Artificial Intelligence:

Propositional Logic: The Tableau Method

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Module – 02

Lecture – 06

So in the last two classes we saw some direct proof methods. We looked at frege's proportional calculus and then we looked at Hilbert style direct proof in which you can make assumptions which you can end up using shorter proofs. So let me give you a small exercise to begin this class. Show that this formula is true. You can use any method that we have studied so far; the importance of this formula is to show that; is useful in the sense that if you want to convert everything in to freges style system then if you are given some problem where you are given A B C D and so on and you want to prove some let say P then according to the deduction theorem we can reduce this showing that A and B and C and D implies P is a topology but this still contains the and symbol essentially inside that whereas freges proportional calculus uses only implication so this exercise that I have given you will give you some hint as to how to convert the sentence of this kind where you have a set of premises and a conclusion in to a formula which can be handled in freges system. It is also an exercise to try out the proof method essentially. But today we want to look at indirect proof methods and in particular we want to look at a very well-known method called the Tableau method.

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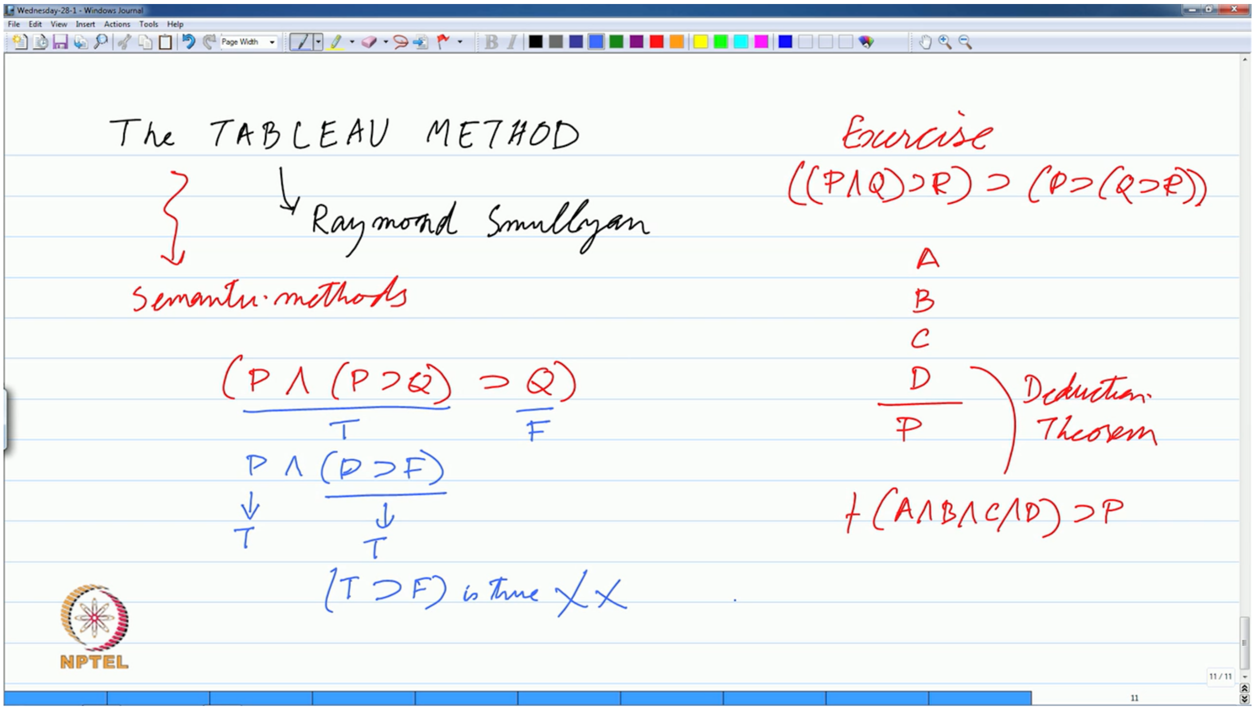

If you look up any book of one of my favorite authors which is Raymond Smullyan who has written a lots of book on puzzles and logic, he is a logician essentially; this particular book that I am talking about is called logical labyrinths, but he has written lots of books; a lady and a tiger, what’s the name of this book and all kinds of interesting puzzles essentially. He was one of the people instrumental in popularizing the Tableau Method essentially and therefore I have taken this particular material from his book essentially. Now interestingly this Tableau method has its roots in semantic methods and its only recently that people have realized that actually gives you very nice proof system essentially. So what’s the difference between the semantic methods and the proof system is in semantic methods you are reasoning with truth values whereas in proof system you are not reasoning with truth values but you are simply replacing formulas with new formulas and you have some procedure which will only look at the form all the formulas and decide whether you have terminated or not essentially.

So if you recall we had said that when we were trying to show that this formula P and P implies Q implies Q is a tautology, we had said that lets try and show it is false actually and if we are not able to show it is false then we will be forced to conclude that it is tautology, so you can see that it is the approach of proof by contradiction and moreover we are trying to satisfy its negation and trying to see we can find a value which will make it false and if we are not able to show that then we are forced to conclude that we cannot find any value which will make it false and therefore it must be a true formula so that’s the kind of flavor of tableau method. So what did we do that time, so let me just recall. We said that if you want to make this formula true so must make the left hand side, if you want to make the formula false sorry, you must make the left hand side true and the right hand side false essentially. So it means Q is false and whatever is there is on the left hand side is true which means you have to show that P and P implies false is true, we want to show that this formula is true essentially. Now if you want to show that this formula is true then you must show that this is true and this is true, right because it is an AND, AND is true only when both the components are true so if P is true then we are forced to show that T implies false is true, unfortunately that is not possible so we cannot make this formula false so we are forced to conclude that original formula which we wrote in red is true.

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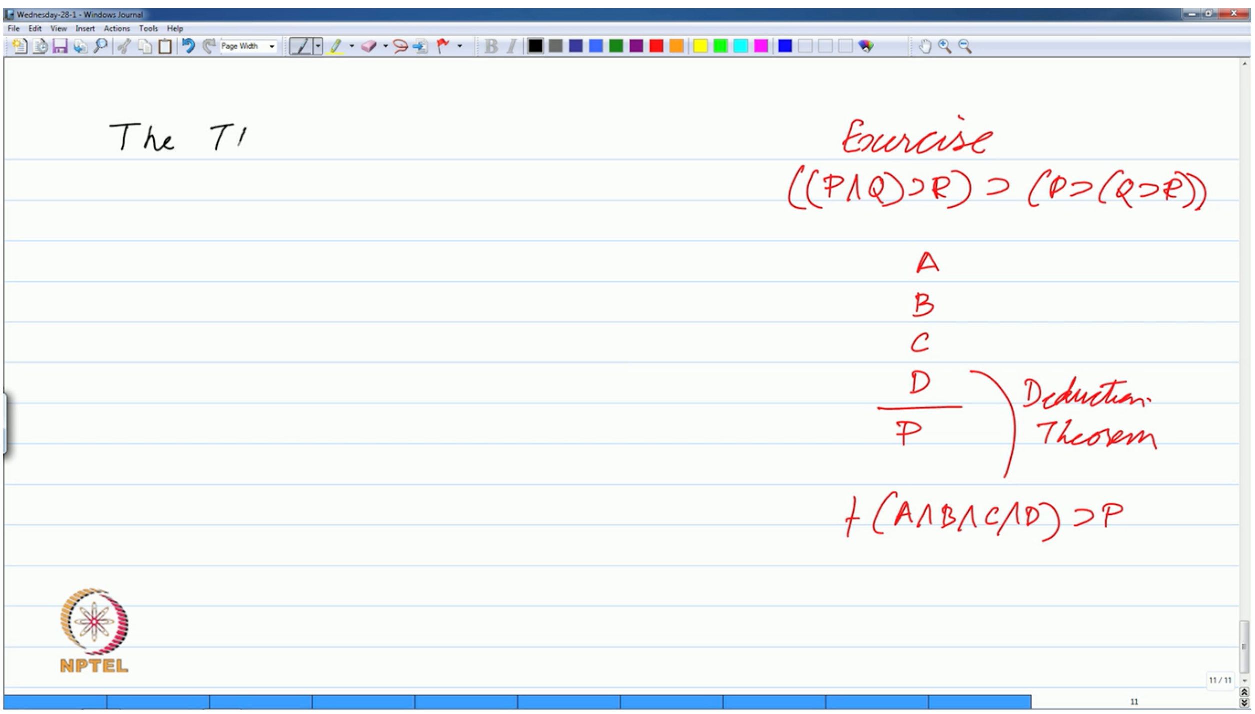

Now the tableau method is very similar to this except that it is used to show that something is unsatisfiable and in a kind of corollary we try to show that it is satisfiable and when we fail to show that it is satisfiable we are forced to show that it is unsatisfiable. So remember the connection between unsatisfiable formulas and tautology. If you want to show that something is tautology you can take its negation and try to show that it is unsatisfiable. And now we are saying that to show that something is unsatisfiable try to show that it is satisfiable, if you fail then it must be unsatisfiable essentially. Let’s look at the tableau method and we will start with this example to see how this works essentially. So the tableau method also works with the set of rules; incidentally the tableau method became popular because people found first of all it can easily be extended to first order tautology which is what we are interested in; but not only that but also things like modal logic which we may or may not get time to look at very much; but also things like description logic which we will look at much later in the course.

It can be shown that these methods carry forward to others specialised kind of logics and we will see it's a very simple method to implement. It is a very straight forward algorithm so it makes the job of writing a proof system much easier as opposed to the direct proof methods where we have to either somehow make guesses about what assumptions to make or we have to somehow we have to figure out what instance of tautology we have to start working with, so if you remember the proofs that we did in previous lesson we had to guess that instead of A implies B implies A we wrote for example that A implies A implies A implies A so we only need the proof for A implies A so this kind of guess work is totally done away with in indirect method and these are very simple methods which can be used in algorithms.

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So what tableau methods try to do is simplify the formula. We will use the term formula and sentence interchangeably so it's a well formed formula of your given logic but we also treat them as sentences so we will use the two terms interchangeably. And what do you mean by simplify that they can be broken down into atomic parts and if we have both P and not P as part of the simplified problem then we cannot satisfy because if your knowledge base has P and not P there is no valuation which will make this anomaly as true because one of them will necessarily be false. So the combination criteria of tableau method and also for resolution method that we will see later is that you are arriving at a contradiction.

You are trying to satisfy something and arriving at a contradiction. So what are these rules for simplification. We will work with only a few operators and or implication not but you can write similar rules for other operators. So the rules are as follows that you can replace negation negation of P with P. that's the rule for negation. Then rule for disjunction there are two rules one is to do with not P or Q you can replace not P or Q not P or Q very well not P or Q will be true when not P is true or not Q is true and not Q is true right. So both must be true so you can replace this with not P and not Q. so the way to read this is on the top of the line is what you are working with and on the bottom of the line is what you have simplified. so we have actually thrown away the disjunction connection or the or connective and produced two formulae but both of them must be true. So when not P is true and not Q is true only then not P or Q will be true. On the other hand if we have P or Q then this formula would be true either when P is true or when Q is true right.

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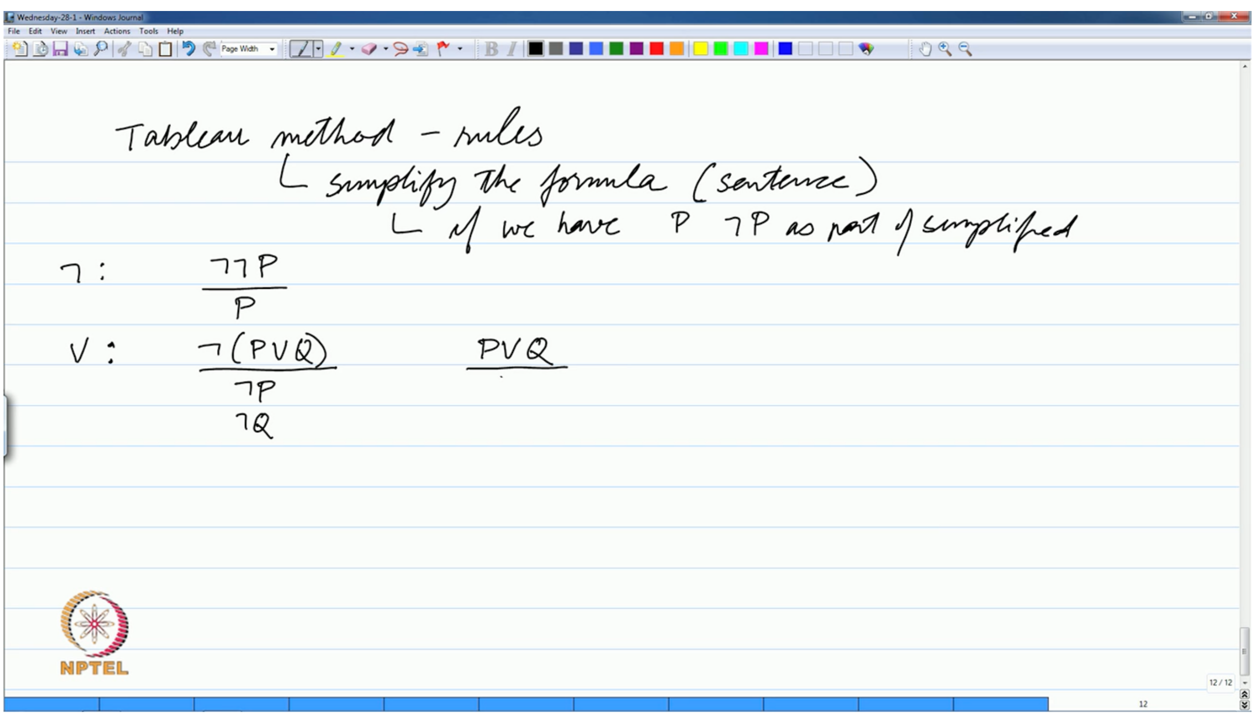

Or both obviously so we write this as followed. We introduce a branch here so we split the knowledge base into knowledge base in one we put P and in other we put Q. remember that the goal of this exercise is to find valuation. So we are saying that if we can find P to be true then our formula and all other formulae will be true. So P or Q can be made by either by making P true or Q true so we introduce that. Then let's look at and so then if we have P and Q then it's simple we have to make both true. But if you have not of P and Q then you can replace it by two branches not P not Q. so at least one of them is false then not of P and Q will be false. And finally the implication if we have not of P implies Q then we must show that either P sorry both P is true and not Q is true that's what we were trying to show when we showed that implication is true show that the left hand side is true and the right hand side is false then it will be true

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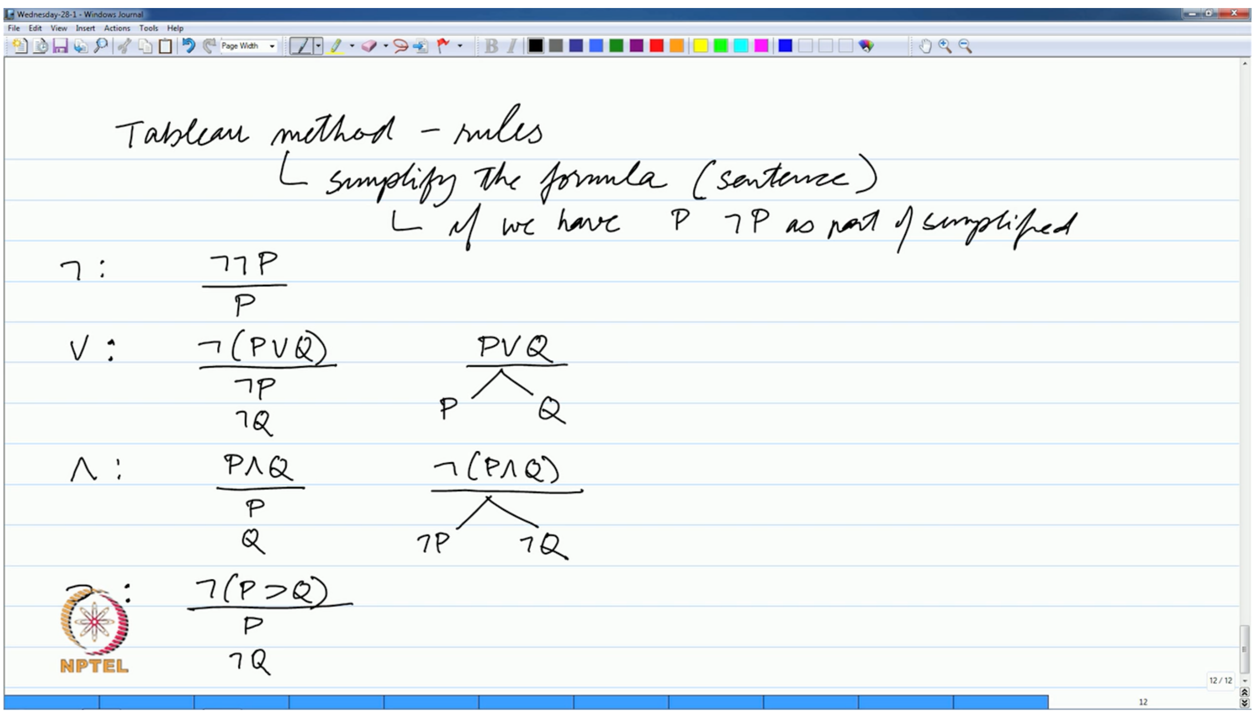

But to show that it is false the left hand side must be true and the right hand side must be false. So if you want to show that P implies Q is false which is to show that not of P implies Q is true P must be be true and Q must be false only then is not P implies is true. Whereas if we just look at P implies Q we can make it true by making P false or by making Q true. So if you look at the truth table for P implies Q you can see that. There are three rows in which P implies Q is true and in one row not P is true in one row Q is true and in one row both are true so those are the three rows and there is one row where P implies Q is false which is when P is true and Q is false. So these are the seven rules that we will need in tableau system using these three four connectives.

And our goal will be to take a negation of a formula and try to show that it is false and then we will try to show that it is not false. So let's work with the example that we were looking at which is this P and P implies Q then whole thing implies Q which is a tautology on which modus ponens is based. So let's try to show that this formula is true by the tableau method so if you want to show by tableau method you must take its negation which I will write in blue here and then try to show that try to simplify the formula. So I am trying to show that it is false and I will eventually fail in showing that it is false. And we have these 7 rules so the first rule that we can apply is the negation of the implication rule which says that left hand side must be true and right hand side must be false. Left hand side is P and P implies Q right hand side is not R. Once we have dealt with the formula we can actually throw it away but since we want to keep the proof you would not throw it away may be we will just put a tick mark to say that we are finished with that formula we don't really need it anymore.

Because it has been replaced by two equivalent formulae which is here. Now there is little bit of heuristic that you can use whenever you want to select which formula to simplify it makes sense to select one which doesn't introduce branching because that way the proof will be more compact. If you introduce branches in earlier part of the proof then you will have to handle each of those branches separately. So as long as you can do away with branches just do the so so what I have written on the left hand side here so they have no branches they don't introduce branches into your structure we trying to write it in a structured pattern. So we would prefer those rules as opposed to rules which in this case we don't really have an option. The second formula we have introduced is already atomic I mean it just has a negation and you can't really do much with that. So we have this other thing which is so let us deal with that so that's done.

So we need to have P and we have P implies Q. so we are done with first two formulae we are left with three formulae now not Q P and P implies Q and remember that we are trying to make it we are trying to satisfy this formula which is a negation of what was. Now we apply the implication rule and we show that we have not P here and Q so we have split the database into two. One which goes down the left branch of tree and one which goes on the right branch of the tree. But we can see that neither is consistent. If you look at this not P here then it conflicts with this not P here so that branch is set to be closed in terms of tableau methods. At the same time this Q here conflicts with this Q here so this is also closed.

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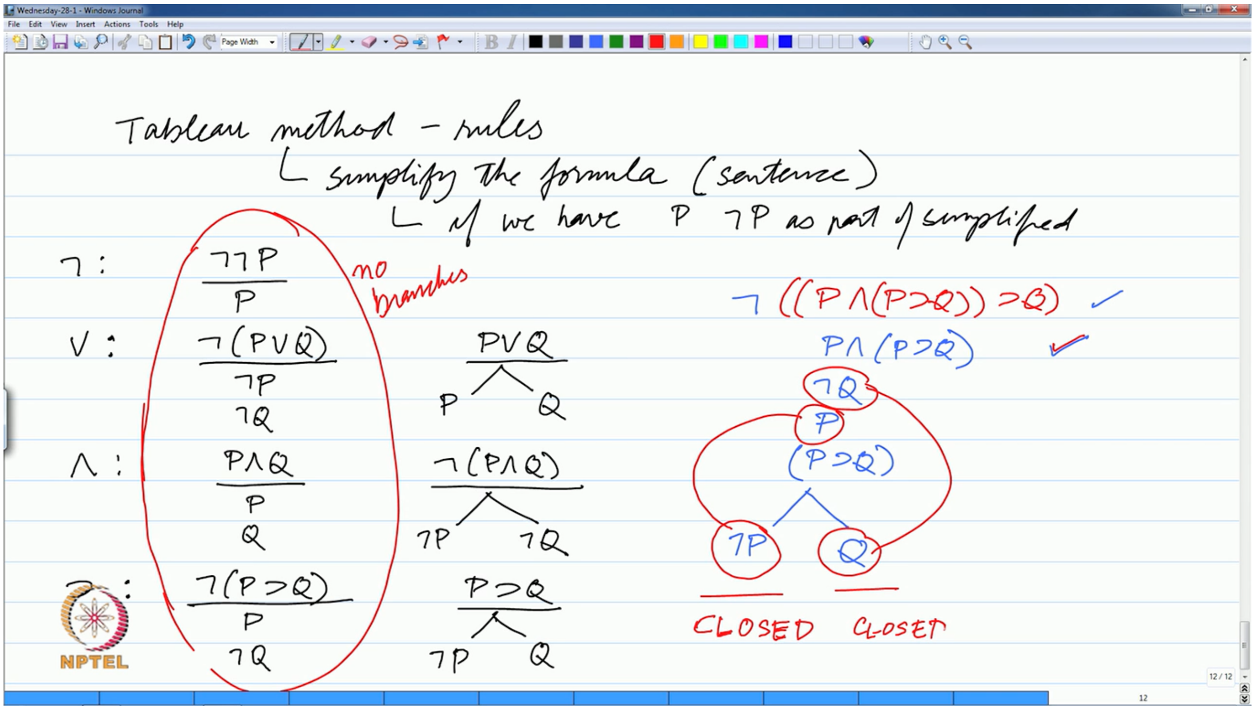

Which means that we were not able to show that the negation of this formula was satisfiable. So it must be unsatisfiable and the negation of the negation which is the original formula we were interested in must be true. So what we wrote on top in red becomes true with the negation sign in blue that is unsatisfiable. So this is basically the flavor of the tableau method so let's look at couple of more examples.

So let's try to prove this formula which I gave you in the beginning of the class which is P and Q implies R whole implies P implies Q implies R. so if you want to show that this formula is true we will work with its negation so we will put a negation sign in the beginning. And then apply the tableau method. There is only thing we can do to start with the left hand side must be shown to be true P and Q implies R. and the right hand side must be shown to be false. So we are done with the first formula. And its better to choose the third formula here because that will not introduce any branches whereas the second formula would introduce branches. So we deal with the third formula first. P must be introduced and not of Q implies R must be introduced. Again it is better to deal with the last formula first because it will not introduce branching whereas the second formula which is still pending with us we will do that later so we will do this one. Q and not R. so so far we have no complication we still have to worry about the second formula which is what we will do now. So there are two branches here one is P and Q must be false or R must be true. So this comes from here. This whole thing comes from here.

So immediately we can see that this branch is closed because it is conflicting with not R. to make not P and Q true we can either make P false or Q false so it introduces two branches. But both are closed because not P is closed with this P and not Q is closed with this Q and not R was closed with this.

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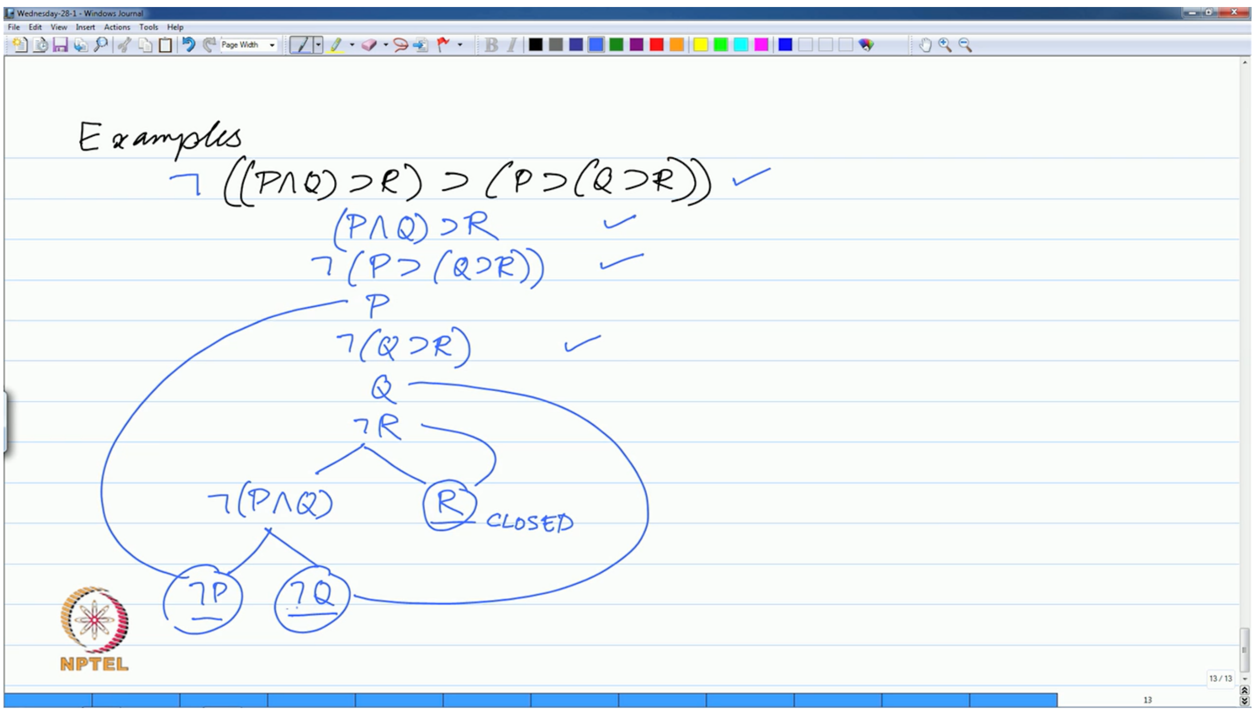

So all three branches we have in database are closed so there is no way we can try to satisfy the negation of the formula so the original formula must be true Okay let me do one last example which is Then 2. So if you remember P implies Q implies R the whole thing implies P implies Q implies P implies R. so as usual because we are using the tableau method we will work with its negation and apply the rules one by one. So initially we must have P implies Q implies R the negation of this P implies Q implies P implies R. so that we are done with the first one. Now we handle the third one again for the same reason that we as an exercise you can try doing it in the order in which you generate them. You will see that you will still get a split knowledge base which is closed but you will end up splitting it into more parts.

Or atleast you will end up doing more work because you will have many branches to look at. So lets finish with this one so P and not R. now we go back to the first formula there are two branches that either not P must be true or Q implies R must be true. Now this not P gets closed because of the fact that P is there. So we are left with only one branch here and we can either look at this one or we can look at P implies Q. supposing we do not P implies Q P implies Q then again we are left with not P which for the same reason will closed because it will conflict with the same P up there and Q. remember that this step is because of the fact that we are simplifying this and then when we are left with Q implies R we get not Q which immediately gets closed and R which also gets closed.

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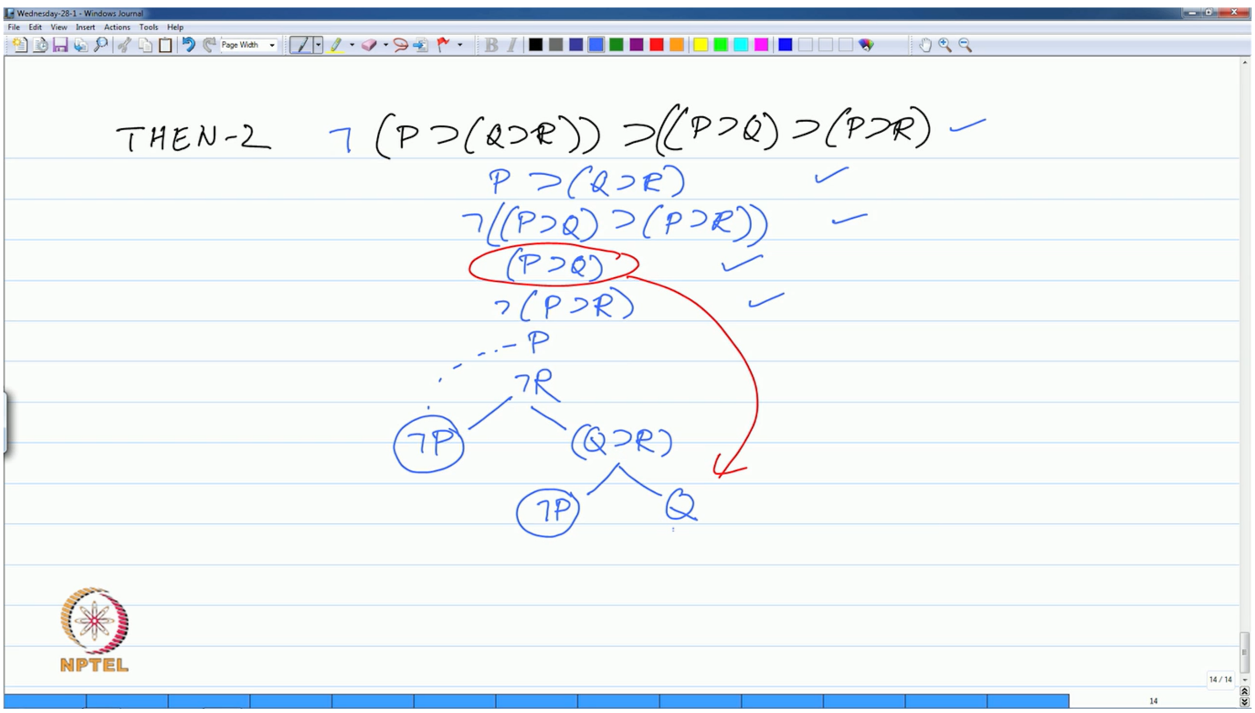

So here is a tableau proof for then-2 problems. So one thing you would have noticed is that this method the tableau method has no axioms. The only thing we worked with is the formula we wanted to be shown to the true. So unlike direct proof methods where you need some kind of axioms to supply the rules here you don't necessarily need that because your goal is not to produce a formula that you want to show to be true but you can in fact work on that formula. And in fact we work with the negation of that formula. So that's one simplification in some sense but it still has as many rules as we have connected.

In the next class we will look at a method called as resolution method which has the property that no axioms and one rule just like Frege's propositional calculus had only one rule of inference which is modus ponens. The resolution method also has one rule of inference which is called the resolution rule which we will see in the next class. And it has no axioms. So you can imagine if as a theorem prover if there is only one rule you can apply all you have to do is to decide which formulae to apply. This method was formed by Robinson and it's a fairly recent method 1965 or so. And since Robinson introduced the resolution method it has actually revolutionized this whole field of theorem proving. It gave a lift to theorem proving and lot of people started automatic theorem proving. So it's a very interesting method and we will really take up resolution method for first order logic when we move to first order logic. And we will see how we can build theorem provers in that. But before we go to first order logic we will study the method in propositional logic because that gives us the feel of the method and then we will do first order logic. So this whole exercise that we have done in the last 5 or 6 classes studying propositional logic is simply so that we get the foundation about logical reasoning in place. We now understand what we mean by deduction what we mean by proof how can we generate proofs and things like that. Propositional logic by itself is not a very powerful language in the sense that you cannot even for example solve the Socratic argument which is All men are mortal Socrates is a man and therefore Socrates is mortal. That you need first order logic but having got the basic machinery for logic in place we will be able to carry forward and we will be able to move much faster over first order logic. So in the next class we will look at the resolution method. May be we will try to prove some of these same formula using resolution method and then move on to first order logic after this.