Discrete Mathematical Structures
Dr. Kamala Krithivasan
Department of Computer Science and Engineering
Indian Institute of Technology, Madras
Lecture # 36
Alegbras (Cont…)

In the last lecture we saw about algebras, what are algebras and also we saw the specific varieties of algebras. Algebra has three components, one is the carrier of the algebra and the second is some operations on the algebra and third is certain constants which have specific features. A variety of algebra is an algebra which satisfies certain axioms. And we have the seen the definition of a semi group, a monoid and a group. We shall recall those definitions again.

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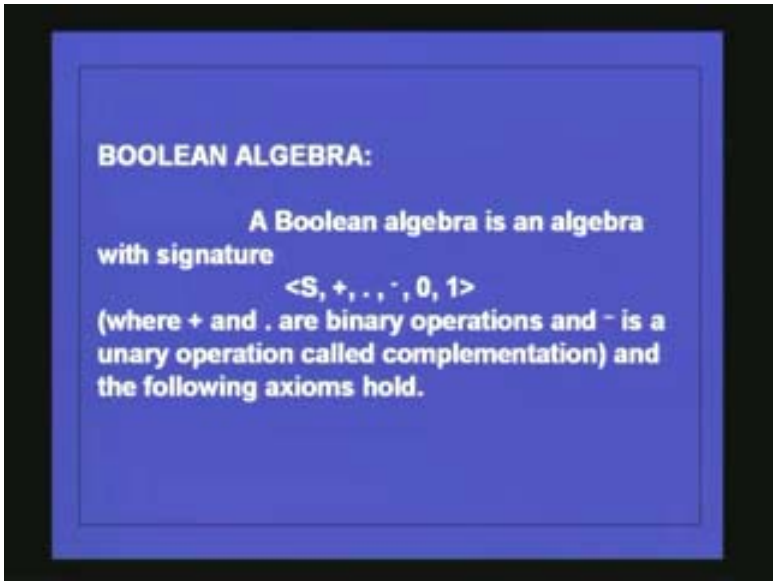

We have also seen what is meant by specific constants like what is meant by 0 element and what is meant by identity element. These are some of the concepts we saw in the last lecture. We shall see about one more variety of an algebra which is called Boolean algebra. So let us see this definition and then let us go back to the definition of semi groups, monoids and groups again.

What is Boolean algebra?
Boolean algebra is an algebra with signature S, plus, dot, this complement notation 0, 1 where S is the carrier of the algebra, plus is a binary operation, this also is a binary operation and this is a unary operation and these two are constants. This is called complementation and the following axioms hold.

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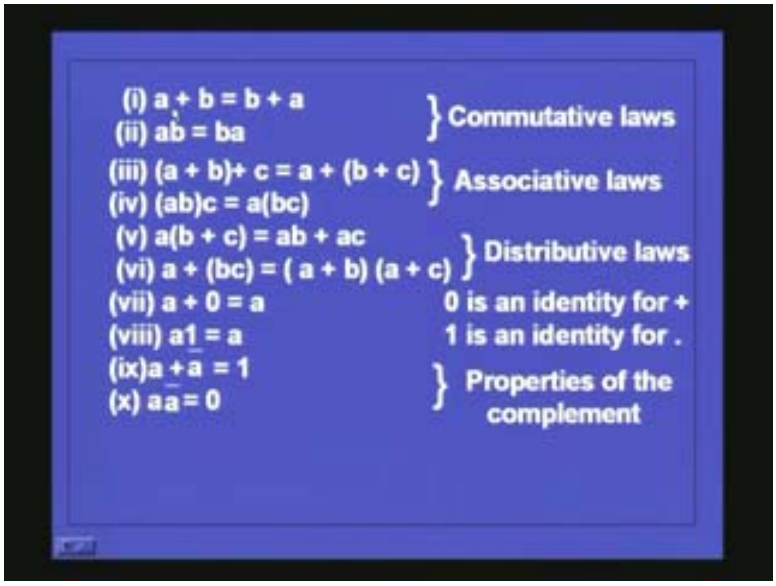

With respect to addition is this binary operation, you have commutativity law a plus b is equal to b plus a and with respect to other binary operation a b means a dot b so a b is equal to b a with respect to multiplication also you have commutativity. And associative laws hold for both addition and multiplication and we shall call the two binary operations as addition and multiplication a plus b plus c is equal to a plus b plus c this is associative law.

Similarly, for multiplication the associative law is a b into c is equal to a into b into c. And multiplication is distributed with respect to plus so a into b plus c is equal to a b plus a c. And addition is also distributed with respect to multiplication so a plus b c is equal to a plus b into a plus b these are distributive laws, one binary operation distributes over the other operation. And 0 is the identity for the addition operation, so a plus 0 is a and 1 is the identity for the multiplication operation that is a dot 1 is equal to a. So 0 is the identity for this and 1 is the identity for this. Then with respect to complementation if you take a and the complement of a a bar a plus a bar will be is equal to 1 and if you multiply a with a bar you get 0. So these are the properties of the complements.

A Boolean algebra is an algebra with this signature with one carrier two binary operations one unary operator and two constants where this is the identity for this and this is the identity for this and it satisfies all these ten axioms. Now, examples of Boolean algebras are, if you take the set 0 1 or true or false if you want and the operation plus instead of plus you have the OR operation for dot you have the AND operation whose complementation is the complementation not operation. Then you have Boolean algebra. So with the set 0 1 that is true or false OR operation, AND operation, not operation and 0 is the identity for OR operation and 1 is the identity for multiplication operation. This is a binary operation, this is a binary operation, this is a unary operation and two constants are there so this forms a Boolean algebra. You can very easily see that all the ten axioms are satisfied. So instead of this plus if you put OR and here you should put AND then you know that the commutative laws hold and the associate laws hold.

One is distribute with respect to the other then a or 0 will be a, and a and 1 will be a, a or a bar is equal to 1 this is a tautology and p or 0p is 1 and p and 0p is equal to 0 which is an absurdity, we have seen these things earlier. So this is an example of a Boolean algebra. Another example will be if you take the power set of a set a, a is some set and you take the power set of set a, union operation, intersection operation, complementation operation, and then what is the identity for addition? Something union 5 will be z set so empty set and the set a.

Therefore, if you take this again you will see that there is a carrier, there are two binary operations, one unary operation, two constants where one is the empty set and the other is the whole set a itself then you find that all these ten axioms will be satisfied. You have the commutative law, here instead of plus you must have union, instead of multiplication you must have intersection and complementation, complementation with respect to the set a then you will see that these axioms are all satisfied.

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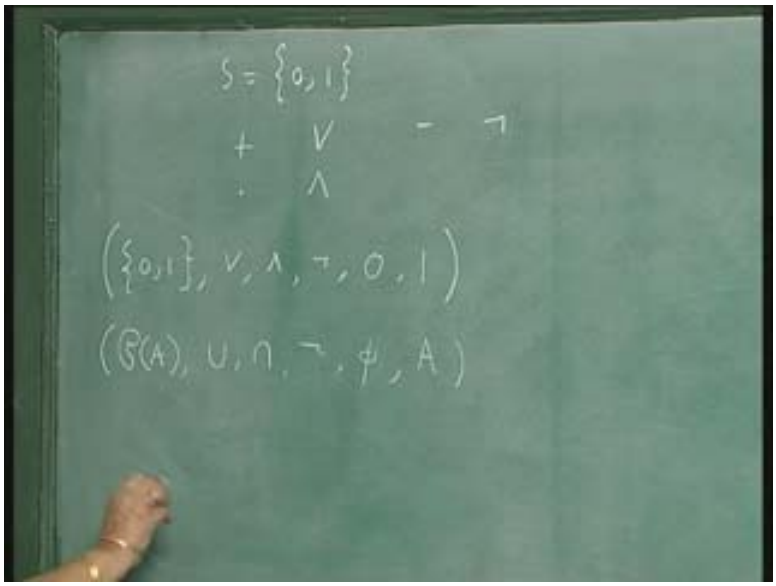

Let us consider one more example. Take the number 30 and all divisors of 30 the set will be 1, 2, 3, 5, 6, 10, 15 and then 30 consider this set. And you have to define two binary operations and one unary operation. The binary operation plus will be l.c.m instead of that least common multiple, a plus b would mean l.c.m (a, b) and dot is multiplication that is gcd greatest common divisor so a dot b will be gcd (a, b). And the unary operation dot or complementation you take as some x bar is 30 by x if you define like this.

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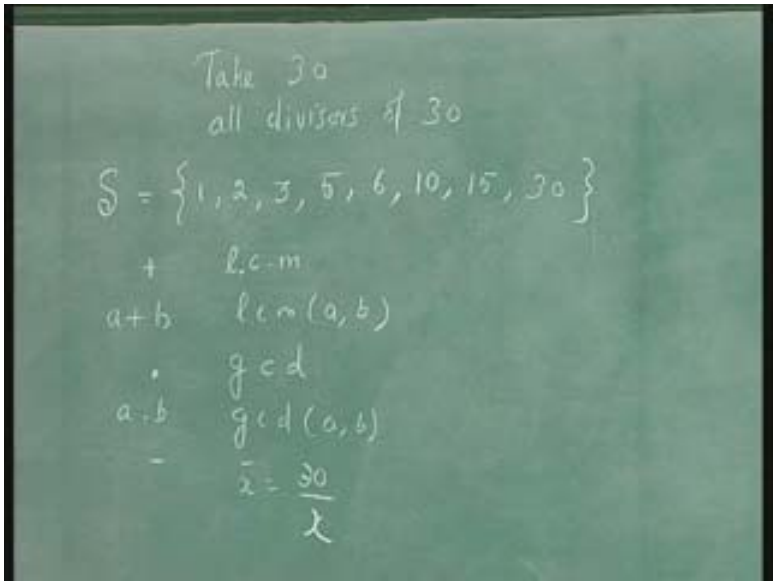

Then I denote this set by S then this set S along with the operation l.c.m, gcd dividing 30 by x and that division I say div and the two constants are 1 and 30, now, if you take like this, this again forms a Boolean algebra. You have to check all the ten axioms. If you take the l.c.m (a, b) this is the same as l.c.m (b, a) so commutative laws can be verified. Similarly, associative law, distributive law and if this is l.c.m a and 1 will give you a, the l.c.m of some number 0 is here 1 1 represents the identity for addition and so on. So with respect to thirty and another number if you take the greatest common divisor that should be the number itself and so on.

So one by one you can check these ten axioms taking plus to be the l.c.m and multiplication or dot to be the gcd. And this complementation is defined as the number of x the complement of x is 30 by x. If you take like this you see that all these ten axioms are verified so this again forms another example of a Boolean algebra. In the last lecture itself we have seen the definition of a semi group, a monoid and a group. Let us recall those definitions again because we know what is meant by associativity.

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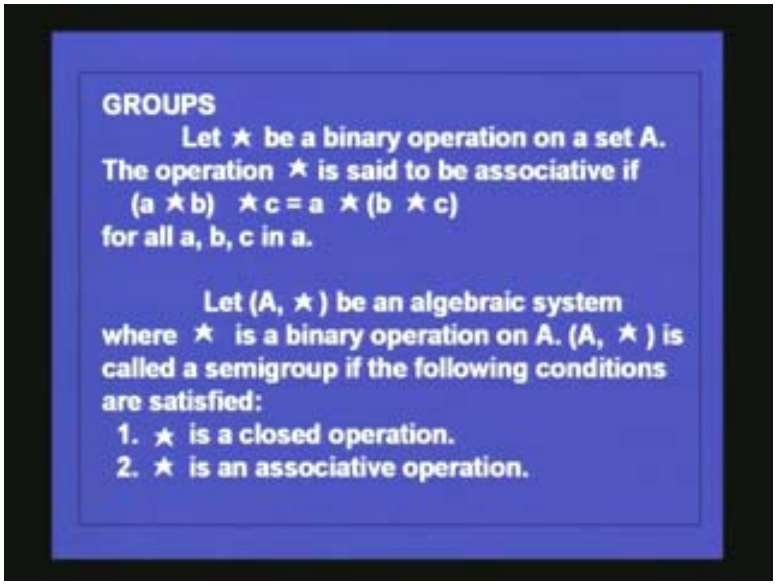

An operation star is associative if a star b star c is equal to a star b star c. That is, you perform the operation on a b first and then with the result you perform c that is equivalent to performing the operation on b and c first and then perform the operation on a with the result obtained. Now let A star be an algebraic system, A is an underlying set and star is a binary operation on A. A star is called a semi group if the following two conditions are satisfied, star is a closed operation, that is, if you take two elements of A and perform the operation star the result should also be an element of A. Second is star is an associative operation. If these two conditions are satisfied the algebra is called a semi group, we have already seen examples of semi group.

If you take a set of integers I and addition operation then you find that if you take two integers and perform the operation addition then you get another integer so it is closed and addition is associative so the two conditions are satisfied so this is an example of a semi group. Similarly, if you take an alphabet sigma and set of all strings over sigma and concatenation of strings as the operator then you find that if you take two strings and perform the operation of concatenation the result you get is again a string so it is closed with respect to concatenation and concatenation is associative. So this again is an example of a semi group. Next let us see what is meant by a monoid. Let A star be an algebraic system where star is a binary operation on A. A star is called a monoid if the following conditions are satisfied, star is a closed operation, star is an associative operation and there is an identity element.

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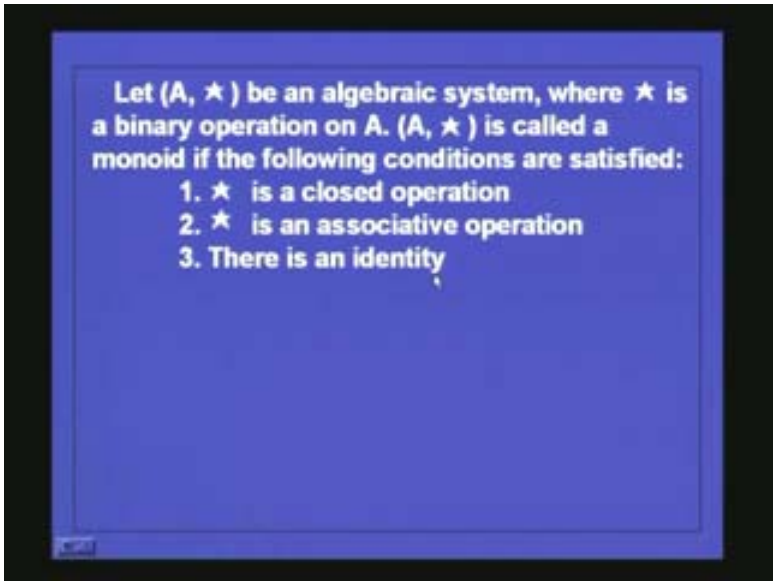

So i plus if you take the identity element as 0, 0 is an identity with respect to plus and sigma star concatenation and a empty string or lambda sometimes it is denoted by epsilon and sometimes by lambda, this is a monoid because we have already seen that it is closed and the associative property holds and this is an empty string and you know that x into epsilon is equal to epsilon into x is equal to x for any string. This is an identity element. Similarly, if you add 0 to any element you get same element so 0 is an identity element for that.

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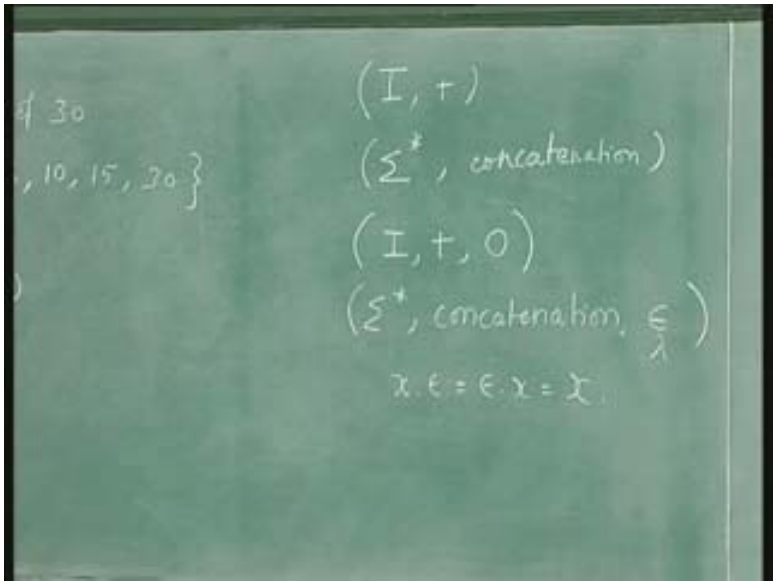

So these two are examples of monoid as well as semi groups and the other condition also holds so they are also monoids. Now what is a group? For group one more condition holds because

there is a unary operation also called the inverse. Let A star be an algebraic system where star is a binary operation. A star is called a group if the following conditions are satisfied, star is a closed operation, star is an associative operation, there is an identity and every element in A has a left inverse. We have seen that if an element has a left inverse and a right inverse they are equal. So actually in this case every element A has a left inverse that left inverse because of associative will also be a right inverse and a unique inverse so four conditions hold.

For example, if you take I plus then the unary minus that is inverse then 0 as the identity then the closure property holds because if you take two integers and perform the operation addition the result will be an integer. And if you take associative property addition is associative and for x plus 0 is equal to 0 plus x is equal to x so 0 is the identity element and for x the inverse is (minus x). So you get x minus x is equal to 0 identity element. Actually this is x plus minus x is equal to 0.

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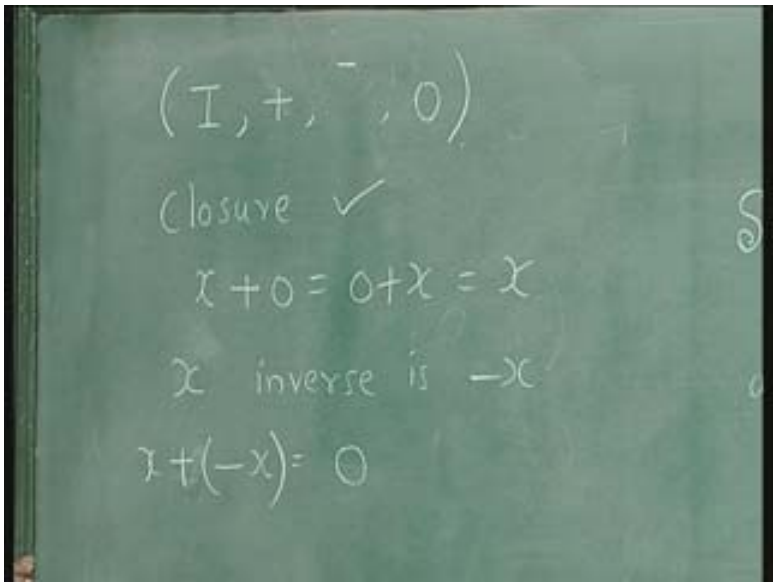

Now, if you have a left inverse that is for x the left inverse is this, this is the identity element denoted as e for this so x inverse x is equal to e the identity element left inverse exist, now if this also has a right inverse you will see that the left inverse and the right inverse are the same. We have seen this but once again we can see this. Suppose an element x has w as the left inverse and e is a identity element this is some operation identity element w is left inverse and y is right inverse then you have inverse of x this is of x so you have x, this operator y is equal to e so from this what do you get? You get w is equal to w dot e is equal to (w dot x) dot y and because of associativity you can write it as w dot (x dot y) and from here you can see that that is the identity element so e dot y is equal to y.

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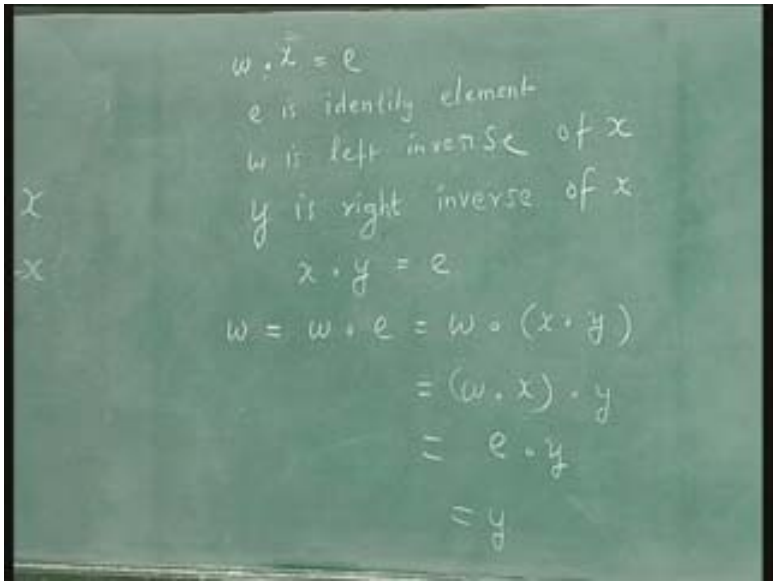

So even though you say the condition that every element A has a left inverse that element will also be a right inverse. And not only that the inverse will be unique. Suppose the inverse is unique, suppose a has two inverses b and another inverse a this is the identity element suppose a has two inverses b and c then we shall see that b is equal to c, why? It is because if b dot a is equal to c dot a they are equal to the identity element multiply on both sides by b so b dot (a dot b) is equal to c dot(a dot b). That is, if you put parenthesis it will be like this. Now, if I multiply by the inverse of this element a dot b is d then b dot d is equal to c dot d, so b dot d multiply both sides on with d inverse c dot d dot d inverse which will be d dot e is equal to c dot e that is b is equal to c.

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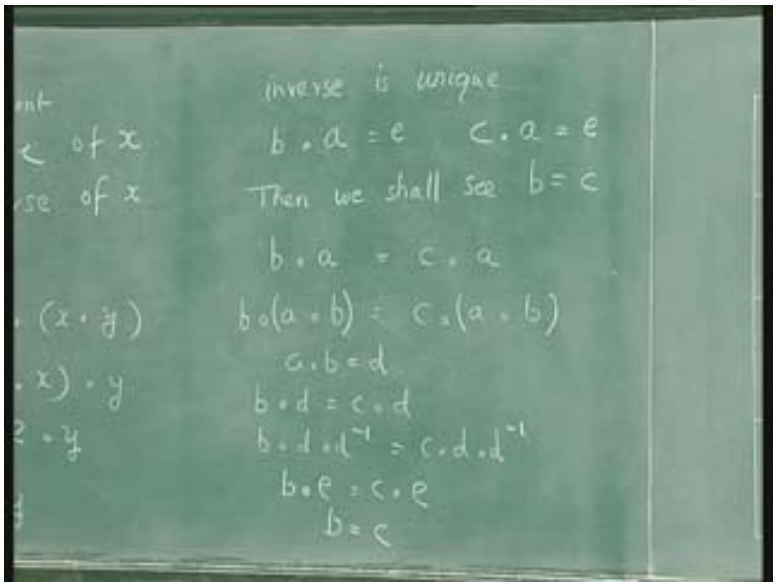

The inverse is unique if it exists; it has to exist in the case of a group. We shall see some more examples of groups. Take the set of even integers and odd integers and addition as the operation then you have the following group. Usually you write the group as a table like this even, odd when you add two even numbers you get an even number, when you add two odd numbers also you get an even number, when you add an even number to an odd number you get an odd number and when you add an odd number to an even number you get an odd number. So this is a group with two elements, this is the identity element because when you add even to even you get even, when you add even to odd also you get odd and so on. So this is the identity element, inverse of even is even and inverse of odd is also odd.

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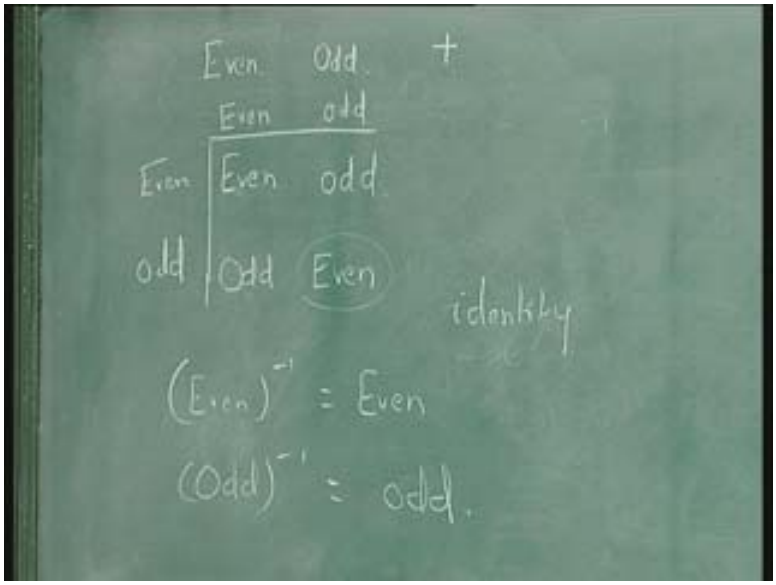

Now because of the existence of inverse left cancellation and right cancellation can take place in a group. Actually that is what we have seen. Suppose if you have b dot a is equal to c dot a then because of the existence of inverse from this you can conclude b is equal to c. The right cancellation can take place, you can cancel of this and say b is equal to c, why? It is because from this b dot a is equal to c dot a so b dot a dot a inverse will be c dot a dot a inverse and this is b dot and a dot a inverse is the identity element e and from this you conclude b is equal to c.

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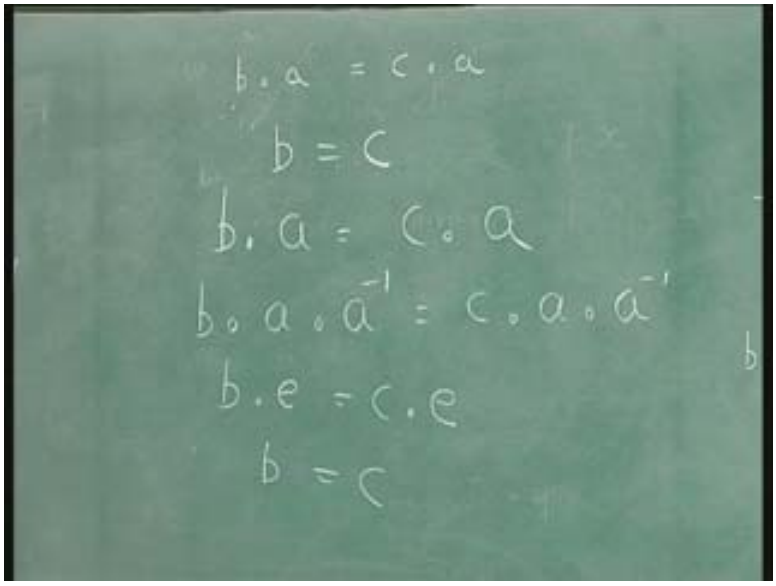

So if you have b dot a is equal to c dot a then by right cancellation you get b is equal to c this is permitted in the case of groups but not in other semi group and other things. For semigroups this cancellation may not hold at all. But you have to be careful while dealing with problems. Similarly, left cancellation can also take place. Suppose a dot b is equal to a dot c then from this you can conclude b is equal to c because if you pre-multiply with a inverse you will find that b becomes equal to c. So left cancellation and right cancellation will hold for groups. Now because of this property if suppose I have group three elements a set with three elements a, b, c then a group table is drawn like this and the operation is like this so you have a b c and a b c. I am just drawing this for three elements but in general it will hold for any number of elements.

Suppose I have a set n with n elements a_1 a_2 an if it is a finite set a_1 a_2 a_n then the group table will be like this a_1 a_2 a_n star a_1 a_2 a_n. So, if this is a_i and this is a_j the ith row and the jth column intersect in this place and here the ijth element has to be again one of these elements a_1 a_2 a_n. Now in a group you will notice that each row is a permutation of these elements. Each element will occur only once it can occur in any order. Each row will be a permutation of these elements and similarly each column will be a permutation this. Two elements cannot be the same in a row, why? Suppose I have the same element b here and same element b here that would mean a_1 dot a_2 is equal to a_1 dot a_j which is not true because of right cancellation it will mean that a_2 will be the same as a_j. So you will find that if you draw the group table each row will be a permutation of the n elements and each column will be a permutation of the n elements and this is because of a left cancellation and right cancellation rules.

Thus, if you look at this set with a b c how does the table look like? The table will look like this, suppose I take a has the identity then because a is the identity element I can fill this like this a dot a is a b dot a will be b c dot a will be c so I can fill this portion of the table. Now what happens here? Suppose I write a here then because each row is a permutation I have to write c here which is not possible in that case c occurs twice in this column which is not correct. So the only way I can fill this table is c and then a here because all the three elements should occur. Similarly, you

will also find that c then here you have to write a. If you write b here you have to write a here then two elements will be the same in one column which is not correct so you have to write only a here and then b here.

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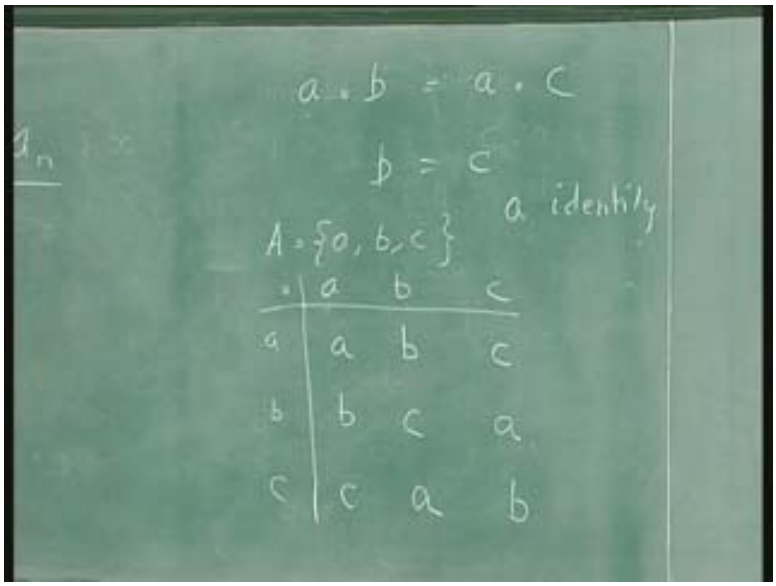

So, this is the table for a group having three elements. Now what about four elements? Of course it is possible to draw this table when the set is a finite set. So when you have a group with an underlying set A and a operation star then the cardinality of A is called the order of the group. It can be finite or infinite in the case of I plus it is infinite order and in the table which we considered now the order of the group is 3 and it represents the number of elements in the underlying set. Now, if you take order four, four elements, the set A consists of a, b, c, d four elements and without loss of generality take a as the identity element, then how can we draw the table? The table has to be something like this a b c d operation a b c d.

Before going into this let us consider the group of order three once more, a is the identity element so inverse of a is a itself thus it is the identity element. Now you see that b dot c is a and c dot b is a so inverse of b is c and inverse of c is b and the other two elements are inverse of each other. Now in the case of group with four elements let a be the identity element so this portion of the table is okay you can very easily write this, then you must see that all the a b c d should occur in each row and in each column and it should also be a permutation of a b c d in each row and in each column.

Now, the other possibilities is b c d you have three elements b c d, one possibility is b is its own inverse, c is its own inverse and d is its own inverse this is one possibility. In that case you have a here b into b is a because b into b inverse is the identity element but because b inverse is b you have this, similarly this is a and this is a. Once you fill these portions the other portions can be very easily filled. The table for this will b you cannot have c here so it has to be d c and similarly you have to have d c here that leads with b here and b here this is one way when you have the inverse of b as b and inverse of c as c and inverse of d as d.

Now the other way to take is take b inverse as d and take c as its own inverse, take d inverse as d that is c is its own inverse and b and d are inverse of each other in which case the table will look like this a b c d a b c d and operator because a is the identity element you can fill these rows very easily. Now c is its own inverse so c dot c will be a but b and d are inverses so b dot d is a and d dot b is a and that leads the other portion to be filled b a you have here you can write c d here, if you write c d here that will work out, if you write d you may get into problem, so if you write c and d here this gives you a permutation of a b c d and here again b c if you write d here this gives you a permutation which also tells you that c dot c is a and then d dot b is a and that is again correct. So you have c d a here so the only thing that is remaining is b and here you cannot have a and you cannot have c here because two elements will be the same so you can only have b so d a b c this is the way you fill the table.

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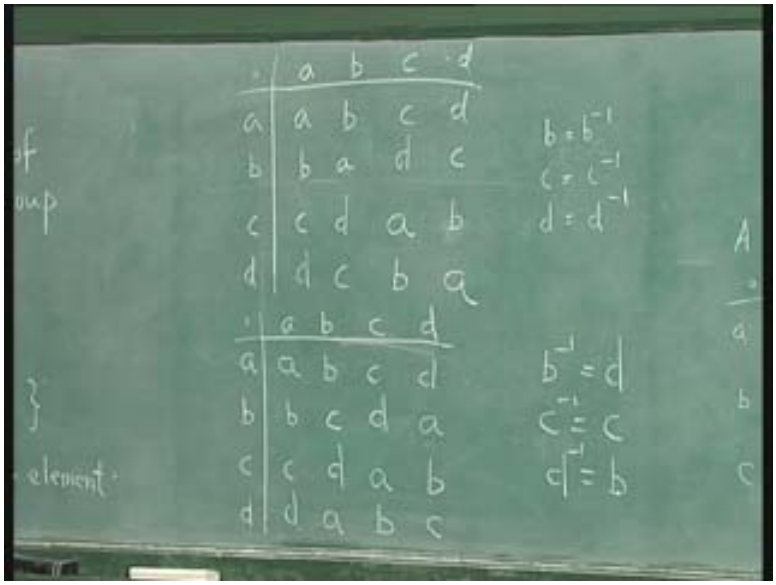

So there are two groups of order four that is I may also have a group like this. For example, I have a set A where A is a b c d and underlying operation is dot and the table is given by this, I can also have another set b with elements alpha, beta, gamma, delta and an operation star. These two groups, this is a group this group is denoted by A dot and this is denoted by B star these two groups B star are isomorphic. If there is a mapping from A to B and if there exist mapping such that this is a bijective mapping when you say isomorphic group both should have the same number of elements and this mapping should be bijective.

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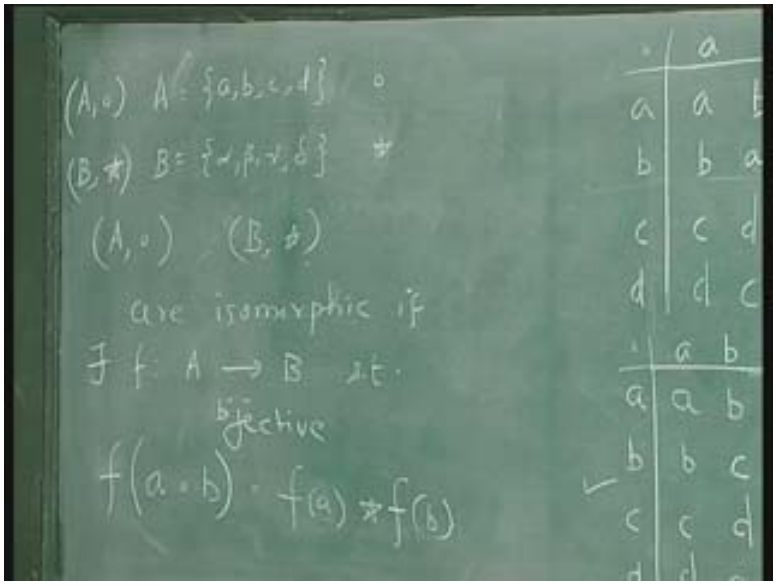

Not only that, suppose I take some a dot b b and find its mapping that should be the same as taking the mapping of a and performing the operation on the second set with f(a) f(b). If this condition is satisfied then you say that the groups are isomorphic. How do I get isomorphic groups? Suppose instead of a b c d I rename it as alpha, beta, gamma, delta then that group is isomorphic to this. And in a group you can interchange the rows and the columns.

For example, suppose b and c are interchanged I can interchange b and c here and then I will get another table but it will represent the same group as this. So, there is no reason why I should write here, for example there is no reason why I should take c is the inverse of c and b and d are inverse of each other. I could have taken b as its own inverse and c and d inverse of each other, then the table will change a little bit but essentially it is the same, essentially it is an isomorphic group. So these are some of the concepts about groups we have seen. We shall see some more examples and a result. Another example of a group is this; take rotations of the plane about an axis perpendicular to it.

Take rotation through 0 degree, 60 degrees, 120 degrees, 180 degrees, 240 degrees and 300 degrees take rotations of the plane about an axis perpendicular to it through an angle either 0, 60, 120, 180 and so on. This is the underlying set A, rotations through 0, 60, 120, 180, 240, 300 degrees is the underlying set. What is the operation? The operation is x star y would mean rotate through angle x and then rotate through angle y. Therefore, suppose I take 60 degrees and then 180 degrees first I rotate through 60 degrees then 180 degrees degrees which will result in a rotation of 240 degrees. Now you can very easily see that this star is a closed operation.

You take any angle rotate through that and take any other angle and rotate through that then the result will be rotation through angle. For example, if you take 240 and 300 that would mean 240 plus 300 is 540, then you have to subtract 360 degrees. Once you perform a rotation through 360 degrees you come back to the same position so 240 plus 300 will give you 540 that means it is essentially a rotation through 180 degrees.

Hence, if you perform a rotation through one of these angles followed by another angle the resultant will again be a rotation through one of these angles so this operation is closed. And you can see that very easily that it is associative operation, if you rotate through x and then through y and then through z that is equivalent to saying first you perform the operation corresponding to y and z and then perform the operation corresponding to z that will also be the same. So the associative property will hold here. In a sense x star y star z would represent rotation through an angle represented by x plus y plus z and if it exceeds 360 degrees you have to subtract 360 degrees so the associative property holds.

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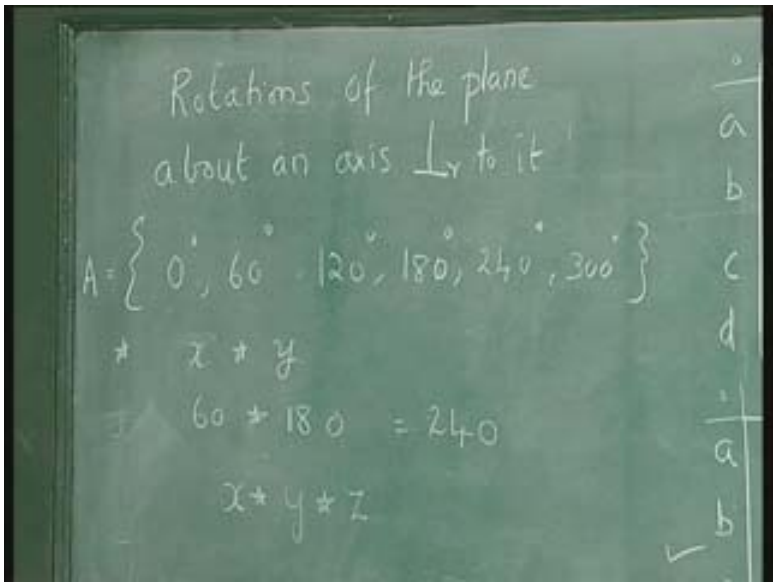

What is the identity element?
This is the identity element so whenever you rotate through 0 no change takes place. So if you rotate through 0 and then 60 it will be essentially 60. If you rotate through 60 degrees and then 0 degrees that is essentially rotating through 60 degrees only so this is the identity element. And what is the inverse of each of these elements? Here 0 is its own inverse, for 60 degrees the inverse is 300 degrees because if you rotate through 60 and 300 degrees you get back to the original position because you have rotated through 360 degrees.

Similarly, for 120 the inverse is 240 because they add up to 360 and for 180 the inverse is 180 itself. So 180 plus 180 is 360 and it gives you an example. Another example of a group would be this; take the set of numbers Z_n it is 0, 1, to n minus 1 and the operation is mod in addition that is a plus b is equal to a plus b if a plus b is less than n that is equal to a plus b minus n. If a plus b is greater than or is equal to n. In that case you see that all the properties hold. Again if you perform the addition you will get a number which is between 0 and n minus 1 because if it is greater than n you are subtracting n from that so it is closed and you can very easily check the associative property.

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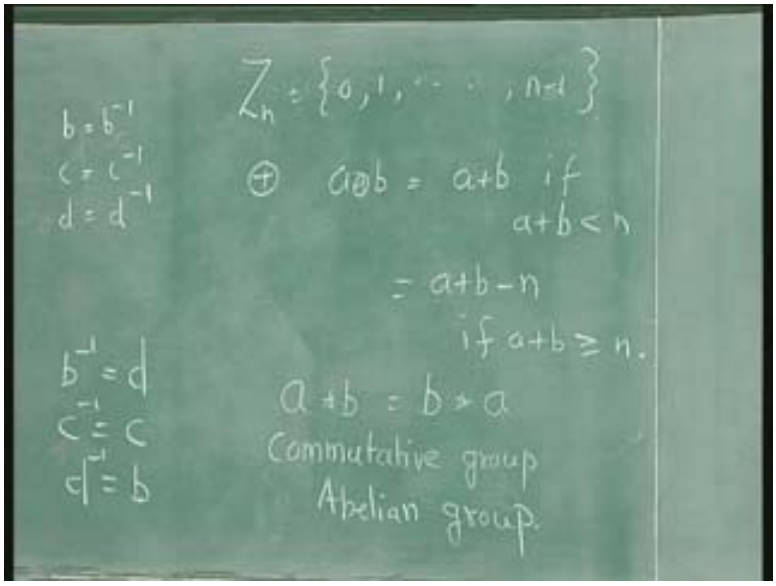

The inverse of 1 will be n minus 1, the inverse of 2 will be n minus 2 and so on so inverse element exist and 0 is the identity element so this again forms a group, this is another example of a group. Now in all these examples whenever you have a star b you also have b star a this is the commutative property where I am denoting the operation by star. Then if this commutative property holds the group is called a commutative group or Abelian group. In all these examples it so happens that all of them are commutative groups, whatever you have considered so far mainly they happen to be commutative groups. For example, take this rotation if you rotate through 60 and 120 it is the same as rotating through 120 and then 60.

Can you think of an example of a group which is not commutative?
In the last lecture towards the end we considered n by n singular matrices. Now I will consider 2 by 2 non singular matrices. Consider all 2 by 2 non singular matrices that forms the set A then the operation star is matrix multiplication. So the product of 2 by 2 non singular matrices is again another 2 by 2 non singular matrix so it is closed under this operation and you know that matrix multiplication is associative so associative property holds. For A the 2 by 2matrix I is the identity element because this is the operation where I is this identity matrix. And the inverse of the matrix A is the inverse matrix of A. A inverse is the inverse matrix of A. Now with this property it forms a group. But you know that matrix multiplication is not commutative, A B is not equal to B A in general in matrix multiplication so this is an example of non commutative group.

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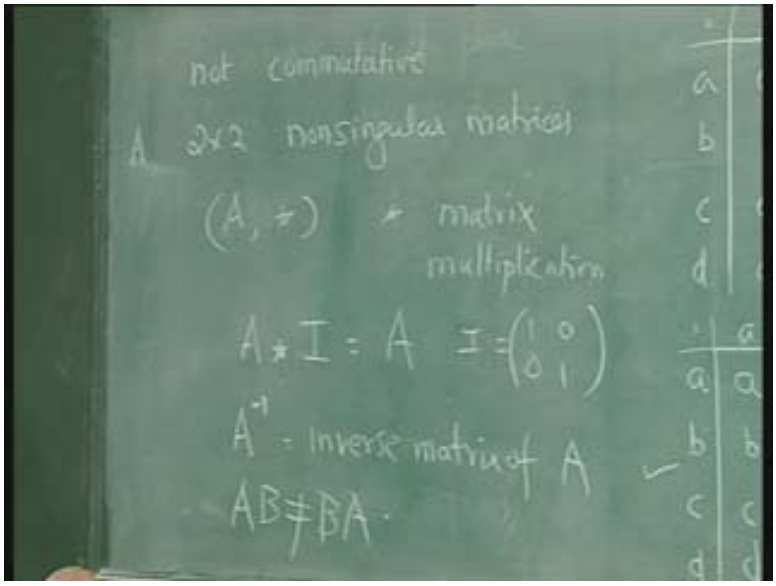

Now having considered groups let us consider subgroups and see what are meant by subgroups. Let A, star be an algebraic system and B a subset of A. The algebraic system B, star is said to be sub system of A, star, this is like sub algebras.

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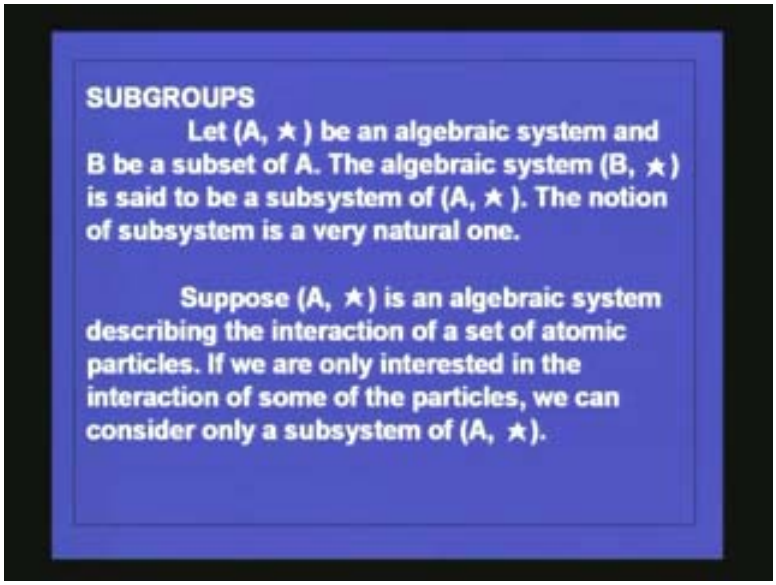

We have used the word sub algebras earlier. The notion of the sub system is a very natural one. Suppose A, star is an algebraic system describing the interaction of a set of atomic particles then if we are only interested in the interaction of some of the particles. We can consider only a subsystem of A star for example in this rotation you can very easily see that we were considering rotations a plane through 0 degrees, 60 degrees, 120 degrees, 180 degrees, 240 degrees and 300

and we saw that it formed a group. you may just consider 0 degrees, 180 degrees this al1 will form another group or you may consider just 0 degrees, 120 degrees and 240 degrees, this also will form. This is a group with two elements having a table like this 0 180 0 180 0 0 180 180 like this. And this will form a table there is only one table with three elements 0 120 240 0 120 240 so 0 is the identity element so you get like this and here you get like this. If you rotate through 120 and again through 120 you get 240, if you rotate through 120 and 240 you get 360 that is this. So 240 120 you get this and 240 240 you get 120, you get 480 minus 360 that is 120.

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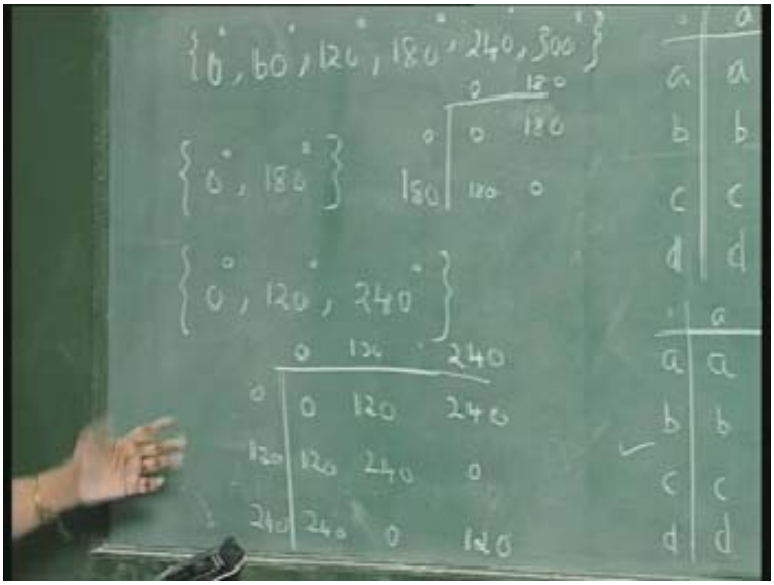

So this is the table you get for this. This is a group and this is a group these are examples of sub groups, this is a subset of this so this is a sub group and this is again a subset of this so it is a sub group. Another example is; Let N, plus be an algebraic system describing the addition of natural numbers. Clearly e, plus is a subsystem of N, plus set of natural numbers with addition operation, the set of even numbers is the addition operation is a subset and that is a sub group because it is closed.

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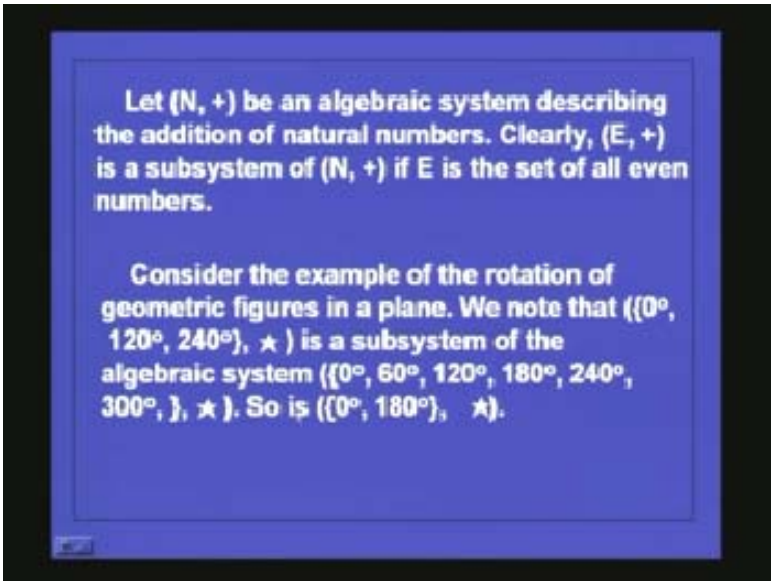

This is what we have considered just now. Let A, star be a group and be a subset of A. B, star is said to be subgroup of A if B, star is also a group by itself.

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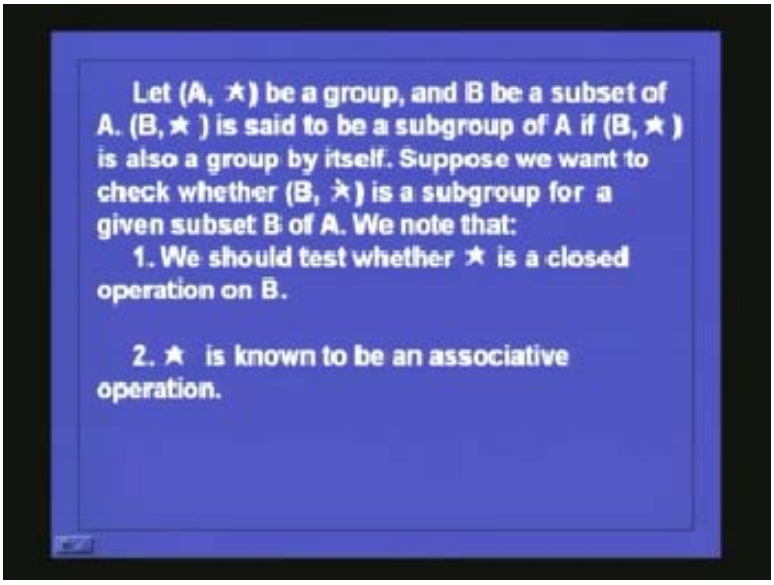

Suppose we want to check whether B, star is a subgroup then what are the things we have to check?
First we have to check whether star is a closed operation on B then associative property we need not check because that already holds for the original group. Then you have to see that the identity element belongs to the subgroup. Since there is only one element identity element e in A such that e star x is equal to x star e is equal to x for all x in A.

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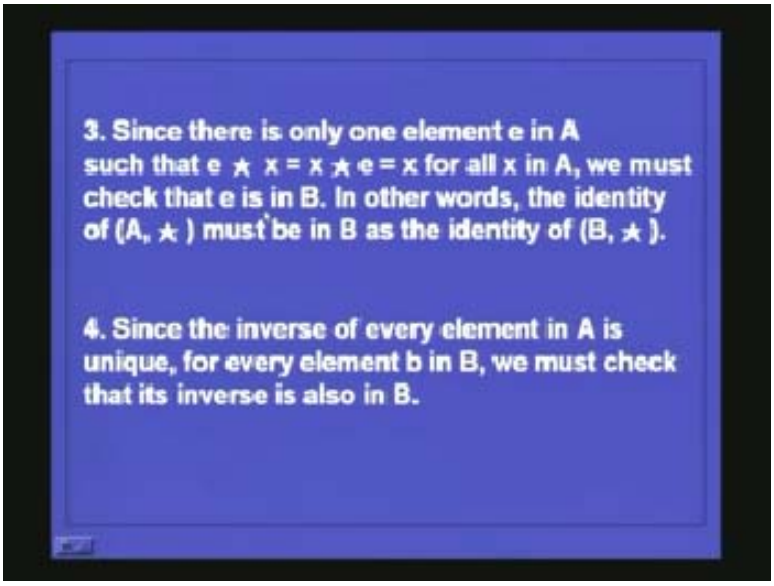

We must check that e is in b otherwise it will not be a sub group. In other words, the identity of A, star must be in B as the identity of B, star. In this case of rotation we have seen that 0 is the identity element here it is also there it is also here. Since the inverse of every element of A is unique for every element B in B we must check that its inverse is also in B. In this example you can see that 0 is its own inverse 180 is also its inverse. In the case of elements with 0 120 240 120 and 240 degrees rotation they are the inverse of each other, 0 is its own inverse. So the inverse of this is also present here the inverse of this also present here and that is what we have to check. But really we will realize that we need not have to check all these four conditions. It is enough if we check that the subset which we are considering under the operation star is closed, this is what we have just seen.

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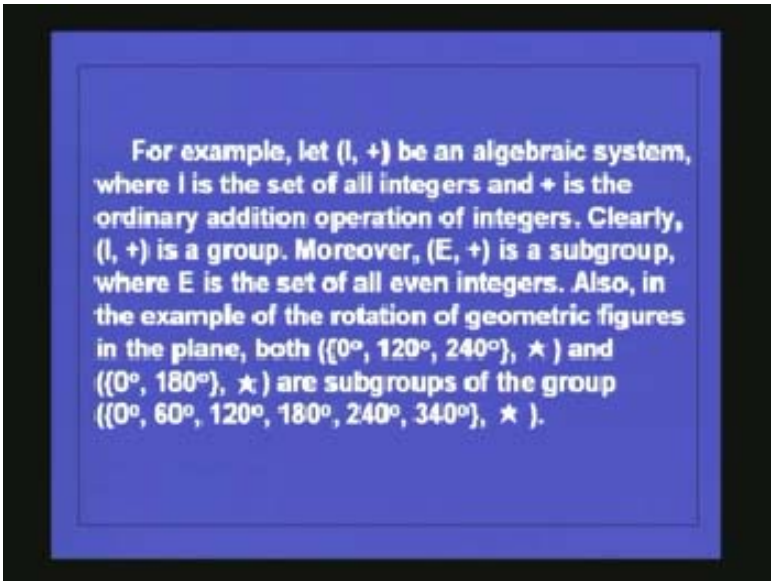

Let A, star be group and B a subset of A. Now if it so happens that B is a finite set the proof will may not hold if it is infinite. If it is a finite set then B, star is a subgroup of A, star if star is a closed operation on B.

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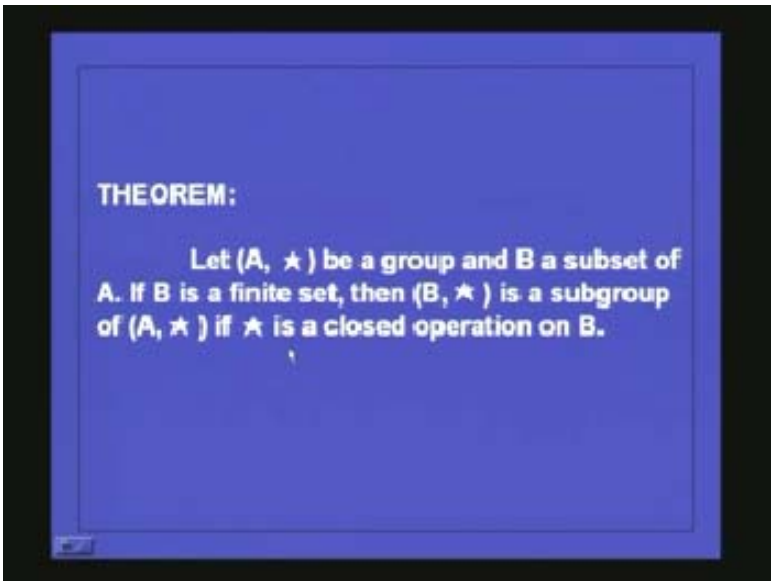

How do you prove this? The proof is very simple, look at the proof like this: What this theorem says is it is enough when you consider a subset whether that subset is closed with respect to that operation. If it is closed then all the other properties will follow, that is the identity will be there, inverse will be there and so on.

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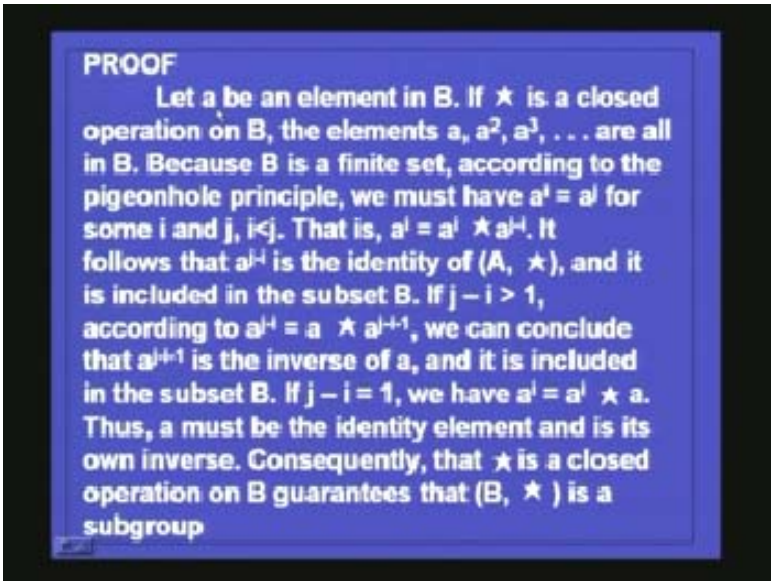

How do you go about the proof?
Let A be an element in B. If star is a closed operation on B then the elements a, a square, a square by a square I mean a star a by a cubed I mean a star a star a they are all in B but B is a finite set. So, according to the pigeonhole principle we must have a power i is equal to a power j for two different i and j and without loss of generality assume i is less than j. that is a power i is equal to a power i, this is a power i this is a power i so what you have is a power i is equal to a power j where j is greater than i then I can write a power i is equal to a power i star a power j minus i. So this gives you the idea that a power j minus i is the identity element.

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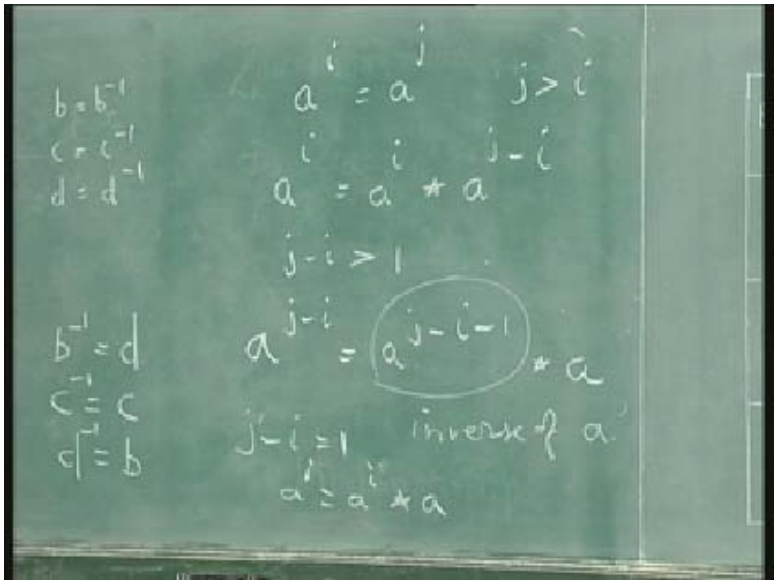

Hence, this shows that a power j minus a i is the identity of this and it is included in the subset of B. If j minus j minus i this is again j minus i is greater than 1 if j minus i is greater than 1 then what happens?
a power j minus i is equal to a, a power j minus i is equal to a power j minus i minus 1 into a so this is the inverse of a. If j minus i is equal to 1 then you have a power i is equal to a power i star a so a is the identity element and so its own inverse. So if j a minus i is greater than 1 the inverse of a is this and that belongs to a subset and if j minus i is equal to 1 you have a power i is equal to a power i star a that shows that a is the identity element and the identity element is its own inverse so that also belongs to the group.

So you realize that if you want to check whether something is a sub group then it is not necessary to check all the four properties, it is enough if you check the first property alone. Hence, suppose I have a set A with a set star the binary operation star and B with the same operation where B is the subset of A now if B is finite then it is enough if you check the closure property alone. That is, check the closure property that is take any two elements and perform the operation star and the result should also belong to B.

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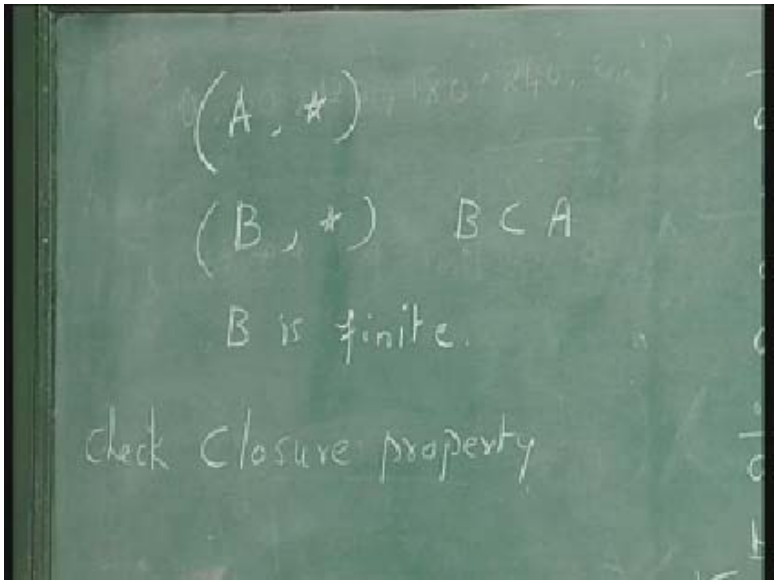

If it is closed then this will become a sub tree automatically because of the properties and the other things like associatively which will automatically hold and therefore the existence of an identity element, inverse element etc will also automatically hold. Thus we have seen what a group is and what a sub group is and how to check for a sub group and so on. Hence, there are some more results about groups and also we shall see the definition of a ring, integral domain, field, etc and we shall consider a few more examples of groups also in the next lecture.