Discrete Mathematical Structures
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Lecture - 40
Lattices

Today we shall consider some properties about posets that is the partially ordered sets. Especially when a poset can be called as a lattice and as some properties of lattices? And also what is a chain and an anti-chain and a decomposition theorem?

So let us recall our definition about our partially ordered set. A set a with a binary relation this binary relation is called a partially ordered set if it is reflexive, antisymmetric and transitive. If these three properties hold for this set under the binary relation it is called a poset or a partially ordered set. It is represented usually by what is known as a poset diagram or Hasse diagram.

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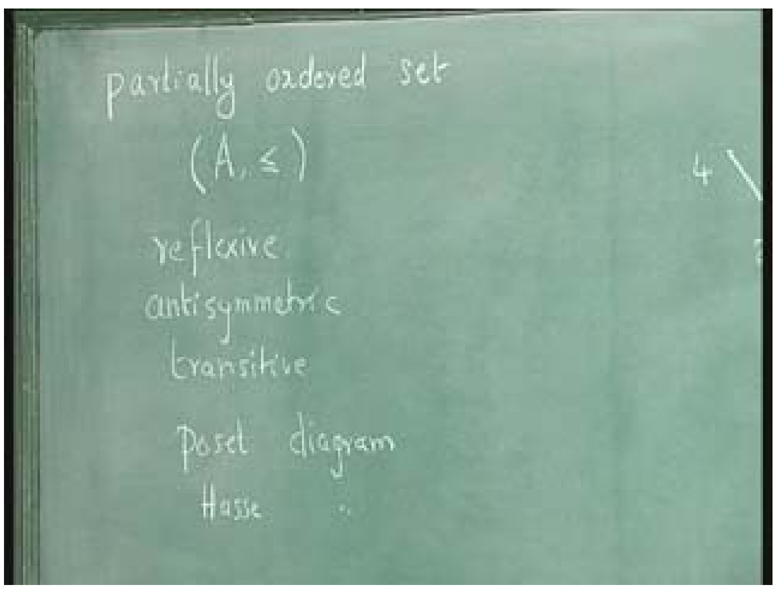

Let us take an example; underlying set is the set of numbers 1, 2 to 6 and the binary relation is a divides b. Under this the poset diagram will look like this, one divides 2, 3 and 5 and 2 divides 6, 2 divides 4 and 3 divides 6 and so on. All the self loops will be omitted here in this diagram. And also the transitivity 1 to 3 and 3 to 6 and 1 divides 6 also but these two together will imply this. Now this is a poset diagram, we have also seen what is meant by least upper bound greatest lower bound etc earlier.

In this particular example 1 is the least element because it is less than everything else there is no greatest element in this poset. And 4 and 6 and 5 are maximal elements because there is nothing higher than that 4, 6, 5 are maximal elements and 1 is the minimal element. For 2 and 3 6 is the least upper bound and 1 is the greatest lower bound. For 3 and 5 there is no least upper bound but there is a greatest lower bound which is 1. For 4 and 3 also one is the greatest lower bound. So you can talk about maximal elements, minimal elements, greatest element, least element, least upper bound and greatest lower bound etc, this we have already seen. Now, we shall see when a poset can be called a lattice so we shall be studying about lattice.

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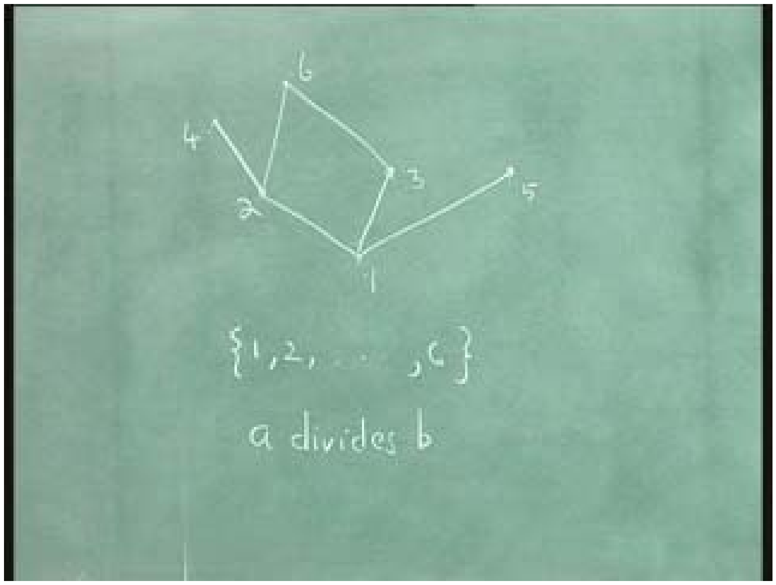

Let P less than or equal to be a poset that is this binary relation be a poset and let a b be two elements of the set P. then an element d belonging to P is called the greatest lower bound or meet of a and b d is equal to a and b when d is less than or equal to a and when d is less than or equal to b and x is less than or equal to a and x is less than or equal to b imply x is less than or equal to d. That is, you have two elements and we are also taking d to belong to the set P.

In general, the least upper bound and the greatest lower bound need not exist and if you take generally a poset and take a subset of it the least upper bound and the greatest lower bound of the subset taking two elements may not belong to the subset. But here we are defining the least upper bound as it belongs to the set d belongs to P so d is less than or equal to a and d is less than or equal to b and x is less than or equal to a and x is less than or equal to b imply x is less than or equal to d. This is called the meet or this is the greatest lower bound.

In this manner similarly dually the element s belonging to P is called the least upper bound or join, greatest lower bound is called meet least upper bound is called join and in symbols it is denoted as s is equal to a or b. If a is less than or equal to s and b is less than or equal to s and a less than or equal to x and b less than or equal to x imply s less than or equal to x.

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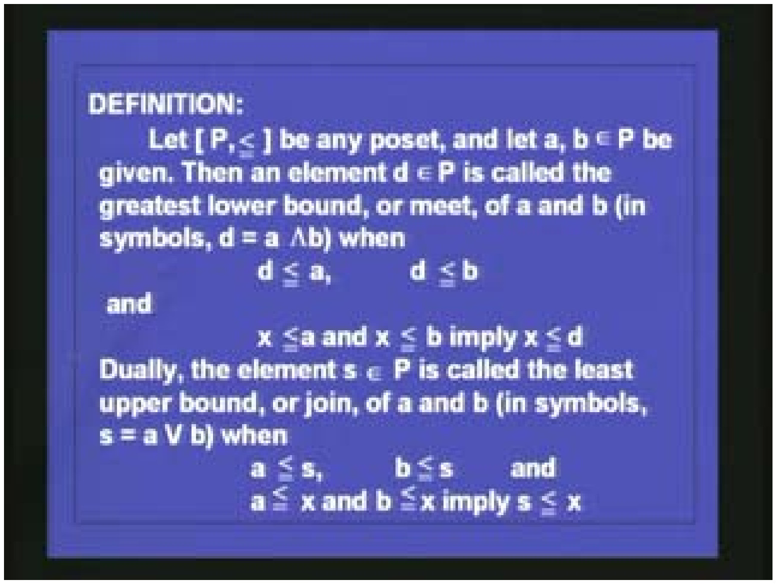

In this example as I mentioned earlier for 2 and 3 1 is the greatest lower bound and 6 is the least upper bound. In any poset P the meets and joins have designated values. So as I told you sometimes the least upper bound and the greatest lower bound need not exist. So if they exist then they will satisfy these laws. What are the laws? This is the idempotent law x and x is x, that is the meet of x and x is x the join of x and x is x, the meet of x and y is same as the meet of y and x commutative law.

The join of x and y is the same as the join of y and x. This you can very easily see and this is associative law, x meet y meet z is the same as x meet y meet z and x join y that is the join of x and y along with z take the join operation, it is the same as taking x and using the operation join along with y join z. These are absorption x and x or y is equal to x and x or x and y is equal to x. Here this is meet operation this is join operation. So x meet x join y is equal to x or x join x meet y is equal to x. These laws will automatically hold if the meet and the join exist. They need not exist in some cases.

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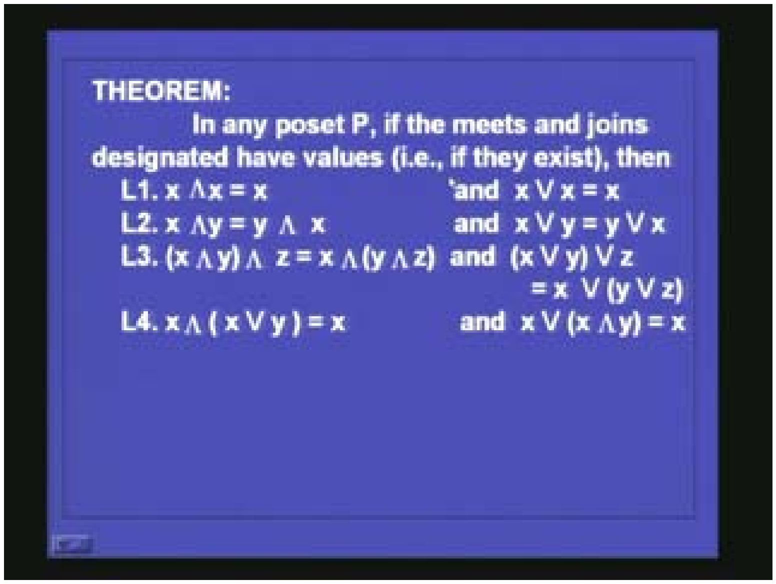

Let S be an arbitrary subset of the poset S so we are not talking the whole set of element but a subset S then the greatest lower bound S of that subset means a is less than or equal to x for all the x and if there is another element new less than or equal to x for all the x in S that means new will be less than or equal to a. That is any other lower bound is less than this greatest lower bound, this a is the greatest among all the lower bounds that is what it means.

Now we will come to the definition of the lattice. A lattice is a poset in which any two elements a and b have a greatest lower bound called the meet and lub called the join of a or b. So, if you take any two elements a and b it has got a meet or the greatest lower bound which is denoted like this and it has got a join or lub which is denoted like this.

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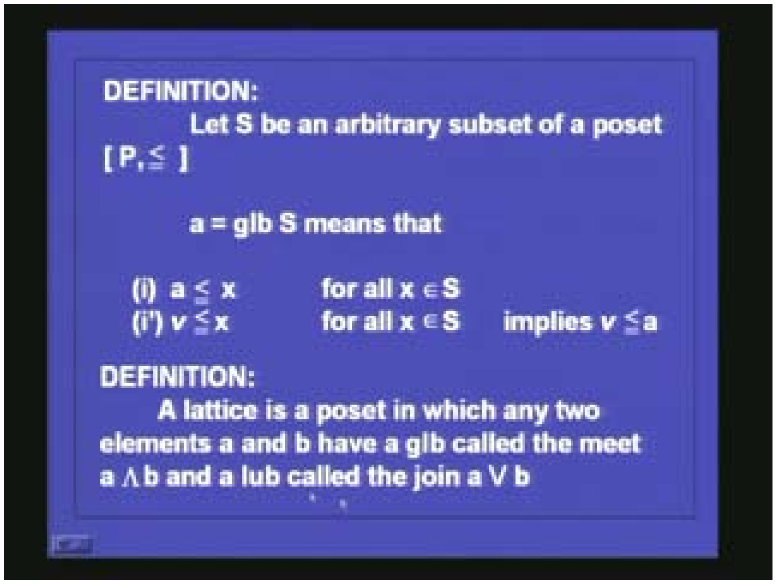

Let us consider some examples. Consider these examples; first of all if you look into this example this is not a lattice, the reason is, if you take these two elements they do not have a least upper bound. The least upper bound should exist and belong to the set. If you take these two there is no least upper bound for this. Consider these examples; are they lattices? Let us check whether they are lattices. Look at the first one which is denoted as M_5. If you take these two elements this is the least upper bound and this is the greatest lower bound. If you take these two elements this is the least upper bound and this is the greatest lower bound. Take any two elements you will find that there is a least upper bound and a greatest lower bound so this is a lattice.

Look at this one; if you take any two elements they have a greatest lower bound and a least upper bound. If you take these two again they have a greatest lower bound and a least upper bound, this is also a lattice. This is denoted as 2 cube and also you can consider as a power set of the set consisting of three elements. Look at this, this is again a lattice, later on we shall see that this is a distributive lattice whereas these two are not distributive lattices. Here again if you take two elements they will have a least upper bound and a greatest lower bound.

Look at this, these two elements have a greatest lower bound. But what about the upper bound, this is an upper bound for them but this is also an upper bound, the least upper bound has the property that any other upper bound should be greater than that. Or the least upper bound should be less than or equal to any other upper bound, that is violated here because this is not less than, they are incomparable you cannot compare this with this and apparently it has got two least upper bounds which is not possible.

In the lattice the least upper bound and the greatest lower bound if they exist they should be unique by the definition. So here because of that again if you take these two elements they seem to be have two greatest lower bounds which is not possible. The greatest lower bound should be unique. So here you are having two of them competing for that and none of them will satisfy the condition given in the definition. So this is not an example of a lattice whereas the other three are lattices.

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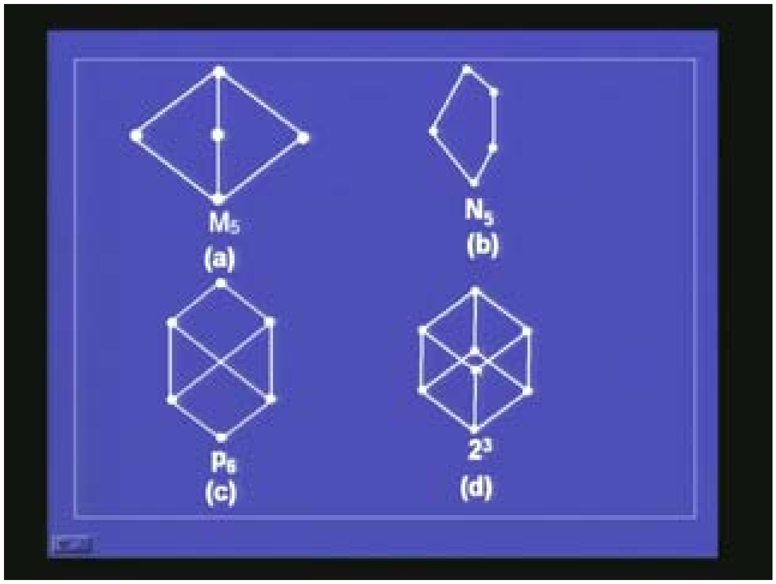

Consider this one; it represents the power set of three elements. Let us consider the power set of a set consisting of three elements a, b, c. The poset diagram will look like this, the empty set a b c ab bc ac then abc this is what we have seen as two cubed earlier, you can see the correspondence. This is the empty set a b c then this is empty set this is a this is b this is c this is ab this is bc this is ca and this is abc. You can see that this is a lattice because if you take any two elements it will have a least upper bound and a greatest lower bound. Any two elements you take it will have a least upper bound and a greatest lower bound so this forms a lattice.

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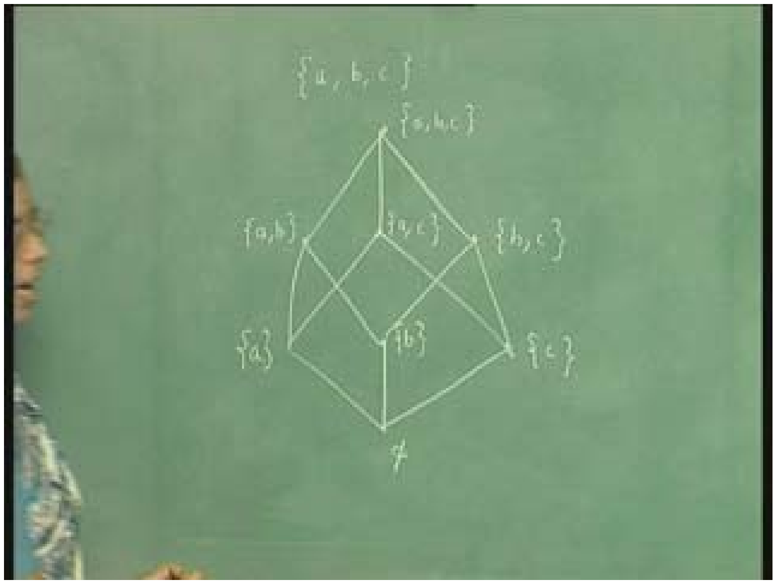

It also satisfies the distributive laws, there are two distributive laws. We saw that for any lattice these four conditions will be satisfied, these four conditions L_1 to L_4 will be satisfied in any lattice. But what about the distributive laws? x and y or z is equal to x and y or x and z or x meet y join of z is equal to x meet y join x meet z and a dual law. Let us see whether such a thing is satisfied in the examples which we consider. If you take this example the distributive laws will be satisfied. But let us take this example and this example and check whether the distributive law will be satisfied.

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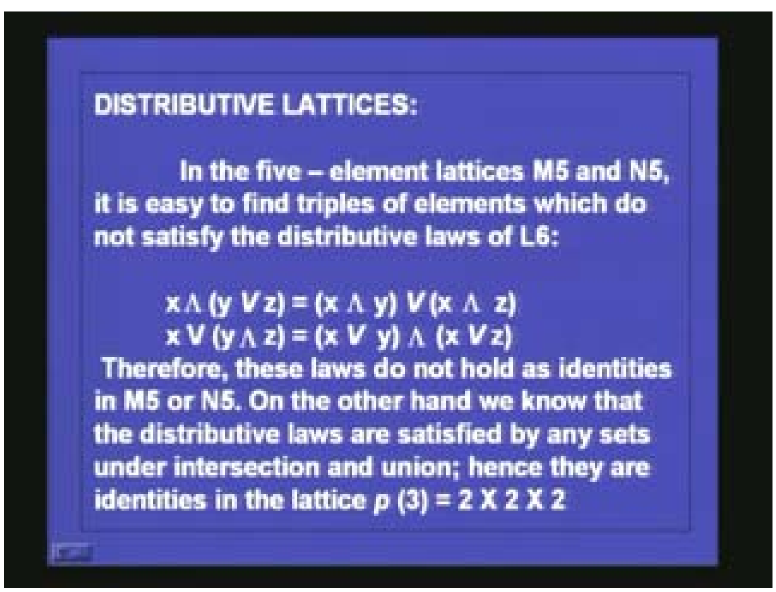

Take this example; call these elements a b c this element. Usually the least element is denoted as 0 and this will be denoted as I. Now what about the distributive laws, let us see. What is a and b or c? If the distributive law holds this should be equal to a and b or a and c. Now, if the distributive law holds, in this example let us see what the left hand side means and what the right hand side means. a and what is b or c? In b or c what is the least upper bound? It is I and what is a and I? The meet of the least upper bound of this will be a.

Look at this the right hand side what is a meet b? The greatest lower bound of a and b is 0. Similarly, for a and c also the lower bound is 0 so this 0 or 0 which is 0. So the right hand side evaluates 0 whereas the left hand side evaluates to a, they are not equal so the distributive law does not hold here.

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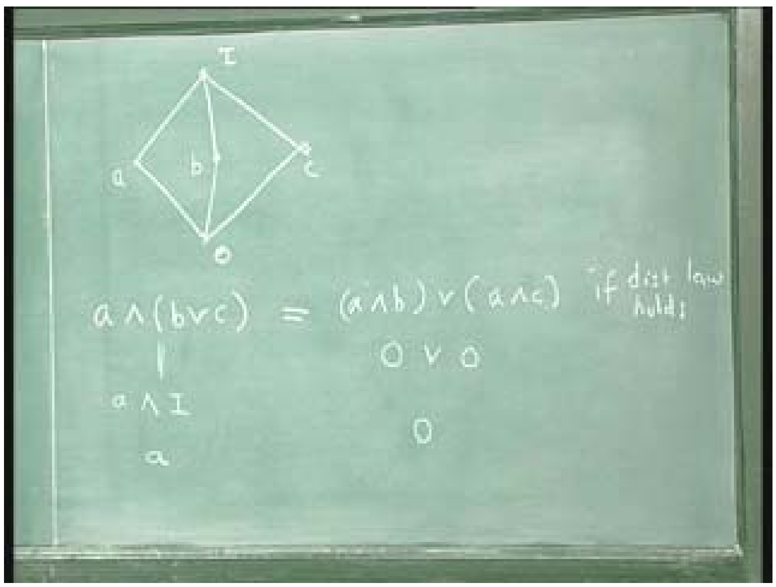

If you take this example take anything you will see that the distributive law holds. You can take any three of them and check you will realize that the distributive law holds.

Now what about the other thing? This is the other example, this is 0 this is I call them as a b c what about a and b or c and a and b or a and c, what is b or c that is, what is the least upper bound? It is I. So, a and I this will evaluate to a, this side will evaluate to a. What about this, what is the meet of a and b? Meet of a and b is 0. The meet of a and c is c and the greatest upper bound or the least upper bound of, or is least upper bound so 0 and c if you take it is c. so the left hand evaluates to a and the right side evaluates to c so they are not equal. So the distributive law does not hold here hence this is not a distributive law.

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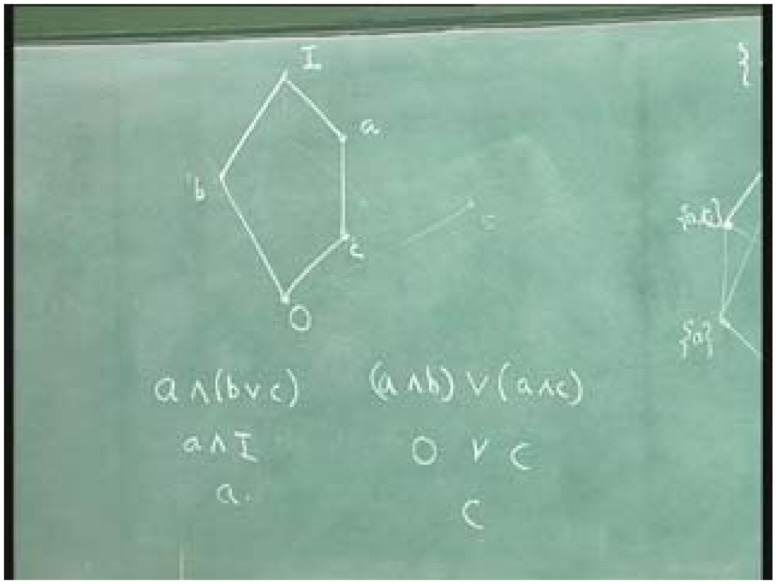

Therefore, in the four examples which we considered this is not a distributive lattice but it is a lattice, this is again not a distributive lattice but it is a lattice, this is not even a lattice this is a distributive lattice. In general, a lattice is called distributive when it satisfies the two distributive laws. Otherwise it is called non distributive. The distributive laws are clearly hereditary from lattice to sublattice. Suppose we have a sublattice, what is a sublattice? We have seen what algebra is and what is subalgebra, we have seen what is a group and what is a sub group and so on. Similarly, for a lattice you can define a sublattice and if the original lattice is distributive the sublattice will also be distributive. If you take the power set of a set U that will be a distributive lattice because for subsets union and intersection the distributive laws hold for them.

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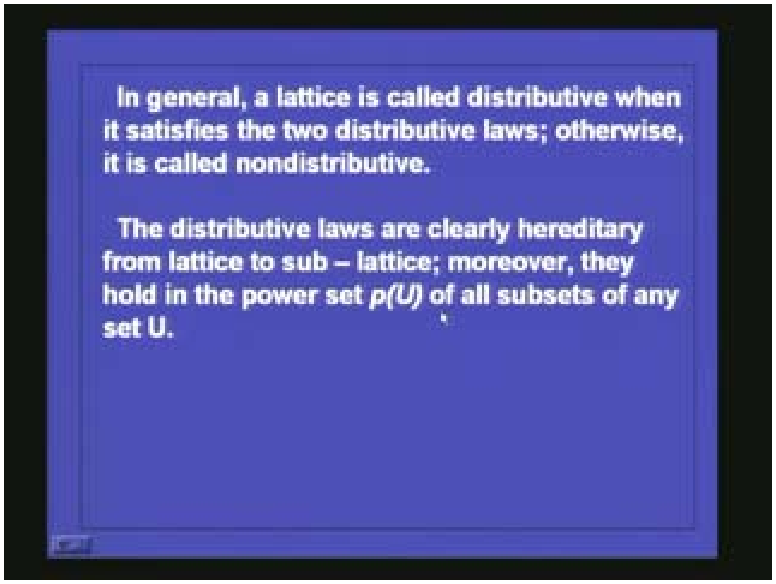

In any lattice the semi distributive laws x and y or z greater than or equal to x and y or x and z actually this means meet and this is join. And this again is less than or equal to this they hold and if one of the distributive laws holds the other will automatically hold. Let us consider this alone, the first alone the other one is similar. We can see that x is greater than or equal to x and y, x is also greater than equal to x and z so x will be greater than or equal to the join of them. And if you take y join z then y is greater than or equal to this and z is greater than or equal to this so y join z will be greater than or equal to the join of these two this is again greater than or equal to these two and this is again greater than or equal to two. So if you take the meet of them that will be again greater than or equal to this. Or this is the greatest lower bound of these two because this is the lower bound this is the lower bound and the meet will be again lower.

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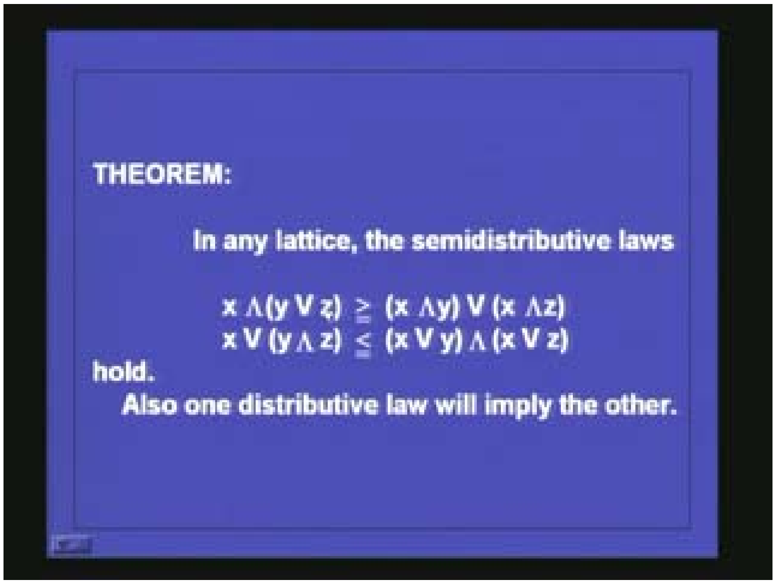

Next, we shall see in any distributive lattice a meet x and a meet y and a join x equal to a join y together will imply x is equal to y. We have the meet of a and x is the same as the meet of a and y and the join of a and x is the same as the join of a and y. In a distributive lattice these together will imply x is equal to y.

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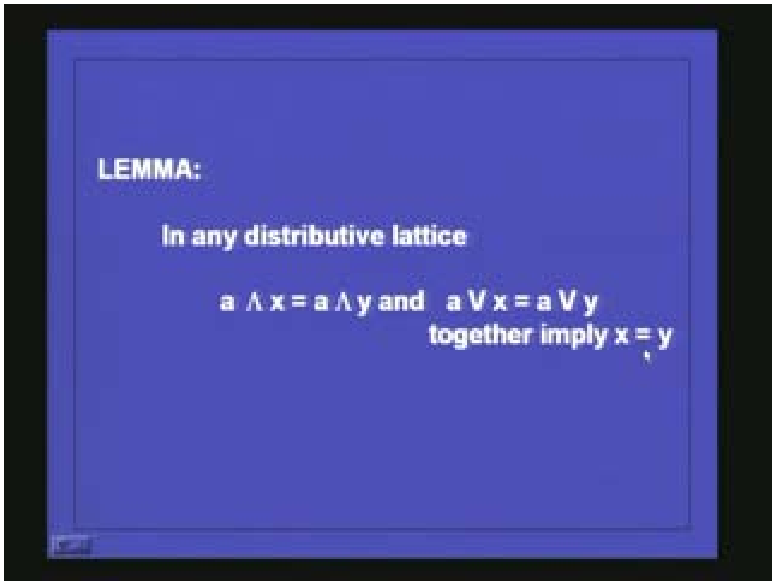

We can also state it like this because of commutative you can say x and a is equal to y and a instead of saying a and x is equal to a and y that is this really represents the meet of x and a, this represents the meet of y and a. Similarly, you also have x or a is equal to y or a. With this you have to show that x is equal to y, how do you show that? You can write x like this because of the first four laws we studied x can be written as x or x and a and instead of the meet of x and a you can put the meet of y and a we know that these two are equal.

Therefore, when you write like this you can expand using the distributive law the distributive law holds because we are considering a distributive lattice. So because the lattice is distributive this is equal to the join of x and y in the meet operation on this and the join of x and a and because of commutativity we can change x or y to y or x. And we know that this is the same, the join of x and a is the same as the join of y and a so this can be replaced by this.

Now, applying the distributive law in the reverse in the other direction this can be written in the form y join x meet a. But we know that x meet a is the same as y meet a so this can be replaced by this so from this we get this. And we know that because of the first rules this will be the same as y absorption rules. So starting with x we have arrived at y and we have come to the conclusion that x is equal to y.

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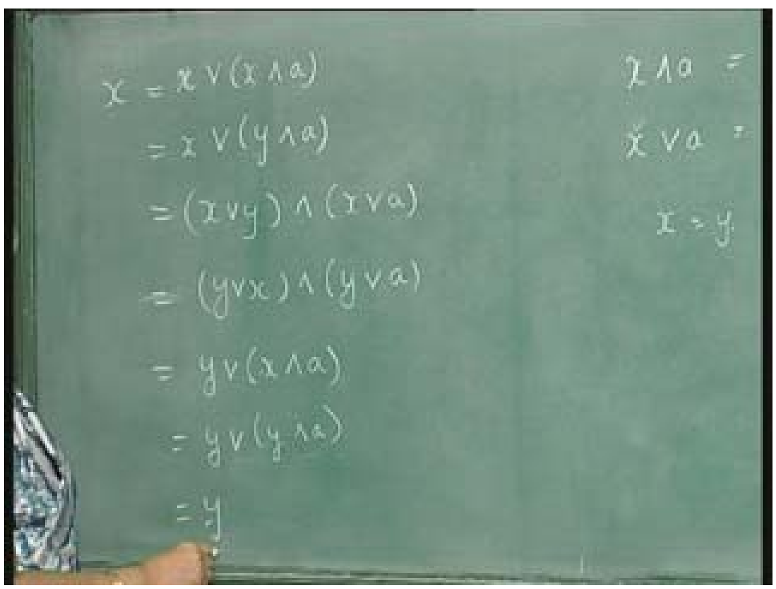

So this lemma holds, that is any distributive lattice if the meet of a and x and the meet of a and y are the same and the join of a and x and the join of a and y are the same that means x is equal to y. Some more results; in a distributive lattice L a given element can have at most one complement. Now, what is a complement?

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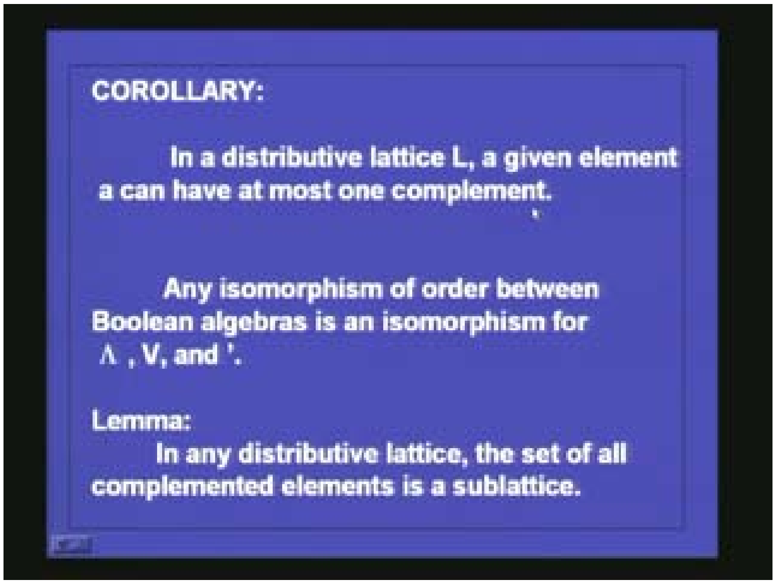

What is a complement of an element?
If you have an element a and another element a dash, a dash is the complement of a, if their meet is the element 0 and the join is the element I. We know that what is the greatest element and this is the least element. For example, if you take this the power set with two sets the lattice will look like this, this is the complement of this because the meet of this is this and the join of this is this, this is I really then this is 0. Similarly, if you take the power set of the set of three elements a, b, c it will look like a cube so this is the empty set and this is a, this is b, this we have already drawn c and so on, sorry this is not b this is c this is b, this is ab this is ac and this is bc, this is abc, this is I and this is 0.

So what is the complement of a?
The complement of a will be bc because for these two the greatest lower bound is this and the least upper bound is this. Similarly, for b and ac the greatest lower bound is this and the least upper bound is this. So for a bc is the complement for b ac is the complement for c ab is the complement. So each of this has got a complement and trivially this is the complement of this 0 and I are complement of each other. So that is the definition of the complement.

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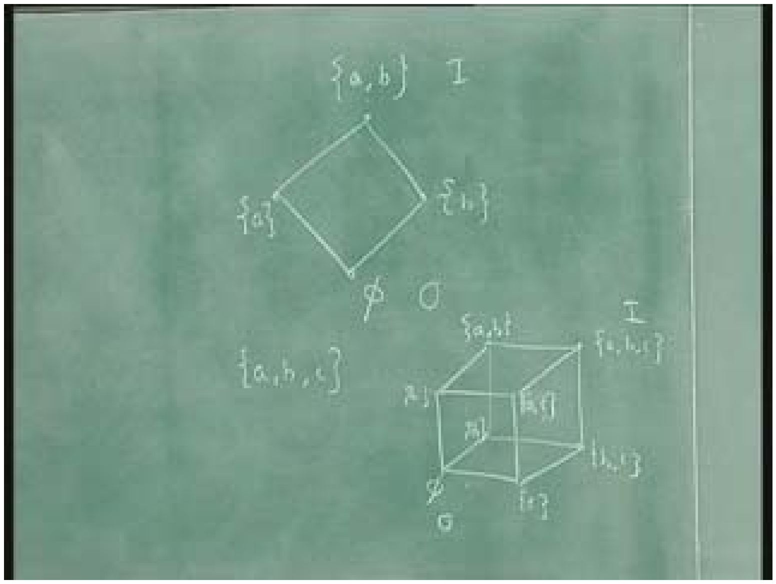

And you can very easily see that in a distributive law lattice L a given element a can have at most one complement. You can prove the uniqueness of the complement in the usual manner. And if you consider a Boolean algebra you consider the and or operation and the complementation. Similarly, in a lattice the corresponding three operations will be meet join and complement. So you can look at it as a Boolean algebra also. That is why the same symbols are used this is for meet. In Boolean algebra you call it as and while reading also sometimes we read it as AND and OR because instead of saying meet and join. Join is represented like this so you can see the correspondence in a Boolean algebra the three operations two binary and one unary operation. In lattice we have two binary operations; one is the join and another is a meet and we have a unary operation namely the complement. And this happens only in the case of distributive lattice.

Now, if you take any distributive lattice then the set of all complemented elements is a sublattice. Take a distributive lattice and take all complemented elements then that will form a sublattice like this.

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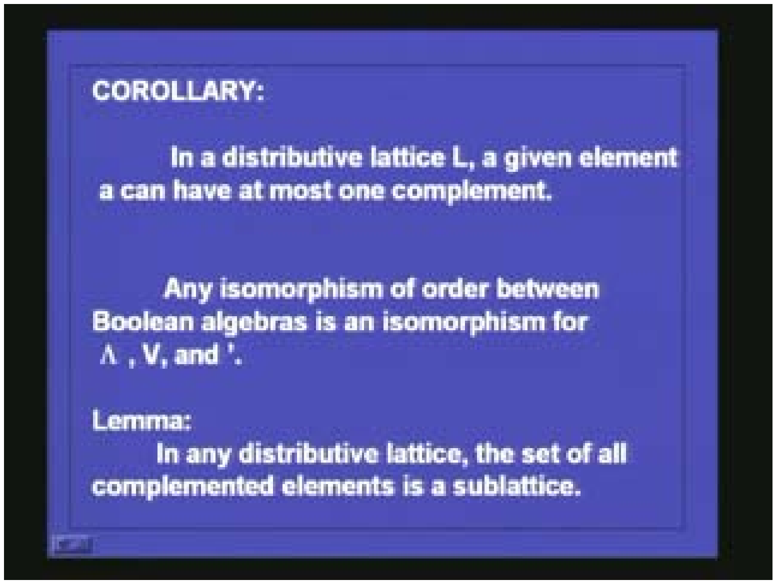

Suppose I have two elements a and a dash, a is the complement of a dash and vice versa, b and b dash are complements of each other. If they belong to a sublattice then the meet of them a and a dash or let me take a and b this is the meet of a and b this is the join of a and b. Now, the complement of, if these four belong to a sublattice then a and b, a or b should also belong to that sublattice because in any lattice the greatest lower bound and the least upper bound should belong to the lattice. So, if you consider a sublattice these two elements should also belong to the sublattice. And what does the sublattice consist of? It consists of all complements so you must show that the complement of this is also in the set and the complement of this is also in the set.

Now take a and b and consider this.
What about the meet of this?
Expand this and it will be a and b and a dash OR a and b and b dash. Now what is the meet of a and dash? That is 0 so the whole thing will be 0 and this will also be 0 so the whole thing is 0. So if you take this element and this element the meet of that is 0 and if you take a and b or a dash or b dash you will realize that this is a or a dash or b dash and b or a dash or b dash this will be I because a or a dash is I similarly b or b dash is I so this is I and I is equal to I. So you realize that this and this are complements of each other. And you can also see that a or b and a dash and b dash are complements of each other. So the complement also belongs to the set. Here also the complement belongs to the set so they will in the sublattice defined by the set of complemented elements.

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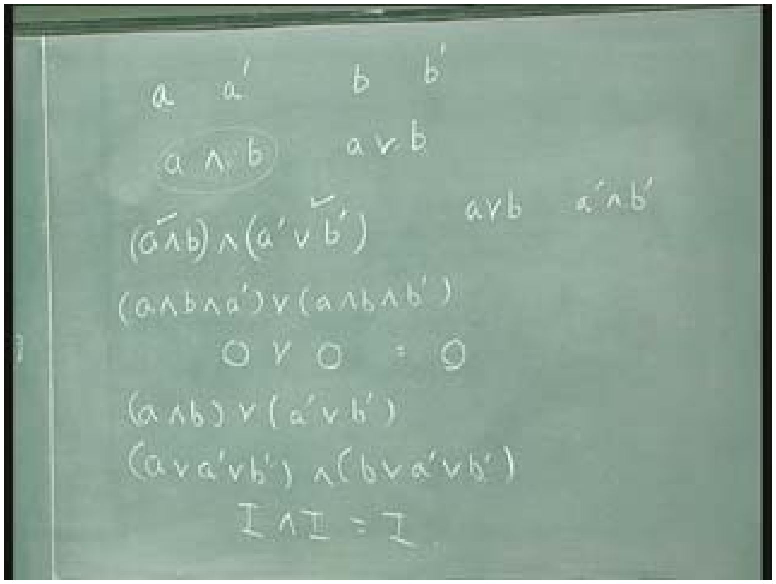

That is what this lemma says: In any distributive lattice the set of all complemented elements is a sublattice. Of course, I and 0 will be in the sublattice they are special elements and they will be in the sublattice also. These are some of the results about lattice and distributive lattice.

Next we shall see about chains and antichains and the decomposition theorem. Let A less than or equal to be a partially ordered set. A subset of A is called a chain if every two elements in the subset are related. Note that because of antisymmetry and transitivity in any chain with a finite number of elements a_1, a_2, ak there is an element a_{i1} that is less than every other element in the chain. There is an element a_{i2} that is less than every other element except a_{i1}. And there is an element a_{i3} that is less than every other element except a_{i1} and a_{i2} and so on.

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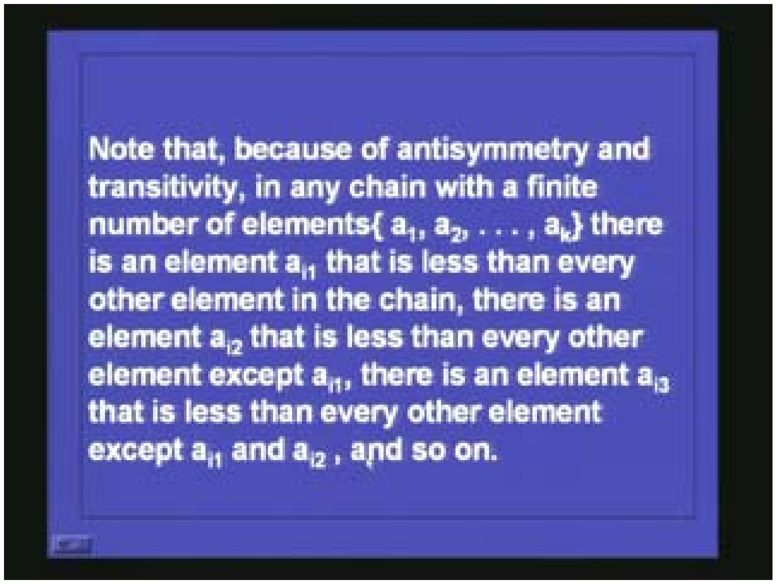

Look at this example; this is a poset and a is the least element here and e is the greatest element. You see that a b c e is a chain and a d e is another chain. Here in this chain a is smaller than all the other elements, b is smaller than c and e but not a, c is smaller than e it is smaller than every other element except a and b and so on. Similarly, you look at this a d e it is a chain.

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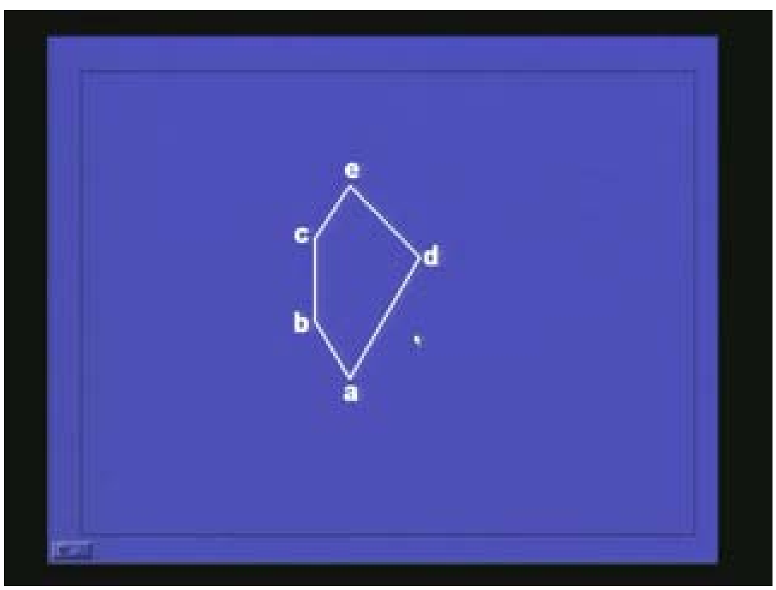

We can use the notation a_{i1} less than or equal to a_{i2} a_{i3} etc as an observation for the list of ordered pairs a_{i1} is less than or equal to a_{i2} and a_{i2} is less than or equal to a_{i3}. That is, a_{i1} is less than or equal to every other element and a_{i2} is less than or equal to every other element except ai1 and so on. We frequently refer to the number of elements in a chain as the length of the chain. Here a b c e is a chain, the length of that chain is the number of elements in that that is 1 2 3 4. Similarly, a d e is a chain the length of that chain is the number of elements in that that is 3 a d e three of them.

A subset of A is called an antichain if no two distinct elements in the subset are related. for example, in the partially ordered set which we considered earlier that is this one c d are antichain they cannot be compared, b d are antichains it cannot be compared. Then a b c e is a chain, a b c is a chain, a d e is a chain and b d is an antichain, c d is an antichain a is an antichain, if you take a single element you can look at it as a chain or a antichain.

Now we have a small result about decomposition. Let P less than or equal to be a partially ordered set. Suppose the length of the longest chain in P is n then the elements in P can be partitioned into n disjoint antichains this is what the result says. You have a partially ordered set and the length of the longest chain is n, then the elements in the set P can be partitioned into n disjoint antichains. Let us see how to prove this result.

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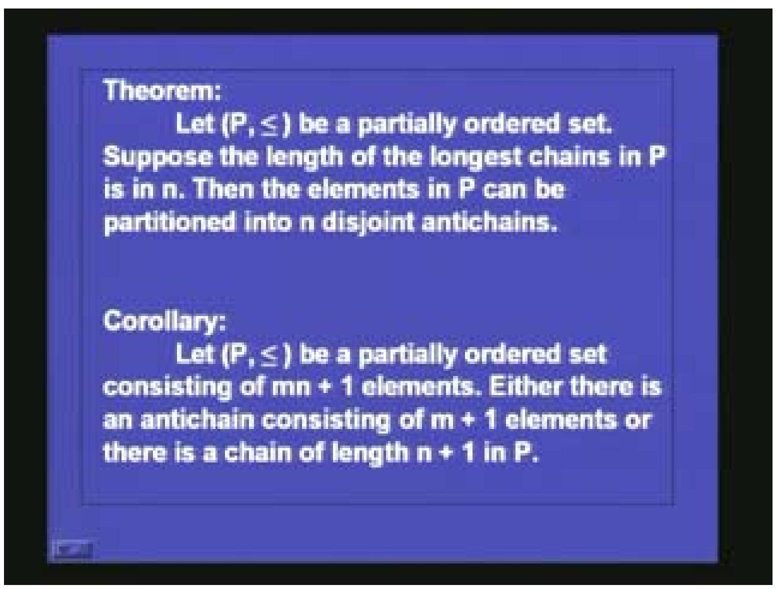

Take n is equal to 1. The proof is by induction proof so the basis part of this is this n is equal to 1 that is you have a poset like this, the length of the longest chain is 1 means the poset will be like this. No two elements can be compared. Now this whole set is an antichain so the whole thing is in a block in the partition it is one antichain. So the result holds for n is equal to 1. Then the induction step is like this; suppose the result holds for n minus 1 that is you have a poset where the length of the longest chain is n minus 1 then the elements can be partitioned into n minus 1 antichains.

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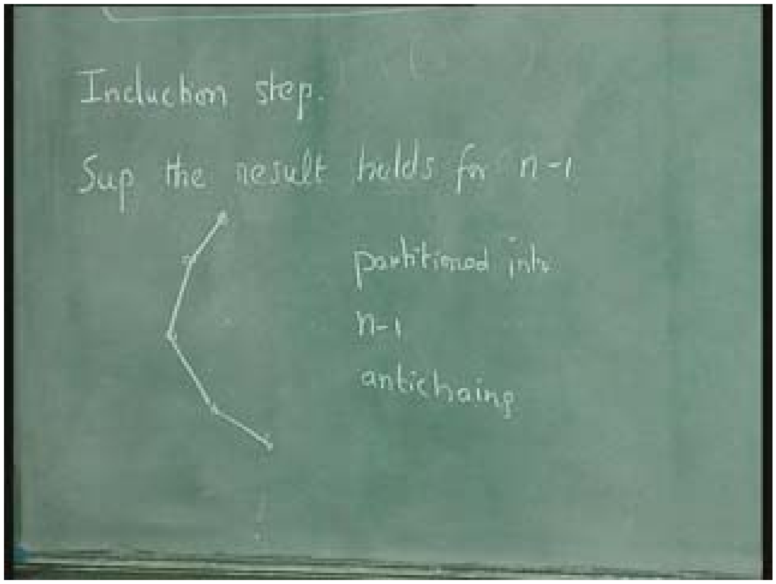

Suppose this holds then prove for n?
That is, the length of the longest chain is n. You may have more than one chain. Now consider all the maximal elements in this.

Here, now you have taken the length of the longest chain that is n. So consider all the maximal elements. These maximal elements cannot be compared with each other so these maximal elements alone form an antichain. You call them as M then consider P minus M remove the maximal elements and consider the remaining one. In the remaining one in this the length of the longest chain or the maximum length of the chain or the chain of maximum length is n minus 1 because you are removing the maximal elements you get the longest chain which will be of length n minus 1. So these elements in P minus M can be partitioned into n minus 1 antichains by induction hypothesis.

By induction hypothesis the elements in P minus M can be partitioned into n minus 1 antichains. And you must remember one more thing, the longest chain in P minus M is of length n minus 1, it cannot be less than n minus 1 also because suppose it is less than n minus 1 then you cannot say that the maximum length of the longest chain or the length of the longest chain is n, you cannot say that. So the length of the longest chain in P minus M is n minus 1. So by induction hypothesis it can be partitioned into n minus 1antichains. This together with the antichain of the maximal elements M forms an antichain and this n minus 1 antichains plus one antichain form n antichains.

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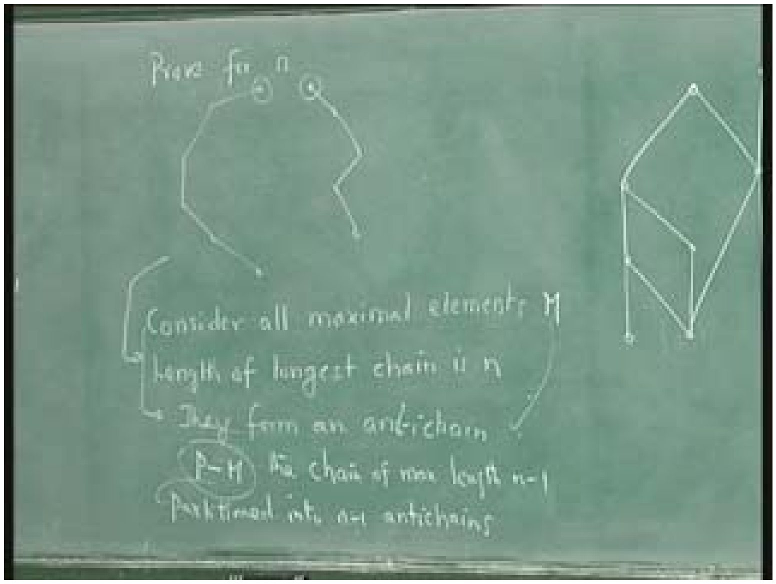

So you get the result that P can be partitioned into n disjoint antichains, so the proof is by induction like this. Look at this example; this is a poset, what are the maximal elements here? These are the maximal elements, these are the minimal elements and what is a chain here? This forms a chain, this forms a chain, this forms a chain and so on.

What is the length of the longest chain?
This is the longest chain, four elements are there so length of longest chain is 4 here. So this can be partitioned, the elements here can be partitioned into four disjoint antichains. for example it can be done like this; these two can be in one block, these two can be in one block, these three can be in one block and so you can see that each one is a antichain. So, the whole set can be partitioned into 4 antichains. This partition need not be unique, for example you may have like this, this alone can be in one partition, these two can be in one partition, these elements can be in one partition and this is one, this is also partitioned into four blocks. So this partition is not necessarily a unique one but you can have them in different phase also. Therefore, number of blocks in partition will be 4 where each block is an antichain.

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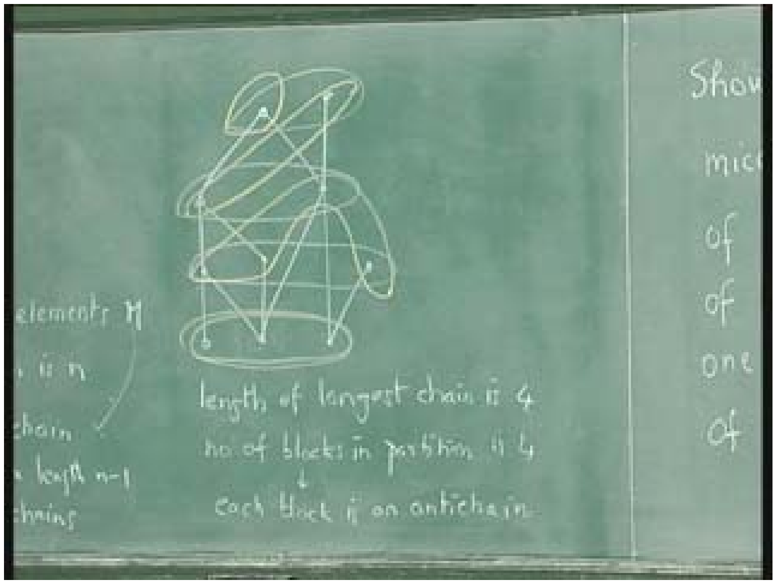

Now following that we get this result; let P less than or equal to be a partially ordered set. It has got mn plus 1 elements where m and n are integers. Then either there is an antichain consisting of m plus 1 elements or there is a chain of length n plus 1 in P. Consider the longest chain in this, if it is of length n plus 1 or more the result automatically holds. So, if you have a chain of length n plus 1 or more in this then the result holds. Suppose the length of the longest chain here is n then the whole set can be partitioned into n disjoint antichains. And at least in one of them there should be m plus 1 elements because if all of them contain m or less than m elements then the total number of elements can be only mn. There are n disjoint antichains at least one antichain should contain m plus 1 element or more.

Otherwise, if each one of them contains less than or equal to m elements total number of elements can be only mn but we know that we have mn plus 1 elements. So by the Pigeonhole Principle or the extension of the Pigeonhole Principle you realize that at least one antichain will have m plus 1 elements or more so this results holds. So, if you have mn plus 1 elements in a partially ordered set then there will be an antichain of length m plus 1 or a chain of length n plus 1, an immediate application of that is this problem.

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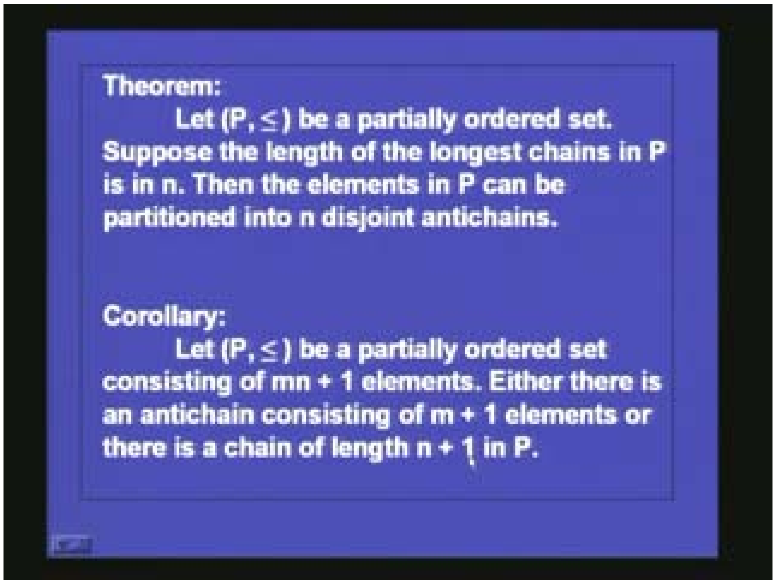

Show that among ab plus 1 white mice either there is a sequence of a plus one mice each a descendant of the next or b plus 1 mice no one of which is a descendant of another. This is a direct application of the corollary.

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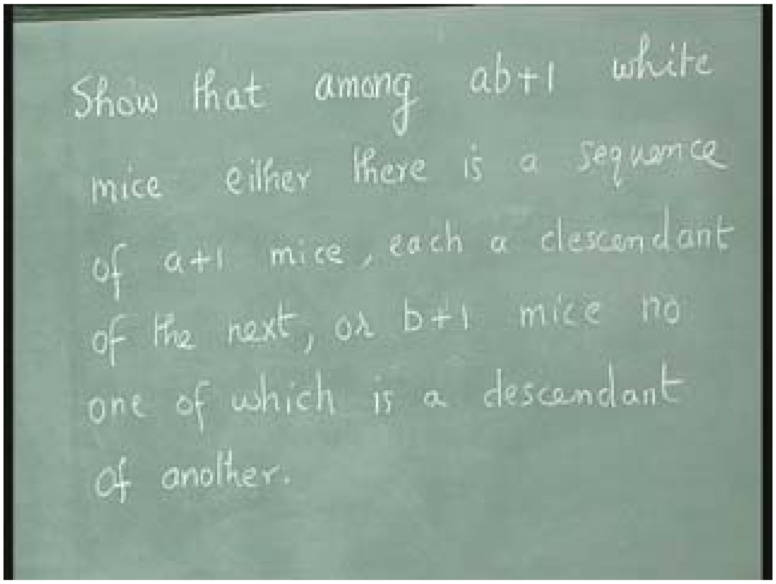

Define the relation like this; a is related to b, a is a descendant of b then the whole thing can be represented as a poset. And the number of elements in that is ab plus 1. So by the corollary you will have a chain of length a plus 1 or antichain of length b plus 1 by the direct application of the corollary.

What does that mean if you have a chain of length a plus 1?
You have like this that means this is a descendant of this, this is a descendant of this, this is a descendant of this and so on. So you have a plus 1 mice each a descendant of the next. Or you have b plus 1 mice no one of it is a descendant of another. If you have a chain this is the result; one is the descendant of the other and so on. If you have a antichain of length b plus 1 there are b plus 1 elements which cannot be compared with any other. That means this is the descendant of the other or this is the descendant of the other and so on. No two are related in any way by the descendant relationship, so the result follows.

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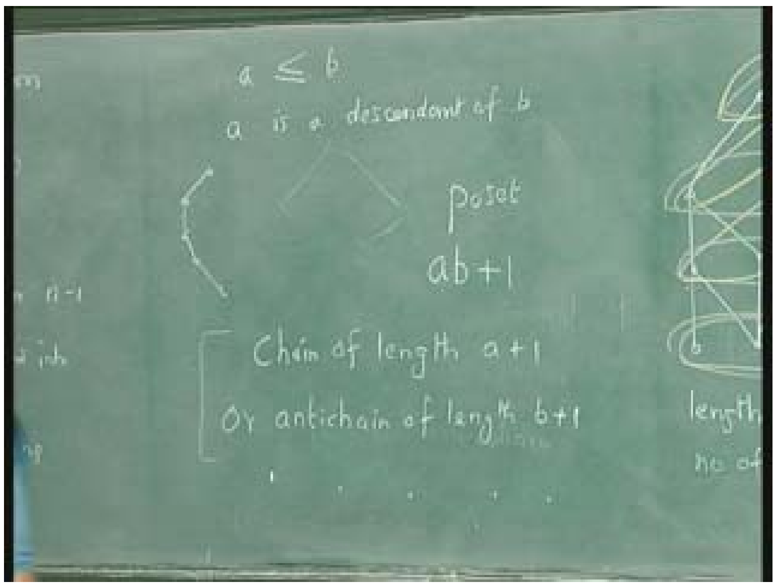

So like that some of the problems can be attempted making use of this decomposition. We saw about meet and join lattices. Usually we talk about join also in relational databases especially in case with tables. In relational databases and n-ary relation is represented as a table like this; for example there are three components the supplier s_1 he supplies part P_1 for project q_1, supplier s_2 he supplies part P_1 for project q_1 and supplier s3 provides for part P_2 for project q_2 and so on. So this is an n-ary relation. This is the ternary relation and it is represented as a table. For example call this table as pi.

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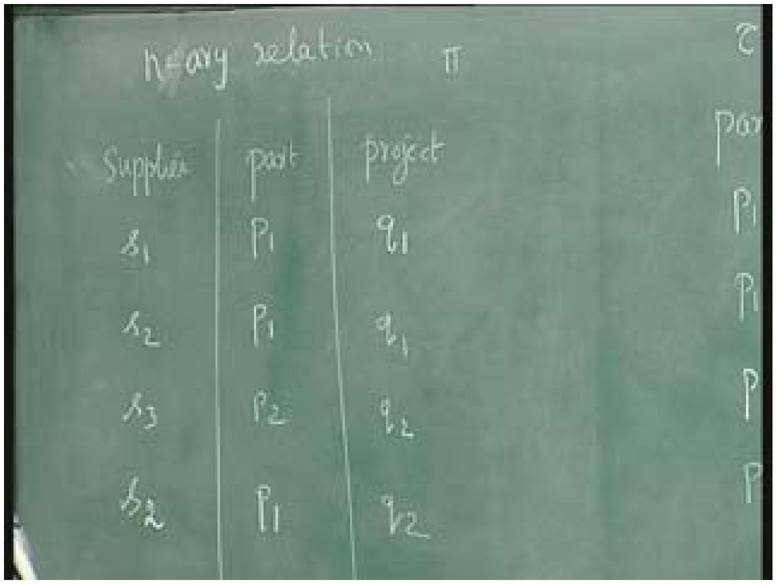

Then there is another relation tow between part project and color. So part P_1 in project q_1 can be in red color or P_1 in q_1 can be in blue color, sometimes it can red and sometimes it can be in blue. And part P_1 in project q_2 takes only blue color and part P_2 in project q_2 takes green color only. So this is another table, this is also a ternary relation they are represented as tables.

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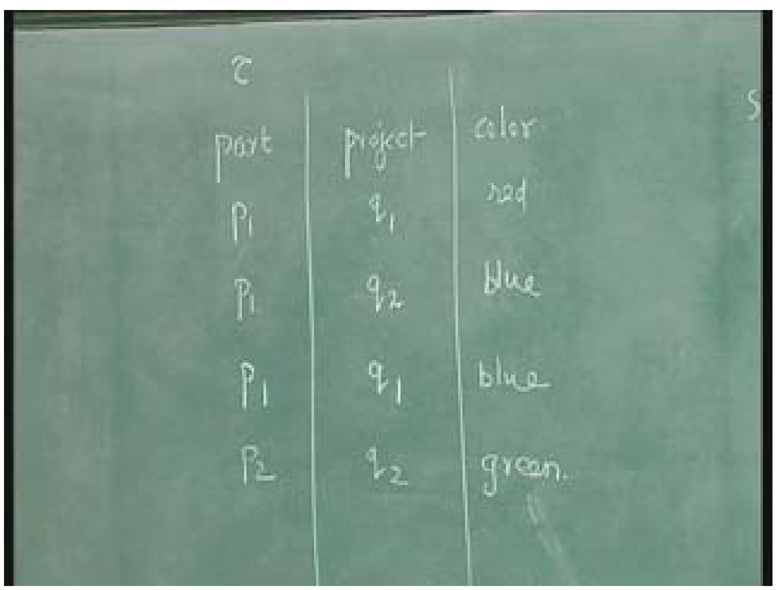

Now you talk about the join of these two, the join of the two tables or two n-ary relations which is called, usually you use the star symbol for that, it is called the join operation, the two tables are joined together. Now, you know that s_1 supplies P_1 for q_1. But P_1 and q_1

has two possibilities red and blue so you have like this s_1 P_1 q_1 red and s_1 P_1 q_1 blue. And similarly s_2 provides P_1 for q_1. So again you will have s_2 P_1 q_1 red and s_2 P_1 q_1 blue. And s_3 provides P_2 for q_2 and P_2 for project q_2 takes green color only. And P_1 for project q_2 takes blue color only so you have only one row for each one of them. So when you combine a ternary relation pi with another 1 2 pi has four rows tow has four rows and when you use the join operation you get six rows or six relations here. The four quaternary relations or each relation is represented by a four tuple here.

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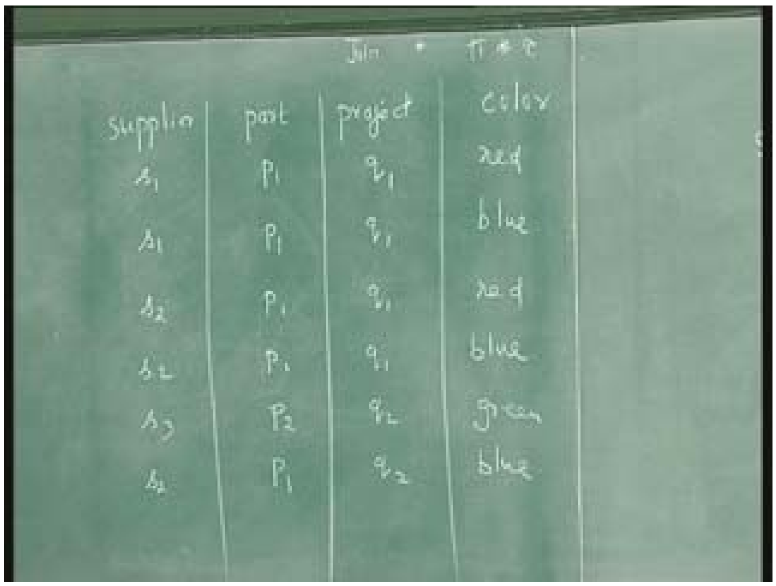

There is also another operation which is called a projection operation. For example here you have three components in each relation; supplier, part and project. You are interested only in finding out which supplier provides which part. You are not bothered about for which project it is provided. Then you have to only consider the projection of this relation that is first two components only you are interested.

Now, you find that when you project it on the first two columns this and this are the same so you need not have to repeat that s_1 P_1 s_2 P_1 s_3 P_2 you do not have to repeat this because it is already there. So when you project you may get less number of rows the projection of the table pi on the first two columns one and two is s_1 P_1 s_2 P_1 s_3 P_2 so you will get less number of rows in this case.

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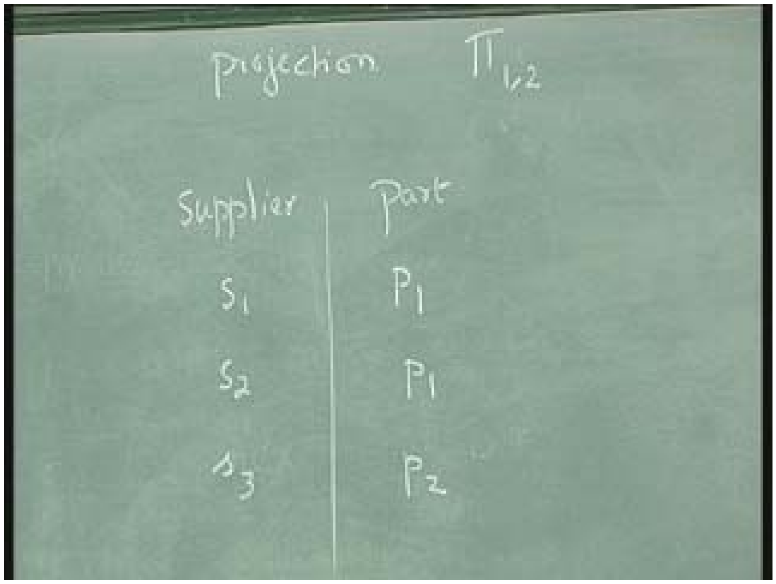

Projection also you can look at it this way, when you have a three dimensional object or when you have a point in three dimension it has got three coordinates when you project it onto a plane it has got two coordinates, when you project it on one axis it has got only one coordinate. Like that you have more number of columns in a relation when you project on to some of the columns you will get less number of rows like that.

Less number of columns of course less number of rows also. Join when you combine the four and four at the most you may get sixteen rows or something like that but in this case it so happens that you are getting only six rows and four columns in the join operation. This joint operation is different from what we considered in the lattice as join though the same name join is used. These are some facts about relations, partially ordered sets, lattices, chains and antichains.

In this course on discrete mathematics we have considered several topics. The topics which are covered are logic, sets, relations, graphs, functions, combinatorics, recurrence relations, algebras and fsa.

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These topics will be very useful when you learn about other topics in Computer Science. This is the beginning and later on when you learn about the compiler design, database theory, artificial intelligence and subjects like that the use of all these topics you will understand. It is like a zig zaw puzzle, the whole course on Computer Science is like a zig zaw puzzle, initially you put some of them and then the whole picture will come clear to you only when you finish the course. So you will understand the use of all the topics which we have covered in this course only when you finish the course because you will find the application in many other fields in Computer Science.

This subject can be very well understood by working out a number of problems. For that we have suggested a number of books. The reference books are; Elements of Discrete Mathematics by Liu, Discrete Mathematical Structures with Application to Computer Science by Tremblay and Manohar, Discrete Mathematics in Computer Science by Stanat and McAllister, Discrete Mathematics and its application Kenneth Rosen, Logic and Discrete Mathematics Grassman and Tremblay, Introductory Combinatorics by Brualdi, Modern Applied Algebra by Birkhoff and Bartee, Mathematical Theory of Computation by Manna, Introduction to Combinatorial Mathematics by Liu, Introduction to Automata Theory, Languages and Computation by Hofcroft, Motwani and Ullman and so on. These are not the only books there are so many other books on Discrete Mathematics.

Select the proper book or the book prescribed by your University and try to read from that book along with these lectures and try to work out as many problems as possible which will help you to understand the subject clearly.