Discrete Mathematical Structures

Dr. Kamala Krithivasan
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture - 4

Predicates & Quantifiers

So in the last lecture we saw how to use the quantifiers for all of x and there exists x. So if you have a statement for all of x p(x), then this will imply p(c) where c is an arbitrary element of the universe.

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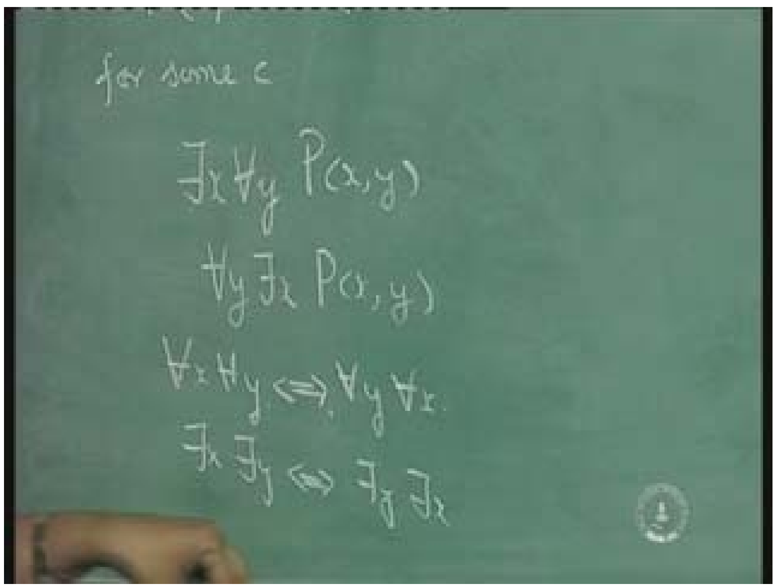

If you have p(c) for some element, for some c then this will imply, there exists x p(x) because there exists x p(x) means for sum of x p(x) is true. Here c is not an arbitrary element it can be true for a particular element. If p(c) is true for some c then this will imply there exists x p(x).

We have also seen that we cannot interchange the operators or quantifiers, there exists and so on. If you interchange and write like this, it gives a different meaning, this is what we saw in the last lecture by giving some examples that the meaning will be entirely changed. But you can always interchange for all of x for all of y to for all of y for all of x when both are universal quantifiers you can interchange the order.

Similarly, when both are existential quantifiers also you can interchange them and it does not make a difference. Let us consider how it is true? Consider the set of natural numbers or non-negative integers as the underlying universe, then universe is the set of non-negative integers. Then look at this statement for all of x for all of y p(x, y). Now you can expand it like this, expanding for all of x it will be for all of y p(0y) and for all of y p(1y) and for all of y p(2y) and so on.

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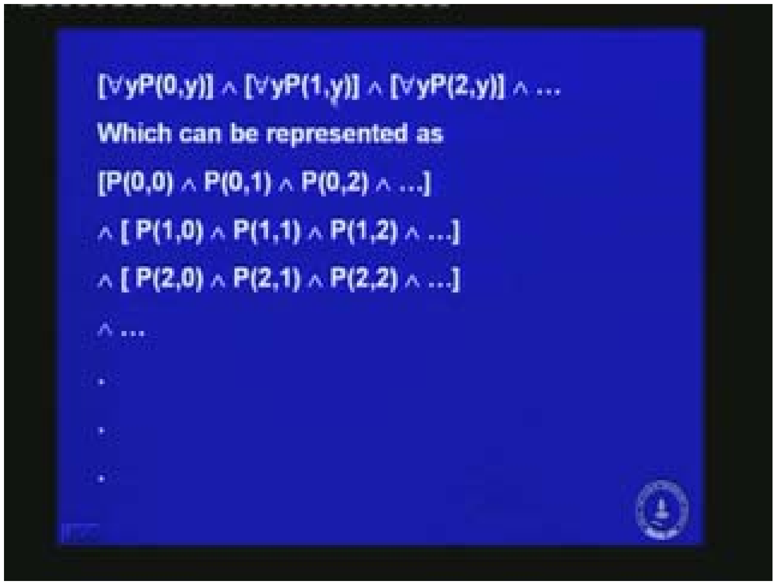

Now this can be expanded as p(0,0), p(0,1), p(0,2) and so on where you are giving all values for y. This can be expanded as p(1,0), p(1,1), p(1,2) and so on. Now the operator AND is commutative and associative, this we have seen earlier. So you can interchange the order in which you have written this p(0,1), p(1, 1) etc.

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You can interchange the order in which they are occurring in a conjunction. Now, if you interchange and write you can write it like this 0 0, p 1 0, p 2 0, p 0 1, p 1 1 etc and so on. That is you are grouping like this, you are grouping like this, you are grouping like this and you are grouping like this. Now this would represent for all of x p(x, 0) and this could represent for all of x p(x, 1) and this would represent for all of x p(x, 2) and so on. And you have for all of x p(x,0, x,1 x,2) and so on. So grouping this together you will get all of y, for all of x p(x), y. So these are equivalent here, we have just taken the underlying universe as the set of non-negative integers, it will hold for any other universe also. Now you can give the same argument for there exists x and there exists y.

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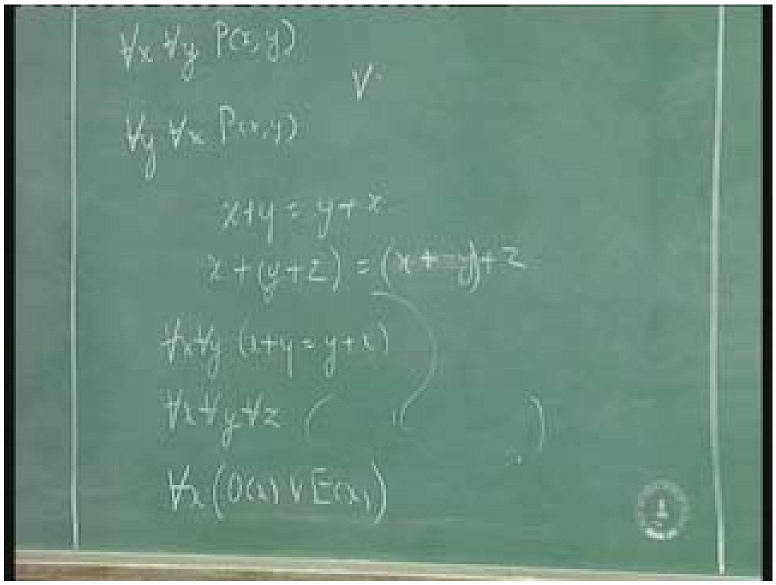

The only thing is you will be using disjunctions. The only thing is instead of AND you will be using OR, the symbol OR. Again OR is commutative and associative and because of that the same argument you can keep changing AND into OR and so there exists x, there exists y is equivalent to there exists y, there exists x. Now let us take some sentences in English and see how to write them in the logical notation using quantifiers.

Before going into that generally in arithmetic you will frequently use something of the form x plus y is equal to y plus x commutative law may be they are for integer arithmetic or Boolean algebra or something, x plus y plus z associative law and so on. What you really mean here is for all of x for all of y x plus y is equal to y plus x and here similarly for all of x for all of y for all of z it is this statement. But usually when you write the rule you omit x for all of x for all of y for all of z which is implicitly understood in many cases. Now let us take some sentences in English and see how they can be written in logical notations using the quantifiers. Let the universe be the set of integers and let n(x) denote x is a non-negative integer, e(x) denote x is even, o(x) denote x is odd and p(x) denote x is prime.
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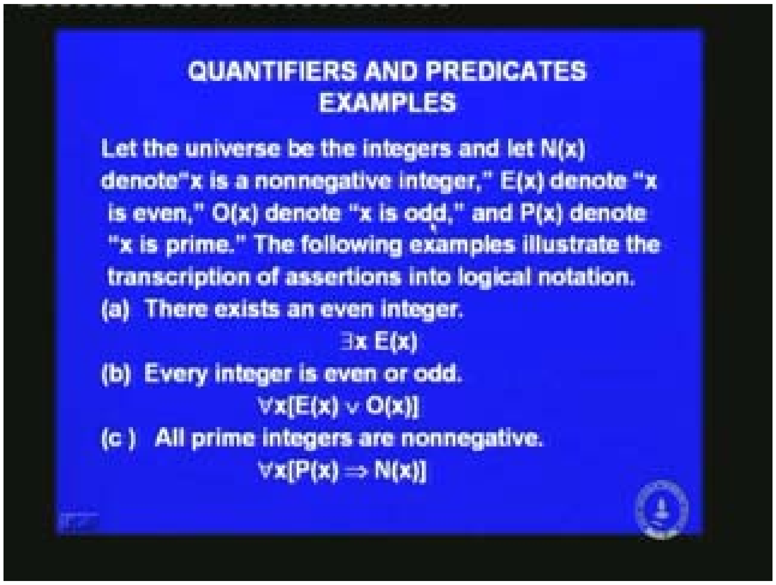

Now let us consider some example and find how will you write this as statement? There exists an even integer and this will be written as there exists x e(x). Every integer is even or odd and how will you write this?

For all of x e(x) OR o(x), even or odd. Then look at this statement, all prime integers are non-negative, then how will you transcribe this in logical notation? For all of x p(x) implies n(x), if x is a prime then it is non-negative. Let us see some more sentences, the only even prime is true for all of x e(x) and p(x) implies x is equal to 2, this will be transcribed like this.

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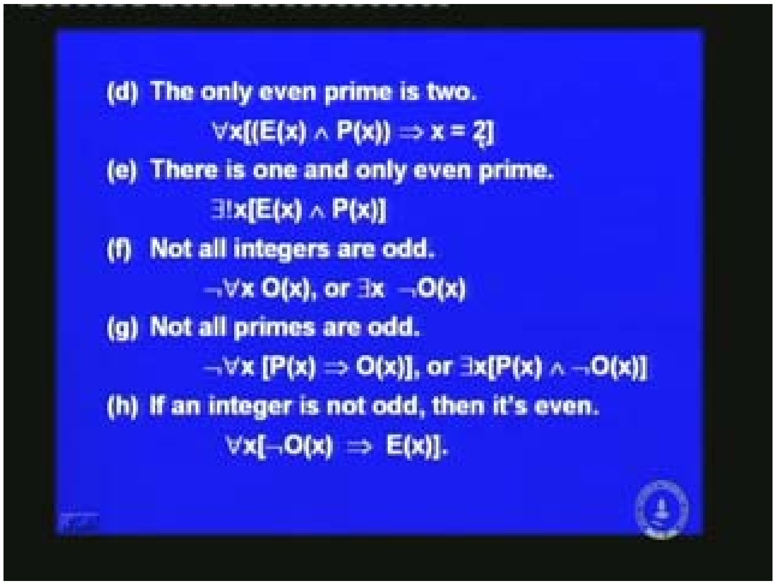

There is one and only one even prime, this can be transcribed as there exists a unique x e(x) and p(x) because in the last class we saw how to write there exist unique in terms of for all and there exists. Not all integers are odd, this can be written in the form not for all of x o(x) or there exists x not of o(x). Not all primes are odd: this can be written in the form not for all of x p(x) implies o(x) or it can also be written in the form there exists x p(x) and not of o(x).

There exists a prime which is not odd and this is what it means. Look at this statement: if an integer is not odd then it is even. This can be written in the form: for all of x not of o(x) implies e(x), because if you expand this using the rule for p implies q, here p implies q can be written in the form not p or q, so if you try to write this in that form it will be for all of x o(x) or e(x) it will become like this. So every integer is either even or odd and that is what it means. Now let us consider some more things, xy is equal to z denotes it by p(x, y, z).

When you bind the variable it is very very important to specify the scope. You have to use parenthesis properly to find which portion is it or which quantifier is binding and so on. If you do not use, the meaning may become entirely different and you may write wrong sentences.

Now if xy is equal to x for all y then x is equal to 0 taking the underlying universe should be the set of non-negative integers, if x y is x for all y that means x should be 0. How will you write this in logical notation? You will write it like this: for all of x for all of y p(x, y) x implies x is equal to 0. Now the scope of the quantifier y is this and the scope of the quantifier x is this whole thing.

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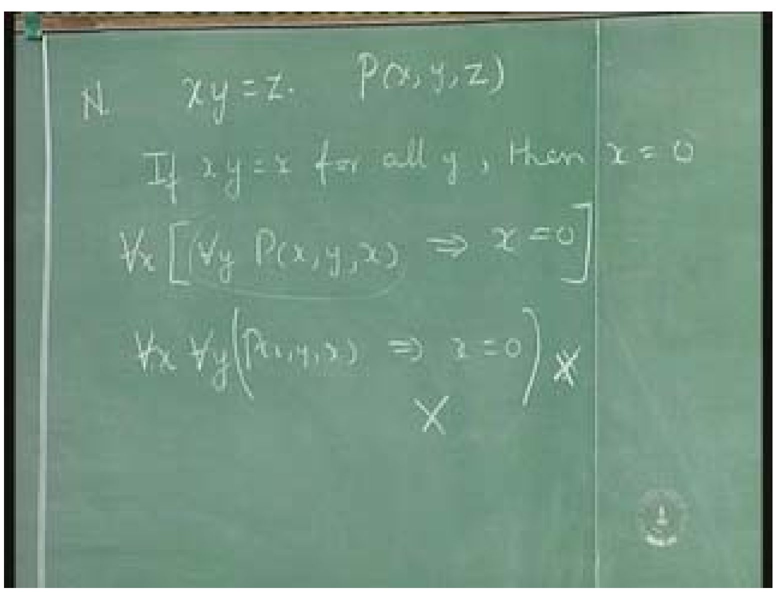

So you have to remember that this is the way it has to be written. Now if you write like this: for all of x for all of y p(x), y, x implies x is equal to 0, this is not correct because you can have x is equal to 1, y is equal to 1 and then this will satisfy xy is equal to x. So there is another value even without x is equal to 0 this p y, x xy is equal to x can be satisfied.

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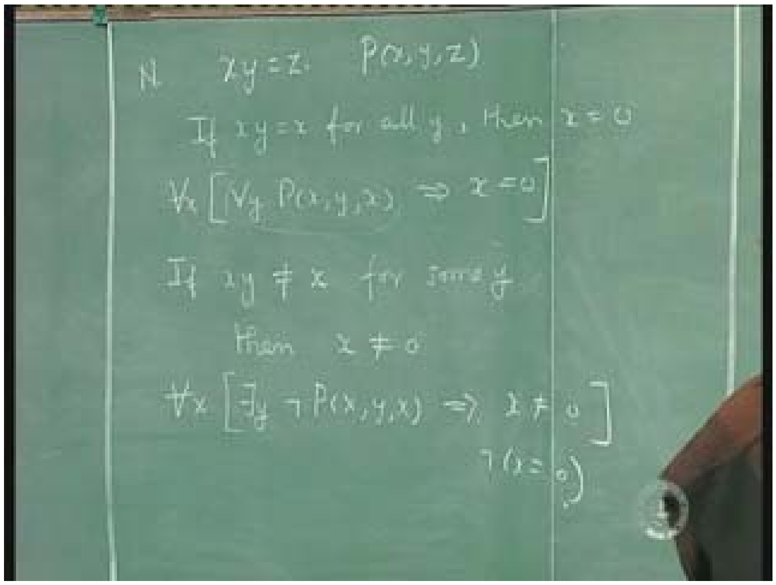

So this is not a correct statement because for all of x you are using only this portion and that is not correct. You cannot write like this also, this is also not correct so you have to specify the scope. Here the scope of this is this and the scope of this is this. Now you can also write like this: if xy not equal to x for some y then x not equal to 0. For all of x there exists y NOT(p(x, y, x)) implies x NOT is equal to 0 you have to write like this. You can write it as x NOT is equal to 0 or NOT of x is equal to 0 and either way you can write. Now an expression involving for all of x p, q like predicate variables, there exists y etc and some quantifiers are called a well formed formula of first order logic or predicate logic or predicate calculus.

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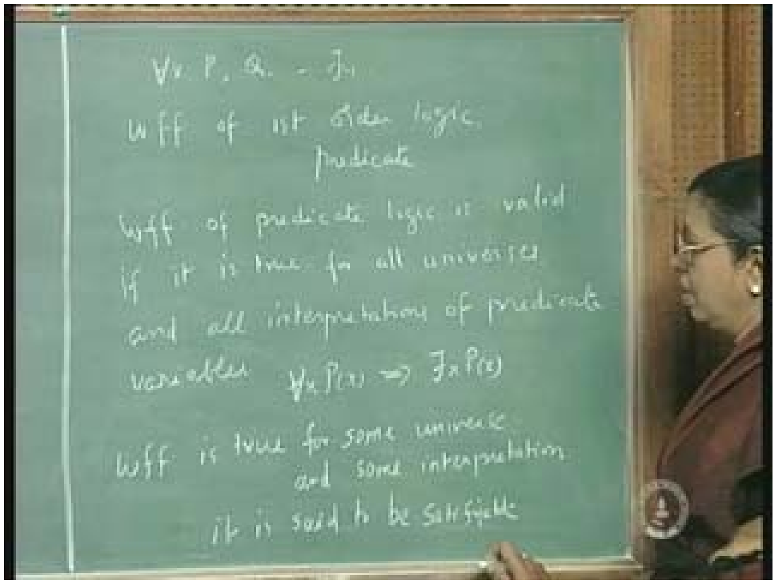

And well formed formula of predicate logic is called valid. It is said to be valid if it is true for all universes and all interpretations of predicate variables. If well formed formula of predicate logic is true whatever may be the universe and whatever may be the interpretation we are going to give for the predicate variable then it is said to be valid.

For example, if you take for all of x p(x) implies there exists x p(x) is always true. Whatever may be p and whatever may be the underlying universe this is always true and this is said to be a valid statement in predicate logic. If this some wff is true for some universe and some interpretation it is said to be satisfiable. If the wff is not true for any universe and any interpretation then it is said to be unsatisfiable, you call it as unsatisfiable.

So, in predicate logic you have logic valid expressions, your satisfiable expressions and you have unsatisfiable expressions, they correspond to tautologies in propositional logic, this corresponds to a contingency which is some times true and some times it is false and this corresponds to a contradiction in propositional logic. Now again coming to the use of this the scope is very important that is there exists x p(x) implies q(x).

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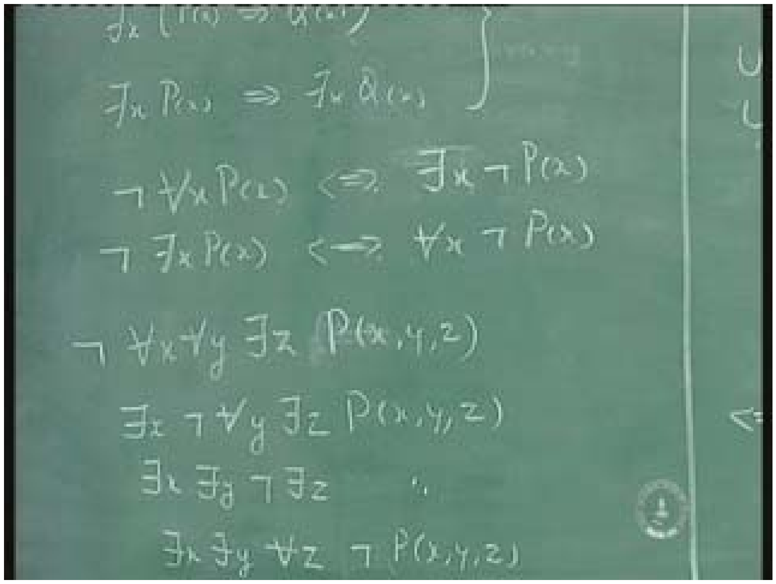

And there exists x p(x) implies there exists x p(x) are these equivalent or not? Later on we will see that they are not equivalent but one will imply the other. So you cannot just bring in there exists inside the parenthesis like this. Here there exists is outside scope of this is this, the scope of this is this, and the scope of this is this. You just cannot use distributive law and then bring it.

Also, when using NOT you have to be careful, for example you may have something like that for all of x p(x), what does this mean? This means that it is not true for all of x p(x) is true, then it means that there is a value(x) for which p(x) is not true. So you can write it equivalently like this, this is equivalent to NOT, there exists x NOT of p(x) has a value for which p(x) is not true. So we are not bringing this inside and then when you try to bring the NOT inside for all x becomes there exists x.

Now you can also have some statement like this: NOT there exists x p(x) which means it is not true that there is a value p(x) for which p(x) is true, what do you mean by that? For all values(x) p(x) is false, so this is equivalent to saying for all of x NOT of p(x). So when you try to bring the NOT inside there exists will become for all and for all will become there exists. So if you have a statement say for all of x for all of y there exists z x plus some p(x), y, you can write like this. Later on we take an example, now if you want to say NOT of this then when you try to bring the NOT inside step by step you can bring it inside and this will become there exists x NOT of for all of y NOT of for all of there exists z p(x), y, z. And again when you try to bring it inside it will become there exists x, there exists y for all z NOT(p(x, y, z)). And again when you try to bring it, it will become there exists x, there exists y for all z NOT(p(x, y, z)). This is the way you bring the NOT inside the statement. Now look at this statement: x plus z is equal to y.

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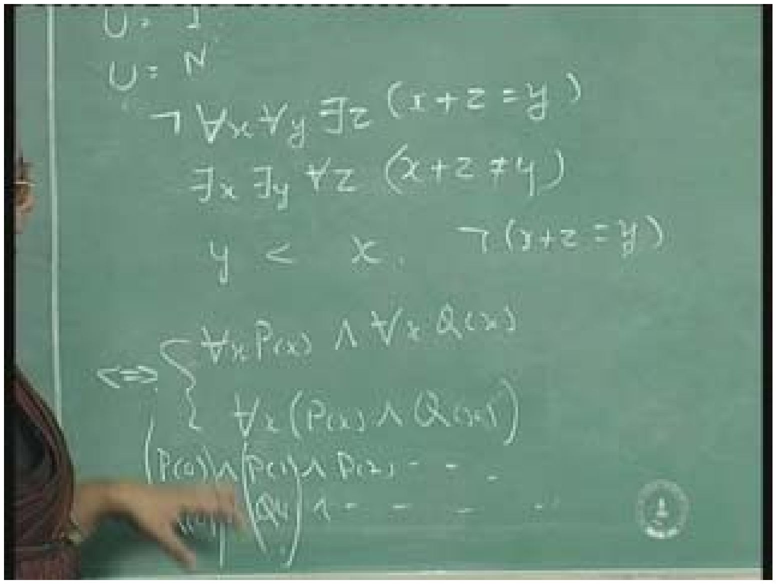

So for all of x for all of y there exists z x plus z is equal to y. Now you should take the underlying universe as the set of integers. This is true for any value(x) and any value of y. You can find the value of z such that x plus z is equal to y. You have to take z to be y – x.

Now if you take the underlying universe to be the set of non-negative integers then this is not true. This will be true only when y is greater than x. If y is less than x you cannot find a non-negative integer z such that x plus z is equal to y, that is not for all of x. For all this is the correct one, you have x plus z is equal to y this is should be the statement but instead of writing this it is easier to write it this way where you bring the NOT inside.

You can write it as there exists x, there exists y for all z x plus z NOT is equal to y. And this is true because you can choose y less than x and in that case you cannot find any value for z for which this will be true. You can write this as NOT is equal to or NOT of x plus z is equal to y is also another way which you can write. Again the scope is very important; you have to be very careful about this. Now we were considering this example, are they equivalent or not? Before proceeding with that example let us see for all of x p(x) and for all of x q(x).

Look at this statement: is it equivalent to saying for all of x p(x) and q(x)? Are they equivalent or are they not equivalent? These two are equivalent because suppose you take the underlying universe to be the set of non-negative integers and then you can write this one as p0 AND p1 AND p2 etc and this you can write as q0 AND q1 AND etc.

Now when you use AND, and then use commutativity and associativity you can group it like this and so this is equivalent to this. Now look at this for all of x p(x) or for all of x q(x) and for all of x p(x) or q(x), are they equivalent? They are not equivalent because here it means that for all of x p(x) is true or for all of x q(x) is true.

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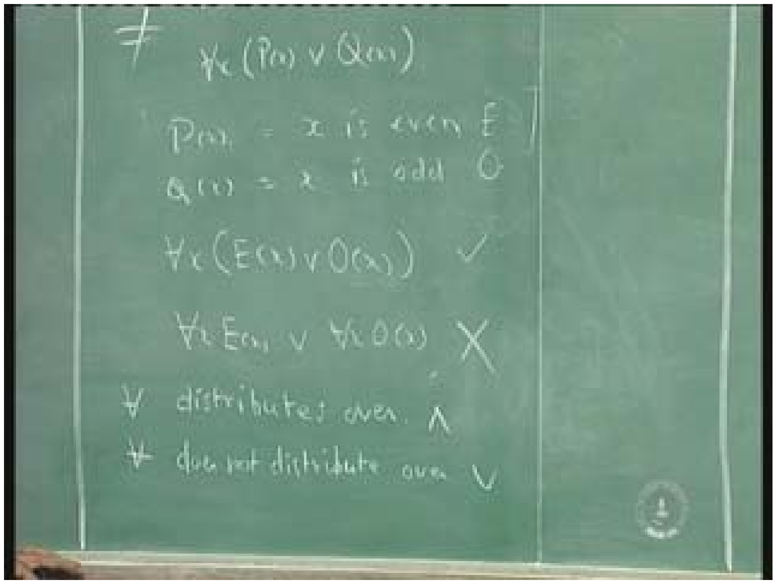

Here it means that for all of x for every value(x) either p x is true or q(x) is true p(x) denote x is even and q(x) denote x is odd so e(x) in that case this would mean for all of x e(x) denotes e denotes o, say e(x) or o of this is true.

For every integer is either even or odd this is true but what does this mean? This will mean for all of x e(x) or for all of x o(x) is not true, this is true, this is not true, that is every integer is even or every integer is odd that is NOT is correct. So one may be true the other may not be true so you cannot equate like this. If you have like this you cannot expand and you use distributivity and expand it like this. So, for all distributes over and for all that does not distribute over or by a similar argument you can show that there exists x q(x) or there exists x p(x).

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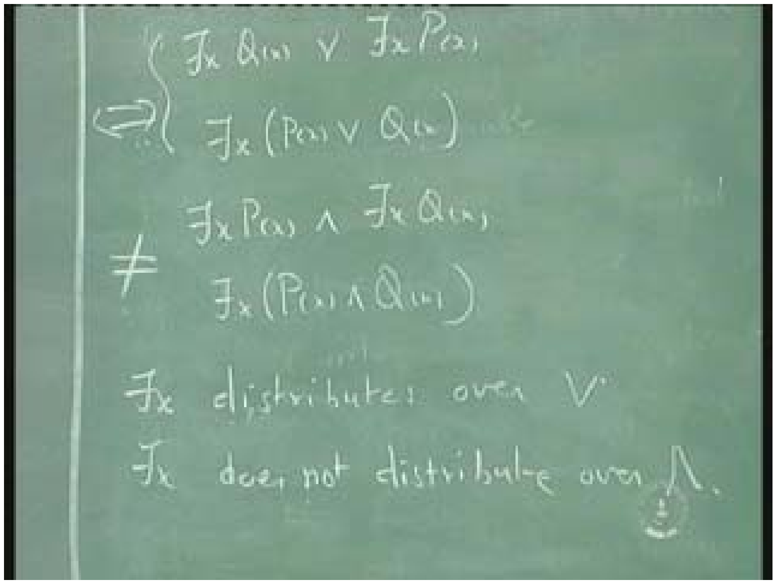

If we consider this and there exists x p(x) or q(x) these are equal, they are equivalent again by a similar argument like we gave. Here instead of this AND you will have OR here and you can group in this way whereas there exists x p(x), there exists x q(x) and p(x) and q(x) this will not be equal. So there exists distribute over or there exists x does not distribute over and now we are considering whether these are equivalent or not, there exists x q(x) and there exist x p(x) implies q(x).

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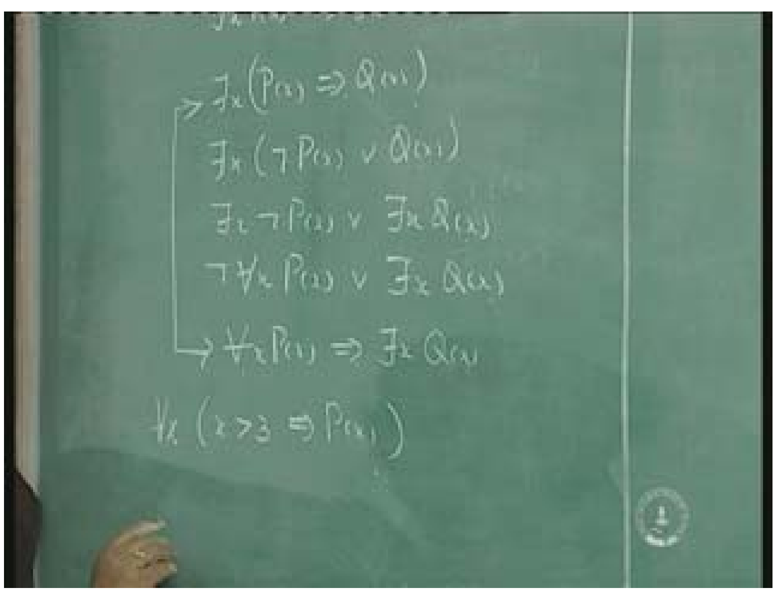

Now you can write this as NOT of p(x) or q(x) and you know that there exists distribute over OR, so you can write this as there exists x NOT p(x) or there exists x q(x) and if you take the NOT outside there exists will become for all. So NOT for all of x p(x) or there exists x q(x) and essentially this means for all of x p(x) implies there exists x q(x). So this is equivalent to this and whether these two are equivalent, they are not equivalent. Obviously you can see that they are not equivalent.

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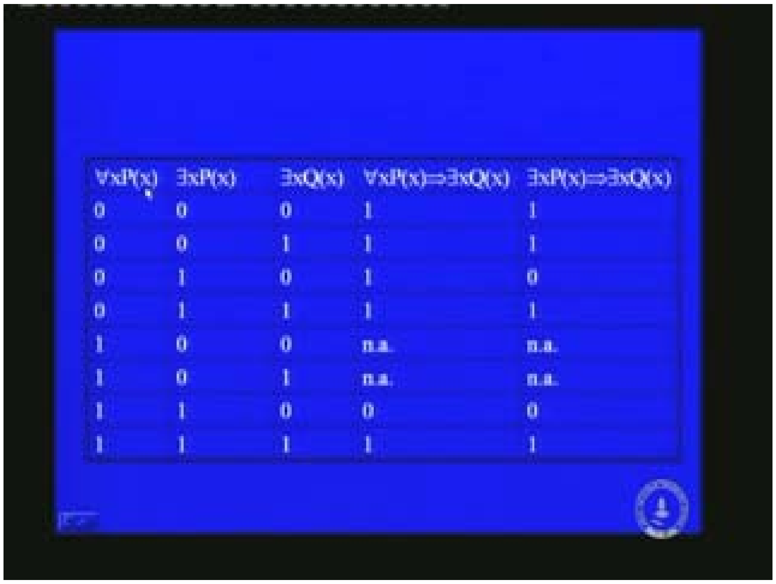

If you try to draw the truth table for this, look at this for all of x p(x) there exists x p(x), there exists x q(x). Now giving all possible values for this 0 0 0 0 0 0 1 1 0 1 0 and so on. You will find that this the truth value for all of x p(x) implies there exists x q(x) because the premise is false, all of this is true, these two lines of the truth table are not applicable because when for all of x p(x) is true you cannot have there exists x p(x) as false. This is not possible so these two rows of the truth table are not applicable. Now coming to this, again this is true and this is false, so the premise is false. And the conclusion is that the premise is true and the conclusion is false so this is false.

Here both of them are true so this is true. Now if you look at this again whenever this is 0 this will be true and it will be false only when this is true and this is false, in these two cases it will be false rest of the case it will be true, these two columns are not identical therefore they are not equivalent but one will imply the other. Which one will imply the other? See, here you can have the antecedent false and the consequence true and that is possible. So this will imply this; this will imply the other statement. So we have some logical connectives, logical relationships involving quantifiers. Let us look into that now.

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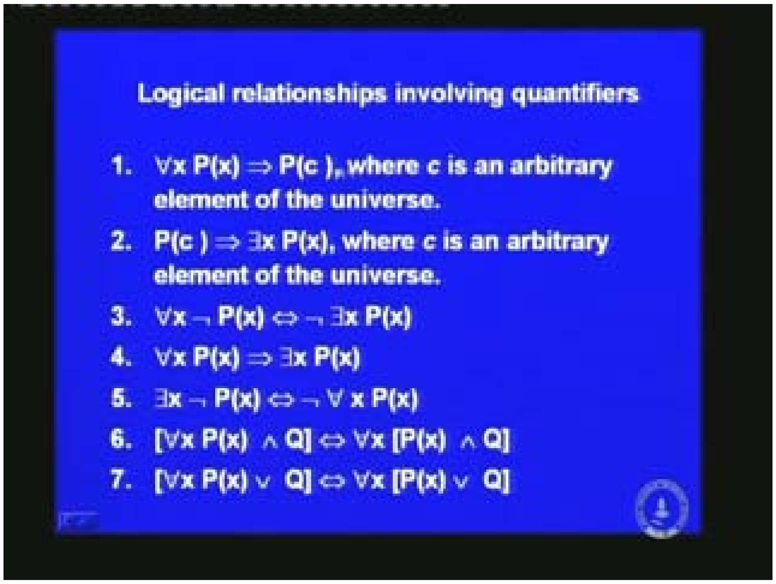

You have this: for all of x p(x) implies p(c) where c is an arbitrary element of the universe and if you have p(c) for some c this will imply there exists x p(x) and again the NOT is inside.

If you take the NOT terms outside for all will become direct and then for all of x p(x) will imply there exists p(x) and again in this case if you take the NOT outside there exists it will become for all. So there exists x NOT of p(x) is equivalent to saying for all of x p(x) and here if q does not involve x whether you bring it inside the bracket or outside the bracket it is immaterial and if q does not involve x for all of x p(x) and q is equivalent to saying for all of x p(x) and q.

Similarly if q does not involve x whether you put q inside the parenthesis or outside the parenthesis is immaterial and these two will be equivalent. Again we have seen that for all of x distributes over and so if you have for all of x p(x) and for all of x q(x) you can put them within the parenthesis and write for all of x p(x) and q(x) these two are equivalent whereas for all does not distribute over and so this will imply they are not equivalent.

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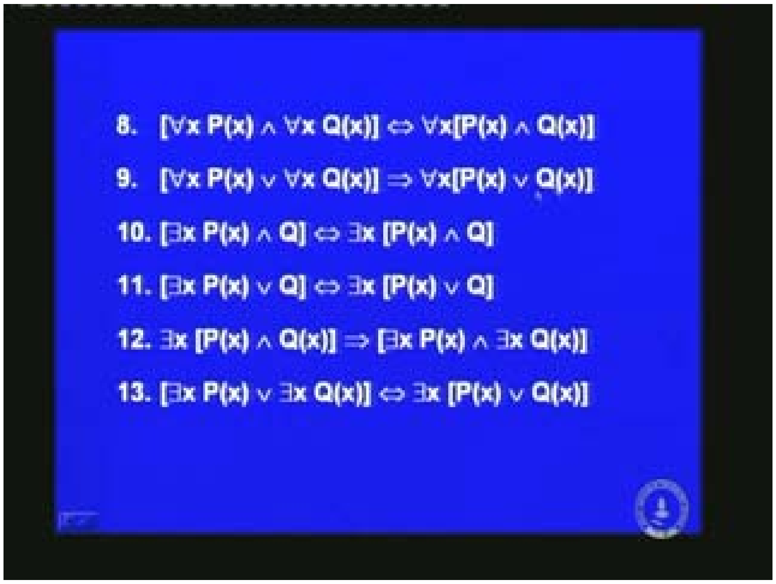

This will imply this similarly, if q does not involve x and if you use quantifier there exists x whether you put it within the parenthesis or outside the parenthesis and whether you use AND or OR is immaterial.

Here q does not involve x so you get these two statements. And again there exists distributes over or so the thirteenth statement is, if you have there exists and or you can put q(x) and p(x) within the parenthesis whereas if you use and it will not distribute but there exists x p(x) and q(x) will imply this but not the other way round. So these are some of the rules or logical relationship involving quantifiers.

Now generally you may have some statements like this: for all of x if x is greater than 3 then p(x) may hold. You may have statements like these. So some result may hold some x greater than 3 and you have to write it in this way but generally you write it in this way: for all of x x greater than 3 p(x), you write this under this so for all values(x) greater than 3 this is true.

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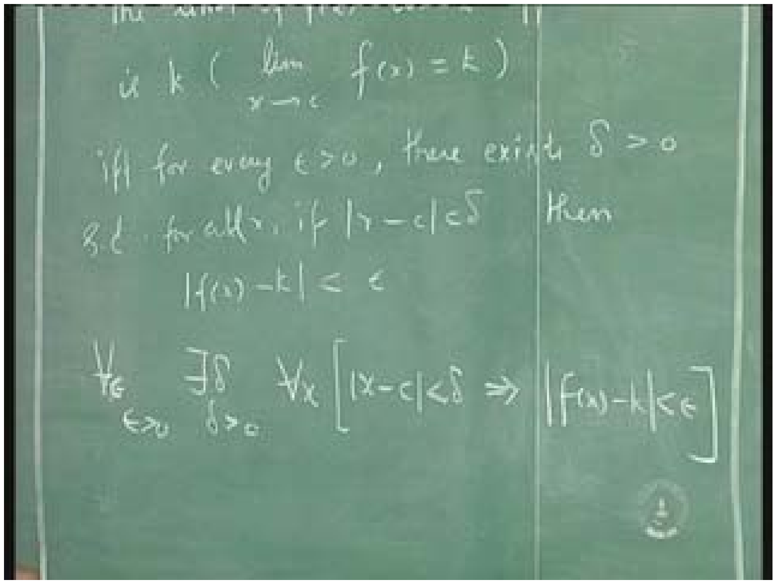

Now this is a general statement which you might have studied in a calculus. The limit of a function as x approaches c is k which you write as x tends to c f(x) is equal to k if and only if for every epsilon is greater than 0 there exists delta greater than 0 such that for all of x if x minus c is less than delta then f(x) minus k is less than epsilon.

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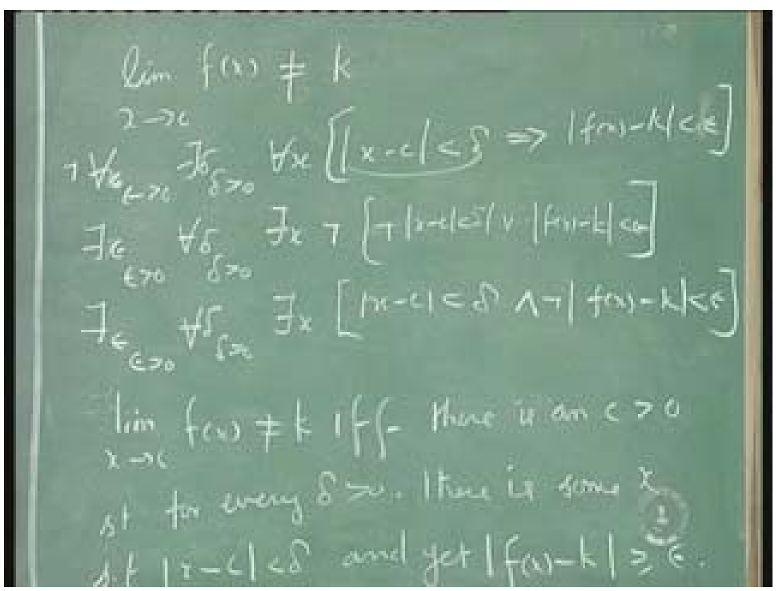

This if you have a function at the value c the y value is k if within a short interval near c the function can take only values which are nearer to k. This is the graphical understanding of this limit. How will you write it in logical notation? You can write it in logical notation like this. For all of x, for all epsilon greater than 0 there exists delta greater than 0 such that for all of x x minus c less than delta implies f(x) minus k is less than epsilon, this is the way you write it. Now when you want to say limit of x minus c f(x) not equal to k you have to consider the negation of this. So it will be NOT for all of x for all epsilon where epsilon is greater than 0 there exists delta greater than 0 for all of x x minus c less than delta implies and so on f(x) minus k less than delta.

And if you bring this NOT inside this will become there exists epsilon greater than 0 for all delta greater than 0 there exists x then NOT of this. So for this you can write the implication as NOT(x) minus c less than delta or f(x) minus k less than epsilon. So using DeMorgan's laws this will become there exists epsilon where epsilon is greater than 0 for all delta and delta being greater than 0 there exists x such that when you bring NOT NOT NOT NOT of NOT is again the double negation and so it will be x minus c less than delta and using DeMorgan's laws or will become AND, this is NOT of f(x) minus k less than epsilon.

So you will read this like: this limit as x tends to c as f(x) not equal to k if and only if there is a c greater than 0 such that there is an epsilon greater than 0 such that for every delta greater than 0 there is some x such that x minus c is less than delta and yet f(x) minus k and the difference between them is greater than or equal to epsilon.

So, writing it in the sentence way in English, you have to write it like this and how do you get this from this using negation and you when you bring the NOT inside for all, it becomes there exists and there exists becomes for all and again for all becomes there exists and this NOT has come inside. You can write this implication in the form NOT p or q and then when you bring this inside it becomes like this and NOT of f(x) minus k less than epsilon. You can write it as f(x) minus k, mode of f(x) minus k greater than or equal to epsilon that is in the figure wise the limit is not k. If this is k if and if you take the small interval here then for some value the difference between f(x) and k will be more than epsilon that is what it means.

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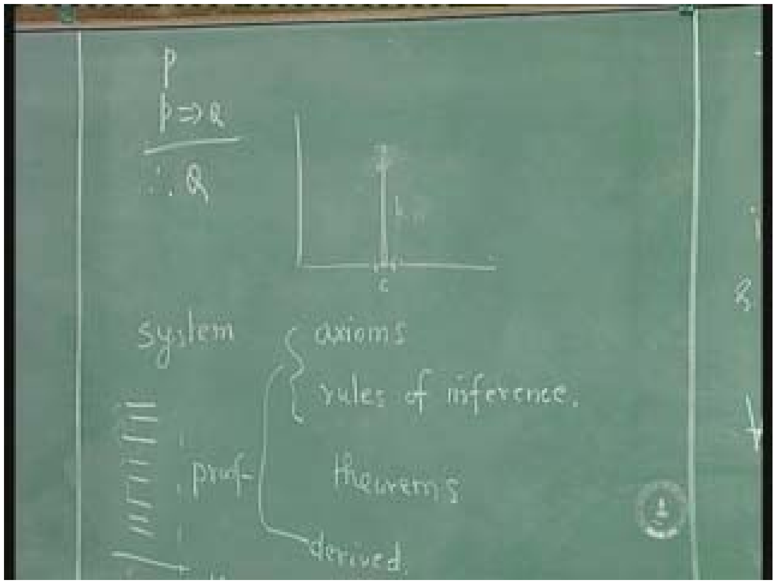

So we have used such statements several times in logic but what do they mean using quantifiers? This we want to see now, so now we see that they have a specific meaning when you use quantifiers and you can express any sentence in mathematics saying a mathematical truth in terms of quantifiers and logical statements. This is called first order logic. Now let us consider some more examples; consider this example, let us try to transcribe some English sentences into logical notations using quantifiers.

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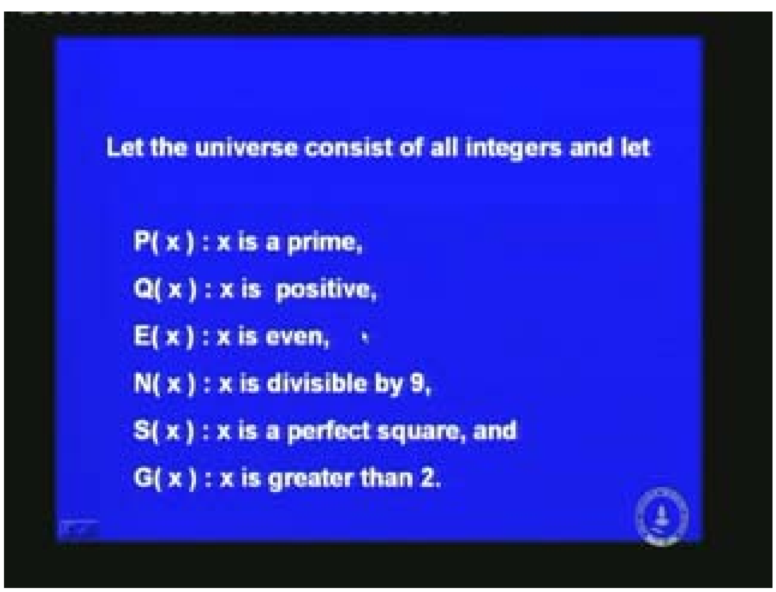
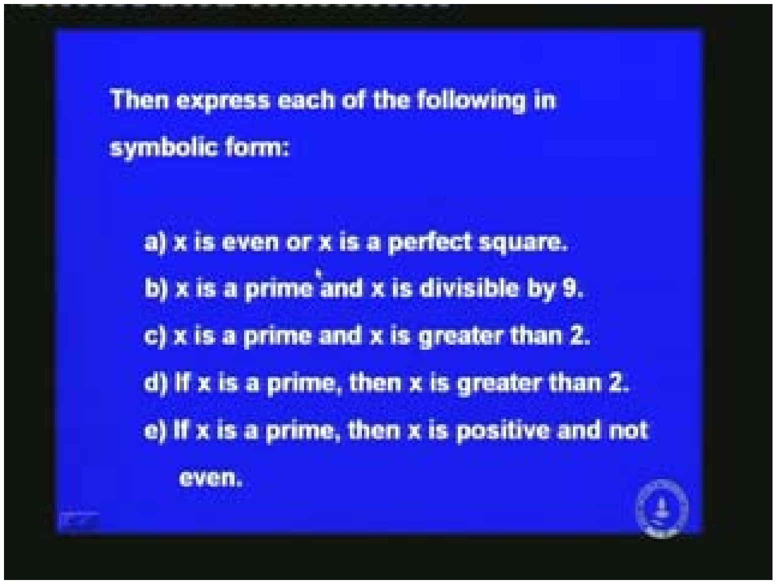


Consider this, let the universe consist of all integers and let P(x) denote x is a prime, Q(x) denote x is positive, E(x) denote x is even, N(x) denote x is divisible by 9, S(x) denote x is a perfect square and G(x) denote x is greater than 2 then express each of the following in symbolic form. Let us take some from this and try to write them using logical notation. (Refer Slide Time: 44:12)



Take the first one x is even or x is a perfect square, how will you write this in logical notation? x is prime or x is a perfect square or s(x) p(x) denotes x is a prime s(x) denote x is a perfect square so x is even or x is a perfect square is denoted by p(x) or s(x). Consider d if x is a prime then x is greater than 2 x is a prime denoted by p(x) x is greater than 2 is denoted by g(x). So, if x is a prime then it is greater than 2 is denoted by p(x) implies g(x) if then else if then is denoted by implication. So if x is a prime then greater than 2 is denoted by p(x) implies g(x), similarly you can write for the other three; b c and e. Now let us consider this: translate each of the following sentences into symbols first using no existential quantifiers and second using no existential quantifiers. First one is not all cars have carburetors.

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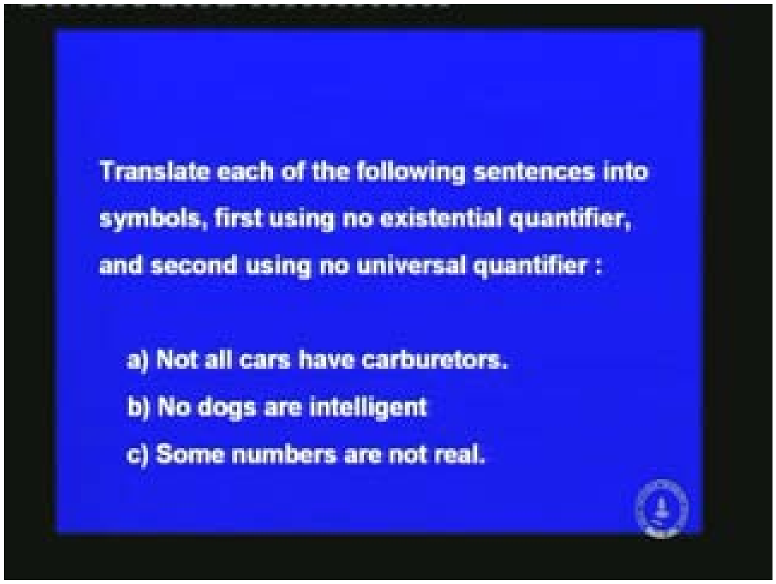

Let car of x denote x is a car and x denote x has carburetors. Now the statement is not all cars have carburetors not for all of x c of x implies car of x implies carburetor x.

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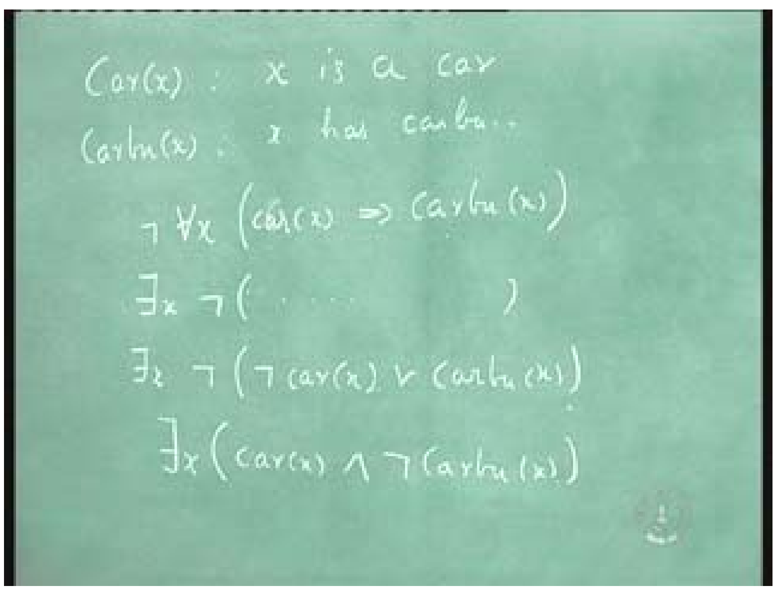

This is the way you write using the universal quantifier, not all cars have carburetors not for all of x car of x implies carburetor, carburetor of x or x is a car implies x has a carburetor now you have written this using universal quantifier only and not existential quantifier. Now we want to write it using existential quantifier but not universal quantifier.

Now if you try to bring NOT inside, for all will become there exists, so this will become there exists NOT, that is there exists x NOT, this implication p implies q you can write as NOT p or q. So this you can write as NOT car of x or now this NOT again you can bring it inside using DeMorgan's laws. So this will become there exists x NOT of car x will become car of x or will become and using DeMorgan's laws and not carburetor x there exists x such that x is a car but it does not have a carburetor. So you are expressing this same thing using existential quantifier but not using a universal quantifier. So let us consider the next one; no dogs are intelligent dog(x) denote x is a dog and x denotes x is intelligent. The statement is no dogs are intelligent.

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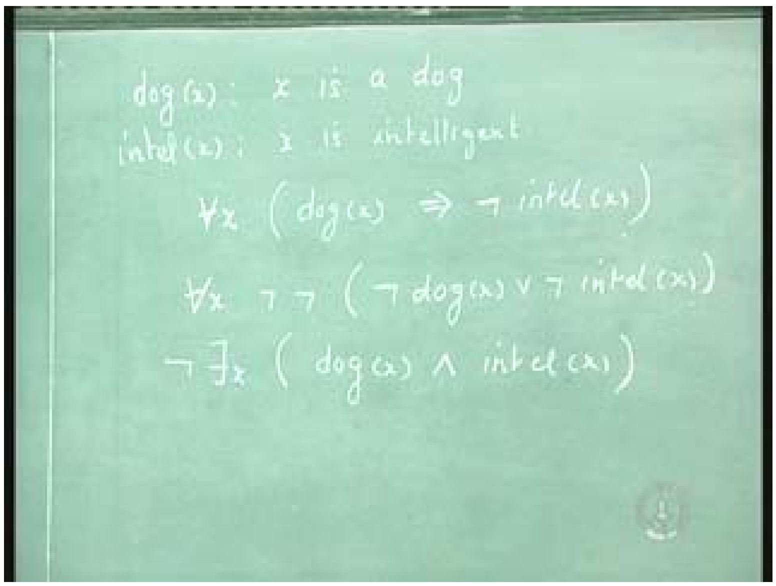

So you have to express it using universal quantifiers but not existential quantifiers. So using universal quantifier alone this can be written like this; for all of x dog(x) implies not intelligent, x is a dog means it is not intelligent, so no dogs are intelligent. So using universal quantifier you can write it in this way.

Now we have to express this using existential quantifier but not universal quantifier, how can you write it? For all of x you can introduce two NOTs here. If you have introduced two NOTs the effect is nullified. So you can use implication for this one as p implies q can be written as NOT p or q NOT dog(x) or NOT Intel of x.

And one NOT you can bring one NOT outside and one NOT you can take one NOT inside, one not if you take one NOT outside it will become NOT there exists x and when you bring the NOT this side for all will become there exists the other NOT you have to bring it inside using De Morgan's laws. So NOT of NOT of dog(x) will be dog(x) and or will become and using De Morgan's laws and NOT of NOT of Intel x will become Intel x. So you have to read it this way; NOT there exists x dog(x) and intelligent x that is you do not have something which is a dog and intelligent. So you are expressing the same statement using universal quantifier but not existential quantifier. The third statement is; some numbers are not real, so num x denote x is a number real x denote x is real.

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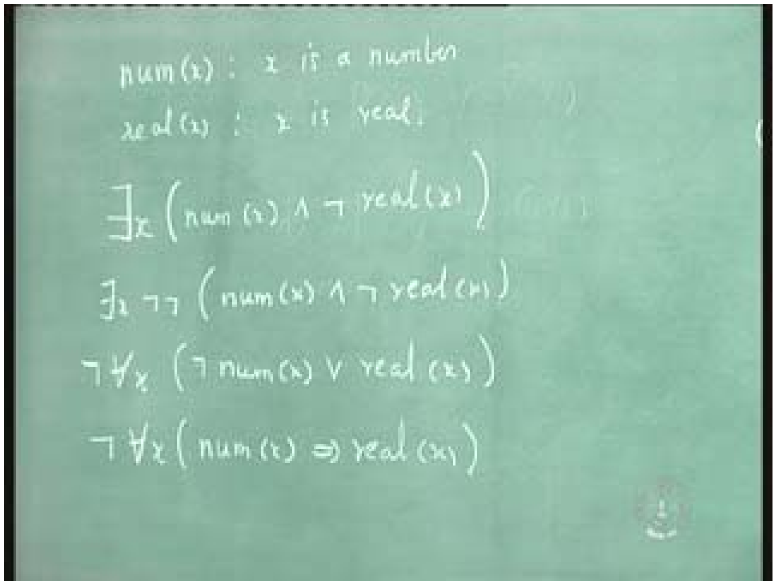

So some numbers are not real, using existential quantifier you can write it like this; there exists x num x and not real x. Now how will you write using universal quantifier? Again the same way you can introduce two NOTs here and you can write this one like this, you bring one NOT outside that is NOT for all of x, you bring one NOT inside that is NOT num x or NOT of NOT of real x that will become real x and if you write this using implication this will become NOT for all of x num of x implies real of x and you can write in this way.

You can also write this in a slightly different way but you can keep this. The next thing we shall consider is logical inference. You have several theorems, you have systems, you have a system of integer arithmetic where the underlying universe is the set of integers, then you have rules and you have some axioms.

What are the axioms? The axioms are commutative law, the associative law and like that the addition, subtraction, etc, then making use of these axioms and rules there are some rules of inference and making use of those rules of inference you can deduce some true statements about that system. So for any systems you have some axioms and you have some rules of inference and starting from axioms and rules of inference you try to reduce more true statements about the system, derive more statements about the system, they are called theorems.

Theorems are derived from axioms by making use of rules of inference, so start from axioms then you will have some more statements. Make use of them, apply rules of inference and ultimately you will arrive at a conclusion which is called a theorem. Now when the argument goes wrong or a theorem is wrong if you start with wrong axioms or even if you start with correct axioms, if you use a rule of inference in a wrong manner then you may arrive at a wrong proof. So when do you say that a proof is correct? The derivation of the theorem from axioms using rules of inference is called a proof. This is called a proof, you are deriving at the proof and this proof may go wrong if you apply the rules of inference in a wrong manner.

Now we have some rules of inference in logic which we have already seen in the first or the second lecture. May be from these are the rules from p we can conclude p or q or in the tautological form which this implies.

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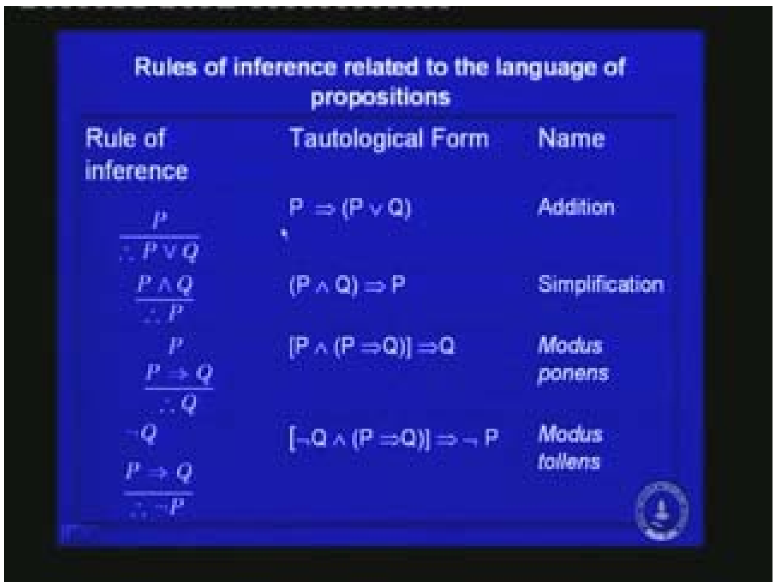

P implies p or q is called the name of the rule is addition. And if you have p and q from this you can conclude p or in the tautological form. This is p and q implies p and this rule is called simplification. This is from p and p implies q, you can conclude q and this is called modus ponens. From NOT q and p implies q you can conclude NOT p from NOT p and q you can conclude NOT p and this is called modus tollens. Then you have this rule from p or q or not p you conclude q and this is called disjunctive syllogism and from p implies q and q implies r you have p implies r.

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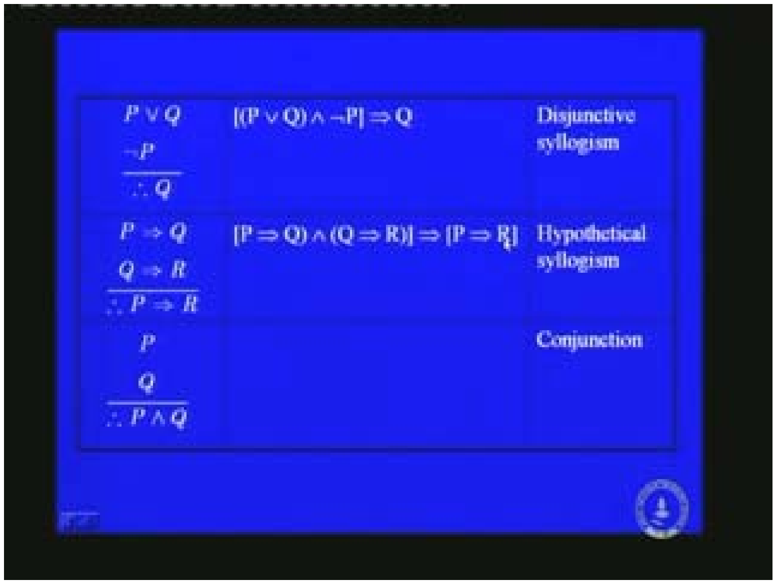

This is called hypothetical syllogism and if you have p and if you have q you can conclude p and q this is called conjunction and if you have p implies q and r implies s and if also have p or r you can conclude q or s in the tautological form.

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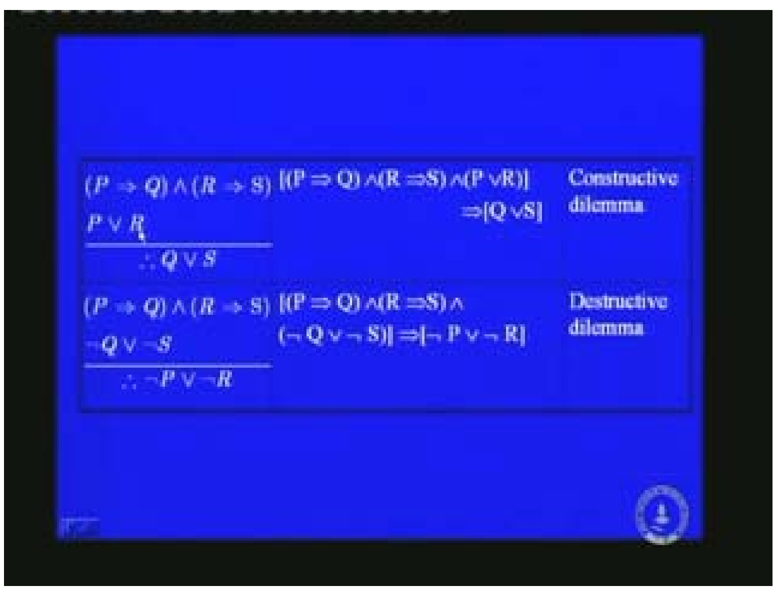

This is read as p implies q and r implies s and p or r this will imply q or s. This rule of inference is called constructive dilemma and if you have p implies q and r implies s and NOT of this NOT of q and NOT of s from this you can conclude NOT of p or NOT of r in the tautological form. It is like this; p implies q and r implies and NOT q or NOT s implies NOT p or NOT r and this is called destructive dilemma. And you can see the similarity between this and modus ponnens and the similarity between this and modes ponnens. These are some rules of inference and in the next lecture we shall see some example where you can find out whether an argument is valid or not by making use of these rules of inferences.