Artificial Intelligence:

First Order Logic: Entailment and Models

Prof. Deepak Khemani

Department of Computer Science and Engineering

Indian Institute of Technology, Madras

Module – 03

Lecture - 03

So in the last two lectures we defined the syntax and semantics of first order logic. Let's continue looking at the semantics in a little bit more detail because first order logic is a language which is the most commonly used language in computer science and it includes all the programming languages that you use, they can be modelled in first order logic and you can see that essentially in a programming language you are dealing with variables and constants and things like that. So let's just have a closer look at what are we really saying when we use first order logic and then we will move towards two sides of it. One is how do we say or how do we express what we want to represent and that is a side of knowledge representation that if you want to represent knowledge in certain domain what can you really say and how do you say it and that's a very vast subject and it has been dealt with many people in great detail but we will try to get the general idea of how to use FOL to start with essentially.

And then we will move towards the proof system we will look at how proofs are generated in first order logic and we will see that essentially they will incorporate some level of search. And then we will come back to knowledge representation and try to really study up how in a real domain we can represent knowledge essentially. So let us first begin with an understanding of FOL semantics. So if you remember a first order language is a language which is defined by a set of relation symbols, a set of function symbols and a set of constant symbols. And given this alphabet we had defined a set of terms. So if you remember variables are terms or constants are terms or if you can take some function symbol and given it an appropriate number of arguments then they are terms. Now terms denote elements of D. What is D? D is the domain we are defining the semantics in terms of. We are saying that there is an interpretation I, interpretation of a set of sentences. So you are saying some sentences in this language what do they mean essentially. So that's one side of semantics as to what does the sentence mean. The other side is of course whether those sentences are true or not.

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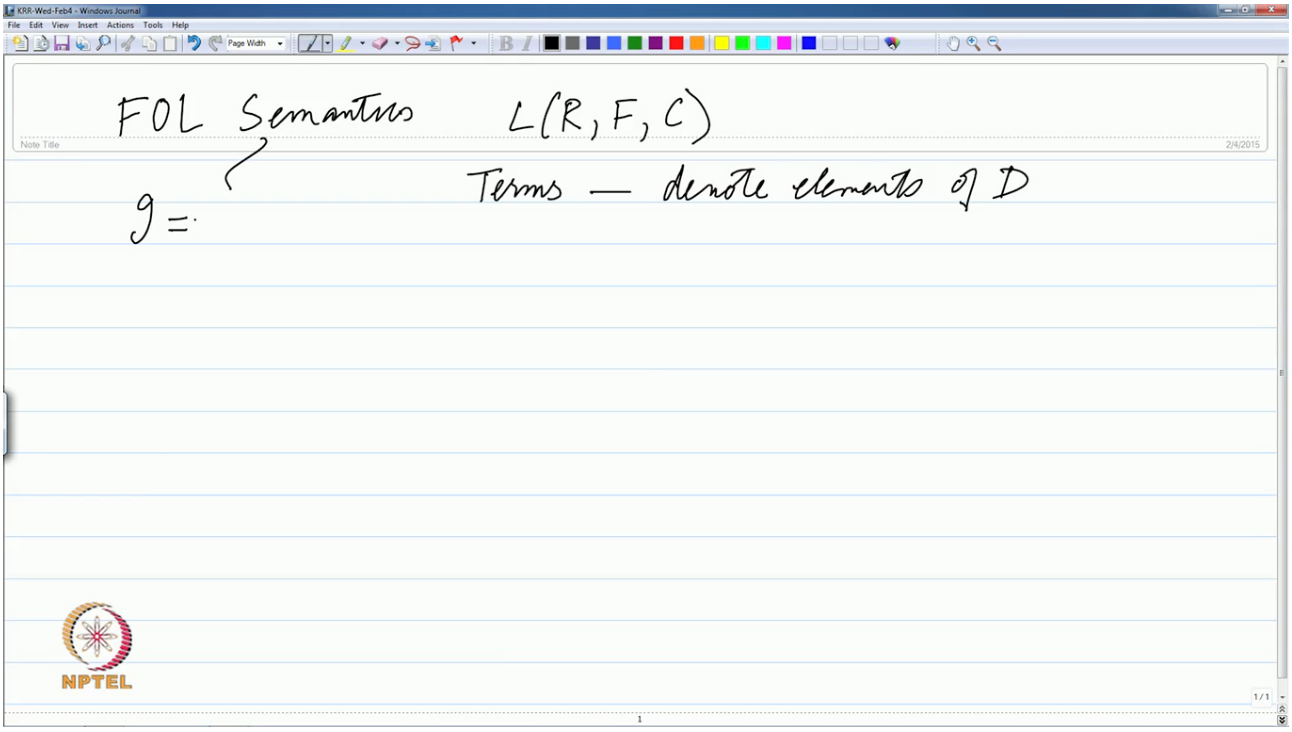

And the interpretation in first order logic is always in terms of the domain D and a mapping I essentially. What this mapping I does if you recall, is that it maps the set R onto relations on D. It maps the set F to functions on D. And it maps the set C to constants of D. And together with this we have the family of terms that we are talking about. So the thing is that terms always denote elements of the domain. So this is the first thing that you must sort of get used to is the idea that the semantic of first order logic is set theoretic essentially, talking about sets.

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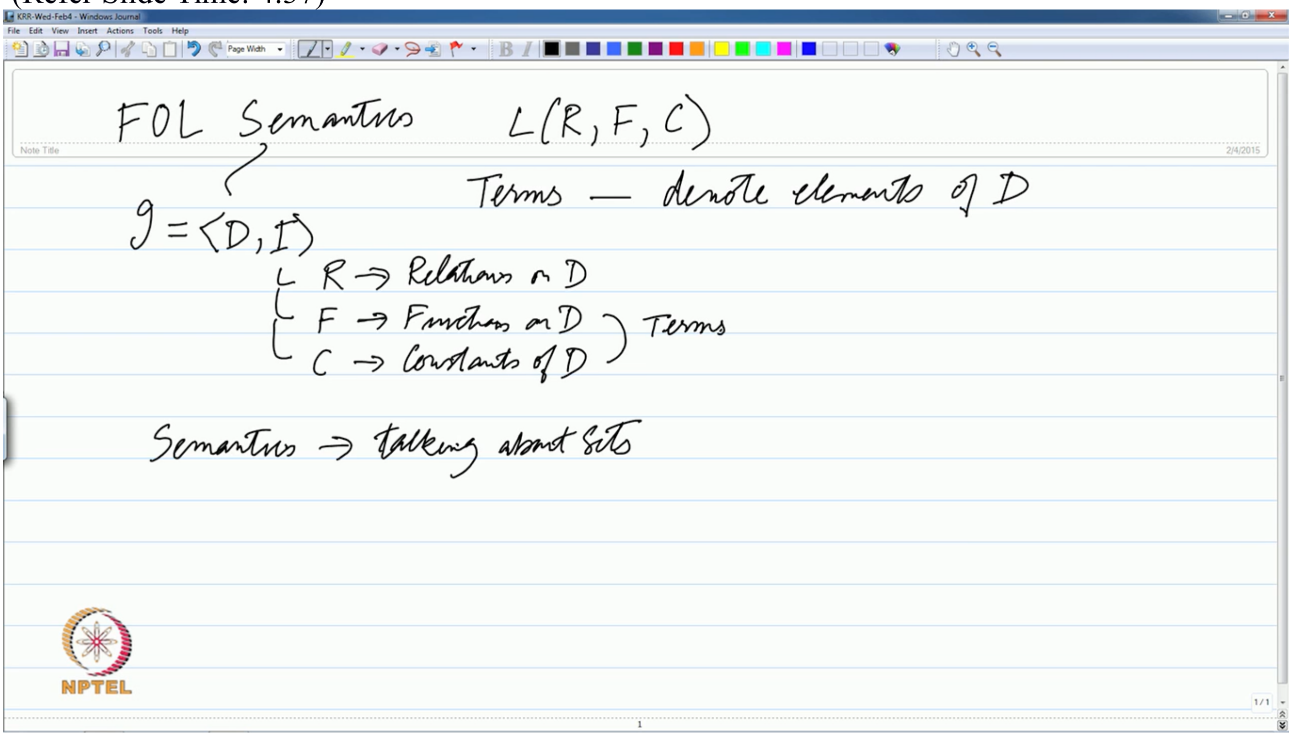

And what is the set we are talking about. The set that we are talking about is the set D which we have called the domain, some books call it the universe of discourse but it is made up of the set of elements and essentially what first order logic is talking about are some properties of the elements in these sets. So terms are used to denote elements, that you can say that x is a term which you don't know and you will be given by assignment or you can say that 3 is an element in my set of numbers or you can say that 4 plus 7 is an element in my number. Or you can say that father of John is an element in my set. So depending on the domain which you have chosen, terms denote elements essentially and relations as we have said denote or predicates denote relations on D and the simplest kind of predicates are usually predicates. So for example we have already said Man x or mortal x. So when its unary it basically means that it takes one argument. We could have said P of x, it doesn't matter. Just like in computer programming the name that you give to a variable is immaterial essentially. It doesnt carry, have any effect on the computation that you are doing. Likewise, in logic the name that you give to a variable or the name you give to a predicate is immaterial. So what really matters is what are the relations between the elements of a set essentially.

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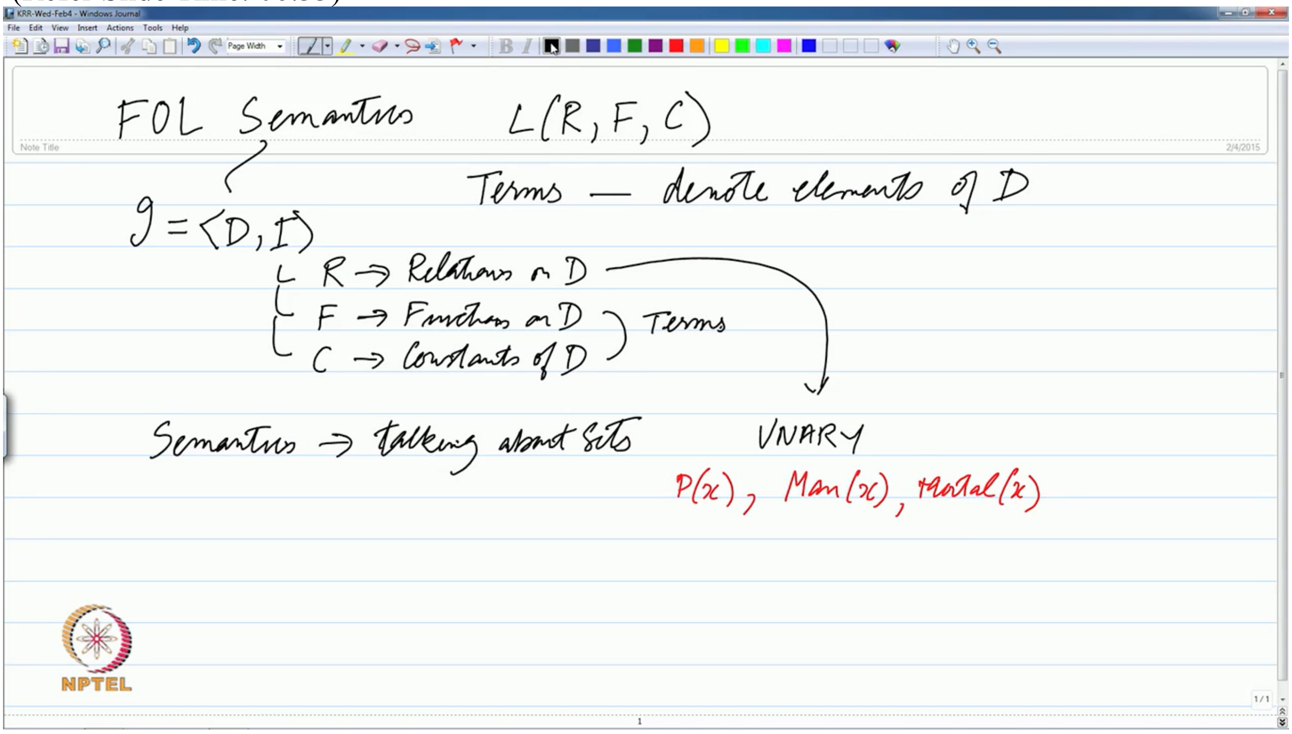

Now as you know unary relations they correspond to subsets. So how do we visualise these predicates. If we were to draw the set D then we are saying that there is some subset of this which is the set which is denoted by man essentially. So this could be man. So every unary relation denotes a subset essentially. Now the first thing to observe here is that in the English language we have different category of things. So you have taxonomy of life forms in which human beings sit somewhere you may have properties such as red, green, blue or the property of being mortal, we don't make this distinction in first order logic, for us they are just subsets.

So if I say red x it means that x is red. It doesn't have the kind of meaning that we use as humans essentially. That is a issue that we will address a little bit later when we get deeper into knowledge representation but as far as current moment is concerned whether they are categories or whether they are properties we will denote them with unary relations or predicates of arity 1 and we will just simply think of them as sets essentially. So man is the set of all men which includes women of course so we say, we use that for humans. Mortal is a set of everything that is mortal, red is a set of everything that is red in fact these predicates define these properties essentially. You may ask what do you really understand by the term red essentially, so if you are a very science kind of a person you might want to say something like you know it's a frequency of light that is reflected by the object and the kind of reaction it makes onto the cornea of your eye based on which there are some signals that happen and they make you percieve the notion of red. All that is an explanation of what is red essentially. We are not interested here in first order logic in all that kind of stuffs, all we are saying is that red is a unary predicate which means it's a subset of the domain.

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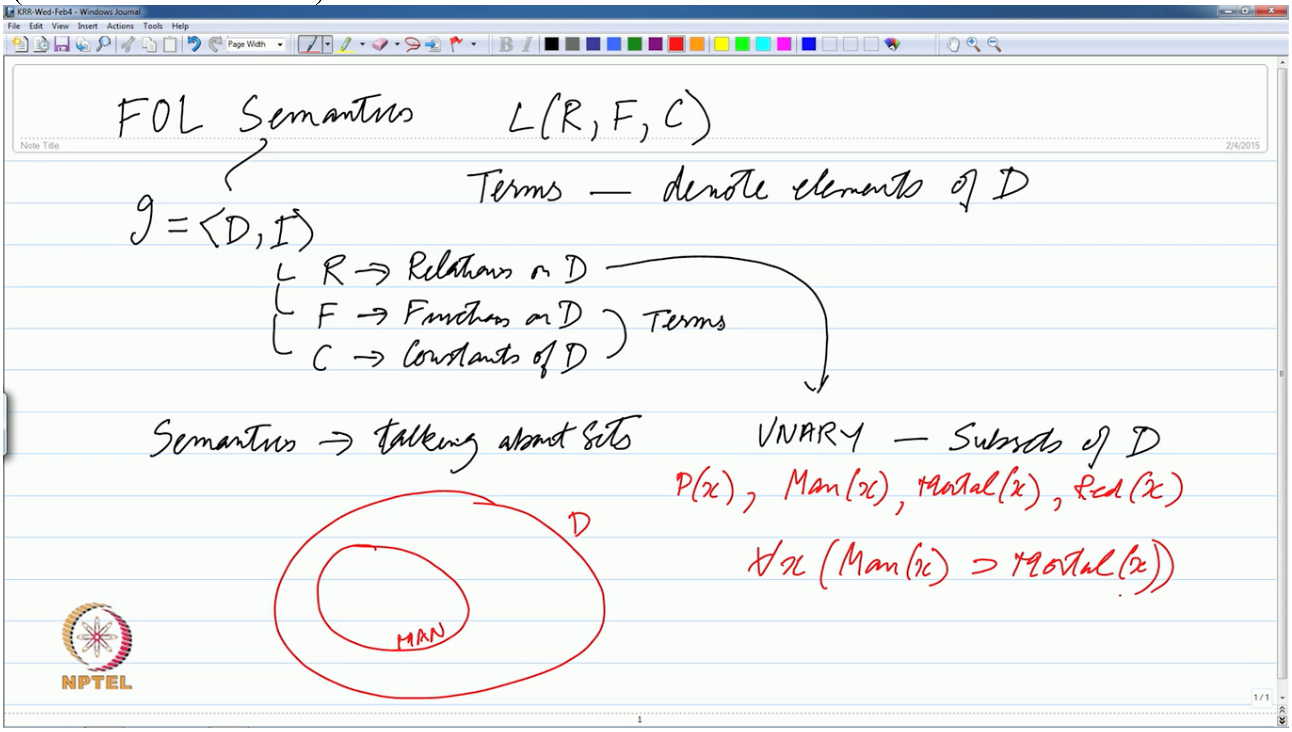

So let's look at the statement which have we started working with says that for all x man x implies mortal x. It's a sentence in the first order logic which will be true of false essentially depending upon the domain. When will it be true or false, the semantics is defined as we said it is model theoretic semantics it will be true if you look at it from the set perspective that essentially we are saying that if x belongs to a set called man, then x must belong to the set of mortals. That's all we are saying in this technique. For all x if x is the man then x is mortal essentially which means if x belongs to man then x must belong to moral which means that my set mortal must be a superset of my set man, so this statement is essentially equivalent to saying, set of men is a subset of the set of mortal things. Let me just use the word things. Nowadays in ontology when you talk about a domain then we say that the topmost level is thing essentially, everything is a thing. So this is a first thing you must understand in terms of semantics of this first order language that we are taking about sets and unary relations are denoting subsets of the domain and logical connectives basically talk about connections between these different sets essentially. So they are relating two sets, man x and moral x essentially. Likewise, relations of higher arity denote or predicates of higher arity denote subset of cross products of an appropriate size essentially.

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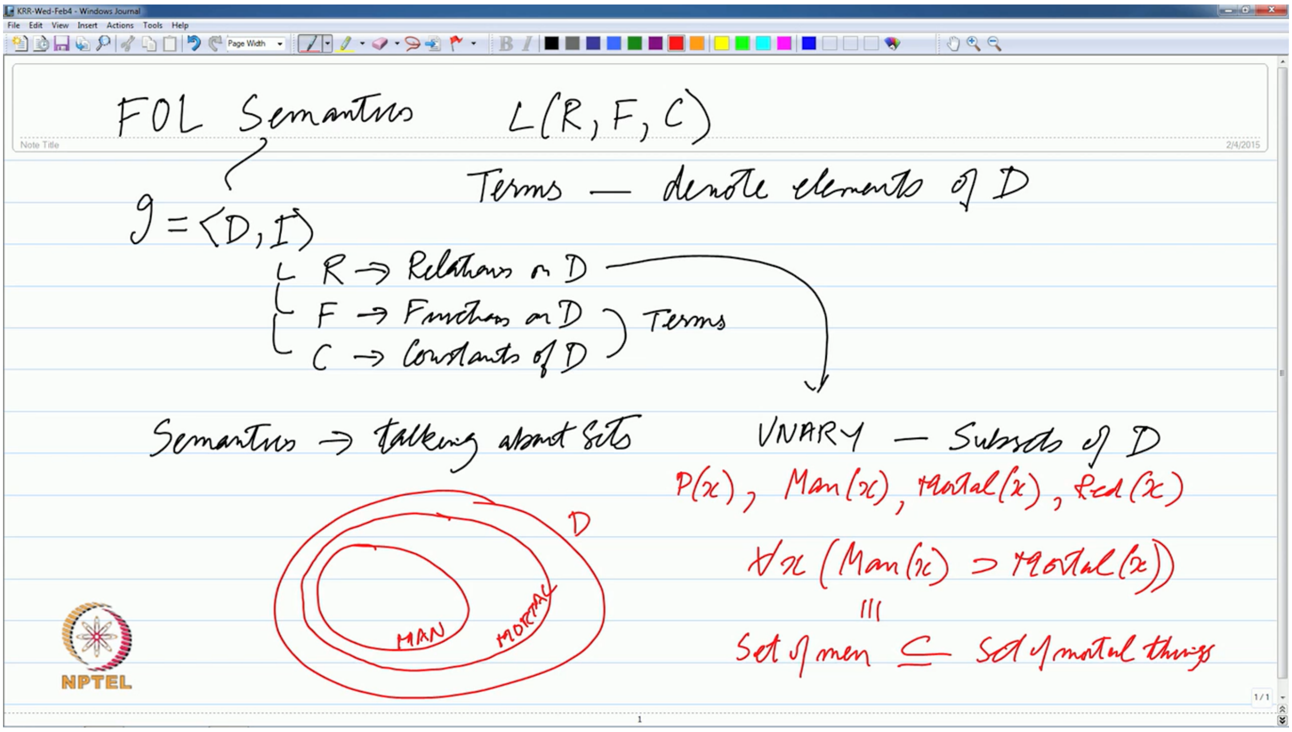

So if you say the relation Friend x y if you say which is a binary relation, it's basically a subset of D cross D where the two elements of the pair are present so which basically means that if the pair is present in that relation, then the predicate is true essentially. So that's how the semantics of first order logic is defined. Now obviously this can be very restrictive in nature

So if you want to use first order logics for knowledge representations there are many things which are difficult to express in first order logic. So one of the restrictions is that, when you look at the predicate take terms so thats the first thing you must realise that predicates take terms as arguments. So whenever you have a precicate so for example if you have P or x y z then x, y, z can only be terms essentially. What is the implication of this? That you cannot how do you express something like this

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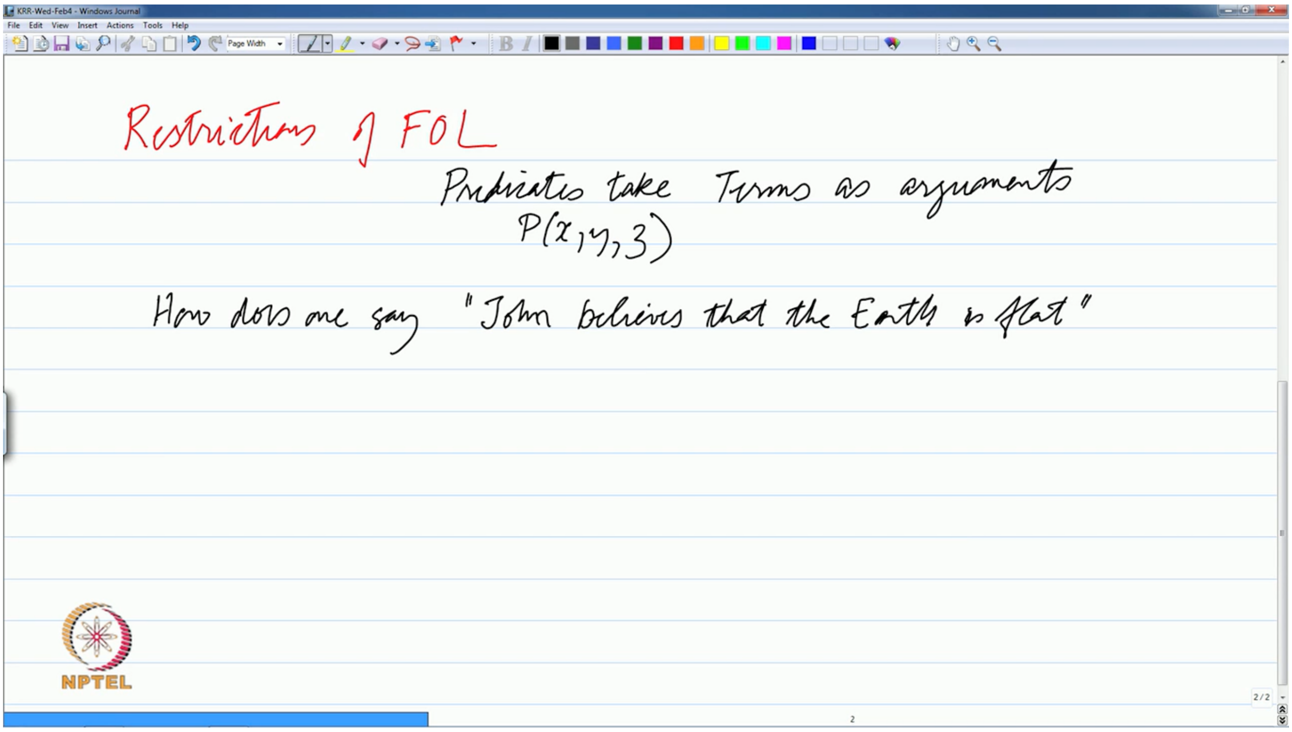

This is an example that I have taken from this book by Charniak and McDermott, one of the popular books on AI and they have a nice section on logic and deduction which I would encourage you to read essentially. How do you express something like John believes that the earth is flat? Now our natural tendency would be to say that there is a precicate for the word believe and the arguments to that predicate would be who believes God, so for example you might want to say something like John, but what does he believe essentially, the earth is flat. So I would write this a something like this. I would want, this is what I would want to say essentially that believe is a relation which captures what John believes in. So the first argument is the person who believes it and the second argument is what that person believes. The second argument is unfortunately in this example a formula

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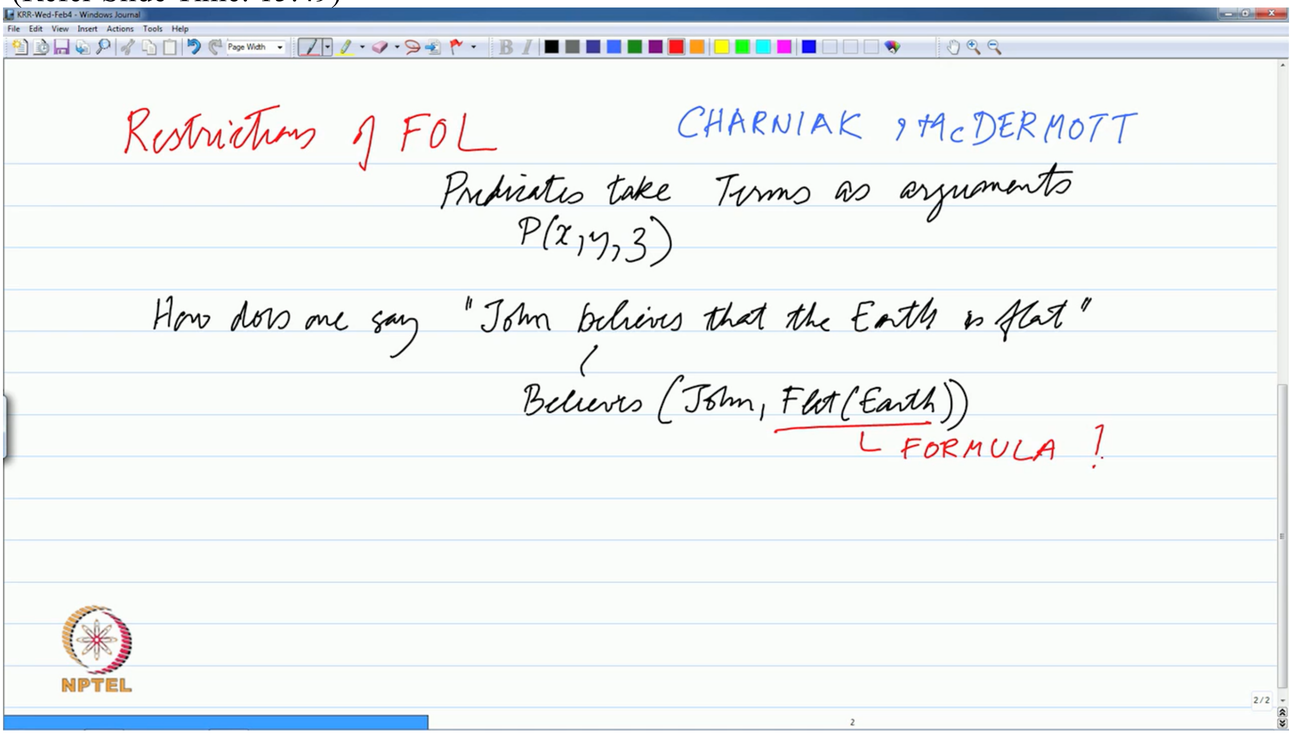

And we have said that the semantics of first order logic is that precicates take terms as arguments and terms denote elements in the domain essentially. So essentially first order logic allows you to talk about elements of the domain. What about sentences when we talk about, and things like that John knows that Mary did this and that kind of stuff essentially. That's the difficulty with first order logic essentially, we will try to see how to get around this fact. So there are different approaches, one is that we can try to twist or extend FOL to somehow include formulae as part of the domain. So that's a process that we will look at later which is called verification. So that we have verified those formulas and that when we say that believes John flat earth we are not really talking about the fact that it's a sentence but that it's a formula somehow.

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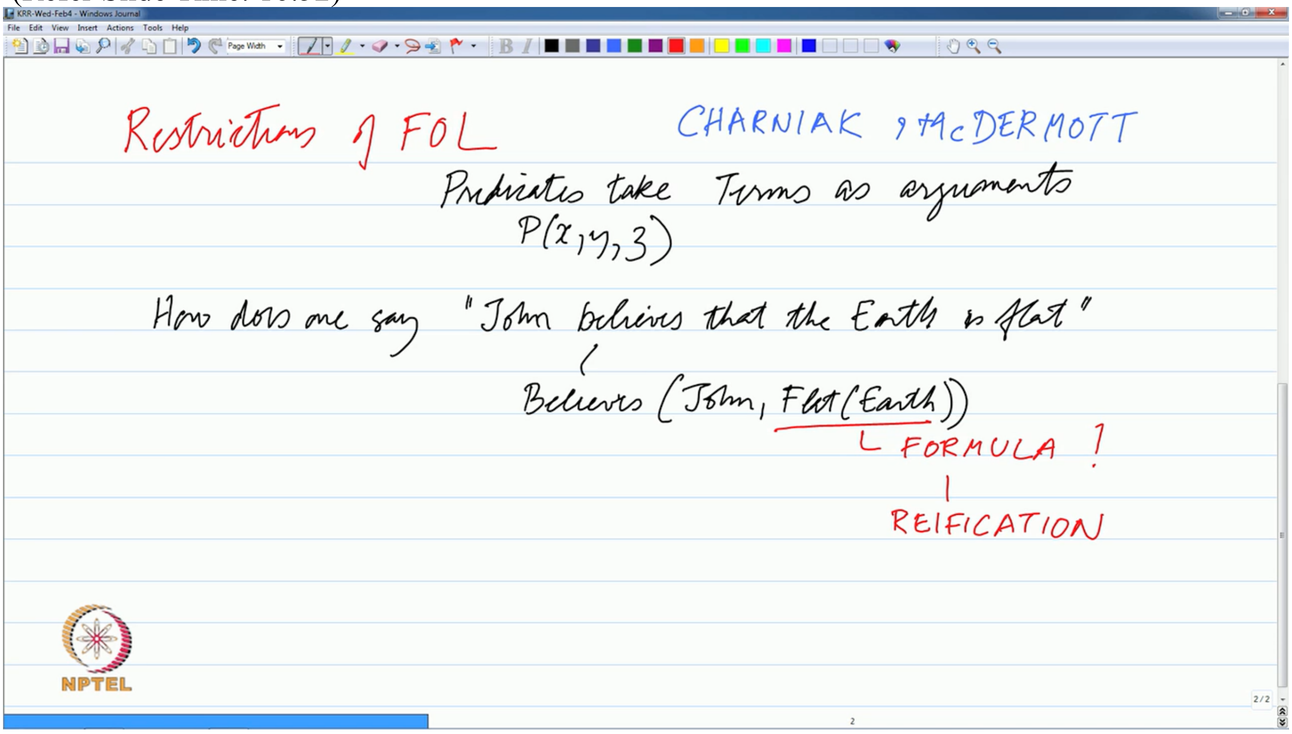

So that's not a neat way of doing things but people have somehow tried to get around this sort of things essentially. So that's a problem with first order logic. Then how does one reason about time and change? Because when we talk about first order logic we are saying that you are given a set of sentences and as a consequence of that sentence, set of sentences certain more sentences are entailed by it. How does one bring time and change into the fact essentially that's a very difficult kind of a thing? So you might, somebody might say that, today they might say that politics is dirty business and tomorrow they might go and join some political party. So how do you cater to change happening in the real world essentially. Or you might say that I have got a block of ice on my table and in Chennai even that doesn't stay for very long so after some time that statement will not be true.

So in a changing world how can we reason about the changing world thats another thing which is a little difficult in approach. The simplest approach that you can think of is that you can include another parameter called time essentially. You can say that this statement is true at this given time. But that also gives rise to as we will see later something called the frame problem. It says that if you said that Pythagoras theorem was true at time t1. How do you know make a statement that the statement is true at time t2 which happens afterwards?

So in fact everything which was true up to a certain time, now you would have to now worry about how to assert whether it is true at a later time essentially. So this kind of difficulties exist essentially. Then how do you reason about continuous things essentially or at least things you perceive to be continuous because we tend to think of a lot of things as continuous whereas in practice they may or may not be continuous. So for example reason about water.

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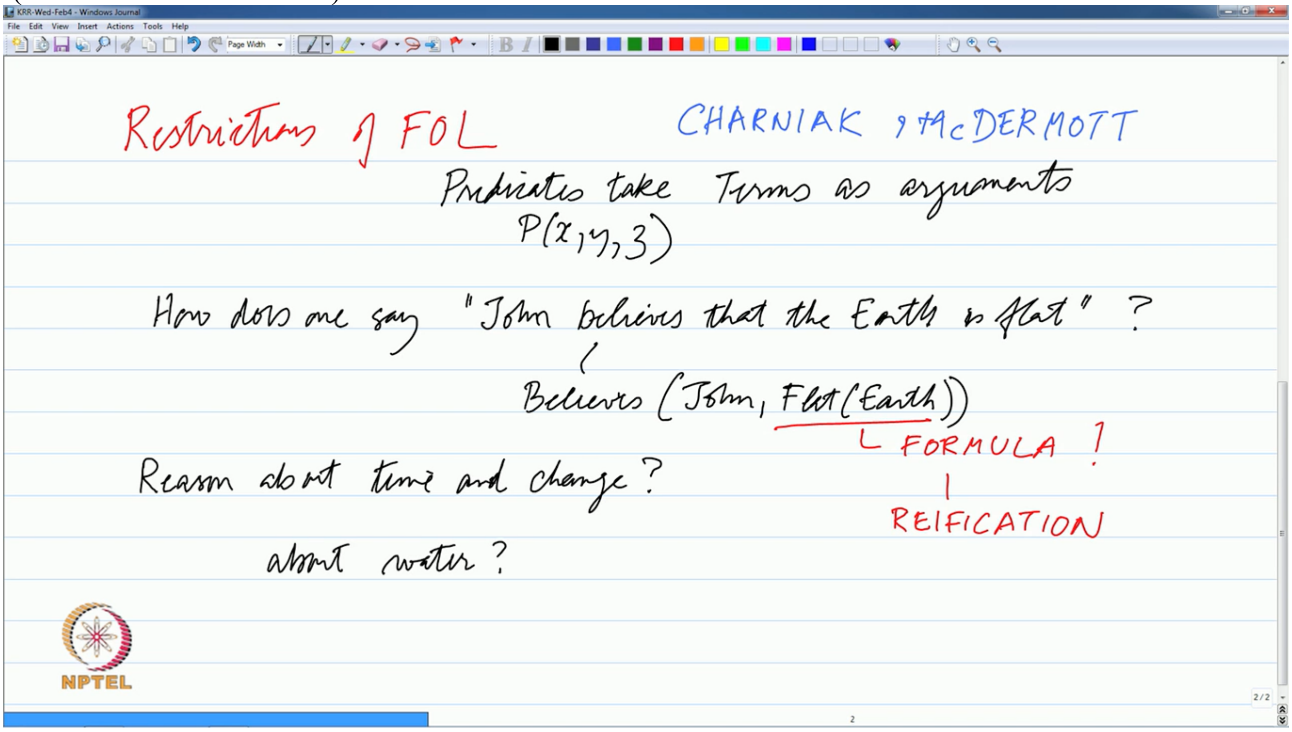

So if you see the kind of statements that we use the term water in, water flows under the bridge, or he drank a glass of water. Now they have different connotations essentially. When you say water flows under the bridge you are making some statement about a category of things and you know things like that essentially, a class of things. Remember that there is a set-theoretic semantics of FOL forces you to talk about elements of the domain and relations between the elements of the domain. Thats about all that you can do. How do you talk about things like water essentially? So if I say there was water in the glass and he drank half of it essentially how do we talk about such things. Because when I say there was water in a glass, if I say something like in glass water.

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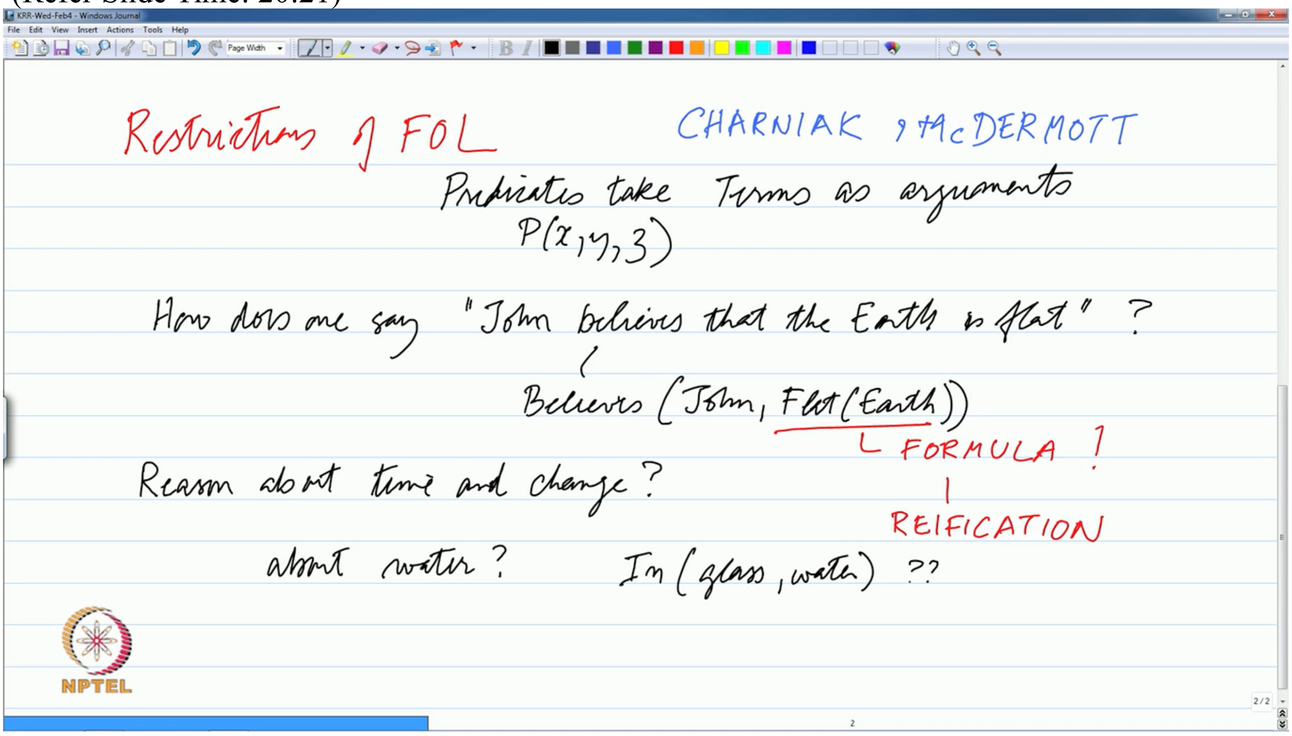

So basically it’s a predicate between two elements, one is called glass other is called water but does that capture what we are trying to say? You know so these are difficulties. How do we reason about space? So the notion of in what we have said is basically a spatial notion essentially. So we can define somehow the semantics of what does it mean to be in if we define some axioms for spatial reasoning and so on. But how you say for example you know that we are all sitting in this room or if you draw a circle and you say that he is within the circle. Or how do you distinguish between the fact that a rabbit may have been swallowed by a python for example in which case clearly the rabbit is in the python. As opposed to the fact that the python is lying in a circle and the rabbit is somewhere in between essentially so do we say that means the rabbit is also in the circle. So reasoning about times, changes, continuous elements is hard in first order logic but we will gradually try to circumvent these things.

Let me come back to the notion of entailment. So we had defined the notion of entailment. We had said when we were looking at predicate in the propositional logic, we had said that a set of sentences entail a formula alpha, whenever S is true alpha must be true. So we carry forward this notion of entailment to FOL and in FOL we say that a set of sentences S entails alpha if whenever an interpretation I entails a set S, so what do you mean by an interpretation I entails a set S, it basically means whenever we have a model for S which means that we have chosen a domain and an interpretation mapping for the domain. So whenever the set of sentences S is true in interpretation then the sentence alpha must also be true in that same interpretation. So that's how we carry forward the notion of entailment to the first order logic.

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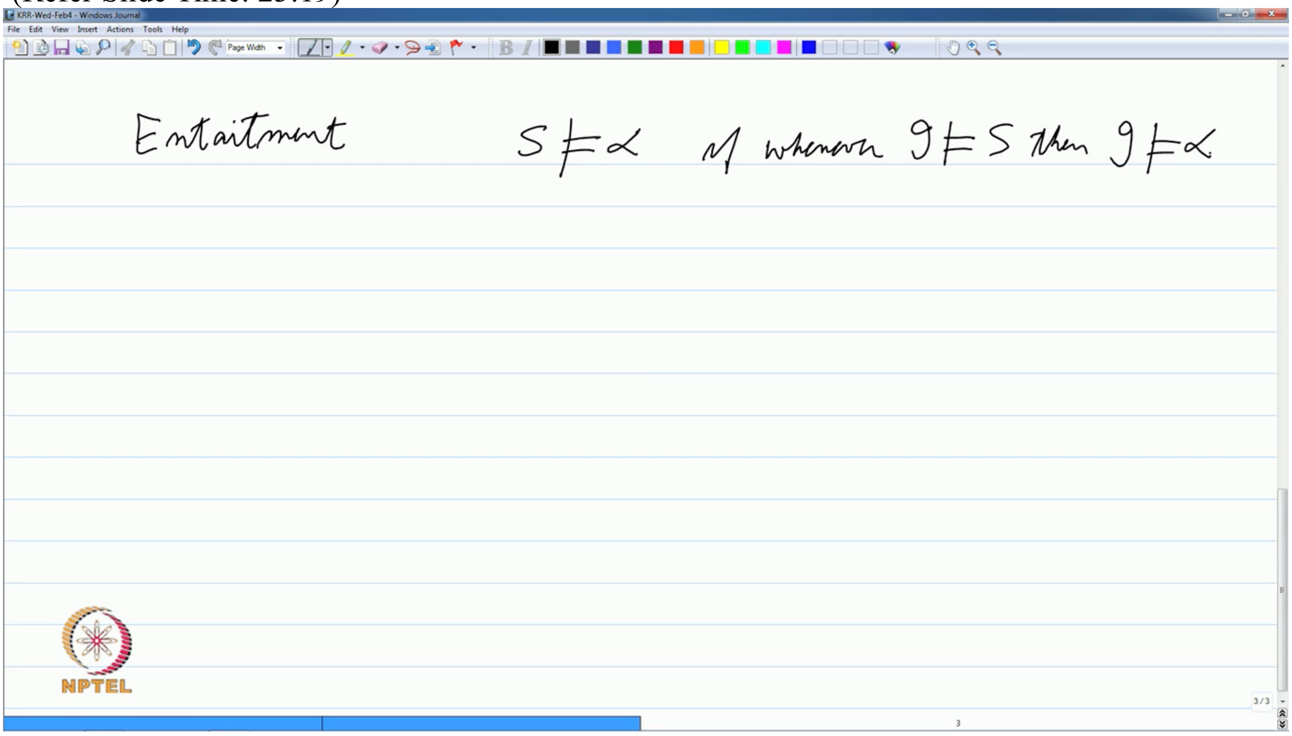

We are saying that if S has a model then alpha must be true in that model essentially. So that's the notion of our entailment. Now let's say we define just to distinguish between the syntax and the semantics part of it. Let's say we define a simple language in which R has two relations let's say G and let's say T, it doesn't matter, and the set F has also two relations let's call it S and C. And the set C has just one symbol, let's just call it Z. So once I have defined these three sets, the set of relation symbols, the set of function symbols and the set of constant symbols. And of course I have the set of variables.

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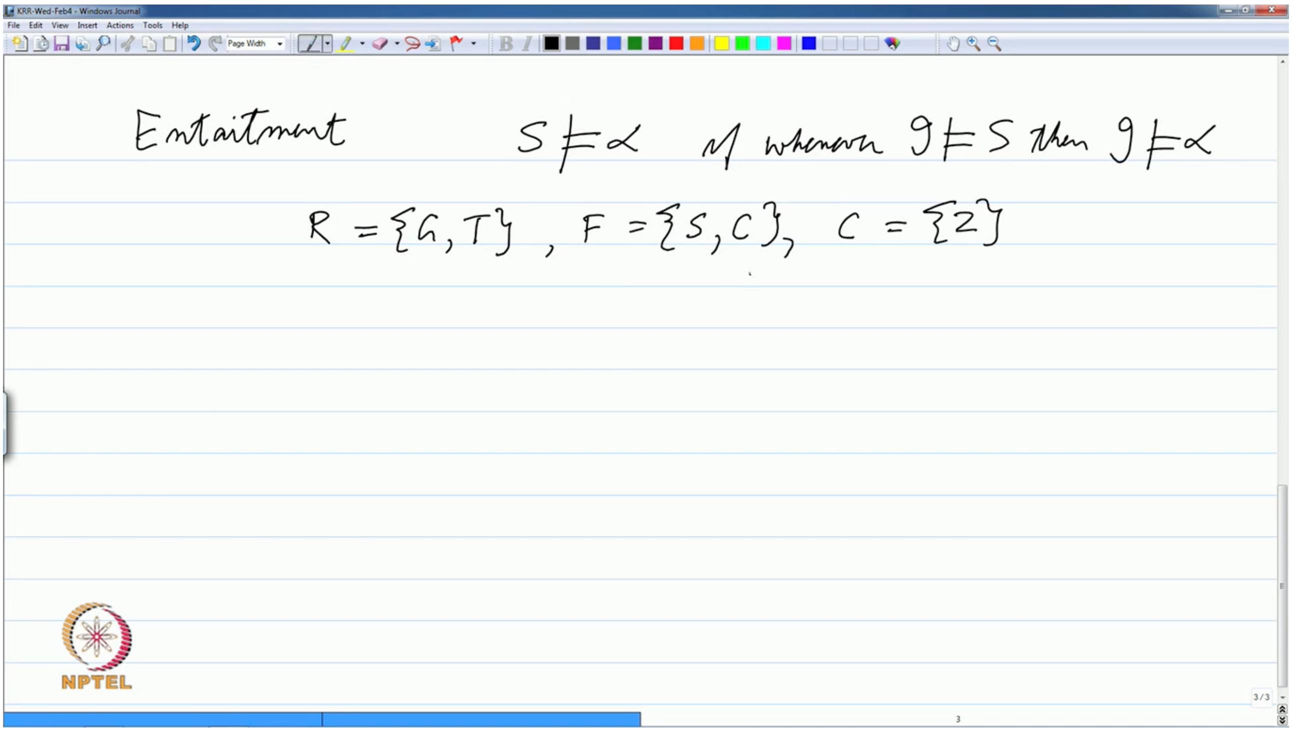

I can always write formulae now I can say for example G assuming its of arity 2, of C sorry not C, x and S of z. I can make statements like this and these are formulae in my logic. If I make sure that there are no free variables, for example if I say that for all x this is true then I have a sentence in my language essentially. So I can of course construct, define the language having chosen this vocabulary. But what about the meaning of these sentences. We had said that the meaning is given by a model essentially. I just want to illustrate the fact that by choosing different models may have a different meaning for the same set of sentences. So what could be the two different models.

So let's say one model, we call it the Z model and in that model let's say D is equal to integers so I am talking about the domain of the integers. Then I say that G refers to greater than and T refers to some relation. So let's say T refers to something, let's say less than, it doesn't really matter. Because I am talking about integers, the notion of less that and greater than is defined on integers. So whenever I use a formula G of some two terms I am essentially saying that the first term is greater than the second term whatever my meaning on the things. I could say that S is the successor which is equivalent to saying plus 1 and I could say that C stands from plus. So C is a function and plus is a function of two arguments so let's say that this is of arity 2 and this is of arity 1 and I could say that Z stands for 0.

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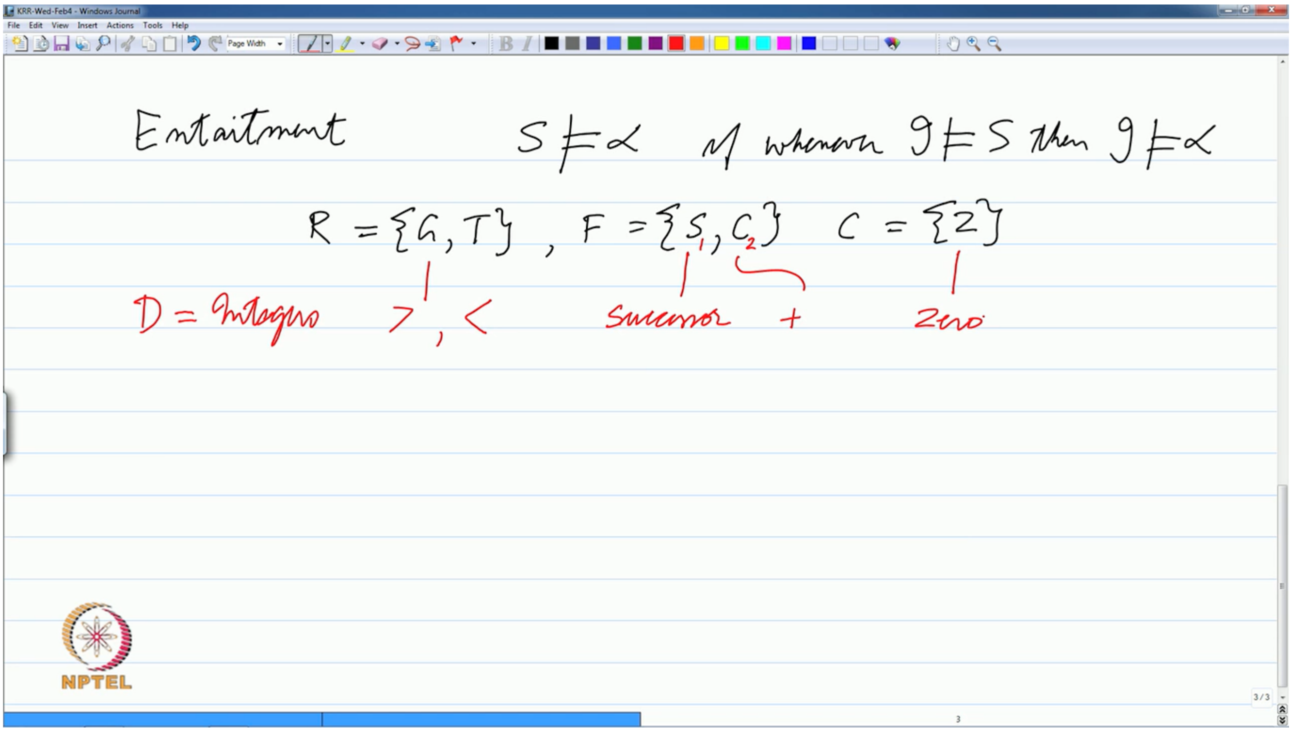

So I have defined a language, I have defined a domain of integers and I can make statements about some number being less than another number or some number being greater than another number and depending on what statements I am making those things may be true or they may be false. So if I have a set of statements for example if I say, S is a very small set for example, it contains let's say one statement that the successor of Z is greater than Z. Let's say some very small set of statements, only one statement in fact. Obviously for this set S my domain D of integers is a model essentially. Because all I am saying is that I need some domain in which the greater than relation which I am saying holds between the first argument and the second argument and the first argument is the successor of the second argument which is true in integers essentially. So in integers this formula is true so therefore this domain is a model for this set of formulae. You can add more formulae for the properties of integers and so on you know. You can define the fact that addition is commutative for example. So you can define it using these kinds of symbols essentially.

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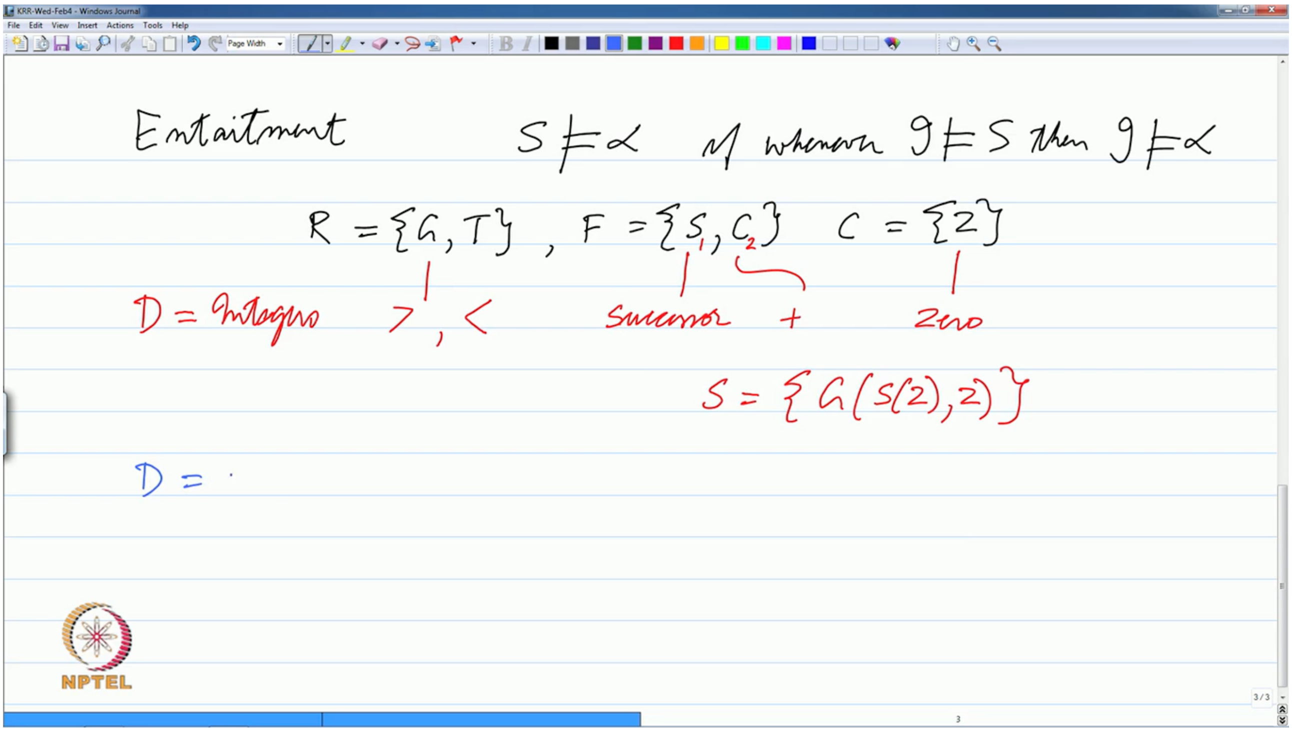

But I could also choose a different model – Strings over the alphabet A let us say. So we all have straight formal languages and I could choose that I am talking about a representation. Now what could be the meaning of my relations. So here G could be for example longer than, T could be let's say shorter than, at this moment I can't think of a different term, but you can think of anything essentially. So the notion of representation is that you are saying what does that relation mean, how does it relate to elements together essentially. S could be a function which also defines a successor which says that the successor of x is x followed by a. I can define the notion of my successor like this.

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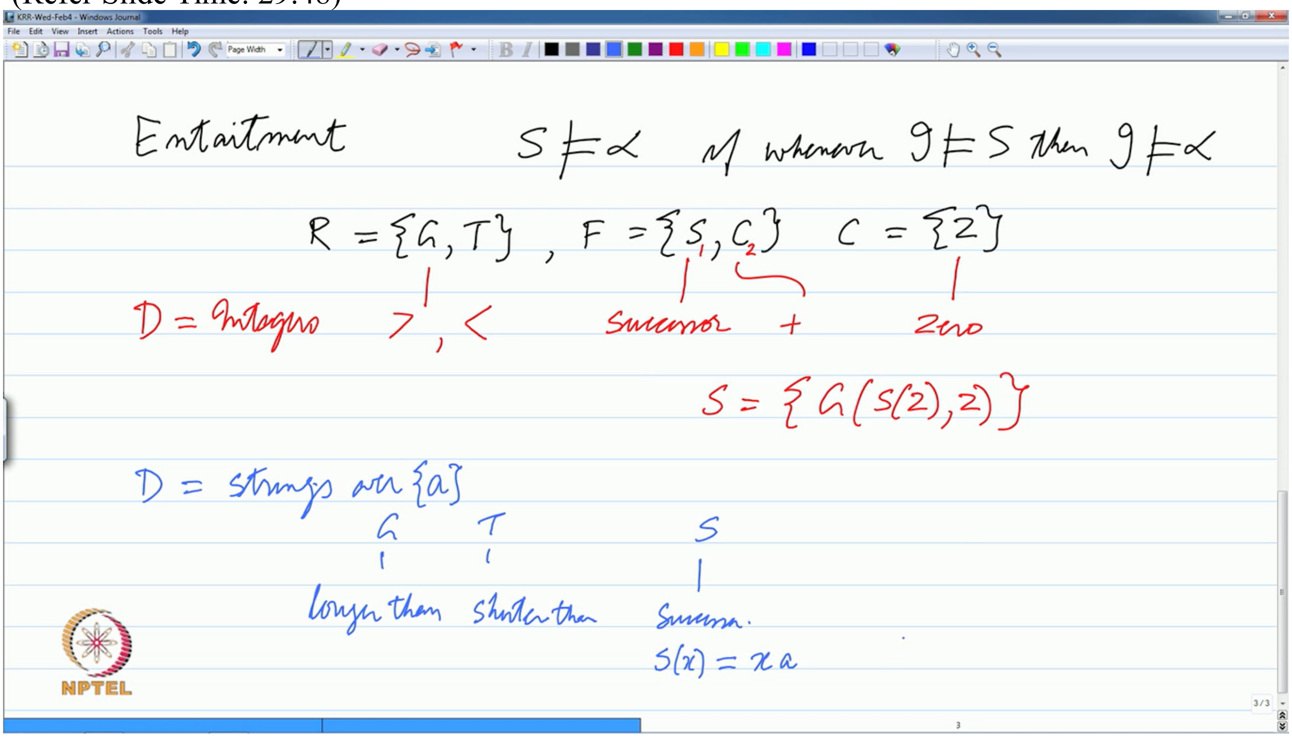

C could stand for concatenation and Z could stand for empty strings or something like that. So anyway the point is not that to look at these two domains but the point is to learn to discriminate between the syntax and the semantics essentially. The syntax is defined by an alphabet and the language we construct over the alphabet so in this case there are two relation symbols, two function symbols and one constant symbol. And we can define a whole set of sentences in this. Whether those sentences are true or false is the kind of thing we are interested in. So if we choose a set of sentences and if we can construct a model for it, it means that we can find a domain for interpretation. What is interpretation? It tells you what is the relation and what are the functions and what does the constant refer to such that the sentence in the set is true or the theory is true, then we have found the model for that. But a given set of sentences may have more than one model. So for example for this very small set which contains only one sentence both these interpretations are models. Because in both those interpretations, in one case the successor number is always greater than the other number and in this case successor a is defined by appending a at the end of it then it will be longer than the other number. This sentence that we have, G of S of Z comma Z is true in both the models. So both are models of this system. So one of the things we would be interested in as a knowledge representation exercise is that if we write a set of sentences which is what we would call as the knowledge base, then is our description enough to assert the fact that the models of the set of sentences is what we intend essentially. In the sense have we said enough in our theory to demarcate the domain that we want to talk about essentially. Because if that is the case then we are really talking about the domain. So we have to say things in precise enough detail and you have to say enough things for your knowledge representation exercise to be meaningful essentially. So when we work with logic we work at the syntactic level, ok we have seen that, proof systems are syntactic in nature. But our goal is obviously semantics we want to do something meaningful essentially. So if you want to create a set of sentences then you want to say enough sentences which a true for a model. So for example you must have heard about things like piano, axioms which are used to define number system. Or you might have heard about Euclid's axioms about geometry. So you make a certain set of sentences which are true and which define a model for which number are a piano systems and geometry is a model for Euclid's system essentially. So the thing you should always remember is that the first order logic is talked about sets and relations between sets and when we write proof systems we are essentially gone to write those proof systems where everything that we can prove happens to be true which means our proof system is sound essentially and hopefully it completes also.

So in the next class we will start by looking at what are the new rules of inference we need in first order logic and then we will try to look at one approach to generating proofs.