Discrete Mathematical Structures
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Lecture # 32
Recurrence Relations

In the last few lectures we considered permutations and combinations. We also considered generating functions and also looked at distribution of distinct and nondistinct objects into distinct and nondistinct cells which also included partition of integers. These are some of the topics which were covered in the last few lectures. Today we shall consider recurrence relations and see how to solve those recurrence relations, what is the use and let us also see how to use generating functions for solving with recurrence relations.

What are recurrence relations?
Suppose you want to ask a person his age or her age whatever it is then you do not feel like asking directly or he may not be willing to tell his age directly but he may say that he is 5 years younger to his brother and his brother is 45 years old, that means he will he his 40 years old, you can calculate his age from his brother's age. So instead of directly telling he can indirectly say that. or suppose you want to ask a person a route to go to the post office you ask a person how to go to the post office, he may say go a few yards from here then turn left then five ten yards then turn left then hundred yards turn right then two hundred yards turn left and like that he can give instructions which will be difficult for you to really memorize and then go on keeping the route in mind to reach the post office.
Whereas he may give the route like this from a particular shop you have to go right and turn left then you will reach the post office.

Then how do you go to the particular branded shop?
From a particular school you move north and then east you will go to that particular shop.
Then how do you reach that school?
From here you go hundred yards towards the east and then turn left or something like that. So step by step he can give the direction how to reach the post office instead of telling go right left right etc he can say that from the branded shop you have to go like this to reach the post office and to reach that particular shop you have to go from the school in this direction and to reach the school you have to go from here like this. So something like that he can explain. That is the underlying basic consideration for the structure of a recurrence relation. They are very useful we shall consider one or two examples.

Let us consider a very simple example of a recurrence relation. You know what is known by tower of Hanoi problem. That is, you have three pecks A B and C and on one of the pegs you have n circles n blocks the lower one is bigger then the next one they are arranged in the decreasing order of their width and circular objects are placed on this peg. Now you want to transfer all this circular disks to B in the same order ultimately you should have these circular disks in the same order. And for transferring these I have drawn 4, in general there can be n disks. You have to transfer them from A to B and at the end you should have them in the same order here. But while doing that we can use peg C in the intermediate steps but you have to do it in such a way that at no time a smaller disk is below the bigger disk. That is, I cannot say this if I number them as 1 2 3 4 I cannot transfer 1 to C then 2 to C then that means 1 will be below 2 and 2 is bigger than 1 that cannot happen. So at all times the smaller 1 should be above the bigger 1. Can you transfer like that from A to B and if so in how many number of steps?

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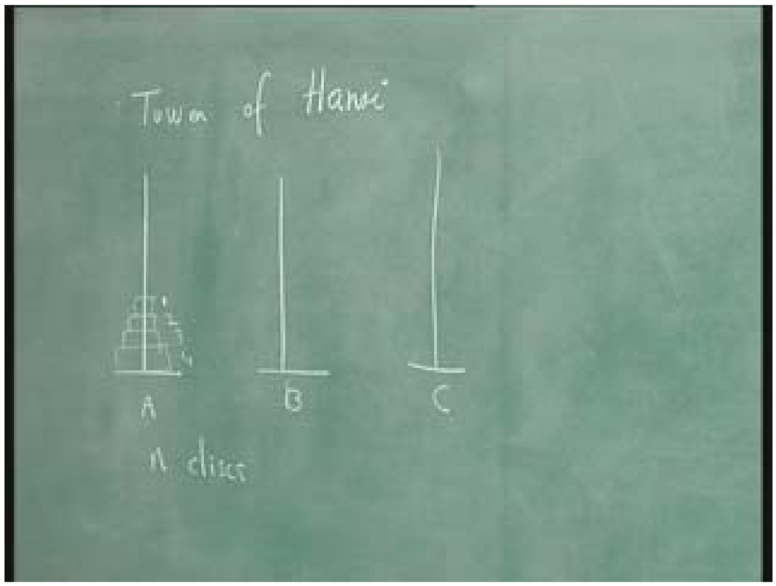

If you have n disks in how many number of transfers you would require to do that?
For example; if I have only 1 it will take just 1 step, if you have only 1 obviously you can transfer like this. If we have two the first 1 we can transfer to C then the second 1 to B then the first 1 from C to B, in 3 steps we can do this transfer the two discs from A to B. In general if you have n disks what is the number?
Suppose I denote number as (a, n) what I have to do is I have n disks like this smaller n disks so first I transfer n minus 1 of them to C in this order the same order smaller 1 upper and then bigger 1 lower Using B as a intermediate peg, in that case how many steps it will take.

If I denote this by a n transferring n disk I denote by a, n transferring n minus disk is (a_n minus 1) so what do you do from A you transfer (n minus 1) disk in that order to C using B as a intermediate peg which will take a (n minus 1) steps then the last disk you can transfer from A to B which is 1 step and then after transferring the last strip here last disk here these (n minus 1) disk again you can transfer from C to B using A as the intermediary peg again that can be d1 in a n minus steps because we denote by a n the number of steps required or number of transfers required to transfer from 1 peg to another in the same order such that at no time a bigger disk less on the smaller disk.

So you can see that the number of steps required to transfer n disk will be twice the number of steps required to transfer (n minus 1) disk plus 1 so this is called arecurrence relation, this type of an equation is called recurrence relation. And the number of steps is really given by 2 power minus 1 which you can see. So once you have obtained this relation how do you solve this? There are two methods 1 is an ordinary method and the other is by the use of generating functions.

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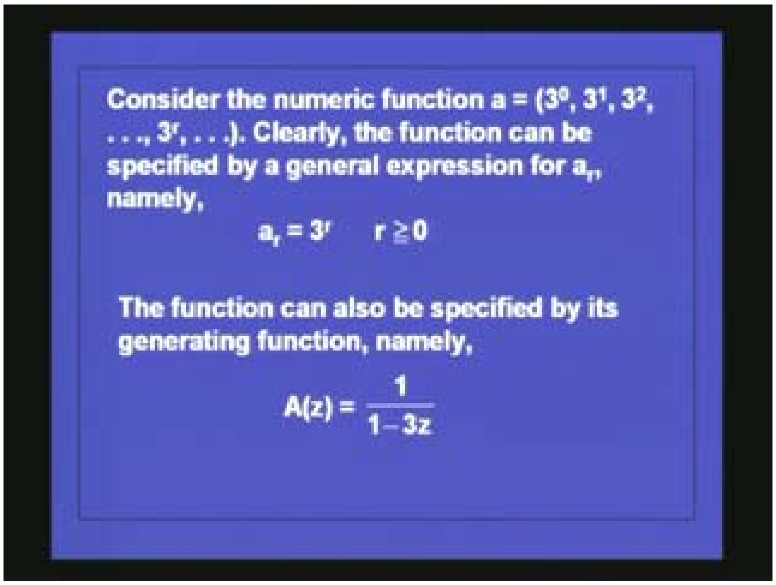

So as an example let us look at this; consider the numeric function a is equal to 3 power 0, 3 power 1, 3 power 2, 3 power r clearly this sequence is a function and ar is given by 3 power r. The function can be specified by a general expression for ar namely ar is equal to 3 power r or greater than or equal to 0. Now we will see later how to solve such equations by generating functions. This function can also be specified by its generating function namely A z is equal to 1 by 1 minus 3z. If you expand this it will be 1 plus 3z plus 3 square z square etc. So the coefficient of z power r general term will be 3 power r z power r so the coefficient of z power r will be 3 power r and that is what is given by this. So the generating function for this is 1 by 1 minus 3z.

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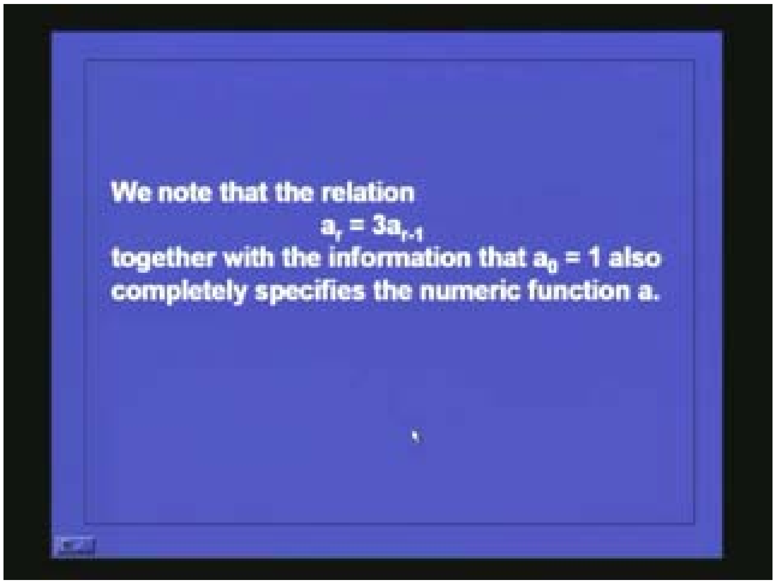

And the recurrence relation is a_r is 3 times a_r minus it is a geometric progression really. But you can very easily see that a_r is equal to 3 into $a_{r \text{ minus } 1}$. So either way you can specify it as a_r is equal to 3 power r or A z the generating function is this or a_r is equal to 3 $a_{r \text{ minus } 1}$ with that information that a_0 is equal to 1. To start with the first point is a_0 is equal to 1 this is known as the boundary condition. Also this completely specifies the numeric function a.

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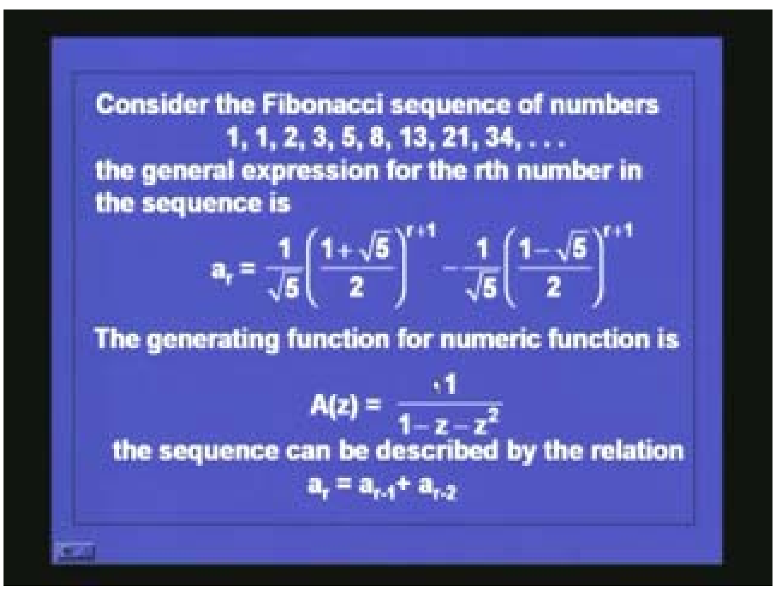

You also know this example which is very common. The Fibonacci sequence of numbers is given by 1 1 2 3 5 8 13 21 etc. The first two are 1 and 1 the third 1 is obtained by adding the first two the 4th 1 is obtained by adding these two 1 is equal to 2 is equal to 3, 2 plus 3 is equal to 5, 3 plus 5 is equal to 8, 5 plus 8 is equal to 13 and so on. This is the way you get the Fibonacci sequence. And it is given by this expression actually a_r is equal to $a_{r \text{ minus } 1}$ plus $a_{r \text{ minus } 2}$ with a_0 is 1 boundary condition a_0 is 1 and a_1 is 1. So a_2 will be 2, a_3 will be 3, a_4 will be 5 and so on. But a closed form expression for this is given by this a_r is equal to 1 by root 5, 1 plus root 5 by 2 to the power of r plus 1minus 1 by root 5 etc.

Just check, put r is equal to 0 the Fibonacci sequence in that expression put r is equal to 0 then a_0 will be starting from 1 r is greater than or equal to 1 put r is equal to 1 then what is a_1? a_1 will be 1 by root 5 into 1 by 2 into 1 plus root 5 minus 1 by root 5 into 1 minus root 5 by 2. So first r is equal to 0 we have to consider, r is equal to 0 a_0 will be 1 by root 5 into 1 plus root 5 by 2 minus 1 by root 5 1 minus root 5 by 2 because you put r is equal to 0 there in that expression. This will be 1 by root 5 into root 5 by 2 plus root 5 by 2 is equal to 1 a_0 is 1 you can see that. What about a_1 r is equal to 1 this is the whole square and this is a whole square, a_1 is given by 1 by root 5 this and this, let us simplify and see this.

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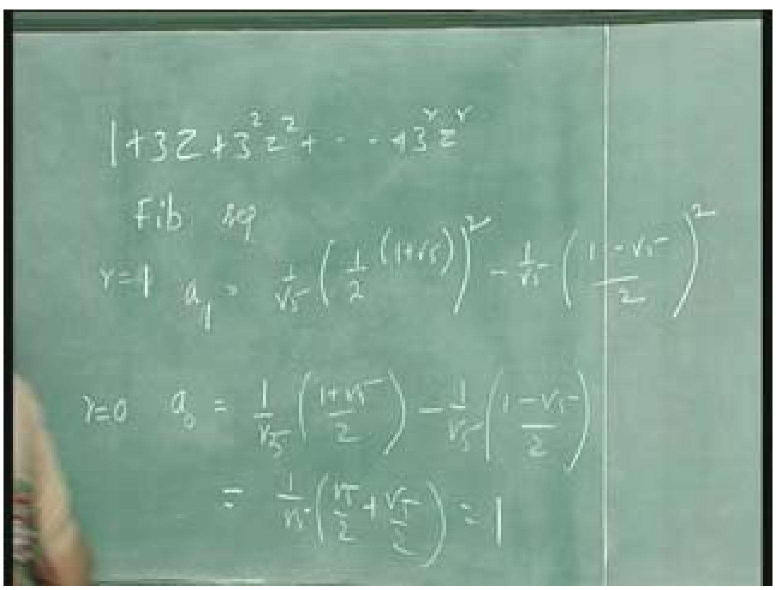

A_1 will be 1 by root 5 into 1 plus root 5 by 2 the whole square minus 1 by root 5 1 minus root 5 by 2 the whole square which will be 1 by root 5 you can take out you can also take 4 out then it will be 1 plus plus 2 root 5 plus root 5 the whole square minus 1 plus 2 root 5 minus root 5 the whole square expanding. So this will cancel with this, this will cancel with this, this will give you 1 by 4 root 5 is equal to 1. So you can check that a_0 is 1 and a_1 is 1 and similarly you can check that a_2 is given by 2 and a_3 is 3 and so on.

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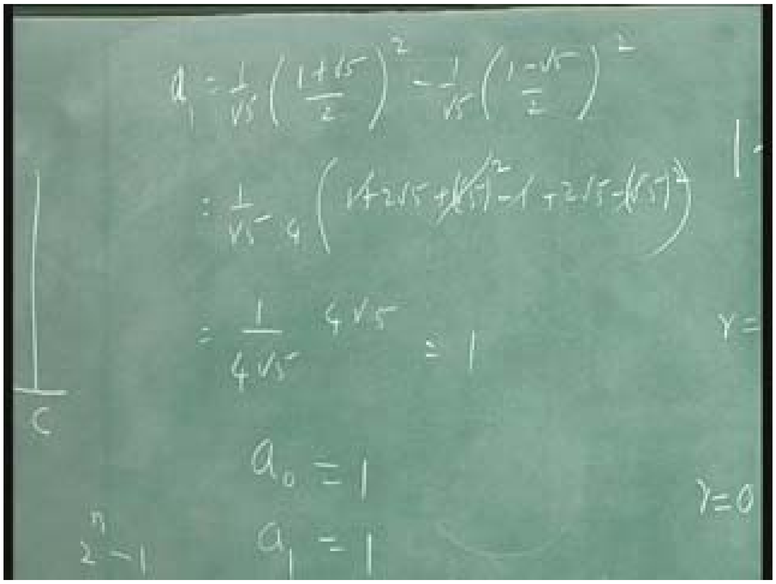

The generating function for that is A z is equal to 1 by 1 minus z minus z square. So, this Fibonacci sequence can be specified in 3 different ways this is the sequence a_r can be given by this closed form expression or you can specify it by the difference equation the generating function like this. The sequence can also be described by this relation a_r is equal to $a_{r \text{ minus } 1}$ plus $a_{r \text{ minus } 2}$ with the condition that boundary conditions are a_0 is equal to 1 and a_1 is equal to 1. And this together with the boundary condition will specify the equation in a unique manner. This is called the recurrence relation for the Fibonacci sequence. Now, when you have a recurrence relation like that how do you solve it? Given the boundary conditions a_0 a_1 etc you can calculate a_2 then a_3 and so on that way you can calculate or you can also find a closed form expression.

Now, how do you find the closed form of generating expression or a simple expression like this in the general case and that is what we want to study in this lecture. Generally, recurrence relation need not be linear but we will concentrate on linear recurrence relation with constant coefficients. So the example of tower of Hanoi and this Fibonacci sequence are examples of linear recurrence relations with constant coefficients. A recurrence relation of the form C_0a_r plus $C_1a_{r \text{ minus } 1}$ plus C_2 $a_{r \text{ minus } 2}$ etc is equal to f(r) is a function of r where C i's are all constants is called a linear recurrence relation with constant coefficients. This form is called a linear recurrence relation with constant coefficients.

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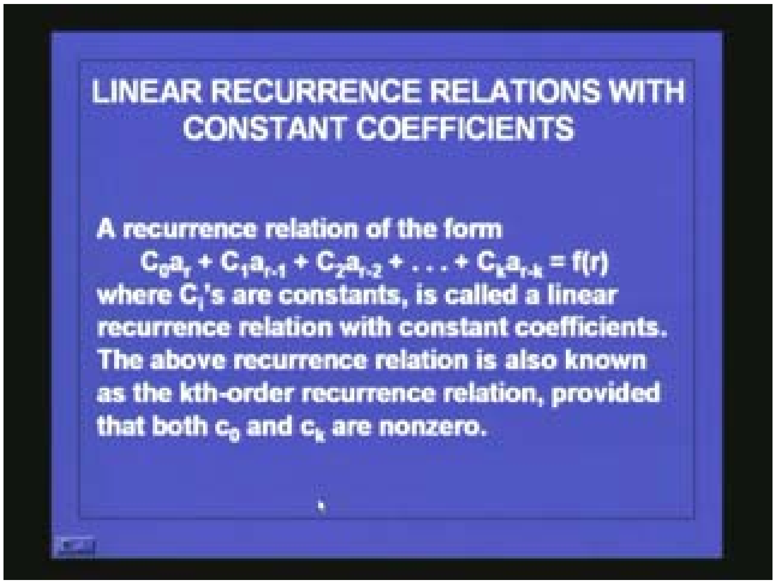

The above recurrence relation is also known as the kth order recurrence relation provided both C_0 and C_k are nonzero. For example, $2a_r$ plus $3a_{r \text{ minus } 1}$ is 2 power r, C_0 a_r plus C_1 $a_{r \text{ minus } 1}$ is equal to 2_r this is a first order linear recurrence relation with constant coefficients. If you have something like this the Fibonacci sequence a_r is equal to $a_{r \text{ minus } 1}$ is equal to $a_{r \text{ minus } 2}$ this is the Fibonacci sequence or you can write a_r minus $a_{r \text{ minus } 1}$ minus $a_{r \text{ minus } 2}$ is equal to 0 C_0 is 1 C_1 is minus 1 C_2 is minus 1 this is a linear recurrence relation of the second order. Second order linear recurrence relation with constant coefficients. If you look at the expression for tower of Hanoi problem it is an is equal to $2a_{n \text{ minus } 1}$ plus 1 you can write it as $a_{n \text{ minus } 2}$ $a_{n \text{ minus } 1}$ is equal to 1 C_0 is 1 C_1 is minus 2 f(r) is 1 and this is linear recurrence relation of the first order with constant coefficients.

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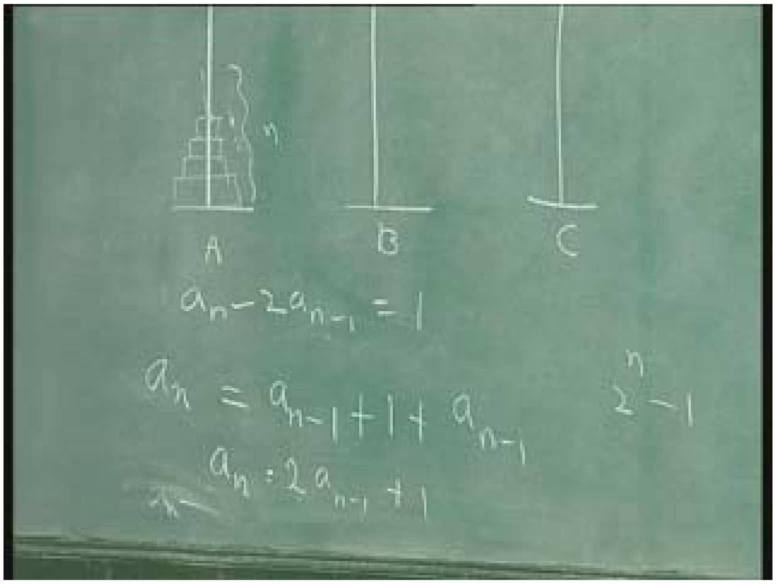

So in general this particular example if you take, this is a first order linear recurrence relation with constant coefficients. Now, when you have kth order linear recurrence relation you can also call this recurrence relation as difference equation, sometimes they are also called difference equations. If you have a kth order linear recurrence relation with constant coefficients of course with constant coefficients it is of the form C_0a_k plus $C_1a_{k \text{ minus } 1}$ etc plus C_ka_0 is equal to f(r). Now, if you have consecutive values like a_i $a_{i \text{ plus } 1}$ etc $a_{i \text{ plus } k \text{ minus } 1}$ k consecutive values of the function this will give you the boundary condition from this you can find out all the other values. Suppose I have the values for a_i $a_{i \text{ plus } 1}$ etc $a_{i \text{ plus } k \text{ minus } 1}$ I can find the value of a_{ik} from this i can find the value of $a_{i \text{ plus } k \text{ plus } 1}$ and so on.

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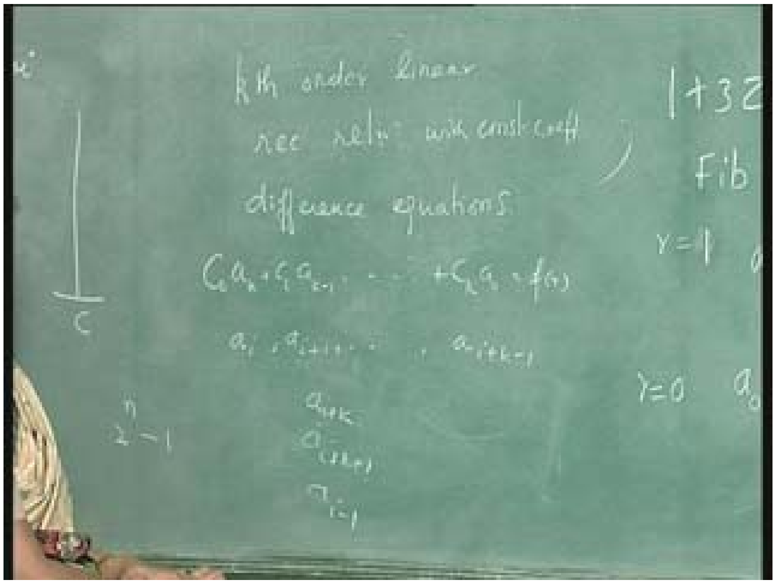

Also from this I can find the value of $a_{i \text{ minus } 1}$ by rewriting the equation in a proper manner and solving it. So, given k consecutive values of the function will give you the unique solution for any value, any value you can find out in a unique manner by rewriting this equation in a proper way and solving that.

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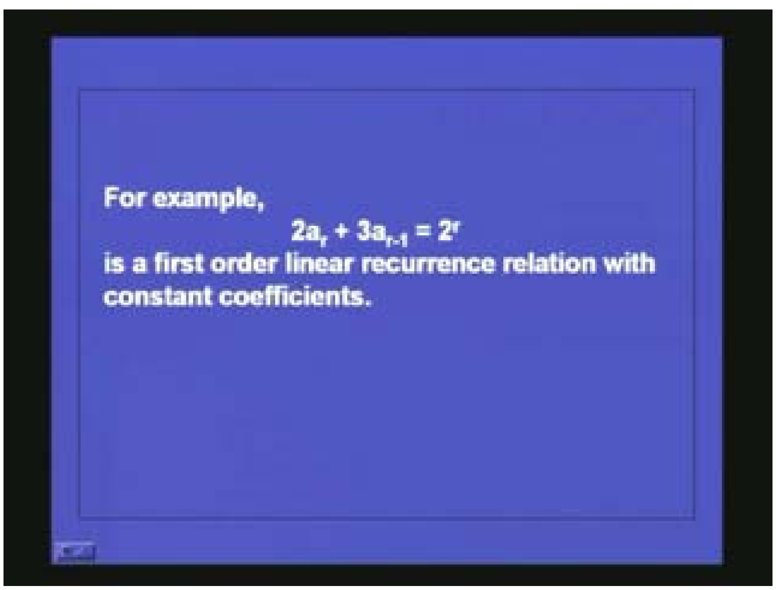

For example; take the Fibonacci sequence, it is a_r is equal to a or I will write it as $a_{r \text{ minus } 1}$ $a_{r \text{ minus } 2}$ is equal to 0, if I know that a_3 is equal to what is the value of a_3 here? Look at the Fibonacci sequence, what is the value of a_3? A_0, a_1 a_2 a_3 is 3 a_4 is 5 I have two values a_3 is 3, a_4 is 5 and from this you can find a_5, a_5 will be from this you can write it as a_3

plus a_4 so that will be 8. You can also find the value of a_2 from this, how to you find a_2? a_2 will be a_4 minus a_3 that will be 2 and a_1 will be a_3 minus a_2 that will be what is a_3 is 3 so it is 3 minus 2 is equal to 1. Similarly, a_0 you will able to find as 1. So by rewriting the equation properly you can get all the values.

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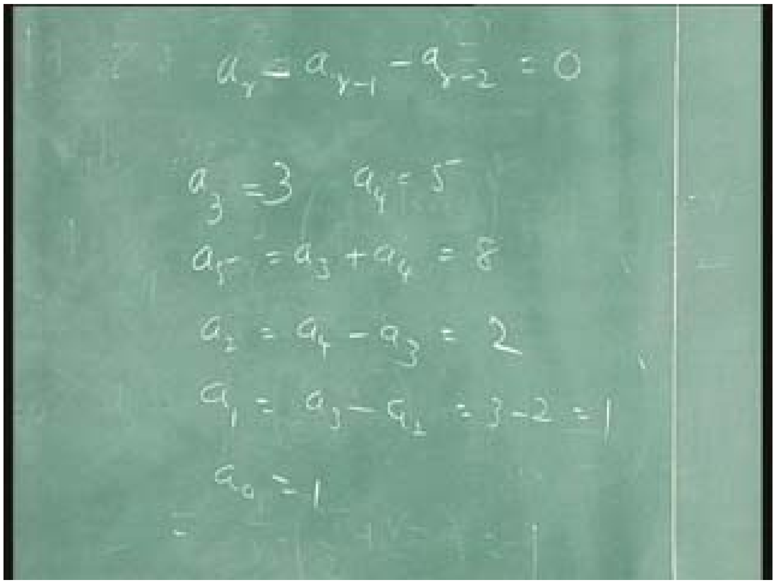

Look at this recurrence relation a_r plus $a_{r \text{ minus } 1}$ plus $a_{r \text{ minus } 2}$ is equal to 4 this is a linear recurrence relation of the second order. If you get two consecutive values like a_i and $a_{i \text{ plus } 1}$ if you have the values you can calculate all the other values in a unique manner. Suppose I am given only one value suppose you are given a_0 is equal to 2 then the sequence cannot be determined uniquely. For example, you can see that 2, 0, 2, 2, 0, 2, 2, 0, 2 this is one sequence. a_0 is 2, a_1 is 0, a_2 is 2 this will satisfy this recurrence relation. Another one is 2, 2, 0, 2, 2, 0 this also will satisfy the recurrence relation, add these three and you will get 4, add these three you will get 4, add these three you will get 4 and so on so the solution will not be unique. Given one value if it is a kth order recurrence relation you must have consecutive k values to get the proper sequence.

If you are given more than one value suppose I am given a_0 is equal to 2, a_1 is equal to 2 then it uniquely represents this sequence. Now, if a_2 is also given it should be given in a proper manner satisfying the equation a_2 should be 0 otherwise if some other value is given means it will not satisfy the recurrence relation this is not the correct thing and this is wrong. If a_2 value also has to be given it has to be given in a proper manner and adding a_1 and a_2 satisfying the recurrence relation. If you are given fewer than that value it is not unique or if you are given more than the require number of values it has to be properly given otherwise it may become inconsistent.

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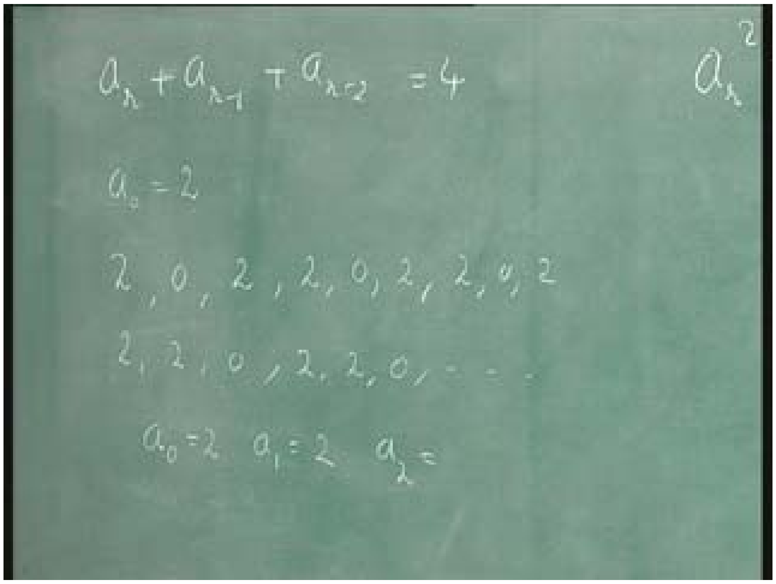

Now, if it is not a recurrence relation linear recurrence relation if it is of the form a_r square plus $a_{r \text{ minus } 1}$ is equal to 5 in that case this is the first order so one value, if it is linear one value should give you a solution uniquely but this is not linear. So giving a_0 a particular value will not give you the solution in a unique manner. Suppose a_0 is equal to 1 then 1 2 but will this satisfy this equation 1, 4 plus 1 is equal to 5 satisfies this equation. The third one can be a_3 squared is equal to $a_{r \text{ minus } 1 \text{ plus } 1}$ so it could be root 3 will this satisfy the equation, this squared 3 plus 2 is equal to 5 so this will satisfy this equation.

You can also see that 1 2 minus root 3 sequence also will satisfy this equation because you are taking the square root here so this square plus this is 5 this square plus this is 5 that is again satisfied so this equation will also satisfy the given recurrence relation. So if it is linear one value should give you the sequence in a unique manner but because it is not linear one value does not give the sequence in a unique manner. But we are concerned only with the linear recurrence relation with constant coefficients so this sort of a thing does not bother us anyway.

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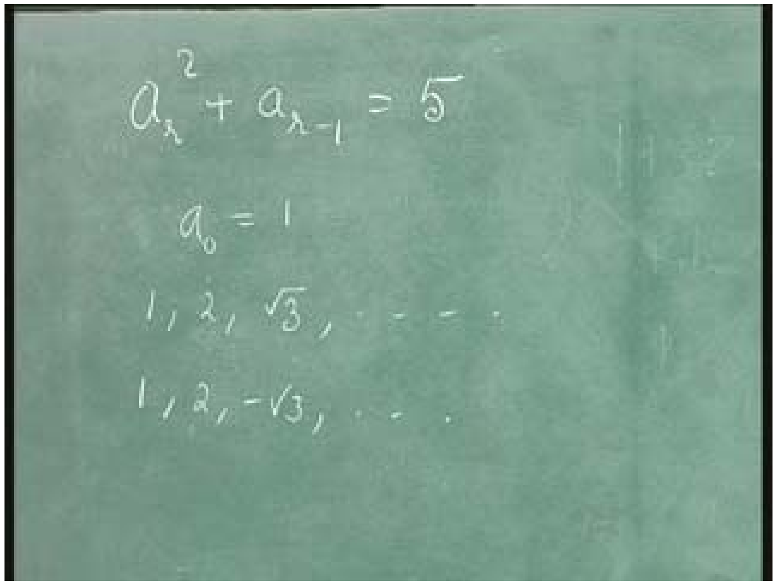

A linear recurrence relation with constant coefficients this is of this form we have already seen this.

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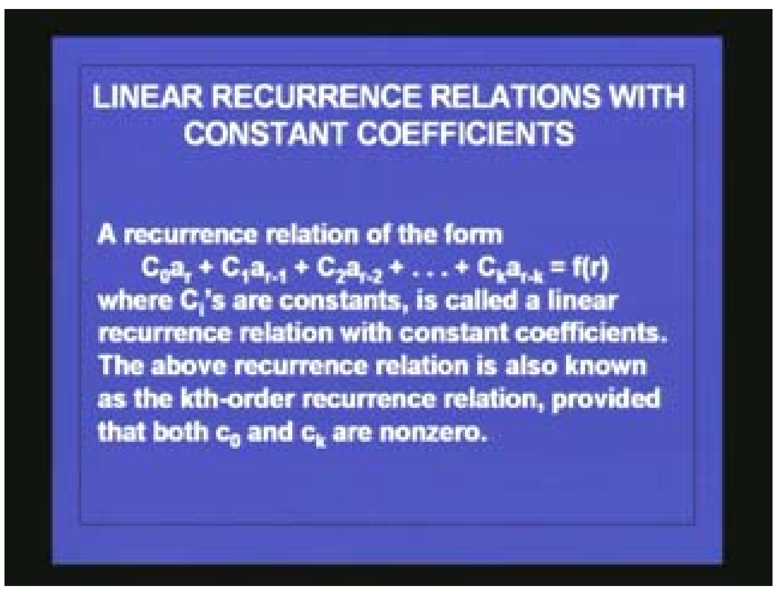

Now, before going into the solution of such linear recurrence relation which has got two parts the homogenous solution and the particular solution we shall first see how to find the homogenous solution and how to find the particular solution and so on.

But before going into that let us consider one more example for formulating the recurrence relation. If you have a problem sometimes it is easier to formulate it as a recurrence relation and then solve it. For example, in the case of tower of Hanoi problem the answer is 2 power n minus 1 which is not very obvious. Whereas if you write it as a_r is equal to $a_{r \text{ minus } 1 \text{ plus } 1}$ that is obvious and it is easy to write this sort of a recurrence relation. So what we would like to do is given a problem try to formulate the recurrence relation and once we formulate the recurrence relation then we can solve it.

Let us consider one more example; in the plane you draw ovals so that any one oval cuts the other ovals in exactly two places. So if you have two ovals it will cut like this and if you have third oval it will cut like this and so on. In how many regions the plane will be divided by such intersecting ovals? Another condition, there are two conditions, that is, every oval cuts any other in two places. That is, it is not something like this, for example it is not like this, this cuts in four places it is not like this it is like this. Then secondly no two points of intersection coincide, whether all points of intersection are distinct two of them do not coincide with each other like that.

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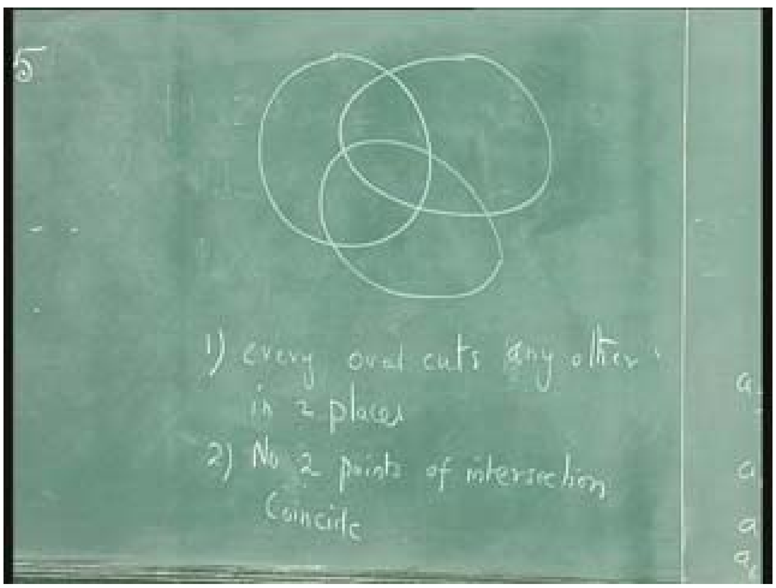

In that case what is the number of regions created? What is the number of regions into which the plane is divided that is what we want. For example, a_0 does not have any meaning, a_1 is one oval if you draw, if you draw one oval it divides the plane into two regions so a_1 is 2. What is a_2? If you have two ovals it will divide the plane into four regions a_2 is 4. If you have 3 ovals how many regions you will have? You will have 8 so a_3 is 8, what about a_4? You can have something like this; number 4 is here 1 2 3 4 5 6 7 8 9 10 11 12 13 it is divided into 14 regions and a_4 is 14. What is the number for a_r or can you get a recurrence relation for this. The recurrence relation for this will be like this. suppose a_r is the number of regions into which the plane is divided when you have r ovals, a_r is the number of regions into which the plane is divided when you have r ovals as $a_{r \text{ minus } 1}$ will be the number of regions into which the plane will be divided when you have r minus oval, what is the relationship between them. You see that after drawing r minus 1 ovals the rth oval you are drawing and it will intersect each one of r minus 1 ovals in two places.

So how many intersections will be there?
There will be 2 (r minus 1) intersection for the rth oval. You have drawn (r minus 1) ovals and that has divided the plane into ar minus 1 parts. Now we are drawing the rth oval and it will cut each one of the r minus 1 previous ovals into two parts. So totally there will be 2 (r minus 1) points of intersection for the rth oval. That is because we assumed that any two points of intersection coincide. The points of intersection are all distinct. In that case the rth oval is divided into 2 (r minus 1) arcs. When you draw the rth oval it is divided by the point of intersections at 2 (r minus 1) places that is this oval is divided into 2 (r minus 1) arcs.

Now you see that if you take this arc this arc divides the earlier region, this region is divided into two parts by this arc. So each one of this 2 (r minus 1) arcs each one of them will divide one region into two regions. So that will be 2 (r minus 1) new region will be created. so the recurrence relation will be the number of regions into which r ovals will divide the plane is what you already have by drawing (r minus 1) ovals plus the rth oval has been cut into 2 (r minus 1) places and it is divided into 2 (r minus 1) arcs each one of the arc divides one of the original region into two regions with that 2 (r minus 1) new regions are created. So a_r is given by this expression a_r is equal to $a_{r \text{ minus } 1}$ plus 2 (r minus 1). What will be the value for a_5? The value for a_5 will be a 4 plus 2 into 5 minus 1 that is 14 plus 2 into 4 is equal to 22. And similarly you can calculate that a_6 32 and so on.

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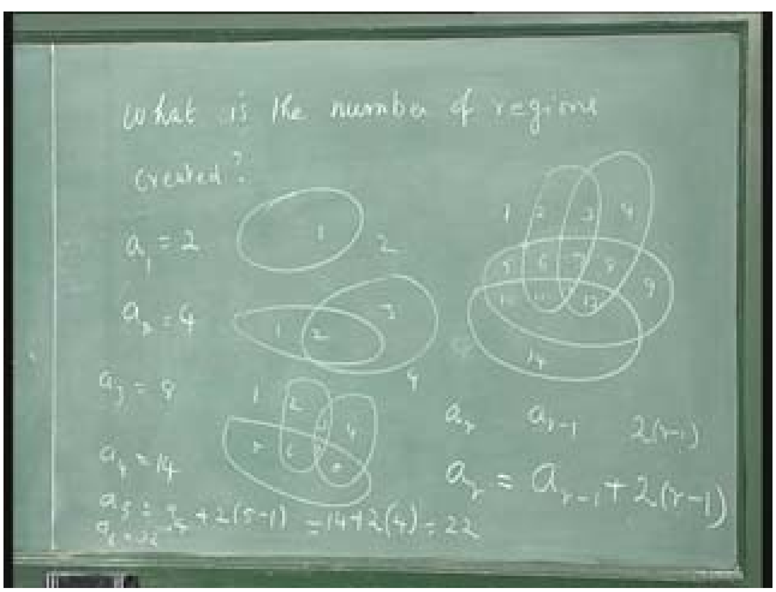

So, this is the recurrence relation, it is a linear recurrence relation with constant coefficients. Let us consider another problem, it is like this: for n is equal to 0, 1, 2 etc let

$h_2(n)$ denote the number of regions into which a plane is divided by n lines in general position. That is, every pair of lines meet in exactly one point but no three lines meet in a point.

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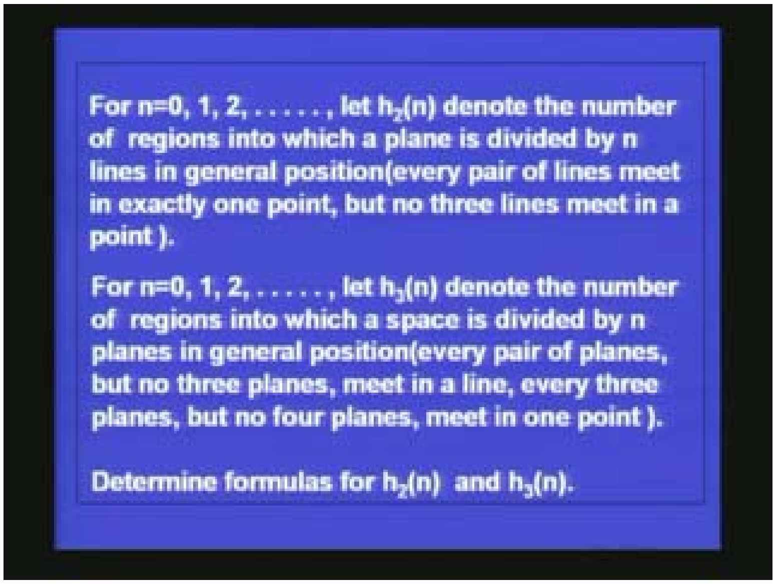

For n is equal to 0, 1, etc let $h_3(n)$ denote the number of regions into which a space is divided by n planes in general position. By general position we mean every pair of planes but no three planes meet in a line, every three planes but no four planes meet in one point. Determine formulas for $h_2(n)$ and $h_3(n)$ this is the problem. So let us see how we can form a recurrence relation and find a solution for this. So $h_2(n)$ means, in a plane you are drawing n lines and find out the number of regions into which the plane is divided. You can see that $h_2(1)$ will be if you draw one line then the plane will be divided into two regions so $h_2(1)$ will be 2 and $h_2(2)$ will be if you draw two lines the plane will be divided into four regions.

Now, instead of considering h_2 if we consider h_1 the one dimensional case, a line is divided by points. So if you have one point $h_1(1)$ this line will be divided into two parts, if you have two points the line will be divided into three parts $h_1(0)$ there is only one that is 1 and if you have n points the line will be divided into n plus 1 line segments. The relation here is $h_1(n)$ is equal to h_1(n minus 1 plus 1).

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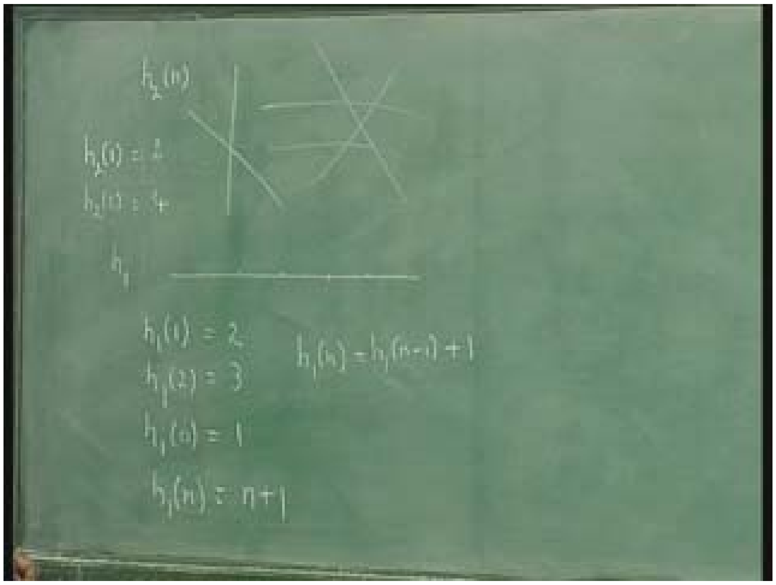

We can make use of this $h_1(n)$ for writing the recurrence for h_2. Suppose you are having (n minus 1) lines then the plane is divided into h_2(n minus 1) regions. Now, we want to express $h_2(n)$ in terms of h_2(n minus 1) something else, what is that?
Now, the plane is divided into some regions, if you want to add one more line then this one more line is cut by (n minus 1) lines so how many line segments the new one will have? The new one will have h 1(n minus 1) line segments. And each one of these line segments will divide an already existing region into two regions so it will add one more region to this.

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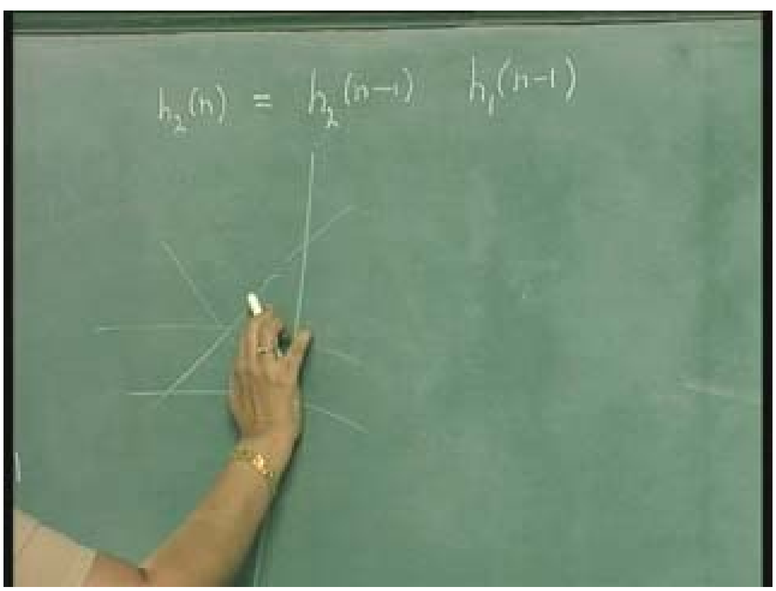

So the recurrence relation is given by this; h_2(n) is equal to h_2(n minus 1) plus h $_1$(n minus 1). Now, making use of this if you try to solve I will just write down like this h_2(n) minus h_2 (n minus 1) is equal to h_1(n minus 1), h_2(n minus 1) minus h_2(n minus 2) is equal to h_1(n minus 2) and so on. Finally h_2(1) minus h_2(0) is equal to h_1(0). Now, if you add them this will get cancelled and so on finally you will get h_2(n) minus h_2(0) is equal to h_1(0) plus h_1(1) up to h_1(n minus 1). Now, what is this h_1(0) is 1 h_1(1) is 2 h_1(n minus 1) is n so this reduces to n into n plus 1 by 2.

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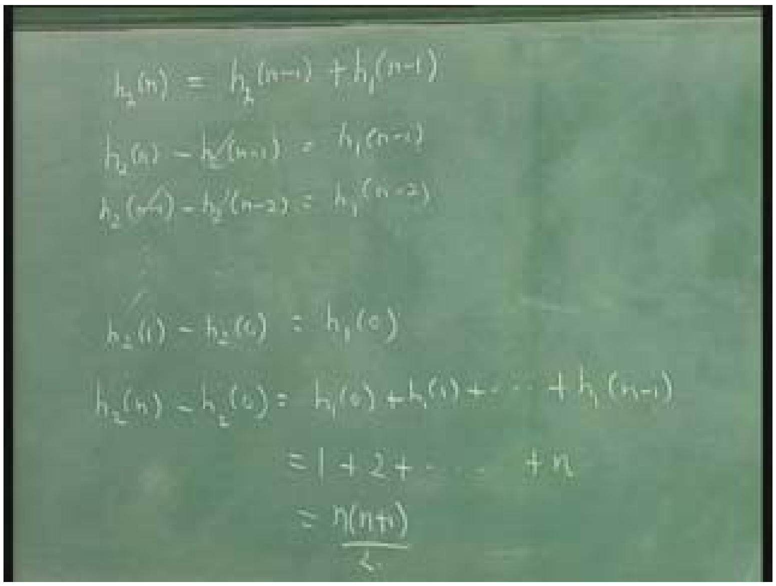

So what can you say about h_2(n)? h_2(n) h_2(0) is when no lines are drawn the plane has only one region. So h_2(0) is equal to 1 so h_2(n minus 1) is equal to n into n plus 1 by 2. And this gives you the answer h_2(n) is 1 plus n into n plus 1 by 2. Now check the answer with 1 n is equal to 1 you put 1 1 plus h_2(1) will be 1 plus 1 into 1 plus 1 by 2 which should be 1 plus 1 is equal to 2 which we have already seen and that is correct. What about h_2(2)? h_2(2) will be 1 plus 2 into 2 plus 1 by 2 that is 1 plus 3 is equal to 4, we have also seen this.

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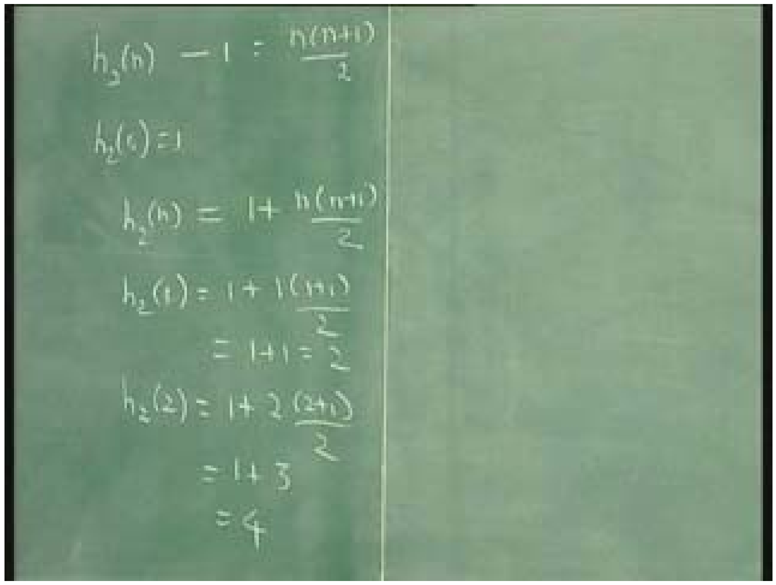

Now, a fact to notice you can write $h_1(n)$ as (n0) plus (n1) that is nc0 nc1 what is this? This will be just 1 and this will be n you know that h1n is n plus 1. If you look at $h_2(n)$ you can also write this as (n0) plus (n1) plus (n2) because what is this? This is 1 this is n plus nc2 is n (n minus 1) by 2 if you simplify this 1 plus 2n plus n square minus n by 2 that will be 1 plus n into n plus 1 by 2.

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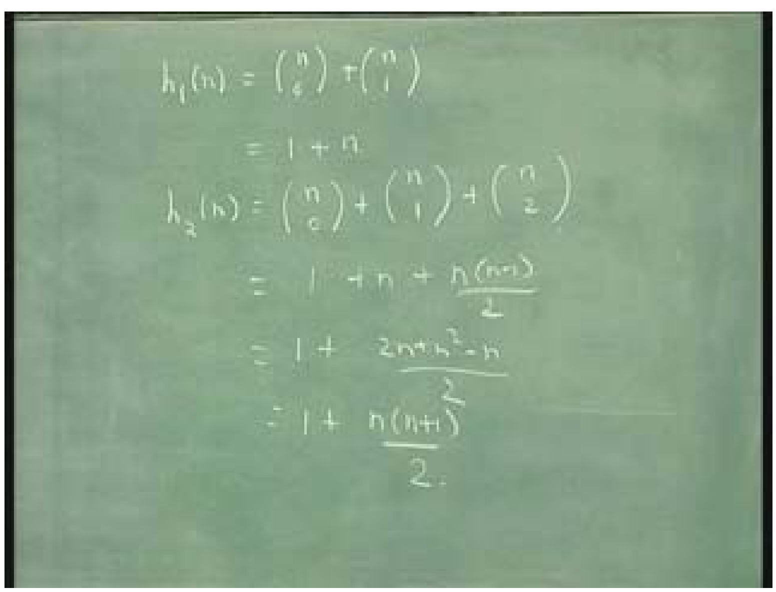

Now coming to the three dimensional case a similar argument will tell you that $h_3(n)$ is equal to h_3(n minus 1) plus that is when you have (n minus 1) planes the space is divided into so many regions that is given by h_3(n minus 1). When you add the nth plane that nth plane will be divided in that particular plane it will be divided into h_2(n minus 1) regions and each one of those plane segment will divide a three dimensional region into two that is adding one more. So the recurrence relation for that will be like this. Again if you use the same argument and try to solve you will see that h_3(n) is equal to (n 0) plus (n 1) plus (n 2) plus (n 3), I will leave the working out as an exercise to you.

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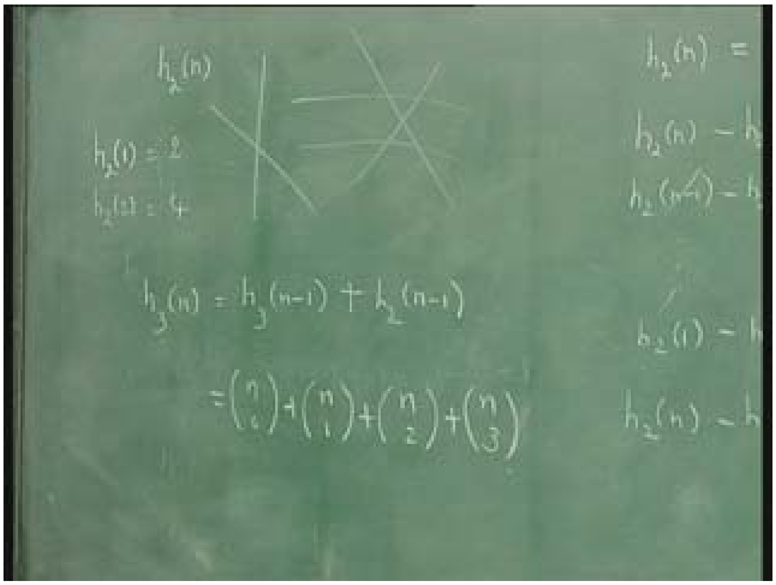

How do you go about solving such a linear recurrence relation? This is what we want to see next. Given the recurrence relation or the difference equation how do we get the expression for a_r?

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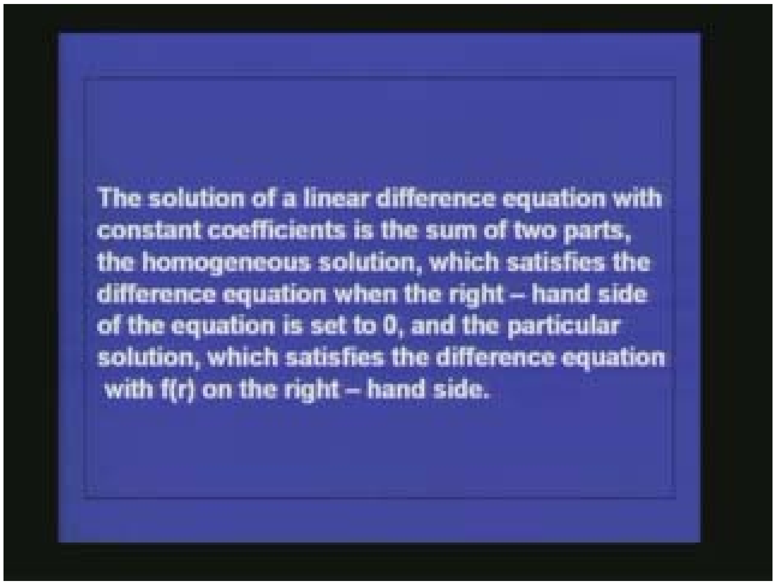

This is known as finding the solution to the linear difference equation. The solution to the linear difference equation with constant coefficients is the sum of two parts, the homogenous solution which satisfies the difference equation when the right-hand side of the equation is set to 0 and a particular solution which satisfies the difference with f(r) on the right-hand side. So the solution consists of two parts; the homogenous solutions and the particular solution. The homogenous solution is obtained when you set the right-hand side to 0 and the particular solution is when you set the right-hand side to f(r).

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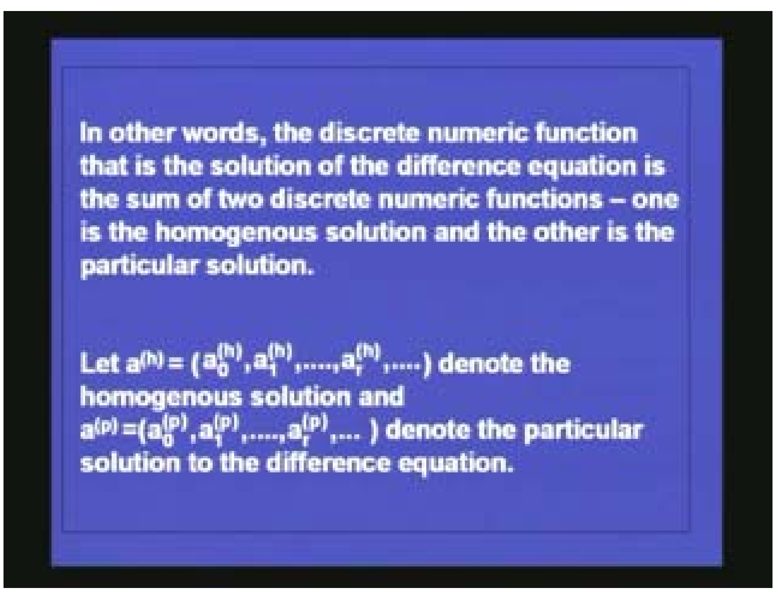

In other words, discrete numeric function that is the solution of the difference equation is the sum of two discrete numeric functions, one is the homogenous solution and the other is the particular solution. The homogenous solution is given like this; let ah where h denotes the homogenous, it is a sequence like this a_0 is a discrete function this sequence a_0 (power h) a_1 (power h) a_r (power h) and so on denotes the homogenous solution. And a particular solution is given by a_p where p denotes the particular solution a_0 to the power p, a_1 to the power p and so on. This denotes the particular solution to the difference equation. The total solution is the sum of these two.

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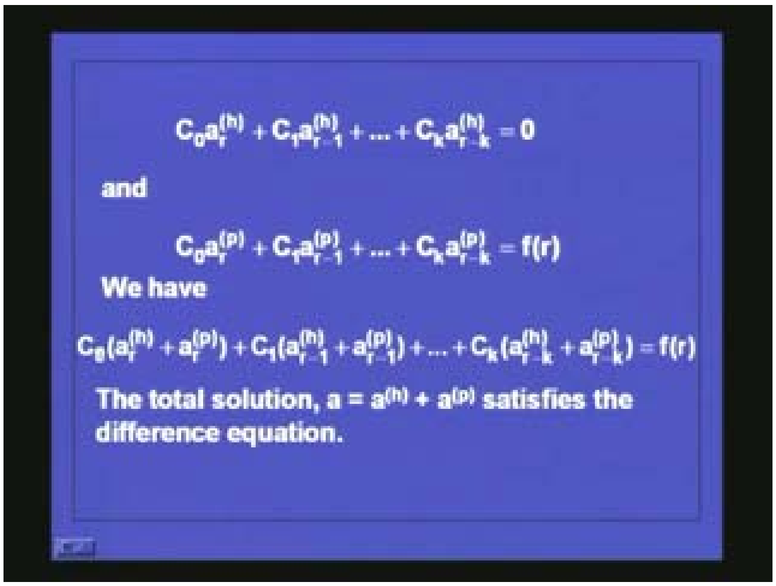

So how do you get the homogenous solution? The difference equation is of this form with the right hand side f(r). But when you want to find the homogenous solution you set it as c_0a_r (power h) plus c_1 a power r minus 1 etc to 0. The right hand side is set to 0 and the particular solution is obtained by putting the equation and f(r) whatever value was originally there or here. So when you add up these two you will get c_0a_r (power h) plus a_rp etc this is the total solution and this is when you add the right hand side this 0 plus f(r) is f(r).

The total solution is a is equal to a power h plus a power p satisfies the difference equation this we can see. But the question is why you have to find the homogenous solution and why do you have to find the particular solution because homogenous solution the right hand side we are making it different.

Why do you do this?

The reason is we have some boundary condition and the boundary condition have to satisfy the equation. And in general only if we take the particular solution alone it may not satisfy the boundary condition but if you take a total solution it will definitely satisfy the boundary condition and so on. The next step is how you go about finding the homogenous solution and how to you go about finding the particular solution.

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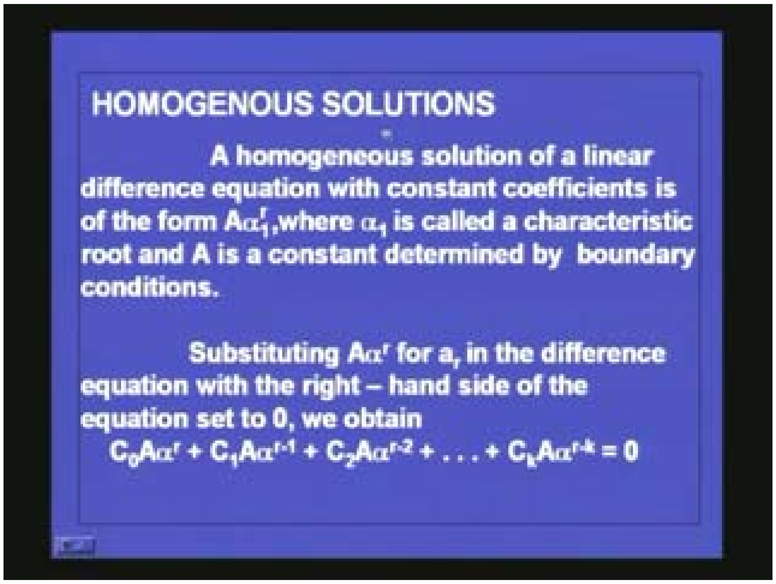

The homogenous solutions are obtained in this manner. A homogenous solution of a linear difference equation with constant coefficients is of the form A alpha 1r where alpha 1 is called the characteristic root and A is a constant determined by the boundary conditions. So the homogenous solution is of the form A alpha 1 power r where alpha 1 is a characteristic root. Let us see how this happens. Suppose you have A alpha r in the a_r then the difference equation with right-hand side equal to 0 then you have C_0A alpha power r plus C_1A alpha power r minus 1 plus C_2A alpha power r minus 2 etc C_kA alpha power r minus k is equal to 0 so you get this if you substitute A alpha r for ar.

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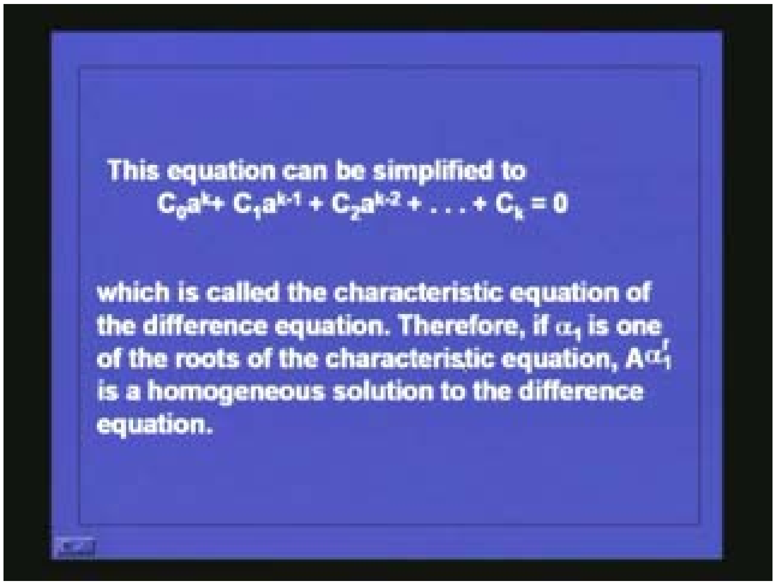

Then this equation becomes C_0a power k plus C_1a power k minus 1 plus C_2a power k minus 2 etc plus C_k is equal to 0 which is a polynomial of degree k and a. And if you find the roots of this equation that will give you the characteristic roots. So this equation is called the characteristic equation of the difference equation. Therefore if alpha is one of the roots of the characteristic equation then A alpha 1 r is a homogenous solution for the difference equation because the root of this equation will satisfy the recurrence relation with the right-hand side set to 0.

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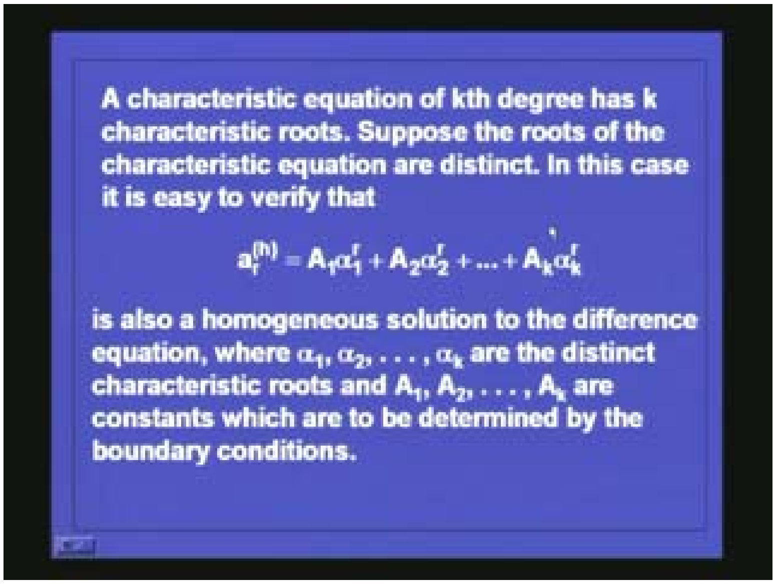

So let us see how this happens. For example; in the Fibonacci sequence we had a_r is equal to $a_{r \text{ minus } 1}$ plus $a_{r \text{ minus } 2}$ or a_r minus $a_{r \text{ minus } 1}$ minus $a_{r \text{ minus } 2}$ is equal to 0. So the characteristic equation is alpha squared minus alpha minus 1 is equal to 0, what are the roots of this equation minus 1 plus or minus squa re root of b squared minus 4ac that is 4 by 2. So the roots are 1 plus root 5 by 2 1 minus root 5 by 2. These are the two roots.

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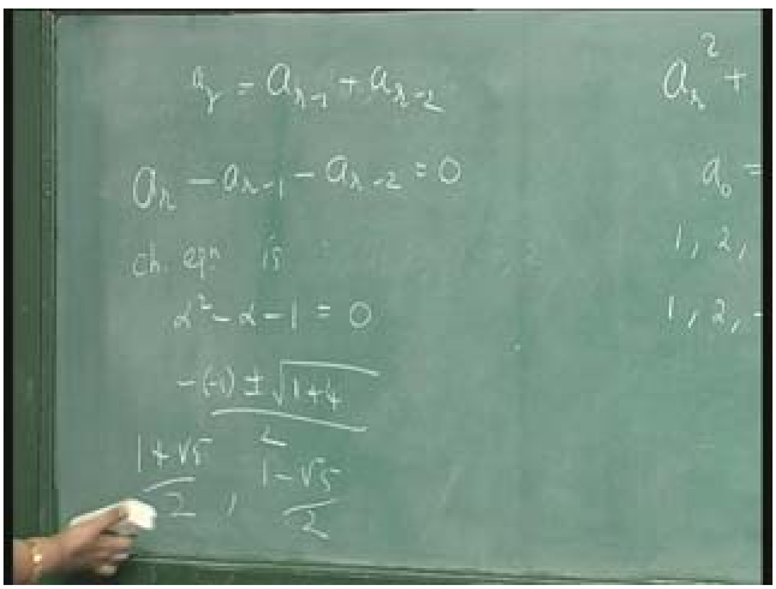

If you have alpha 1 as a root it will satisfy this equation. So what we have is the solution for this, a_r is of the form some A_1 1 plus root 5 by 2 to the power of r plus a_2 into 1 minus root 5 by 2 to the power of r and A_1 and A_2 have to be determined from the boundary conditions.

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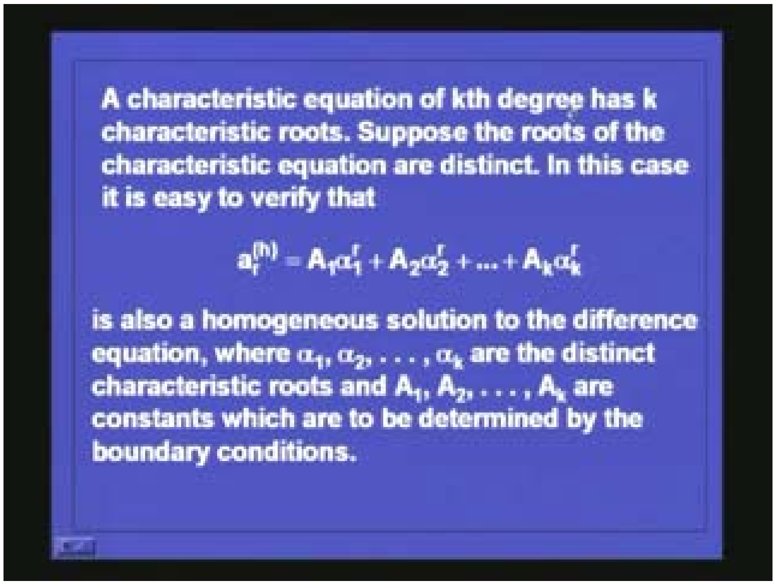

A characteristic equation of kth degree has k characteristic roots. Suppose the roots of the characteristic equations are all distinct in this case it is easy to verify that the homogenous solution a_r power h is A_1 alpha 1 r plus A_2 alpha 2_r etc A_k alpha k_r. This is a homogenous solution to the difference equation where alpha 1 alpha 2 alpha k are the distinct characteristic roots and A_1 A_2 A_k are constants which are to be determined by the boundary conditions.

So in this we can see that 1 plus root 5 by 2 1 minus root 5 by 2 are distinct roots of this equation. This is a second order linear difference equation with constant coefficients. If you write the characteristic equation it is this equation, the roots of this equation are given by this.

So the homogenous solution will be of this form where the constants A_1 and A_2 have to be determined using the boundary condition. Now what is a_0? a_0 is 1 and a_1 is 1 this we know, staring points are a_0 is 1 and a_1 is 1. So putting r is equal to 0 you will get 1 is equal to A_1 plus A_2 and put r is equal to 1 you will get A_1 into 1 plus root 5 by 2 plus A_2 into 1 minus root 5 by 2 that is is equal to 1. But what is A_1? A_1 is 1 minus A_2 so 1 minus A_2 into 1 plus root 5 by 2 plus A_2 into 1 minus root 5 by 2 is equal to 1 which will give you

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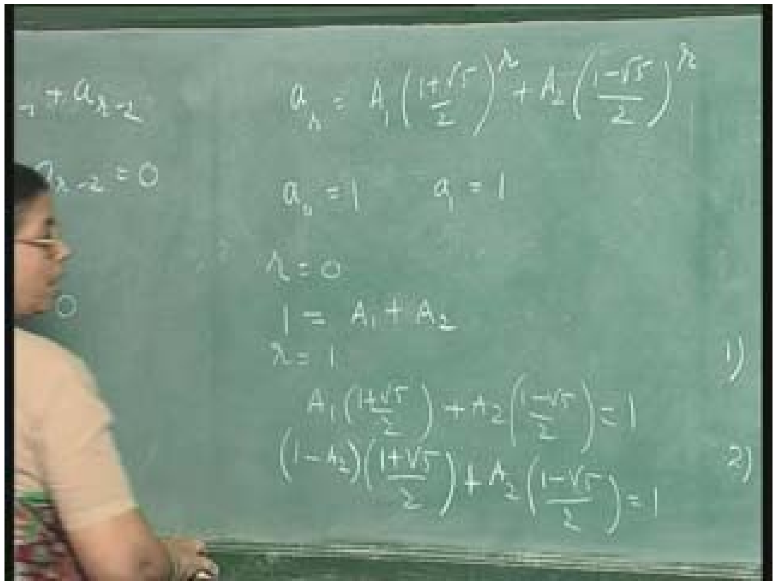

A_2 into 1 minus root 5 by 2 minus 1 plus root 5 by 2 that is the left hand side is equal to 1 minus 1 plus root 5 by 2. So this will give you A_2 into 1 minus root 5 minus 1 plus root 5 by 2 minus root 5 by 2 is equal to 2 minus 1 minus root 5 by 2. That is A_2 will be 1 minus root 5 by 2 root 5 minus with a minus sign. So A_2 will be 1 by root 5 minus 1 by root 5 1 minus root 5 by 2.

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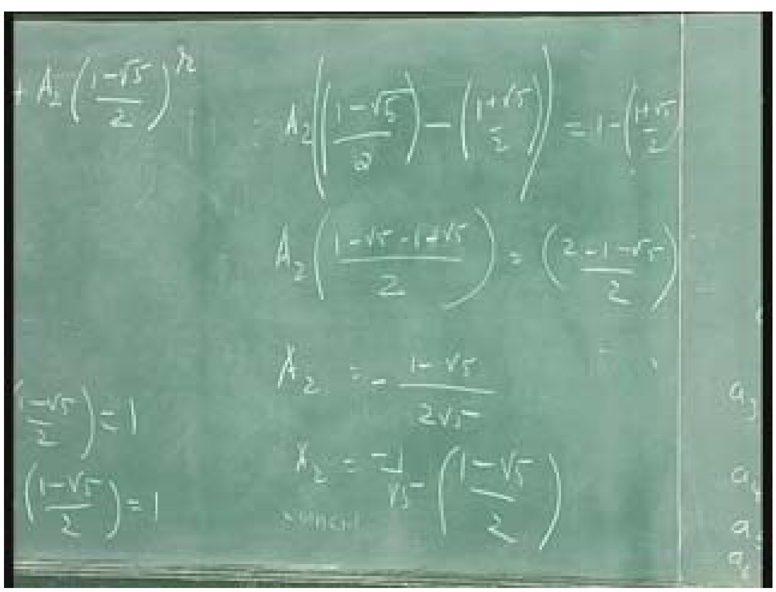

And similarly if you calculate a_1 will be 1 by root 5 1 plus root 5 by 2. So using this value of A_1 like this and A_2 as we have seen is minus 1 by root 5 1 minus root 5 by 2 using this in this equation the homogenous solution becomes a_r is equal to 1 by root 5 1 plus root 5 by 2 r plus 1 minus 1 by root 5 1 minus root 5 by 2 to the power of r plus 1. Here, we had 1 plus root 5 by 2 to power of r but A_1 we found that to be of this value so putting that you get this term and here we have A_2 into 1 minus root 5 by 2 to the power of r and A_2 is this value so putting that here you get the homogenous solution for a_r like this. So this is when the characteristic roots are different.

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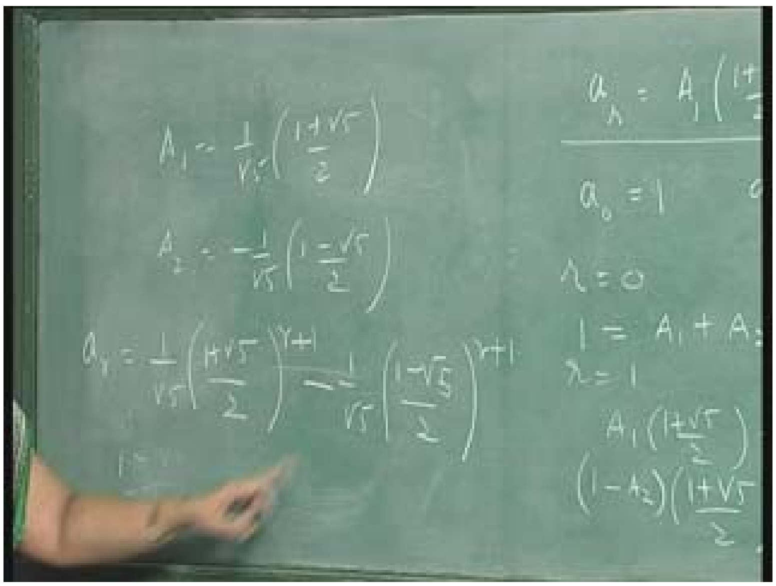

Now, if you have multiple different roots what is the homogenous solution. And when you find the homogenous solution how will you find the particular solution. These are some of the things we have to consider next. So these will be considered in the next lecture.