each of these super vertices, super notes is actually representing larger set of vertices in the original graph. We have to keep track of them. Let us say this super note comprises the set S and this super note comprises the set V minus S. Our claim of course is that this cut induce by S and V minus S will be a minimum cut with probability at least one over (Refer Time: 10:06) two. That is the proof of this claim; we will study that in the subsequent segment.