Artificial Intelligence
Prof. Deepak Khemani
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture No - 43
Prepositional Logic

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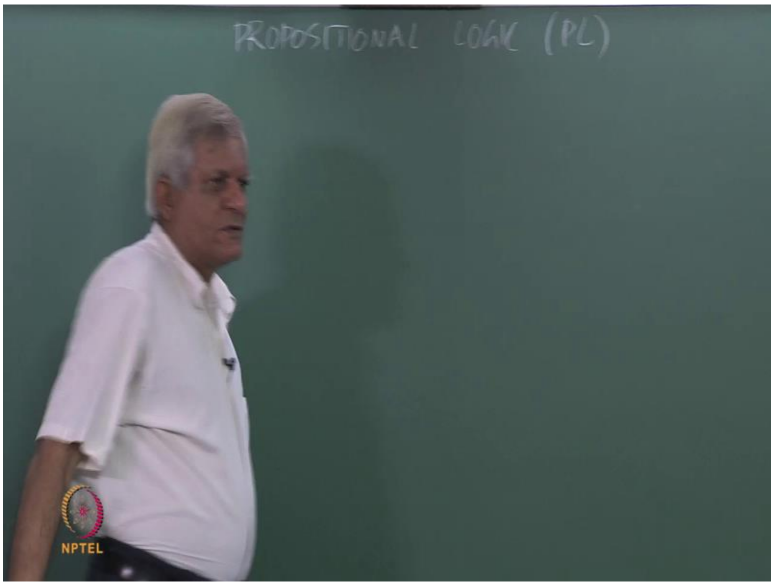

So, we are looking at logic and listening and we are looking at propositional logic, so just to do a quick recap.

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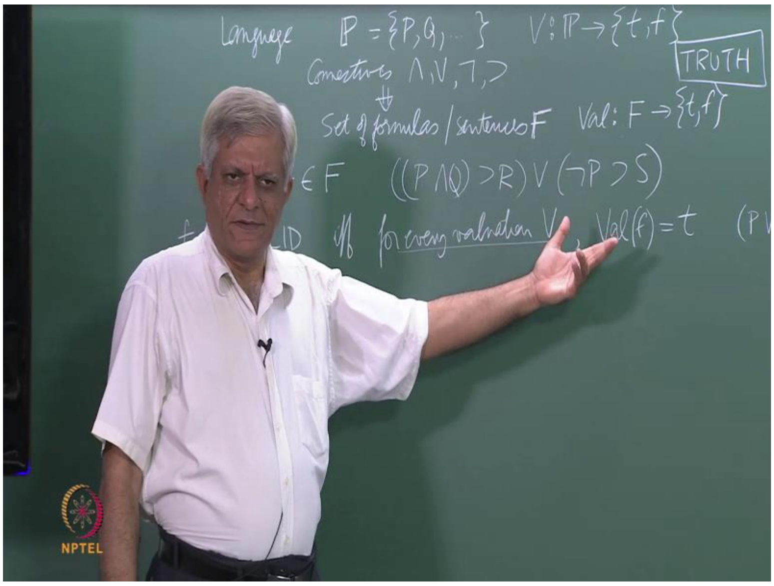

We first look at the language and the language is made up set of propositional symbols I do not remember what the symbol we use anyways. So, something like this accountably, if a set of atomics symbols and each of these symbols stands for a sentence and we do not particularly care about its stands for essentially. Associated with this set of symbols is a function v which map p this symbols to set let say true and false basically a two value set which as far as we are concerns will stands for true sentences and false sentences. Then, we have connectives and or not imply and so on, which gives lies to a set of sentences, so we will call them a set of formulas.

Let us call set f, so we could construct compound sentences from the atomic using the logic connectives and associated with this set of formulas is a function with maps f to the same set true or false. So, this side is the syntax and that side some sentence semantics of this thing and what the semantics captures here is that how do this logical connectives connect the or influence the meaning of compound sentences essentially and that is given by this. So, if you remember for example, a sentence like p and q would map to true if both p map to true and q map to true and so on. So, now, given a sentence f belonging to f or formula f belonging to f, so for example, you might say something like p and q implies r or not.

Then, p implies some such formula re arbitrary formula, which can be constructed using this logical connectives p q r and s are the atomic symptoms and implies of or not implies other connectives. Essentially, we can find a valuation for this if you know what the valuation for p q r and s is, so if you know for example, the p is true, let us say every p q r s everything is true and this true implies true or false implies true. So, this whole thing will evaluated to true essentially, so this was the notion of truth this is the notion of truth. It says that given a sentence f belonging to this set of synthesis there we can compute whether the sentences true or not given a sentence and given a valuation v.

For all the atomic sentences, we can compute whether the given sentence match to true or false essentially and I do not remember whether we discussed this, but in general a formula can fall into three categories one is a valid formula. So, we say f is valid if for every valuation we that we can think of Val f is equal to true, so you are familiar with this notion I just repeating this. So, when you say for every valuation v essentially we are talking about the truths table for the formula. So, this formula for example, we can construct the truth table which will have 16 rows because there are 4 propositional atomic sentences, which means p can be true or false q can be true or false, r can be true or false or and s can be true or false.

For each of these 16 valuations or each of the sixteen rows in the truth table if the last column is label with true then we label then we say the sentence is valid. So, a sentences is valid if is true for all possible valuations a trivial sentence which is valid is for example, p or not p this is of course there is only one variable, but this is a valid sentence also known as the tautology always true.

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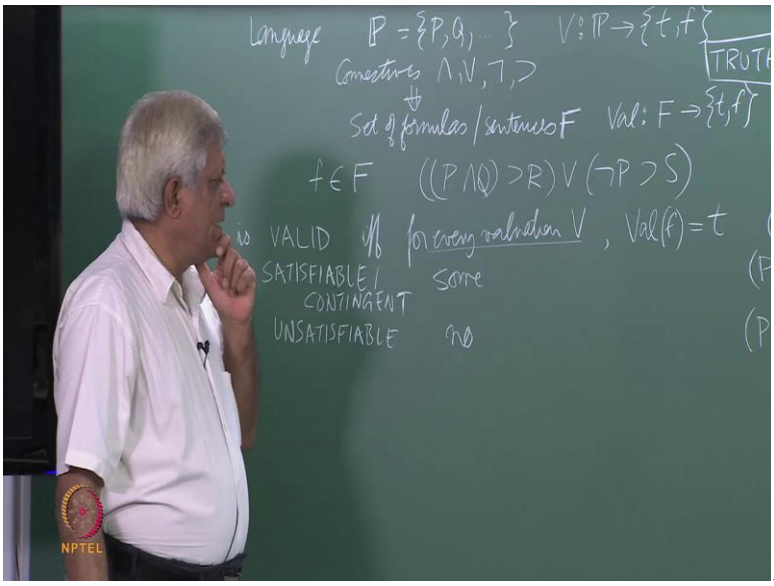

We say that the sentence is satisfiable or the formula satisfiable if for some valuation v this is true the same thing accept that instead of every v use term some. If you can find at least one valuation which will make this true then that sentence is satisfiable for example, I might say p implies q 1 simple sentence not always true for example, if p true and q false this is false. I can find valuations of p and q usually make this true if there are 3 valuations which will make this true such sentences is also sometimes call contingent sentences. So, one is valid the second is satisfiable or contingent and a third is unsatisfiable.

Here, we replace this with no if there is no valuation which makes this sentence true, then it is unsatisfiable example as you can this something like p and not p essentially. So, in general there are three kind of sentences valid satisfiable and unsatisfiable. All are seen notions are used somewhere or the other as we will see later essentially, so when we talk about truth we also say we sue this symbol to say that a formula is valid, you can use say using this essentially.

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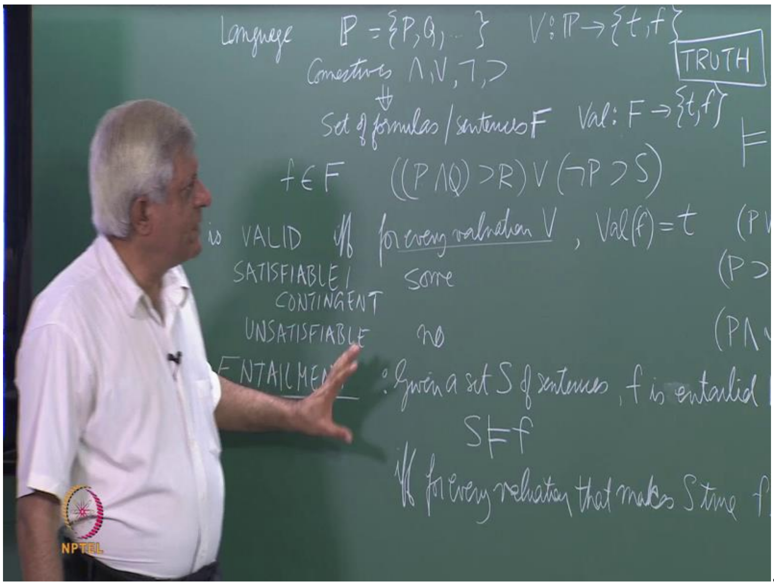

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Then, there is a notion of entailment if now this is an interesting notion because in when we study logic the reason we study logic is to we are able to capture valid forms of argument. Essentially, we are not so much interested in saying can we find the valuation which will make the sentence true or not we are more interested in saying that if somebody has given you a set of premises or a set of axioms. Then, does some other sentence follow that essentially, so given set of s sentences we say that a formula f is entailed by s, which we write as f entails s if the following is true if for every that makes s true and when we say a set of sentences is true essentially.

What we mean is that every sentence in that set is true, so you can think of it either bigger sentence with the sign in between their essentially. So, we are more interested in the notion of entailment that somebody give us the set of premises let us call them s then we want to ask whether a given formula f is true or not essentially. That is notion of entailment essentially, so these concepts are kind of semantics concepts.

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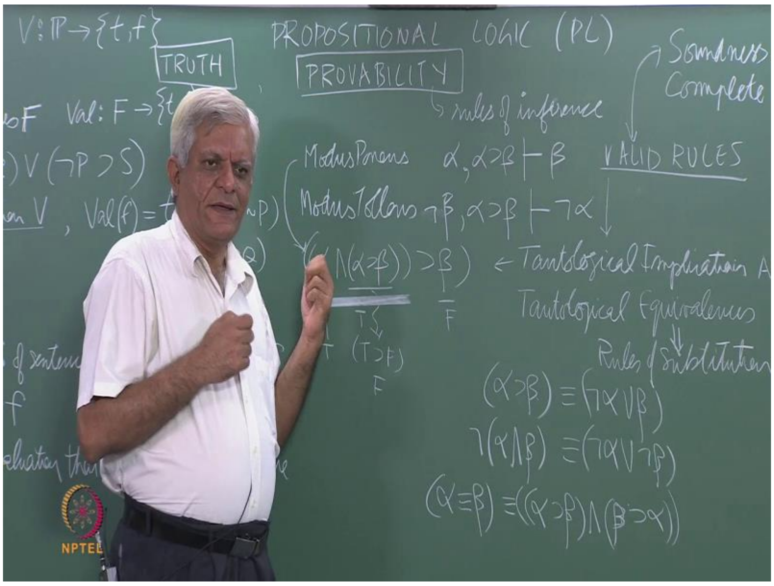

On the other hand we have the notion of provability, so if you recall we did a small proof in the last class notion of provability is tied up with the rules of inference. For example, most commonly used tool of inference call modus ponens and we express it follows like if you have alpha and if you have alpha implies beta then entails beta. So, last time we wrote with as three separate lines alpha and then alpha implies beta below and then beta, but this is just another way of writing it. Essentially, you are saying that if you are given the alpha and if you are given alpha implies beta, then you can derive beta or in some signs you can add beta to your set of sentences.

So, notion of provability is a or the notion of proof is the entirely syntactic concept if simply says that given set of formulas, you can keep adding new formulas to the system and this particular rules says that is you have a formula which matches alpha. If you have another formula which matches alpha implies beta, then you can add the formula which matches beta to the set essentially this alpha and beta could be arbitrarily compound from. They do not have to be atomic form, essentially it is just s it is just a pattern that a pattern the same thing must be here of course alpha can be any formula. If that formula implies beta is present in the data base or nor it be a set of sentences, then you can add this.

So, the proof finding algorithm is basically a simple algorithm which keeps which does the following select a rule of inference select some data to which you will apply it to and add some new formula to your database. You keep doing this repeatedly till you have proved the in formula that you are interested in essentially. So, another rule of inference is for example, Modustollans, which says that if you have not beta and alpha implies beta then you can add not alpha in this set I will show.

You are familiar with this kind of formulas, so the next question I want to ask is what makes a rule of inference a suitable rule of inference essentially. So, if you recall we had this notions of soundness we say that a logical system is sound which says that if s in other words if you can derive a formula alpha from a given set of sentences then if their logic system is sound. Then, this alpha will also be entail by the set of sentence s which can be re express as the following if your set of premises are true then the conclusion will necessarily be true that is the notion of entailment.

Thus, the right hand side this notion of entailment we say that the rule of a logic system is sound if anything that can be derive is entailed as well. On the other hand, a logical system is complete if anything that is entailed can be derived observe this also look likes sentences in a logic. This is a slightly different logic in which we can talk about sentences like s derives alpha or s entails alpha which of course has to be in some different logic essentially, not in proportional logic. So, the question we want to ask is when the rule of inference is sound essentially, so soundness is tide up to valid rule, so rule is valid if it entails right hand side.

So, rule like this is valid if this and this actually entail beta, so look at an example of a rule which is not valid if I say alpha implies beta and beta, I am just writing in the old which is its similar thing this is not really a valid rule. So, we do not have a name for it this rule s says that if alpha implies beta is given to you and beta is given to you infer that alpha is true. I do not know whether we discuss in the last class this is actually the process of abduction, so it is like this for example, you might say that if somebody is drunk then that person staggers while walking.

So, that could be alpha could stands for somebody is drunk and beta will stands for person stagger while walking essentially, then you say somebody staggering while walking essentially.

You come to the conclusion that the person is drunk essentially, now this is not a valid rule of inference because it possible that the person may be drowsy or sleepy or tired or hurt it could be anything, but it is not necessarily follow that that. If you have this fact you assume that to be true that people who are drunk will stagger, it does not mean that anybody who staggers is drunk essentially, whereas if we you were to use that formula here if some you say that somebody is drunk. If you have that somebody is people who are drunk stagger then you can info the other person who is stagger because the rule that actually says that essentially.

In other example, that we might have discusses this problem this process of diagnosis in medical diagnosis a disease causes symptoms. So, disease implies symptoms if you see the symptom then you infer the disease essentially now that is not necessarily a valid inference. It is possible that the symptom could have been due to some other disease as well essentially because there are many diseases for example, cause fever. Then, if you simply say that just because it is fever it is this particular disease not a valid rule of inference. So, how do we distinguish between valid rules, and rules which are not valid rules are based on tantological implications, which means that corresponding to every rule of inference.

There must be tantology which is an imp there must be an implication statement which is a tantology and corresponding to modus phonons for example, the tantology is alpha and alpha implies beta implies beta. So, notice the similarity between that pattern and this pattern here this is one sentence one sentence in my language of logic. There, I am using an additional symbol which stands for derives or something like that which is and extra logical symbol in sense it is not a part of language that I am using, but this is a sentence in my logic and what we are saying now is that rule of inference is valid. If it is based on a tontological implication by this, we mean we should we should have a corresponding sentence here and a sentence must be a tautology.

So, is this sentence are tautology you can construct a truth table to find out and show that this is tautology or you can try and show that this is a tautology by trying to show that it is not a tautology which is kind of proof that you often do proof by contradiction. This would say let say that can we make that sentence false or if you remember the truth table for implication you can make it false only if you make it this part true.

If you make this part false now you can make this part true only if you make this true and if you make this true to make this true you have to either make alpha false or beta true, but if you have made alpha is true here. So, this mean that alpha is true, so now, alpha is become true and beta is become false, so this part becomes true implies false which is actually false, once this becomes false this and this becomes false and once this and this become false, this becomes true. So, we can make this sentence false sorry we cannot make this sentence false because to make that sentence false we have to make this part true. We have to make this part true, but we are unable to make this part true because you have to make this true and this part false.

That is the only row in the implication to table where this implication formula becomes false. So, this has to be false, so that means, this beta has to be false now to make this whole part true this has to be true. So, to make this part true implies false has to be true, but true implies false is not true it is false. So, we are not able to make this false, so we are not able to make this whole sentence false. So, this sentence is a tautology, so as a small exercise, maybe you should just construct the truth table for this and see that this is tautology. Now, there are other kinds of rule of inference which are based on tautological equivalences and they give rise to rules of substitution.

This rules of inference have a sense of direction in the sense that you have to be given the on the left hand side and then you can produce the formula on the right hand side. So, it has a sense of direction you can go from left to right essentially in rules of substitution. You say that two formulas are logically equivalent and therefore, you can substitute one for other at any point of time essentially. So, an example of a rule of substitution is and you must be familiar with many rules like De Morgan's law and so on, but one rule is for example, alpha implies beta is equivalent to not alpha not beta. So, you must have looked at it at some point that if you have not done this, tries to construct a truth table for this.

You will see that this tautology and if it is a tautology we can base a rule of substitution on this, which means that whenever we see a pattern of kind alpha implies beta. We can replace it which the pattern of kind not alpha or the beta and vice versa and we do not have to replace it in the sense we do not have to substitute it we can even add it, but it same thing. So, De Morgan's law for example, you must have studied, so there are many tautological equivalences and you can convert each of them in to rule of substitution you can also see a rule of substitution as a bidirectional rule of inference.

It says that this implies this and that implies that because if you might remember you can say something like alpha is to beta is equivalent to alpha implies beta and beta implies. So, you can think of a rule of substitution s to rules of implies essentially going in both the direction essentially. So, what you have in your to construct a logic machine is a set of rules of inference a set of rules of substitution and you have to ask is now to pick one of this rules and keep producing new formulas till the formula you were that looking for is generated. So, till this formula f we are interested is generated, so if you look if you remember the example we did last time we had something like this.

It was given to us to that p and q were p was something like allies like maps and like music or something like that. Then, p implies r, then r and s implies t, then not q or s and from this you have to show t that was the problem that we looked at last time or something very close to this. So, these are the four premises given to us the set of sentences s and from this we have to show the t is derived essentially. So, we write this s as this it says that this and this and this and this let say this is s 1 s 2 s 3 s 4 then s 1 comma s 2 comma s 3 comma s 4 and t entails this or derives this we can only use the truth term terms interchangeably entailment and derivation if the logic is both sound and completes.

If it is sound, then whatever is derived is true is entail if it is complete whatever is entail has been derive. So, we come to the completeness in a moment, but so far we have address the issue of soundness when my logic machine sound is provided I use valid rules of inference. Valid rules of inference are characterized by the tautological implication or tautological equivalence and corresponding to each tautological implication we can construct a rule of inference. So, for example, if somebody says p implies p is a tautological implication which is trivially true. You can have a trivial rule for this which says that if you have p you can add p to that essentially, but of course that is trivial for any tautological implication.

You can add a rule of inference for any tautological equivalence you can have a rule of substitution. So, before you move on i want to mention one theorem, which is quite well known. So, let me write me here its call deduction theorem this theorem says that if I have a set of sentences s and a sentence alpha and I want to derive the sentence beta if I can do that. So, remember that this most notation stands for this idea of generating a proof that you keep apply in rules of inference till you generate beta essentially.

The deduction theorem says that if you can do that then always the case that you can do this, you can take s to the right hand side sorry you can take alpha to the right hand side and instead of deriving beta you can derive alpha implies beta. Now, this s could be an empty set, which means that if you want to show that beta can be derive from alpha, then you can equivalently show that this formula alpha implies beta is true essentially. So, in other words if you want to show that this is true, this implies this amount is showing that this and this and this and this implies this if I construct the large formula then that is true or tautology. So, you can see that all valid derivation amount to proving this large tautology essentially.

So, let me ask the question what is a relation between these three sentences can you think of a relation if f is valid, then if f is satisfiable or if f is unsatisfiable is there a relation between them, we will come to this, so, just think about that a little bit. So, I want to spend a little bit of time talking about completeness when is a logic system complete. So, you can see that by definition a logic machine or a logic system is complete if it can derive every true formula. In other words, every formula that is entailed a given set of axioms essentially and in the last a couple of hundred years ago, there were lot of people trying to build logic systems and show that they are complete essentially.

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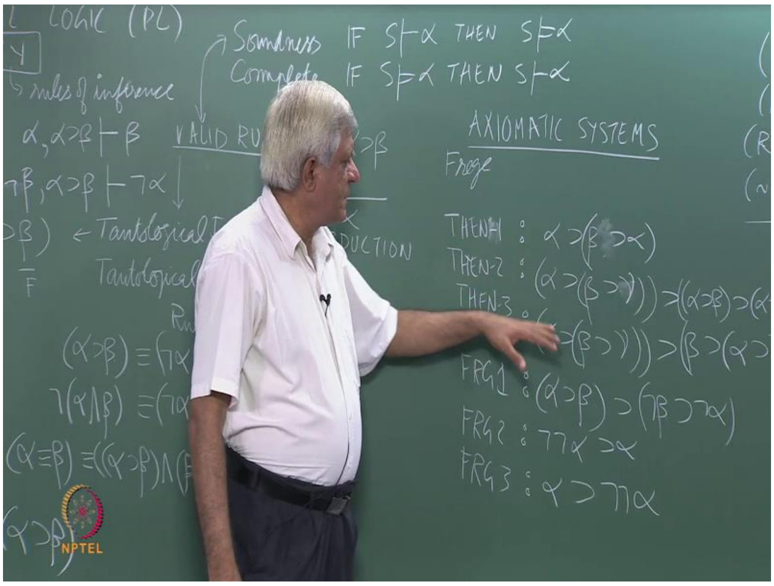

So, this are known as axiomatic systems axiomatic system said there is set of statements you must true to be you must accept to be true without asking them asking about them.

You know just accept them to be true on faith and some rules of inference of choice made is essentially. So, one of these earliest systems, in fact probably the first one which is given by Frege who is also credited with inventing first order logic was the modern form of first order logic that we use. He had six axioms, which were as follows this is a name of the axioms then one it says alpha implies beta implies alpha. Then, if you just look up for Frege axiomatic system, you will find a very nice page on Wikipedia which describes this six axioms and the things which we follow from there.

Essentially, you do not have to write them here, so this Frege gave this axiomatic system which have this six axiom first axiom says alpha implies beta implies alpha second axiom says alpha implies beta implies gamma. The whole thing implies alpha implies beta implies alpha implies gamma the third one says alpha implies beta implies gamma is also the same as beta implies alpha implies gamma. Then, there are three axioms says which use negation sign alpha implies beta implies not beta implies not alpha. So, you can see for example, this one Frege one has connection with this two rules, essentially this tool says that alpha implies beta and if you see that not beta implies not alpha is implied.

Then, if you have to replace this to with not beta, not alpha, then you have something like this. So, you can see the modus Tollens can derive using Frg one essentially and substitute this with not beta implies not alpha and that then it becomes like modus ponens because not beta is given not beta implies not alpha is given and not alpha is given. This from many of us is trivial negation alpha implies alpha is a axiom in Frege system essentially and here one rule of inference which is modus ponens, which we are familiar with this rule here. So, Frege's axiomatic system says that all tautologies of propositional logic can be derived from this set of axioms and this rule of inference.

So, it is a complete system because we are not go into going to proof of this year, but we just take it for granted that here is a definition of a complete logical. After Frege, there were many people who devised different axiomatic systems using different sets of rules of inference and other things essentially now as a small exercise. I will ask you to prove this formula using Frege's axiomatic systems observe that this is not taken for granted in Frege system, when you say an axiomatic system.

These are the only six sentences that you take for granted this are pattern, which means alpha can be substituted with anything essentially and only one rule of inference allow. Using this, can you come up with a derivation of this sentence p implies p, you can try it a little bit while. Then, if you cannot succeed you can go to the Wikipedia site and you will find the proof there. So, I am not going to do it in the board here, but you can find a proof essentially, but the whole idea of developing axiomatic systems was to choose a set of axioms in the given language. This is now a case professional calculus or professional logic a rule of infer some rule of inference and say that this set which is complete, which means all tautologies can be derived in this including this tautology which stands for this.

This implies if you look at the deduction theorem that if I can show that to be a tautology then I can this t follows from those four sentences is equivalent. So, if I can show this then I can show that beta follows from alpha essentially and Frege system can derive all possibilities of course that does not mean that it is a trivial task. So, as you know people have been struggling to find proof of things for example, Fermat’s last theorem to a few hundred years before it was accepted as being solved essentially. So, finding the proof is not the trivial task it is because there is a lot of choice available to you what to use and so on.

One of the kind of proof that we did in the last class is called natural deduction which says you take a rule of inference applied to some premises and add a new one to the set and then again take another rule and apply it and so on. So, we had if you remember we had set that from this we can first derive p and then from p and p and r we can derive r and then from p and q we can derive q then from q and not q or s. We can derive s there is a rule which I am not may be not stated. So, from r and s, we can derive r and s and from this we can derive t. So, there you can start like this derive all this things then derive t, this process is called natural deduction essentially.

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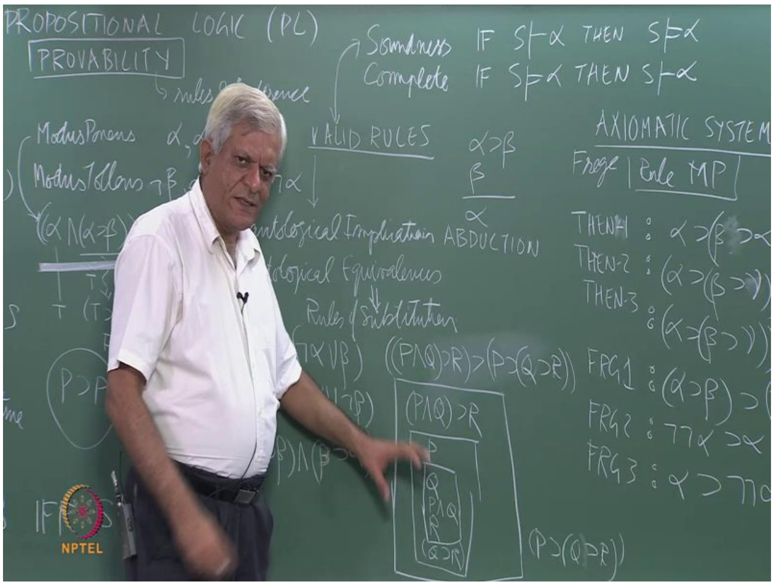

Very often, we can generate rules of inference by showing something like this supposing I want to show that this is a tautology then one way of doing things is to assume the left hand side. So, I assume p and q implies r and in some sense put it in a box, so everything that follow within the box is based on this assumption. Then I make another assumption p, I open another box and third assumption q which I open, another box and now I can say. So, I have assume this three formulas I have assume this I have assume this I have assume this, now I can say p and q then I can use p and q and this and apply modus ponens and assume r or infer r rather.

Then, I can close this box and say I made this assumption of q and so it is a little bit like this application of this deduction theorem that I am closing this box and I am saying. Now, q implies r because r is what I reduced from here then I am saying I will close this box and I will get let me write I t here p implies q implies r. Then, finally when I close this box then I have this implies this, so you see to prove this formula. So, you must be familiar with this kind of proof which you have done quite often assume that this is a case then show that something in this true. So, to show that this is a tautology we said let us assume the left hand side, then we said let us assume q, so we have three assumptions that picked by this three boxes.

So, anything that is entailed by these assumptions are inside the boxes, so p and q is true inside this box because p has been assume q has been assume and because this has been assume r can be inferred because p and q implies r modus ponens. We can apply then we close this box we get q implies r because the only assumption we made in here was q, then we close this box then we get p implies q implies r because the assumption we made was p here. Then, we close this box we get this formula essentially, so that is another way of another rule of inference essentially.

So, the process of finding proof is not a straight forward process and lot of peoples spent a lot of time trying to devise strategy for finding proofs. In 1965 along came a logistician by the name of Robinson who devises the Skema, which you will look at in the next class in which only one rule was of inference was enough for driving all kind of things essentially. Now, observe that when we talk of axiomatic systems we are daily talking of proving all tautology essentially. In the real world we are often more interested in showing that given a set of premises that something else is true. Before we go on to this Robinsons method, there is another observation that we must make which is about the choice of connectives.

Now, if you observe this Frege system, it uses only two connectives the implication and the negation essentially and yet they can made a claim that system is complete which means any true statement that can express in proportional logic can be derive essentially. So, where do the other connectives come from well you cannot derive statements in those exact form, but we have this rules of substitution essentially. So, for example, we have this rule of substitution which says that not alpha or beta is equal to alpha implies beta. So, Frege saying that not that he can derive this form of the sentence, but then equivalent form of sentence here, but logically they are the same they are saying the same thing essentially.

So, there is a notion of a set of connectives also which is complete I am sure you are familiar with this set. So, I am not going to spent too much of time here for example, you must say that this set. So, Frege set this set is complete just use the implication sign and the negation sign and you can express everything can that can be expressed in proportional logic using this connectives. We have all this rules of substitution and you can derive everything that is a tautology in proportional logic using this system of derivation which is those six axioms plus the modus ponens rule.

Essentially, there are other systems which are complete for example, a commonly used system is uses these three signs negation. We will see that Robinson’s method using this set of connectives you can even just work with and or you can even just work with or and negation sign. There are other combination which are possible and some that you must be familiar with the NAND and NOR. So, you must see familiar with the fact that these two connectives NOT and AND, NOT or NAND and NOR. This one single connective enough to express everything you can express in proportional logic which means you choose any set of connectives that we have here and or implies equivalence and so on.

So, express anything using those connective use can device a language in which there is only one connective NAND or only one connective nor or you can express the same thing in this logic with only one connective. What do you mean by you can express the same thing you mean that they are logically equivalent which mean that any valuation which makes any sentence true in the first one will also make the second sentence. The same sentence true in this in this second formulation, in other words if alpha implies beta is true here not alpha or beta will be true here or if this sentence is true, then this sentences that is a meaning of equivalence.

So, anything that can be set in proportional logic can be set with set of complete sentences complete connectives and then you can choose a set of axioms and a rule some rules of inference and you can have a logically complete system. The beauty about Robinson system was that he did not need any axioms he just needed one rule of inference and that was the complete system you could derive everything all tautologies in that essentially. So, as a last thing it is nice to define a complete system which is minimal stance in some sense small set of axiom and small one rule of inference. As you will see if you try to show that p implies p it involves a lot of steps in showing that p implies see because you have to start with some axiom substitute something for there.

Then, you know keep laying with the axiom it is a nice exercise you must try in practice you would not want a minimal as the system you would want a system in which inferences can be made very quickly. This means you can derive sort of complex rules of inference, so you can say for example that and this side in fact this is a rule of inference that is alpha implies beta and beta implies gamma, then alpha implies gamma, you can have a rule of inference like this.

So, you can see its like transivity and f alpha implies beta implies gamma, then you have rules says that alpha implies gamma if you have such a rule in your system then you have to showed alpha implies gamma or something like that. If somebody have given you alpha and asked you to show gamma then you can just take this rule here and this conclusion here and then use modus ponens can derive gamma essentially. So, it makes sense to introduce more rules of inference which would make you inferences shorter, so there is some wage analogy which defining a language here.

So, when people define the instruction sets for machines, they work with more instructions or less instructions and each ahs his advantage essentially. So, theoretically of course it is nice show that a small set is good and complete, but in practice you guy need something bigger unless you come up with something like Robinsons method which was small and yet very efficient essentially. So, we look at Robinson's resolutions method in a next class essentially.