Artificial Intelligence: First Order Logic: Skolemization

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Module – 04

Lecture- 01

Okay so we have been looking at reasoning and we have looked at forward chaining in first order logic. Let's come back to representation a little bit and see how we can actually represent the things that we want to represent. So we are still focusing on FOL first order logic. Let's start by looking at something that we have not done so much which is the existential quantifier. We have looked at the universal quantifier and we have seen that most of the rules that we express are universally quantified in the sense that all men are mortal and things like that. But for the sake of completeness we will also need existential quantifier and then we will see that there are certain things that you can say using only the existential quantifier.

So let's look at that little bit. So if you remember the quantifier symbol is like an inverted E followed by a variable name and followed by some predicate. Predicate may have other arguments it doesn't matter. For simplicity we will just talk of P of x. So what this is saying is that there exists an x such that p of x is true essentially. P is a predicate, some predicate. Remember that unary predicates are subsets of the domain and so on essentially so. For example, we might say something like there exists an x even x. So if our universe of discourse is a set of numbers then we are making a statement that there exists an even number essentially so. Now one thing about the existentially quantified statement is that it is a strong statement in the sense that we are asserting that there is at least one element which satisfies the predicate essentially whatever that predicate is.

Now if you remember the universally quantified statements. So if you said for all x as I said P x implies Q x. Then if you remember the semantics, so this is something you must keep in mind when you talk about representation that the semantics of first order logic that we are dealing with is basically set theoretic in nature and that unary predicate stands for subset of the domain, binary predicate stands for relation between elements of the domain and higher order predicates are similarly relations and so on. So every predicate can be interpreted as the subset of either of the domain or cross product of the domain of certain arity. So if you remember the meaning of this statement is that there is a subset of the domain which satisfies the property Q and what the given statement is asserting is that inside this set Q there is a set P. So what we are essentially saying is that if anything is a P then it's a Q essentially. So for example if anything is man then that thing is mortal.

All students are bright if I say if anyone is student and anyone is bright. So that was the meaning of that essentially. When you talk about an existentially quantified statement so this basically stood for all Ps are Qs. What if you want to talk about some Ps are Qs. So the moment we say some, we need to use the existential quantifier. So we are saying that there exists at least one element in the domain which is a P and which is also a Q. So before we do that before we express that can we just replicate so let me put a question mark here. Can we simply write there exists x P x implies Q x. And the answer to that is no essentially.

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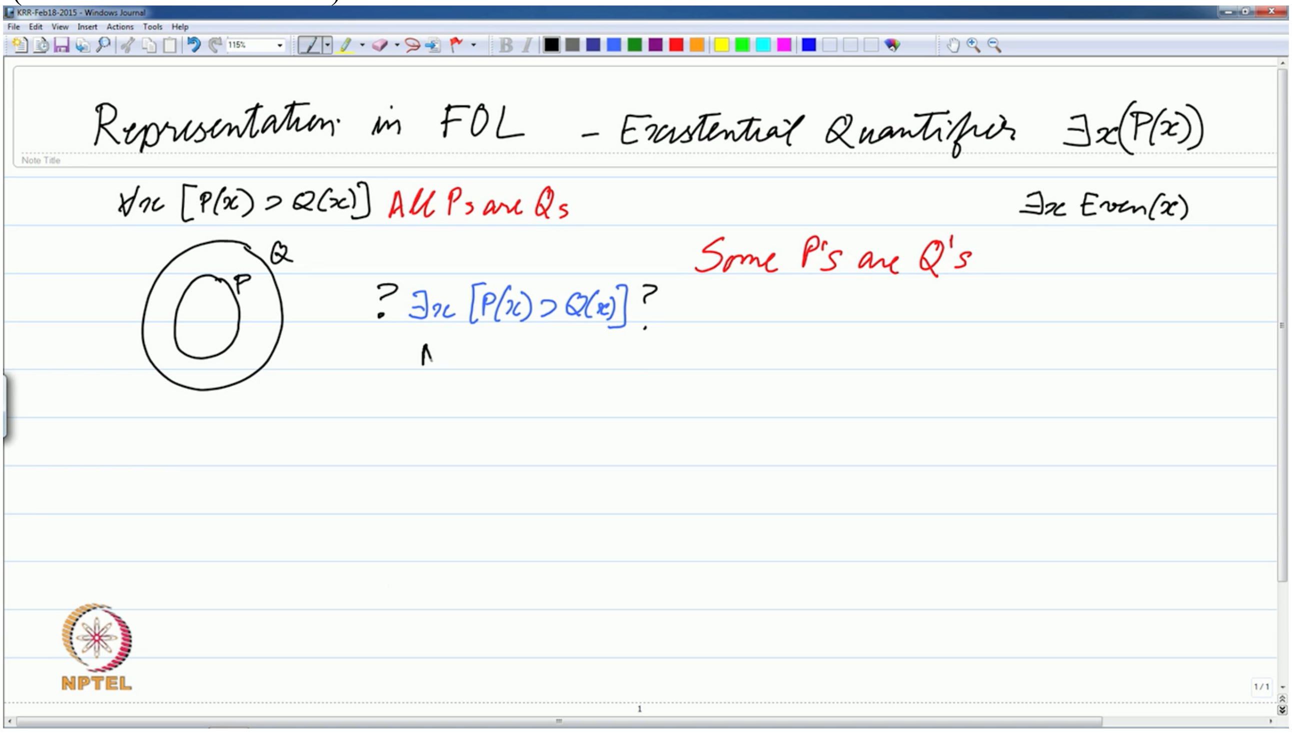

so if you want to represent this way, this is not a correct way of representing and one way of testing whether your representation is consistent, is to try and look at the negation of a statement. So look at the negation. So what is the negation of some Ps are Qs. Its negation is that so what are we saying when we are saying some Ps are Qs, we are saying there is some element which is a P as well as Q. So not we want to say there is no element which is both P and Q at the same time essentially. Now if you want to negate this formula then you basically put a negation sign in front of this. Then if you remember the rules of first of first order logic, rules of substitution, we can push the negation inside. So this will become for all x Px implies Qx. If you remember you can rewrite this as not P x not Qx which we can rewrite as by again pushing the not inside, the de morgan's law will apply, the OR will get converted into an AND.

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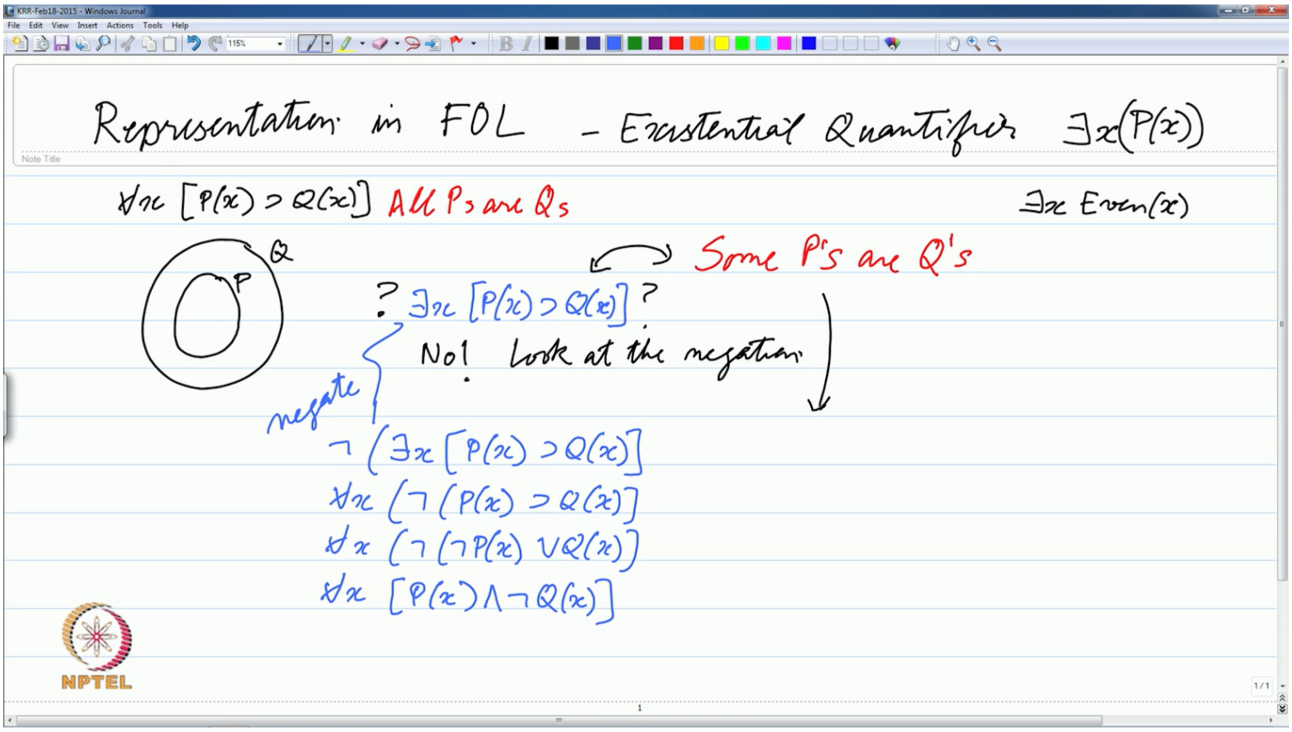

So the negation of the logic statement which says that there exists an x P x implies Qx, the negation of that is for all x P x and not Q x. So if you read as to what is this statement saying. This is saying that everything is a P and not a Q. So let's take an example, if I wanted to say some students are bright, what this is giving us, a negation of that is first of all saying that everything everybody is a student or everything is a student and that student is not bright. So obviously that is not the correct way of representing things so this representation that we talked about is wrong. So what is that right representation, the correct way to do it is to write it as there exists an x P x and Qx which if you were to look at the semantics of the system essentially what you are saying is that there is a set P and there is another set Q and there is at least one element which belongs to both the sets or the intersection of both sets is non empty. And this statement is essentially saying that there is at least one element which is both Px and Qx. So let's negate this one just to try out if this gives something more meaningful.

So if we negate this we again put a negative sign, there exists an x Px and Qx which when you put the negation inside will become for all x, not Px and Qx. Which if you push it further in will become for all x not Px or not Qx. If you put it further in, okay before we do that you can see that this statement corresponds to what we are trying to say. When we say that some Ps are Q, we say there exists an x Px and Qx . The negation of that is for all x it's not true that Px and Qx essentially which is the literal negation of the expression we had essentially. But we can continue to rewrite this as follows.

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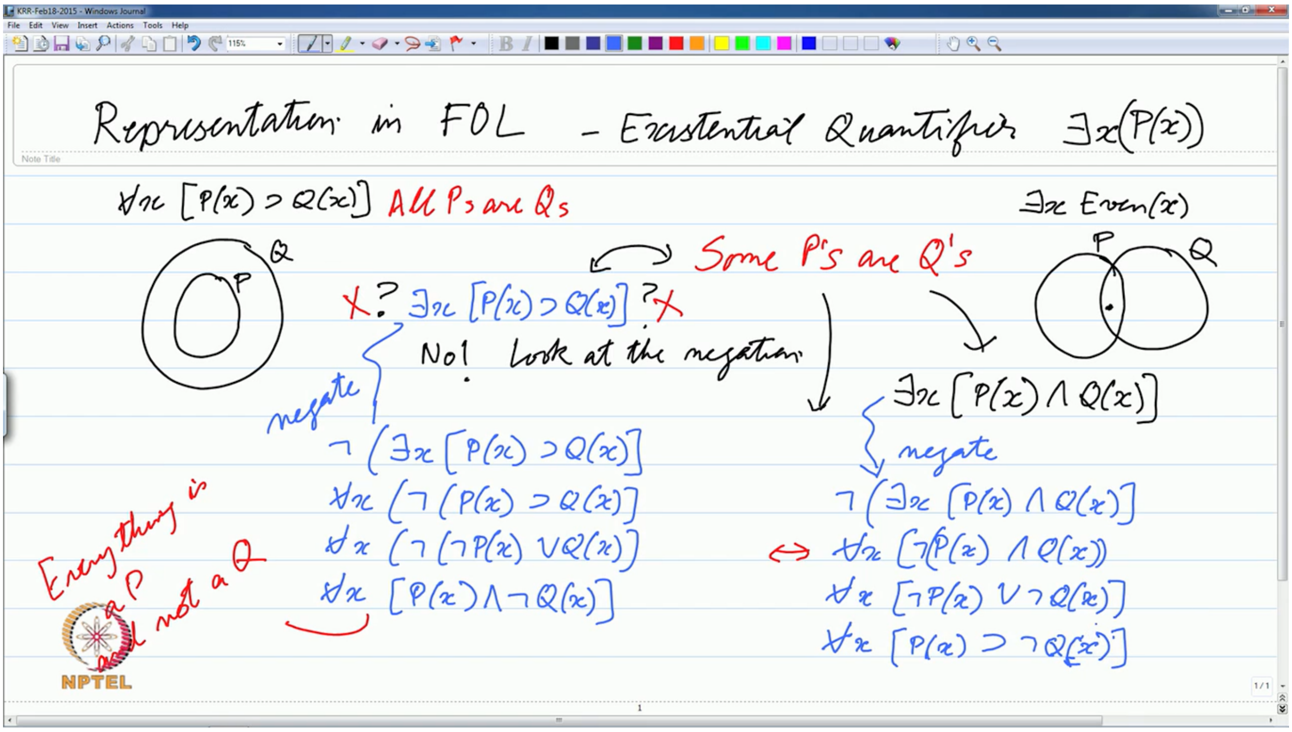

so what is this one saying this last statement is saying that for all x if x is a P it cannot be Q or it's not a Q essentially. Which is why we can see that this representation is the correct representation for the statement some Ps and Qs. Now let's go to the concept of implicit quantifier notation. So remember for a universally quantified variable, we have for all x, we will replace it with question mark followed by x. What do we do for existentially quantified statements? So for existentially quantified statements it turns out that the process is not so straight forward. So let's look at it step by step. Let's take a simple formula, this says for example if you are talking about numbers. So this says there exists n Even n, we are saying there exists a number which is even number essentially. Now what we do is to skolomise this. We say that we know that there is a number remember existential statement is strong in the sense of existence essentially as opposed to the universally quantified statements.

So just to go back to that point, if we said that we have a set P which is inside the set Q and we say for all x P x implies Qx. Then if the set P were to be empty then also this statement is true essentially. So if I had said something like all black apples are let's say pink. Obviously it's a somewhat pointless statement but what we are trying to say is that a statement like this is true is there are no black apples. Or if the set in the left hand side is empty. So a universally quantified statement may talk about such things, all black apples are pink, all honest politicians are bright all that kind of stuff. If the element that we are talking about doesn't exist, then the statement is still true. Whereas if we had a statement like Some politicians are honest. It's a stronger statement because it asserts existence of an honest politician. Which means that this statement will be true only if there is at least one politician who is honest.

Of course luckily for us now a days we can say that this statement is true but it's a stronger statement. It makes the assertion that there must be at least one element essentially. So what we do is convert this into implicit quantifier form. It's to say that okay we will give it a name so we will call this some name and what do we mean by that. Let me introduce a new constant, so instead of saying there exists an n Even n, we will say let's say even N where this is a constant. And so if we have this statement Some politicians are honest, we would, in implicit quantifier form, write it as let's say P stands for politician and I use this term sk-11 and sk-11 where sk-11 is some constant essentially.

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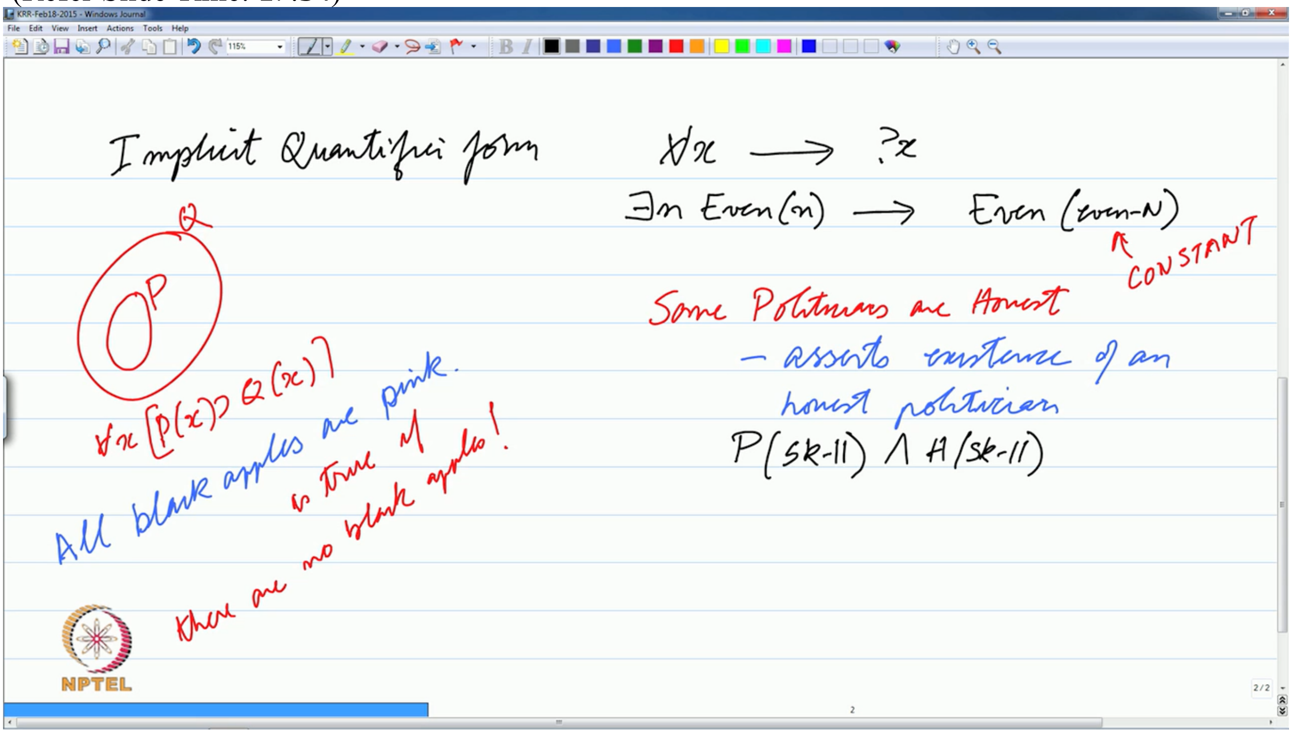

so the important thing is that you must introduce a new constant name, it must not coincide with some existing name or something else. It must be a new constant name. And for that reason we also call it as SKOLEM constant. And this process of converting into implicit quantifier form including universal quantifier, we call it as SKOLOMIZATION. And this is after a logician by the name of Skolen, Thoralf Skolen. So this process of converting to implicit quantifier form is called Skolemization after Skolen and the simplest case is when we have simple existentially quantified statements like there exists even number of there exist honest politician, then we simply say that there is a constant name we will introduce and which is different from our language so far, it doesnt occur in the language and essentially we are saying that is the element we are talking about essentially.

Let’s now look at slightly more complex cases, these cases occur when the existential quantifier comes in the scope of a universal quantifier essentially. So let me begin by an example. So supposing i make this statement Every boy loves a girl. You want to express this in first order logic and then we want to skolomize it essentially. So when you want to express it in first order logic the first question is what do we mean by this statement essentially. Now there could possibly be two interpretations of this statement, one could be that there is a girl that every boy loves and the other could be for every boy there is a girl that the boy loves essentially. So one way of doing this would be to say that there exists g such that capital G stands for a girl that g is a girl and for all b Boy b implies loves g, let’s say L stands for loves g.

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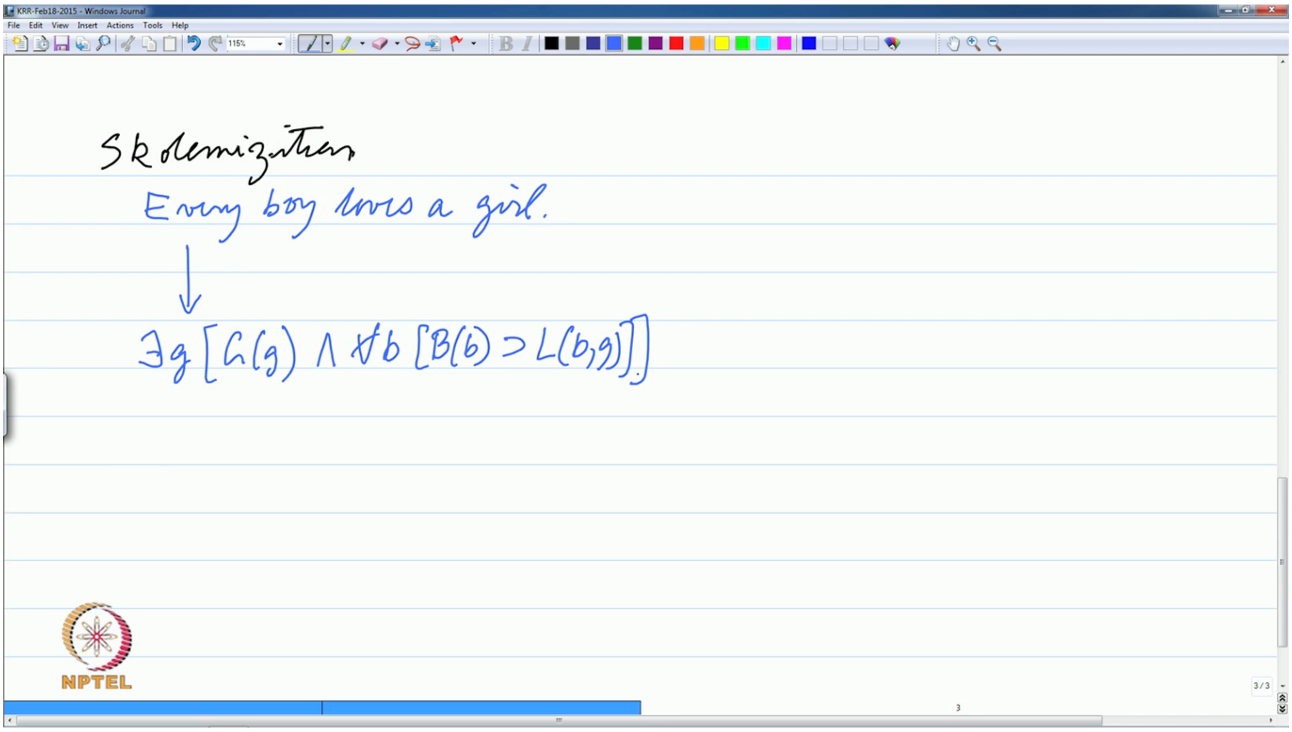

so this is saying that there is one girl whom everybody loves essentially. And if you want to skolomize this we just replace g with a skolem, so let's call it sk.g in the honor of Skolem and replace b with a variable. So you must look at the meaning of this statement, the meaning of this statement is that okay there is some girl we don't know who that girl is so we are identifying that by a skolem constant which is sk g and for every boy, that boy loves this girl essentially.

An alternate way of saying this is that the girl that the boy loves is specific to that boy essentially which is possibly the more normal form of this meaning essentially, so this we can write as the following:

for all b, Boy b implies there exists g, such that g is a Girl and L boy g. So as an exercise you must go back at take the negation of both the sentences that we have and see whether the negation are meaningful or not. That's a way of testing whether your representation is correct or not. Now let's talk about skolemizing this.

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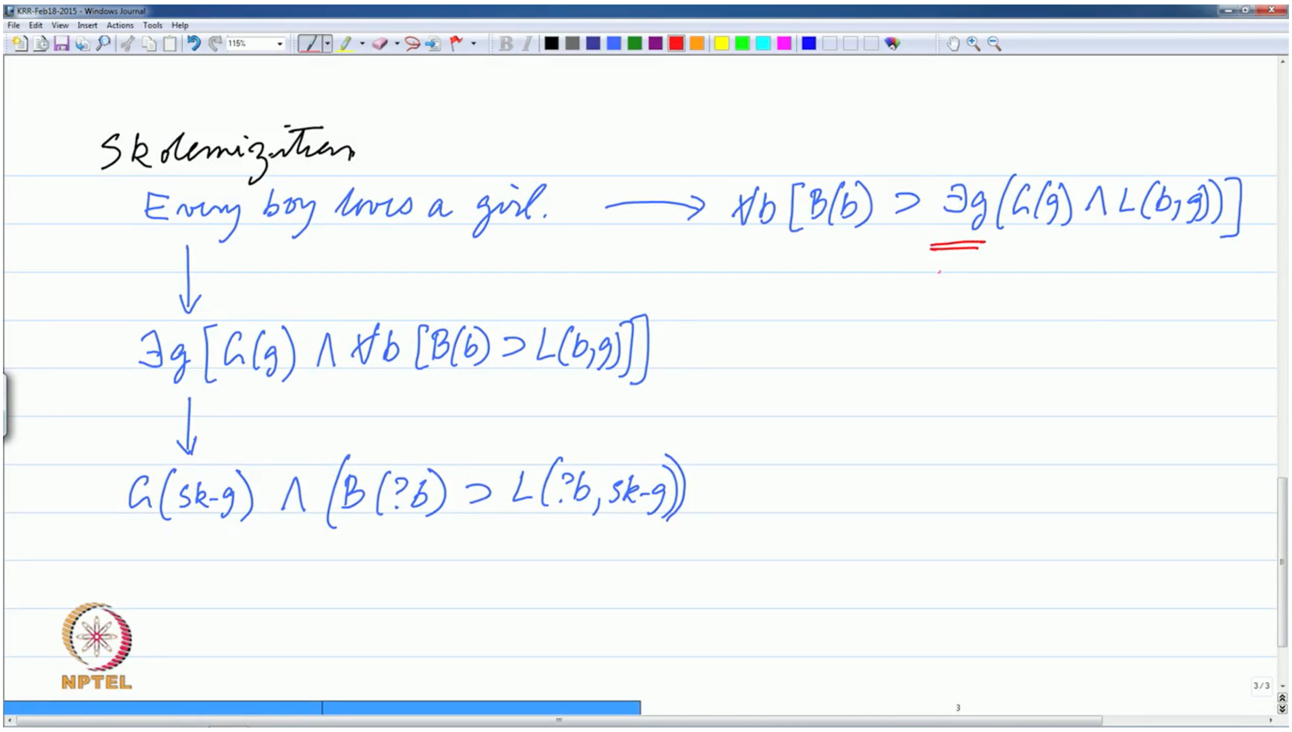

this is in the scope of for all b. So when we say there exists a girl we are not talking about any girl essentially we are talking about that girl who this particular boy loves. So in some sense, g is DEPENDENT upon b, because the girl we are talking about will depend on which is the boy that we are talking about so one boy may love girl a another boy may love girl b and so on so forth. So those girls are not arbitrary girls they are dependent upon b which we say are Skolem functions of b. So just as we introduce Skolem constants we introduce Skolem functions. And then we replace this why? as before the variable b is a universally quantified variable so it will be applying earlier. But this girl is now a Skolem function so let's say SK.L of b. So we are saying that there is some Skolem function SK.L which when applied to b points to, remember that functions return terms to us. So every function denotes a term essentially so this Skolem function also denotes a term. And it denotes that particular term which this particular boy b loves essentially.

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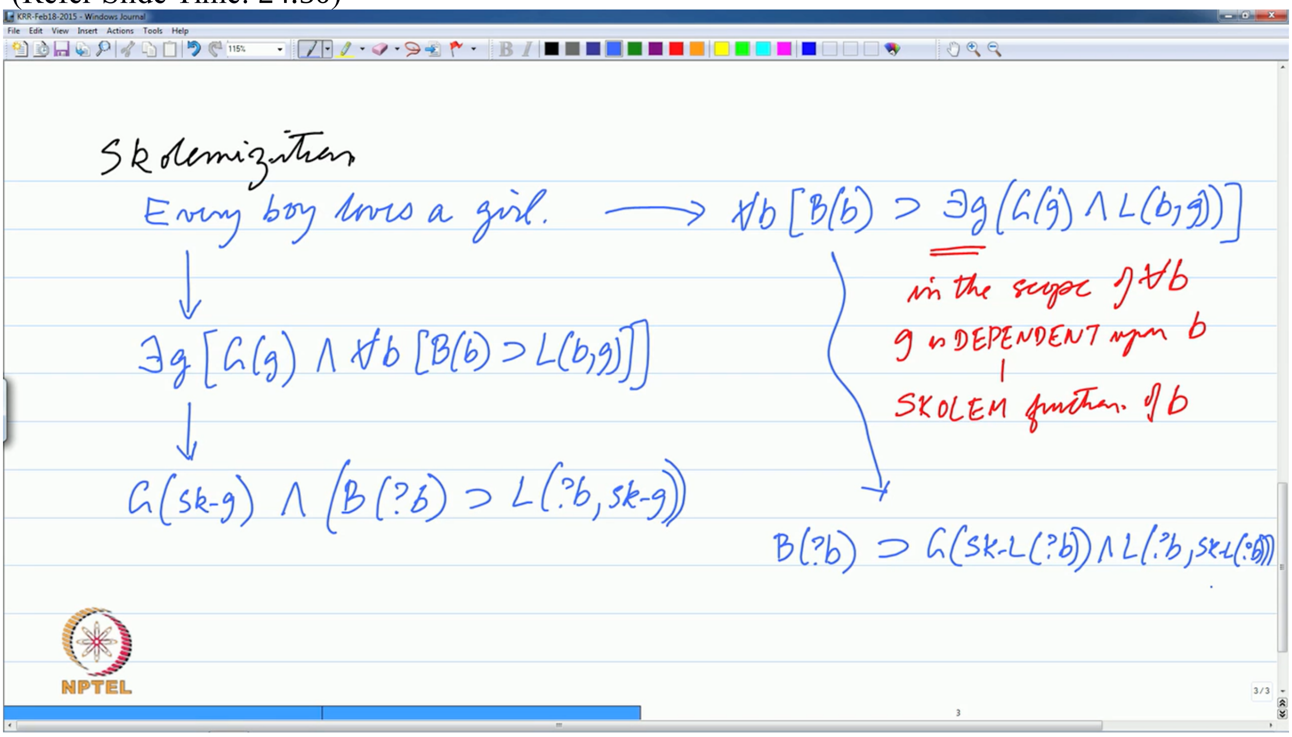

so thats a key idea we have been following here that this is a skolem function of b essentially. So likewise another example if I had said for all n there exists a number m such that m greater than n. Forget about whether this statement is true or not, it depends upon the domain you are working with. But the statement says that for every number there exists a smaller number essentially. Notice that i used the mathematical notation for greater than, I could have written something like GT m n but since we are so familiar with mathematical notation we can coninue to use that sort of a notation.

So what is the role for skolemization. That if there is a set of and inside this group there is an exist p for example, then this p must be replaced with a skolem function of all the universally quantified variables in whose scope it comes essentially. So this p will be replaced by some skolem function, let's call it SK-3 of x, y, z. Skolem function of all variables so actually i should say all universally quantified variables in the scope of which p falls. The rest of the matching unification rules apply as before. So let's take an example. So supposing I say that there exists an x Even x and for all x Even x implies not Odd x. So from this we should be able to infer that there exists an x not Odd x. I will leave this as a small exercise for you to do.

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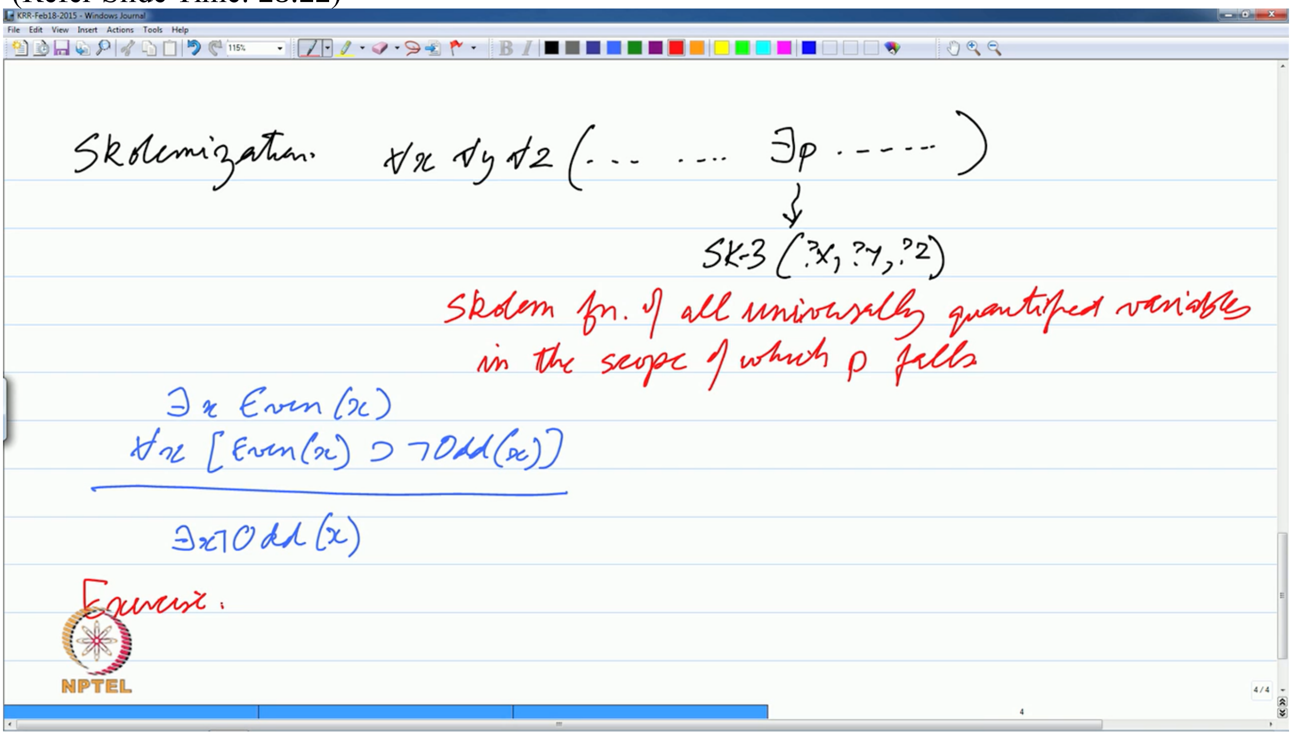

Skolemize these statements and see that you can still apply Modus Ponens. It's a straight forward process, x will be replaced by question mark x in the second case. In the first case it will be replaced by some constant then it will become easy essentially. Likewise, you should be able to show something like this that for example if you say for all x there exists a y, Love x y. And if you say for all x for all y Loves x y imples Caresfor x y you should be able to infer that for all x there exists a y such that Caresfor x y. You can just try the skolemization process and show that this can be done.

Okay i want to end with one last thing in skolemization which is how do you identify the nature of a variable. In the sense we know how to quantify universally quantified variables, how to quantify existentially quantified variables provided we know that it is universally quantified variable or existentially quantified variable. So to illustrate that let me state an example here. Supposing I say it's not the case that. How do you read this statement ..that it's not the case that there exists an x such that x is not mortal, that x is immortal. Now is x a universally quantified variable or existentially quantified variable. Its universally quantified variable why do you say that because the sign there says there exists an x.

It is indeed a universally quantified variable because this is equivalent to saying for all x negation of negation of Mortal x which is equivalent to saying for all x Mortal x. So this reveals the true nature of the variable that it is a universally quantified variable and the way to do that is to Push the negation sign inside. Because we know that when the negation sign is showing up outside a quantifier, when it is pushed inside it will change the nature of the quantifier essentially. So if i had a statement like this that If there exists a black apple for example then let's say the earth is flat. Which i could write as follows, there exists b such that let's say B stands from black apple or BA b. Implies Flat earth. There is only one variable which is b in this. Is b universal or existential? It's a little bit trickier. So let me just quickly rewrite this. We can rewrite this as not there exists b BA b or F E which we can rewrite as, I am pushing the not inside, for all b not BA b or F E. And there is nothing to stop me for putting an extra pair of brackets here which means pulling the universal quantifier out putting F E in the scope of the universal quantifier which doesn't create a problem because it doesn't have the variable x essentially. And once i do that you can see that this is stating that for all b BA b implies F E. So the point that I am trying to make here is that whenever we have a combination like this. So this If contains a hidden negation as you can see here and therefore whenever existential quantifier occurs here in the antecedent of an implication statement, you must treat it as a universally quantified statement essentially. Okay so i stop here. In the next class we will try to see how we express relations between different kinds of categories and things like that in first order logic essentially.