Discrete Mathematical Structures
Dr. Kamala Krithivasan
Department of Computer Science and Engineering
Indian Institute of Technology, Madras
Lecture # 31
Generating Functions (Contd..,)

In the last lecture, we saw about generating functions and what is the ordinary enumerator using some indicator functions for a sequence. And we also saw how they are related to the number of combinations of R objects out of N objects because if you expand (1 plus x) power n, you get the coefficient as CnR. So we saw that connection. So for finding out the number of combinations you can use the ordinary generating functions. This is what we have seen in the last lecture. Today, we shall see how you can use the idea of a generating function for finding out the number of permutations. For this we have to define another type of generating function called the exponential enumerator.

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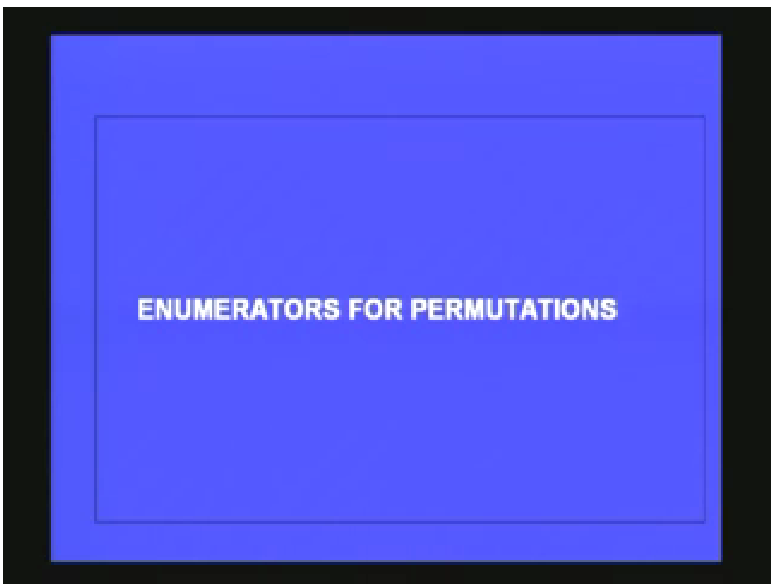

So let us see what it is. What will be the enumerator for permutations? This is what we are going to see today. So a generating function for the permutation, suppose we are having two objects a and b, then it should be of the form 1 plus (a plus b) x plus (ab plus ba) x square.

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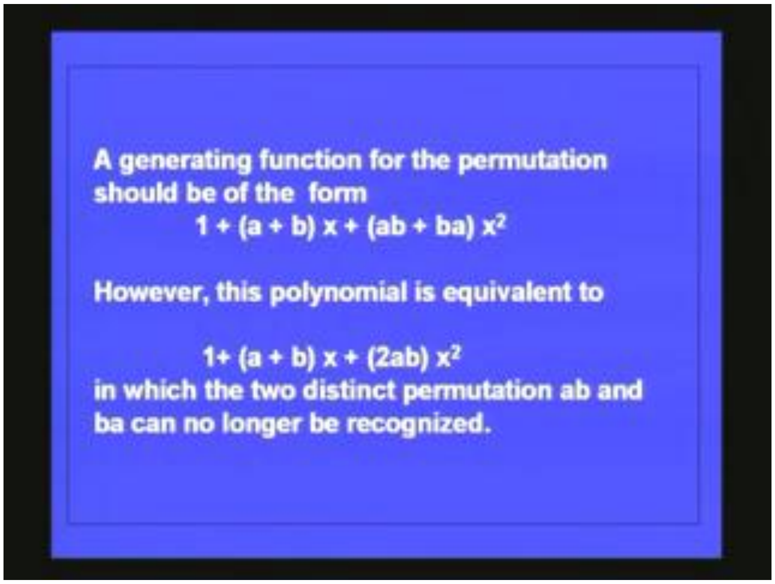

Suppose you are having two objects a and b, then if u do not have any of them, then there is only one way of doing it and if you have one of them either you can put a or you can put b, if you have two of them you can arrange them in this way: either a in the order ab or in the order ba. So if you want to represent this as a generating function it should be of this form. But numerically or mathematically this polynomial is equivalent to this polynomial because numerically you do not distinguish between ab and ba. So this will be represented as (2ab)x square, but that is not what you want, you want ab and ba to distinguish. The two distinct permutations ab and ba can no longer be recognized. This is what you do not want. You want them to be distinguished; ab and ba should be distinguished. So in general, if you want to have an enumerator for the permutations, looking back, then the enumerator for combinations is that you have a formula like this where you have C(n, 0) C(n, 1) etc.

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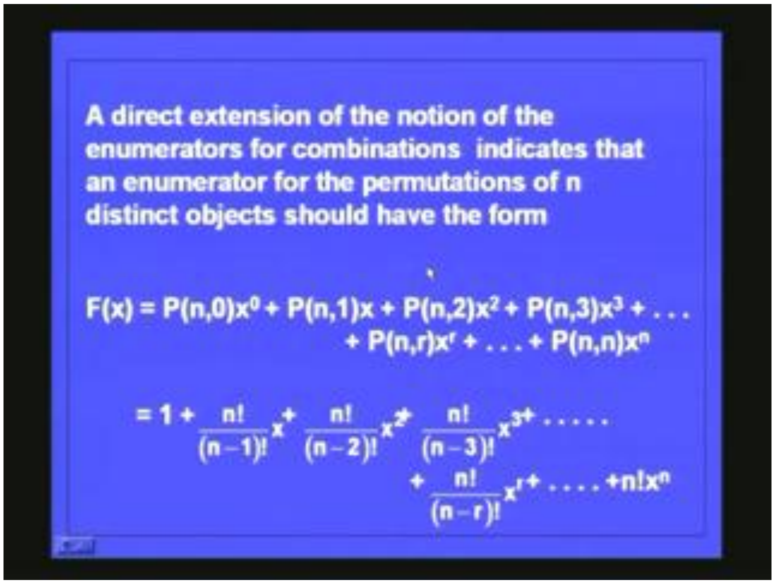

So, direct extension of the notion of the enumerators for combination indicates that, an enumerator for the permutation of n distinct object should have the form like this: F(x) is equal to P(n, 0)x power 0. This gives the number of zero objects number of n objects, this gives the number of permutations of one object out of n object, this gives the number of permutations of two objects out of n objects, and that should be a coefficient of x square if you want the permutations to be defined. And if you write the formula for (n,0), (n,1), (n,2) etc, you know that P(n, 1) is n factorial by n minus1 factorial; P(n, 2) is n factorial by n minus 2 factorial and so on. In that case, you do not have a closed-form expression for this. There is no simple closed-form expression for F(x) if you represent F(x) in this manner. Unless we have a close-form expression, we cannot use it for calculation.

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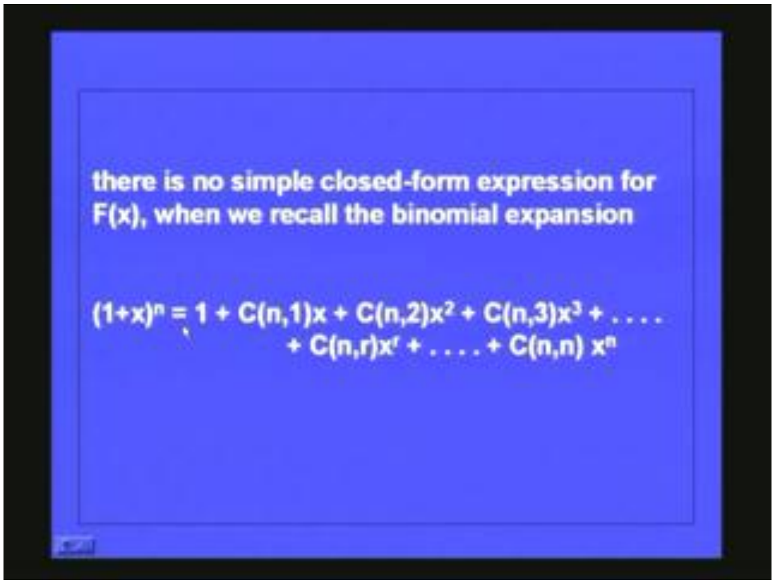

So, it is better to have a closed-form expression but we know that 1 plus x power n is equal to 1 plus C(n, 1)x plus C(n, 2)x square; this is the binomial expansion. From this you get this.

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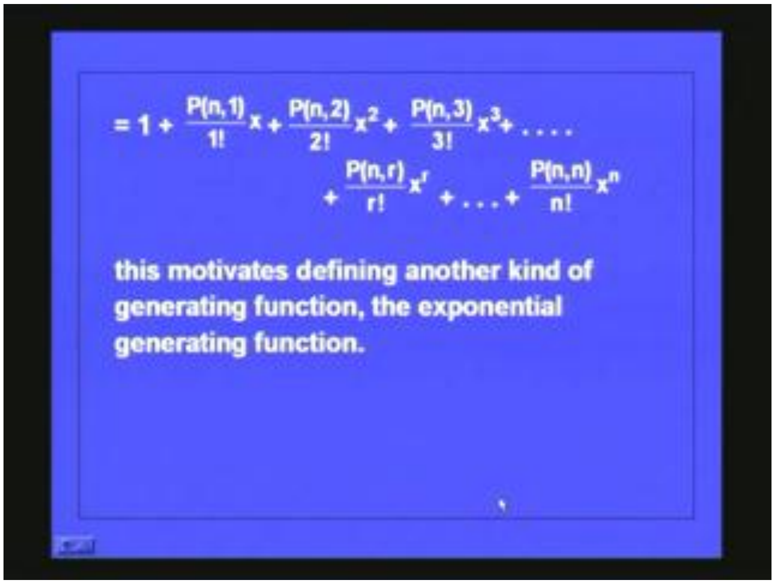

What is C n1?
C(n, 1) is P(n, 1) by 1 factorial, C(n, 2) is P(n, 2) by 2 factorial. The number of permutations of two objects out of N object is to select two objects out of n object and permute them. That can be done in two factorial ways. So, C(n, 2) will be equal to P(n, 2) by 2 factorial. This we have already seen. So this is equal to 1 plus x power n. But here

we are able to get the value of P(n, 1), P(n, 2), P(n, 3) etc. This motivates defining another kind of generating function called the exponential generating function. So, for the porputations we need to define another kind of generating function known as the exponential generating function.

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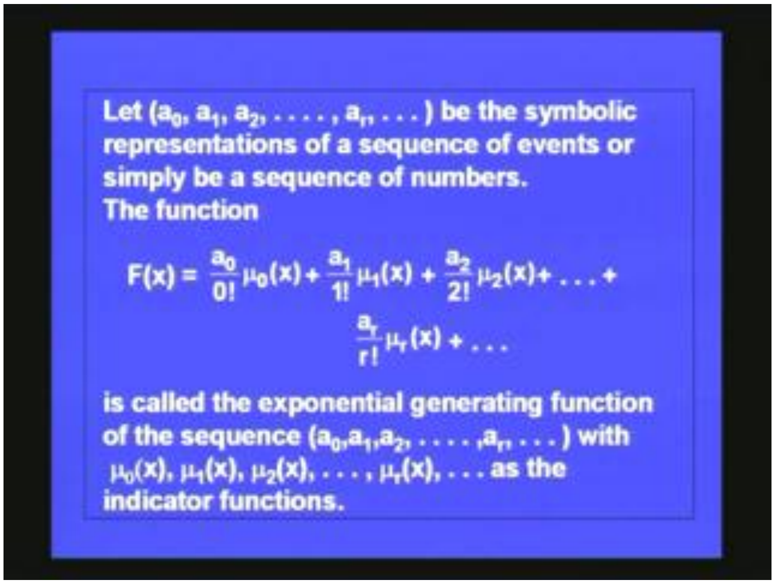

Let a_0, a_1, a_2, a_r be the symbolic representation of a sequence of events or simply a sequence of numbers. Then you look at the function F(x) is equal to a 0 by 0 factorial mu$_0$(x), a1 by 1 factorial mu$_1$(x) a_2 by 2 factorial mu$_2$ x and so on. The general term is a_r by r factorial into mu$_r$(x). This is called the exponential generating function of the sequence a_0 a_1, a_2, a_r, etc with indicator functions mu$_0$(x), mu$_1$(x), mu$_2$(x), and so on. As you see, in the case of ordinary enumerator it will be a_0 mu$_0$(x), a_1 mu$_1$(x), etc. In the case of exponential enumerator it is a_0 by 0 factorial mu$_0$(x), a_1 by 1 factorial mu$_1$(x). Usually it is always advisable to take the indicator functions as 1x, x square, x cube, etc.

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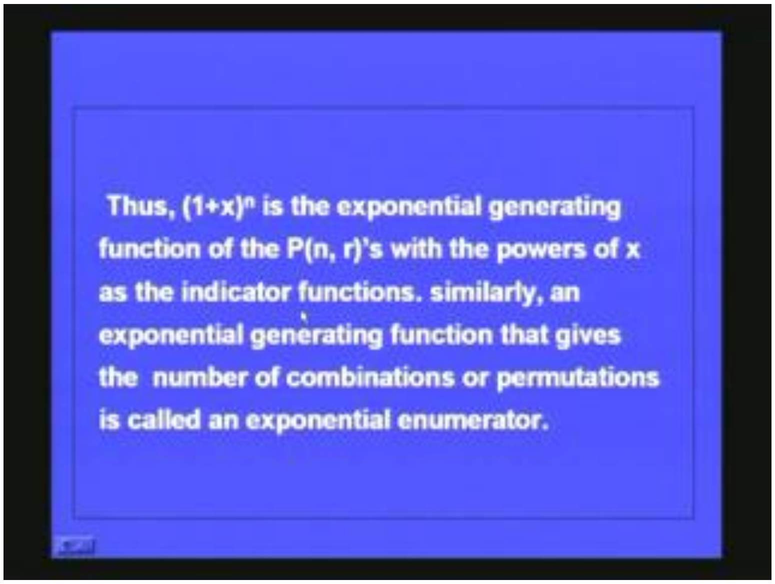

Thus 1 plus x power n is the exponential enumerator or the exponential generating function for P(n, r)'s with the powers of x as the indicator functions. That is, x, x square, x cube, etc are the indicator functions. Similarly, an exponential generating function that gives the number of combinations or permutations is called an exponential enumerator. So for finding out the permutations we use the exponential enumerator.

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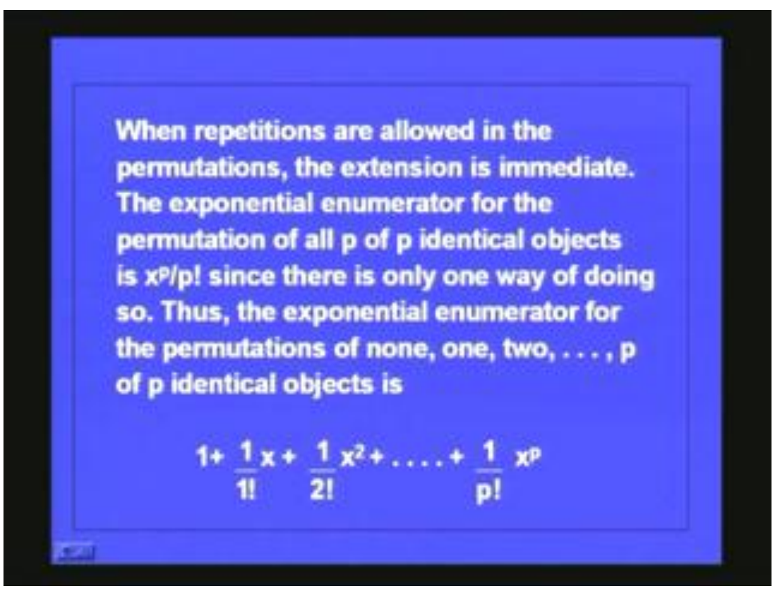

When repetitions are allowed in the permutations, we can see the extension easily. The exponential enumerator for the permutation of all of p objects, all p of p identical objects is x power p by p factorial. That is, the answer is 1. There is only one way of doing it. If you have p identical objects and if you arrange them there is one way of doing it. So for that, the exponential enumerator is x power p by p factorial by our definition of exponential enumerator. Thus, the exponential enumerator for the permutations of none, 1, 2, etc, p(p) identical objects is given by this formula or this expression 1. Whether you take 2 or 3 or 4 identical objects and permute them, the answer is only 1. So, you have 1 plus 1 by 1 factorial x, 1 by 2 factorial x square, 1 by 3 factorial x cube and so on. And 1 by p factorial x power p.

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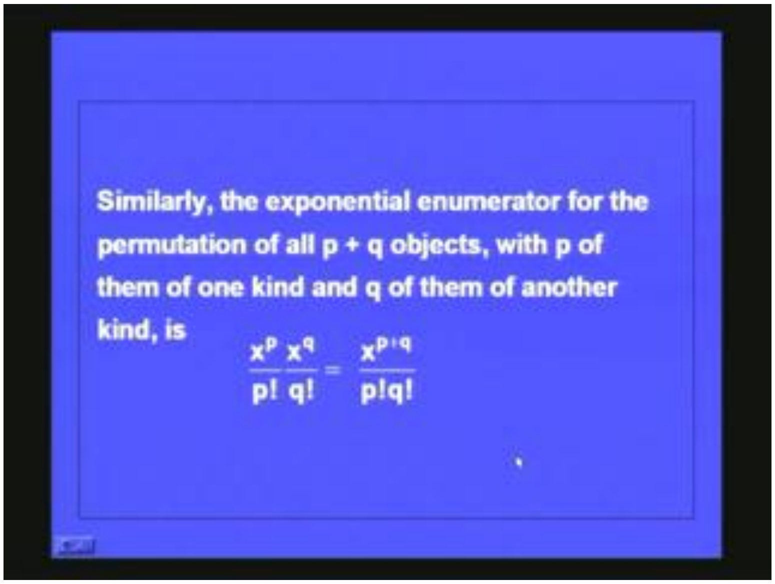

Similarly, the exponential enumerator for the permutation of all of p plus q objects, with p of them of one kind and q of them of another kind is x power p by p factorial, x power q is q factorial which is given by x power p plus q by p factorial q factorial. See, by the rule of product, if you have p identical objects of one kind and q identical objects of the second kind, then the number of ways of arranging these as 1, 2, 3, 4 of these objects or if you take all the p of them, it is given by x power p, p factorial. If you take 1, 2, 3, 4 of them it is given by expression 1 plus1 by 1 factorial x, plus 1 by 2 factorial x square and so on. So when you take all the p of them, the exponential enumerator is this.

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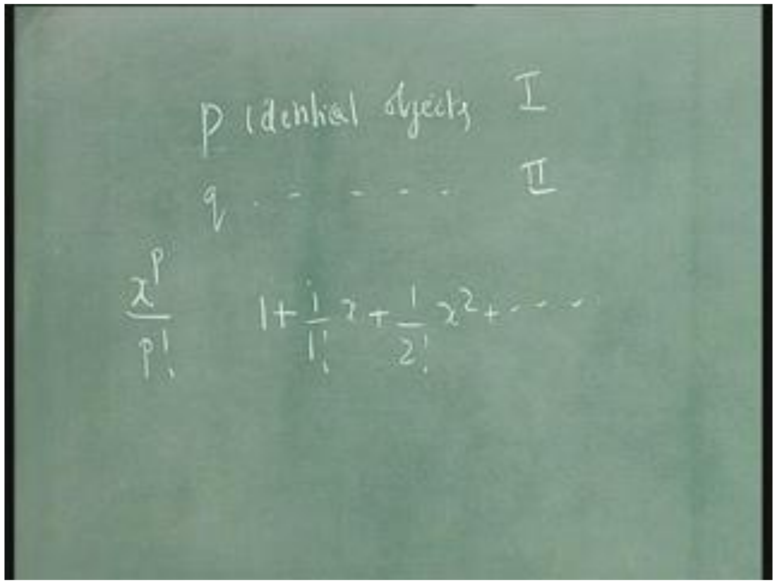

Similarly, you have q identical objects as the second kind. If you take all the q of them and arrange them, the number of ways of doing it is only one, that is, x power q by q factorial. If you take p of them, q of them and p plus q of them and arrange it, it should be this and that is x power p plus q by p factorial q factorial.

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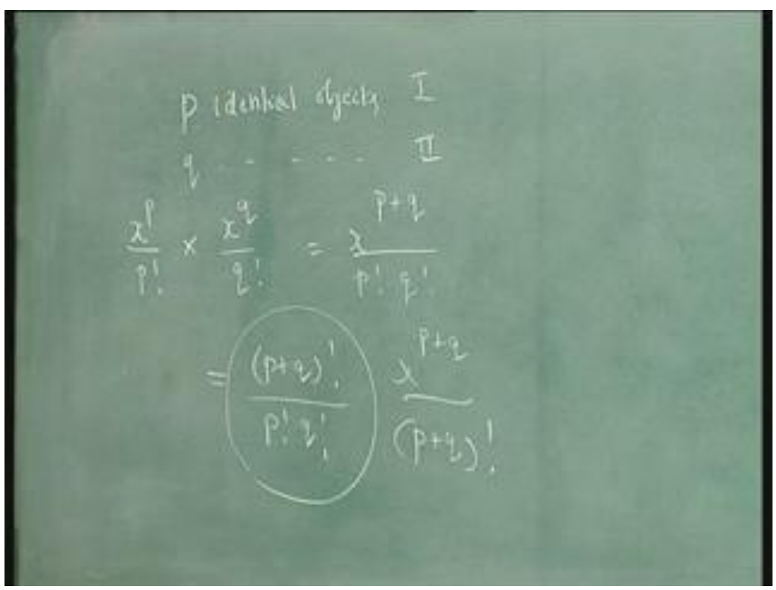

This you can write as p plus q factorial by p factorial q factorial and x power p plus q by p plus q factorial. So, the number of permutations of p plus q objects of which p of them are identical and q of them are identical but of a different type, are given by this formula. We have seen this formula earlier also, so you see how you get this using the idea of an exponential generating function. Let us consider a small problem. What is the exponential enumerator for the permutations of two objects of one kind and three objects of another kind?

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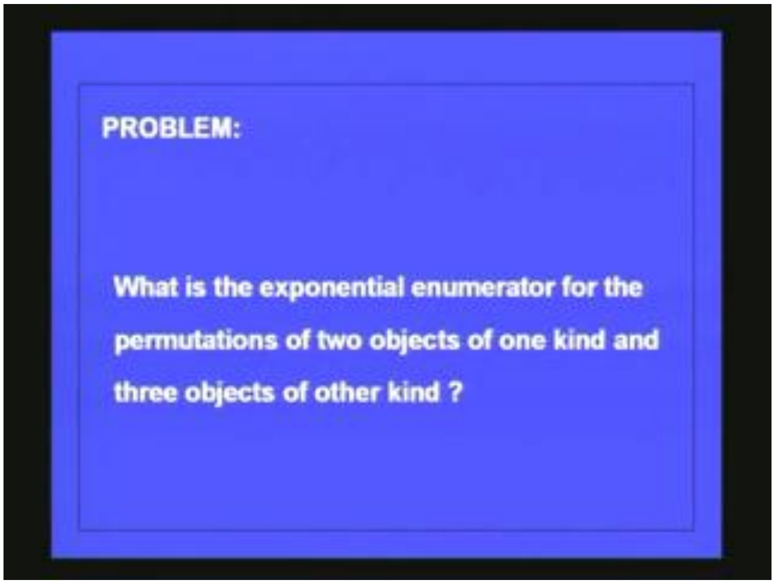

So you are having two objects of one kind and three objects of another kind. What is the exponential enumerator for this? Now for the first one, the exponential enumerator will be, 1 by 1 factorial x by 1 by 2 factorial x square because you are having (()), will be this. Then in the second one, you are having three of them, so the exponential enumerator for that will be, 1 by 1 by 1 factorial x plus 1 by 2 factorial x square plus 1 by 3 factorial and x cube. Now, multiply this, you will get 1 plus.

What is the coefficient of x 1 by 1 factorial plus 1 by 1 factorial x plus 1 by 1 factorial into 1 by 1 factorial plus 1 by 2 factorial plus 1 by 2 factorial x square plus what is the coefficient of x cube?
1 by 2 factorial, 1 by 1 factorial, x square into x, x into x square will again be 1 by 1 factorial, 1 by 2 factorial plus x cube into 1. That is 1 by 3 factorial x cube plus power of 4 and the power of 5 will be 1 by 2 factorial, 3 factorial, x power 5. This is the exponential enumerator. So, suppose I want to pick two of them out of this five and arrange them, in how many ways can you do that? You can pick two like this or you can pick two of them from the other side or you can pick one from this and one from this, in this case you can arrange like this. The answer is 4. Do you get 4 here? It should be the coefficient of this. So, you write as the exponential enumerator, as an exponential enumerator you write like this, the answer should be like this, answer into x square by 2 factorial.

If you write 2 factorial, you have to multiply by 2 factorial here. So, in that case, you get 2 plus 1 plus 1 equal to 4. So, that verifies this. So the exponential enumerator for permutations of two objects of one kind and three objects of another kind is given by this product. And, if you take the coefficient of x power r, some coefficient will be there, that coefficient into r factorial will give you the number of permutations, because you have to express the exponential enumerator in this manner. So the coefficient of x power r in this product multiplied by r factorial will give you the number of permutations.

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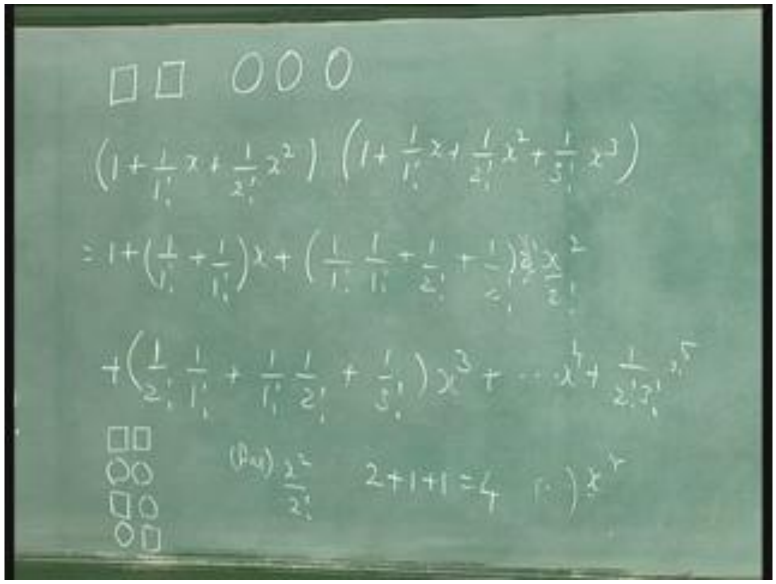

So the number of r-permutations of n distinct objects with unlimited repetitions is given by the exponential enumerator like this.

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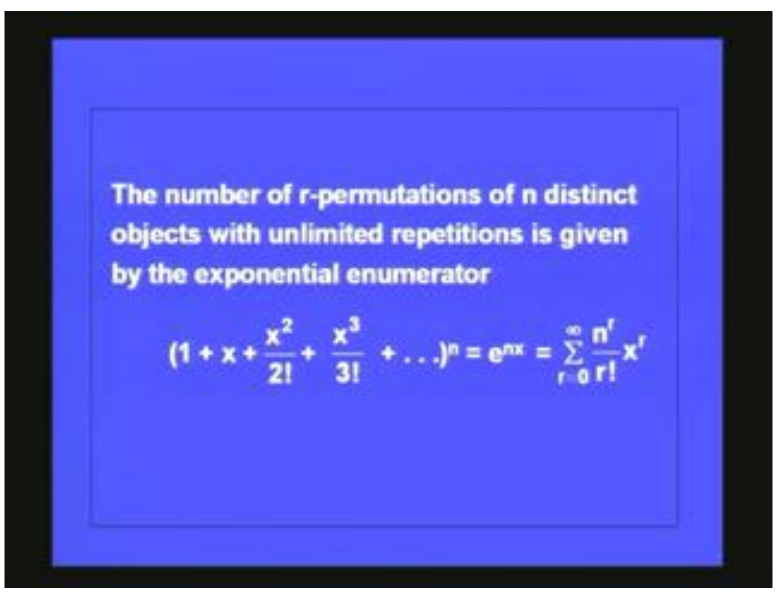

So you are having n objects and each object you can repeat any number of times. So for the first object the exponential enumerator will be 1 plus x plus x square plus 2 factorial etc. The second object again is the same. Like that for each one of the objects. So this is power n. But we know that the closed-form expression for this is e power x. So you have e power x into n or e power nx and the power of x power r in this given by n power r by r factorial. You can write e power nx as sigma r is equal to 0 to infinity n power r by r factorial into x power r. So looking at it as an exponential enumerator, this gives the number of ways of permuting r objects out of n objects with unlimited repetitions. This you can very easily see. Suppose, I have to fill r places and I have unlimited n objects say a_1, a_2, a_n, I can have unlimited reputations. Then, the first place can be filled in n ways and the second place can be filled in n ways and so on. The rth place can filled in n ways also. So the answer you get is n power r. And that is what you get here, n power r.

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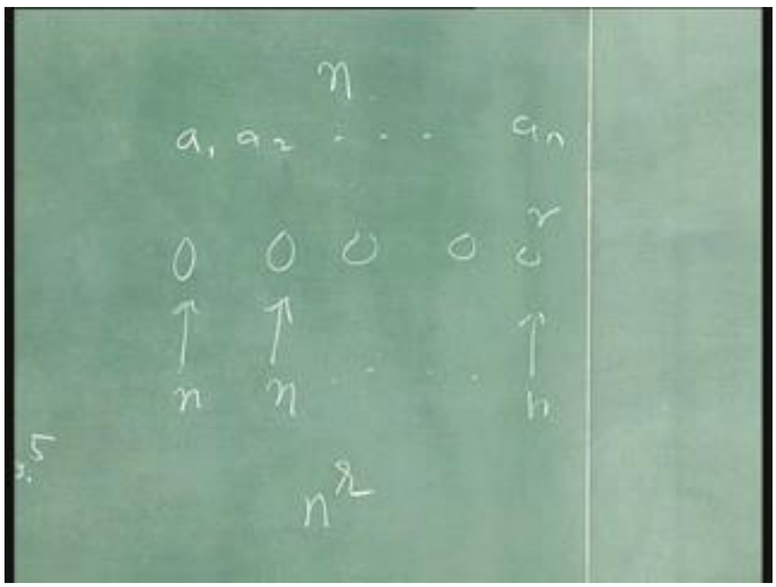

So, the same results which we obtained earlier in the ordinary manner now, to get the same answer, we are using the exponential enumerator. Now, this idea of exponential enumerator we can use for finding the distribution of distinct objects into nondistinct cells. Earlier, we considered the distribution of distinct objects into distinct cells and we also considered the distribution of nondistinct objects into distinct cells. So, what we considered earlier is this, distinct or nondistinct objects into distinct cells, both we have considered earlier. Nowyou can use this idea of generating function to find the distribution of distinct or nondistinct objects into nondistinct cells. So the next thing we will consider is, distribution of distinct or nondistinct objects into nondistinct cells. So, first we consider the distribution of distinct objects into nondistinct cells and how you can use the idea of exponential enumerator.

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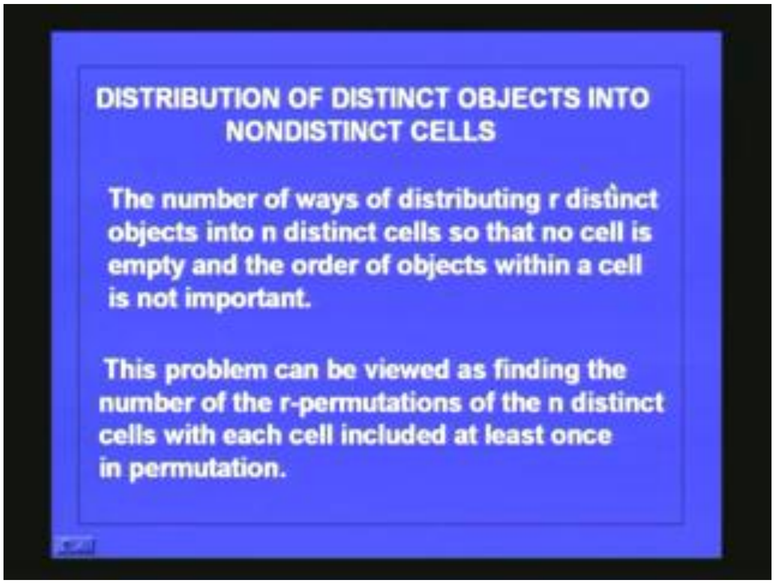

The number of ways of distributing r distinct objects into n distinct cells, first we considered r distinct objects and n distinct cells then we will consider the nondistinct cells. So that no cell is empty and the order of objects within a cell is not important, this is what we will consider first. This problem can be viewed as finding the number of the r-permutations of the n distinct cells with each cell included at least once in permutation. So the exponential enumerator for the permutation is given by this.

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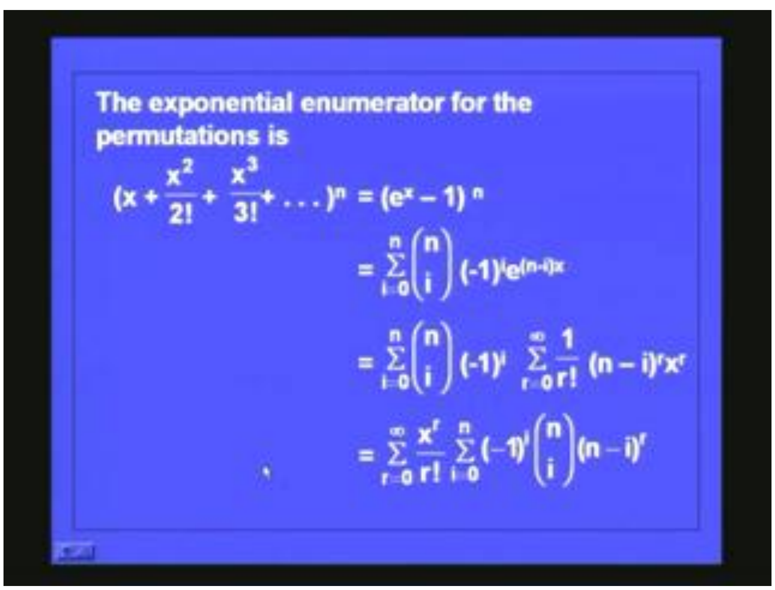

Each cell has included at least once. So you do not have to start with 1, you start with x straight away. So, it is x plus x square by 2 factorial, this is for one cell. So for n cells you have this. And the simplification of the expression within the bracket is e power x minus 1 and if we expand it will be sigma i is equal to 0 to ni minus1 power I, e power n minus i into x. And again simplifying this, this will be this, this, and if you expand this, r is equal to 0 to infinity, 1 by r factorial n minus i power r x power r, rearranging x power r by r factorial here, taking this and this here and by arranging and rearranging, you will get this.

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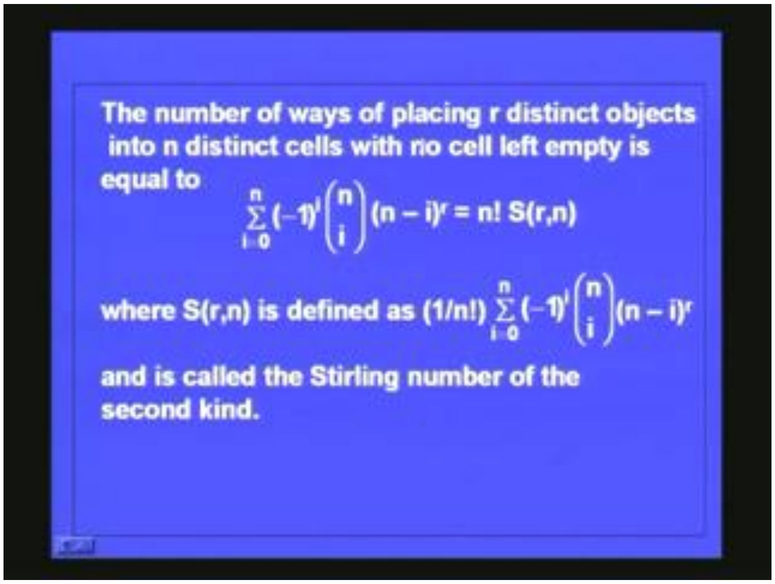

The number of ways of placing r distinct objects into n distinct cells with no cell left empty is equal to this expression. That is, you have to take the answer. The answer into x power r by r factorial gives you sigma r is equal to 0 to infinity and the coefficient of this x power r, the term for x power r is, x power r by r factorial and this and this will give you the required answer. So, we are not considering nondistinct cells now, we are considering distinct cells. So, this expression gives you the answer. That, you can write as, n factorial into S(r,n). This is for distinct cells.

So, when you want to say that the cells are all identical or say that they are nondistinct cells, you have to divide it by n factorial. So, this should be divided by n factorial to get you the answer when you want to distribute r distinct objects into n nondistinct cells. And this whole expression is written as n factorial into S(r,n) where S(r,n) are known as Stirling's number of the second kind. And S(r,n) is 1 by n factorial, you bring n factorial this side, sigma i is equal to 0 to n minus1 power i C ni n minus i power r. The Stirling numbers are calculated for different values of r and n and for example S(r,n), if you take S 8, 4, the value is 1701.

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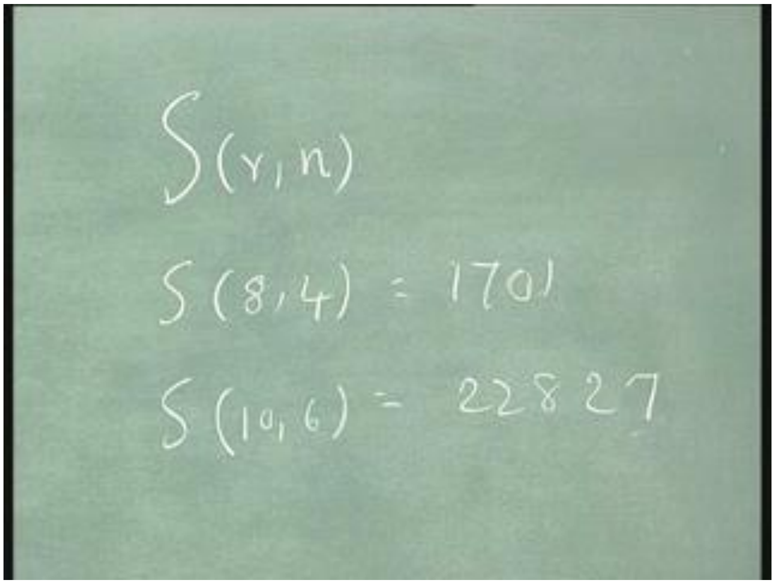

If you take S 10, 6, the value is 22827 and so on. There is a table which gives you the Stirling numbers for different values. So, what we have considered just now is the distribution of distinct objects into nondistinct cells. So if you want to distribute r distinct object into n distinct cells with no left cells empty, the answer is given by the Stirling number which is denoted as S(r,n).

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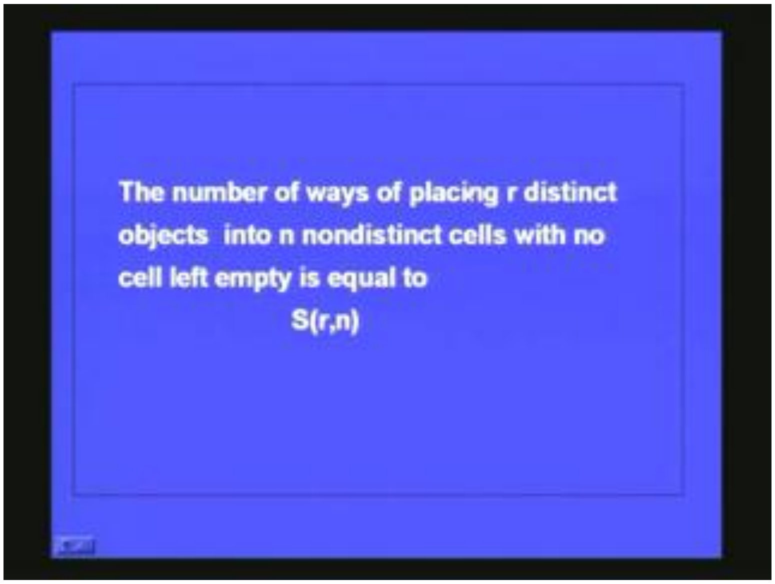

The number of ways of placing r distinct objects into n nondistinct cells with no cell left empty is equal to S(r,n). And we have already seen that, S rn is this. Bringing this n factorial this side, it is 1 by n factorial sigma i is equal to 0 to n minus1 power i C ni n minus i power r. Next is, we have considered the distribution of distinct objects into nondisitnct cells. Next we should consider the distribution of nondistinct objects into nondistinct cells. That is really known as partition of integers. So we will consider that next.

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What do you mean by partitions of integers?
A partition of an integer is a division of the integer into positive integral parts in which the order of these parts are not important. So it is equivalent to having distribution of nondistinct objects into nondistinct cells.

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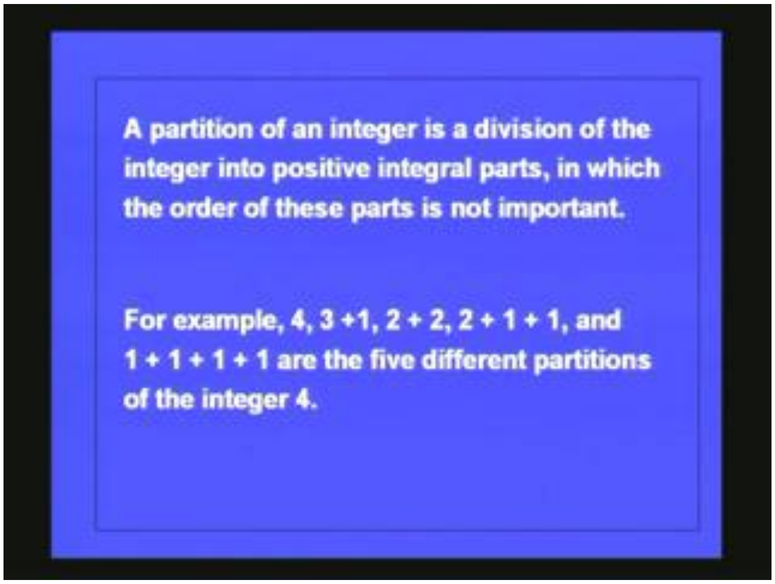

For example, if you take the number 4, it can be partitioned as 4 plus 0 that is just 4 alone or it can be partitioned as 3 plus1 or it can be partitioned as 2 plus 2 or 2 plus 1 plus 1, or it can be partitioned as 1 plus 1 plus 1 plus 1. These are the five ways of partitioning the integer 4. Either you can have just 4 alone or you can split it as 3 and 1, or split it as 2 and 2, or split it as 2 1 1 or 1 1 1 1. So, the number of different partitions is 5 here. The answer is 5 here. This is also the number of ways of distributing different r nondistinct objects into nondistinct cells. Also, some of the cells can be empty here.

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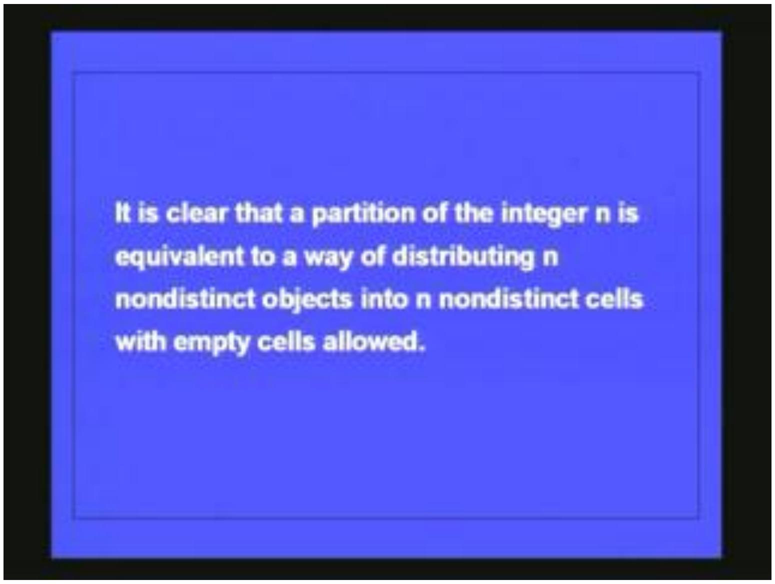

It is clear that a partition of the integer n is equivalent to a way of distributing n nondistinct objects into n nondistinct cells with empty cells allowed. So, this is what we are going to consider. And let us see how you can use the idea of generating function for this. Consider this expression 1 by 1 minus x, this is equal to 1 plus x plus x square x cube plus x power 4 etc. The number of ways you can have k ones in a partition is 1. So, you can have, say for example 4and you can split it as 2 plus 1. The number of ways of you can have two one's in a partition is the coefficient of this. The number of ways you can have three ones is the coefficient of 1 and so on. So, in any partition, the number of ways you can have k 1's is only one way. So that is the coefficient of x power k here.

Consider this; this is 1 plus x square plus x power 4 and so on. The number of ways, if you split into 2 plus 2 or something like that, need not be before some integer you want to split, the number of ways you can have, say, one 2 is the coefficient here, two 2's is the coefficient here, three 2's will be the coefficient here. You can have it in only one way. The number of ways you can have one 3, this is x cube plus x power 6. The number of ways you can have one 3 is this, two 3's will be the coefficient of this and so on. So, in general, if you expand this, say, take 1 by x into 1 minus x square into minus x cube 1 minus x power 4. Similarly, it is for 1 power 4. If you expand this what will you get? 1 plus x plus x square and so on. Those three terms are there. So I will write for 1 by 1 minus x power 4 that is 1 plus x power 4 plus x power 8 and so on. This has a meaning, for example when you consider the coefficient of x power 4; that will give you the value correctly up to 4 only. If you consider for 5 and all that may not have any meaning. See, you want to consider a partition of 4, then the fifth and sixth coefficient may not have any meaning because if you take 4 you can split it like this 1plus1plus1 plus1. So there is no way you can have five 1's right? So when we consider an integer n, powers greater than x power n do not have any meaning.

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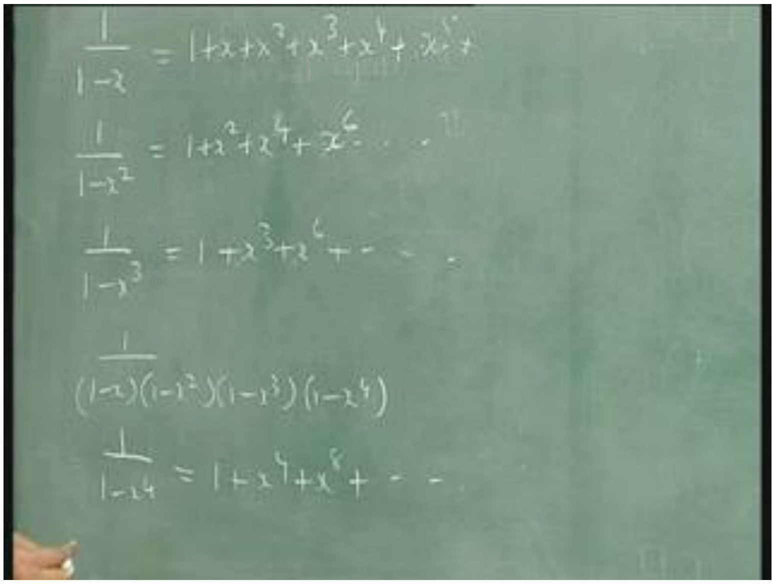

So if you consider the expansion of this you have to multiply this, this, this, and this. So what will you get? You will get the constant term as 1 and the coefficient of x can be multiplied with this, this and this. What will be the coefficient of x square? You can take x square from here and multiply with this and you can take x square from here and multiply with the other one. So it will be 2 x square.

What will be the coefficient of x cube?
You can pick x cube from here and 1 from the other three. That is 1. You can take x square, then you cannot multiply so that is ruled out. If you take x and x square and 1 1 from here you get one value. If you take 1 and x cube from here you will get one value, so that is 3.

What will be the coefficient of x power 4?
You can take x power 4 from here and 1 1 1 from here. You can take x power 4 from here. x power 4 from here, then x power 4 from here then 1 1 1 here, x power 4 from here 1 1 here, you can take x square x square from here, you can take x and x cube from here. So these are the five ways of getting it and this 5 is the number of ways in which you can partition 4. This we have seen. How do we get this? Four expressions are there. So, from the first you choose x power 4 and the second one you take the constant term or you take x square from the first one and x square from here and the constant term from here. Or you take x from here and x cubed from here and the constant term here. Or you take x

power 4 here and 1 from the other, or you take x power 4 from here and the constant terms here. This amounts to the partition, partitioning into 1's only having components 1 and this amounts to the partition 1 plus 1 plus 2. This amounts to partition 1 plus 3, this amounts to partition 2 plus 2 and this amounts to partition 4. So, this coefficient will give you the number of ways you can partition an integer.

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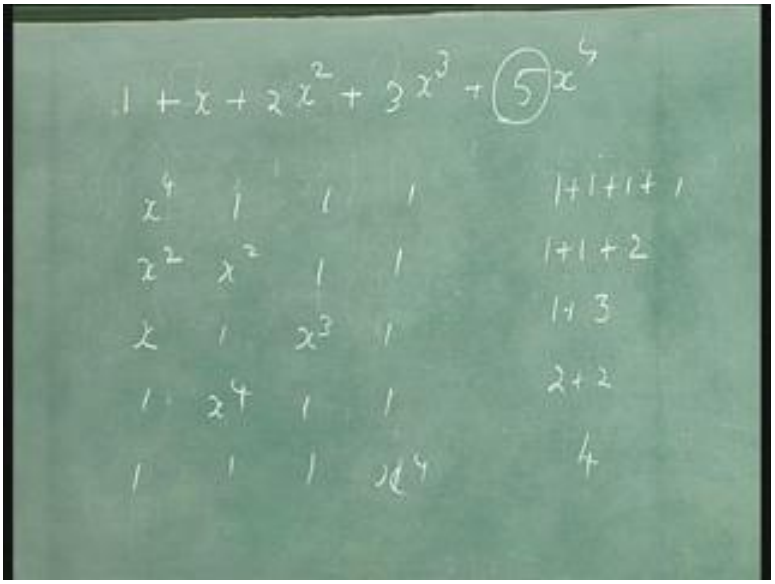

You can look at it as the partition of the integer n is equivalent to a way of distributing n nondistinct objects into n nondistinct cells.

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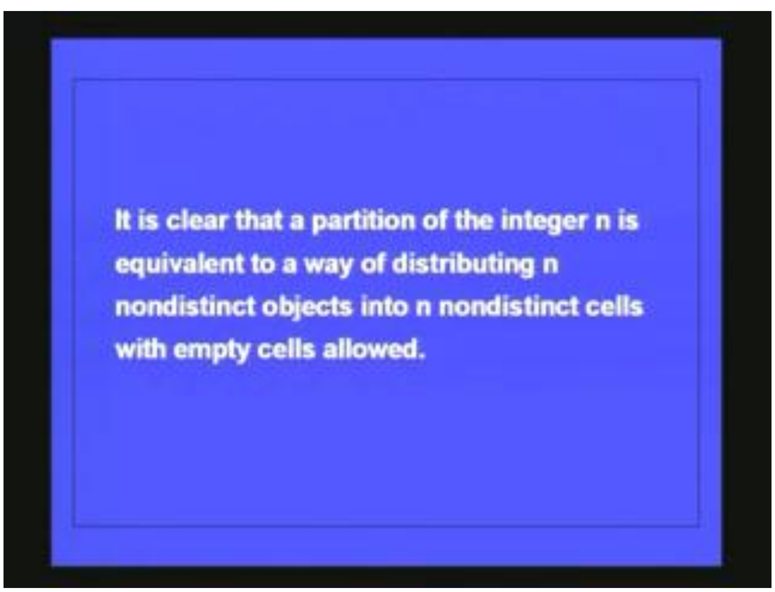

This can be graphically represented with dots and that is known as Ferrers graph.

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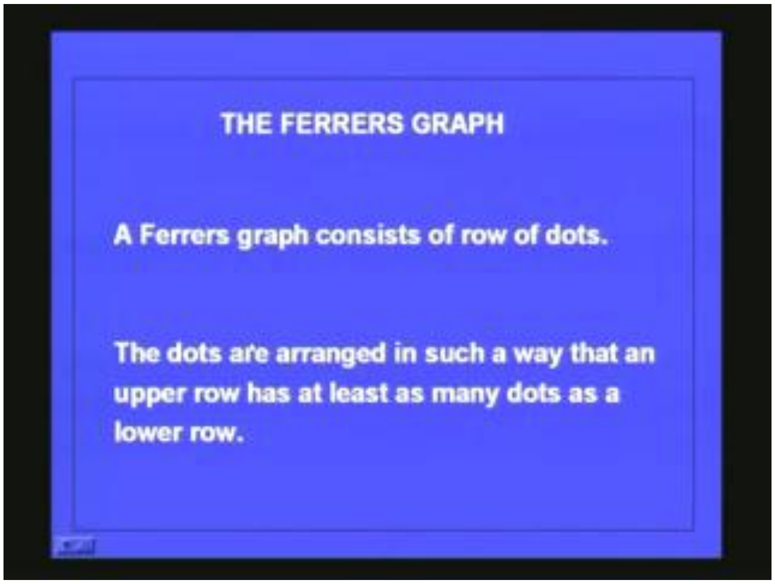

A Ferrers graph consists of rows of dots. The dots are arranged in such a way that an upper row has at least as many dots as a lower row. So suppose I take the number 14, it can be represented like this.

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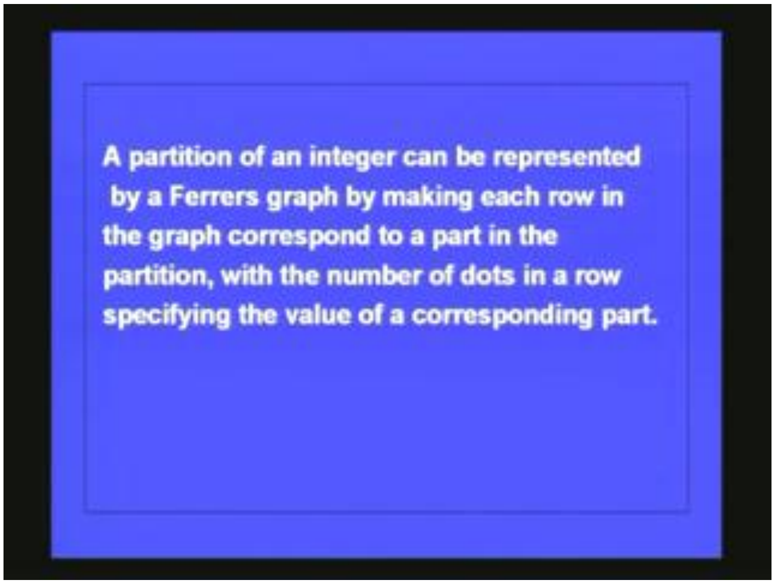

A partition of an integer can be represented by a Ferrers graph by making each row in the graph correspond to a part in the partition with the number of dots in a row specifying the value of a corresponding part. For example, if you take 14 and split in like this, 6 plus 3 plus 3 plus 2, the Ferrers graph corresponding to this is like this.

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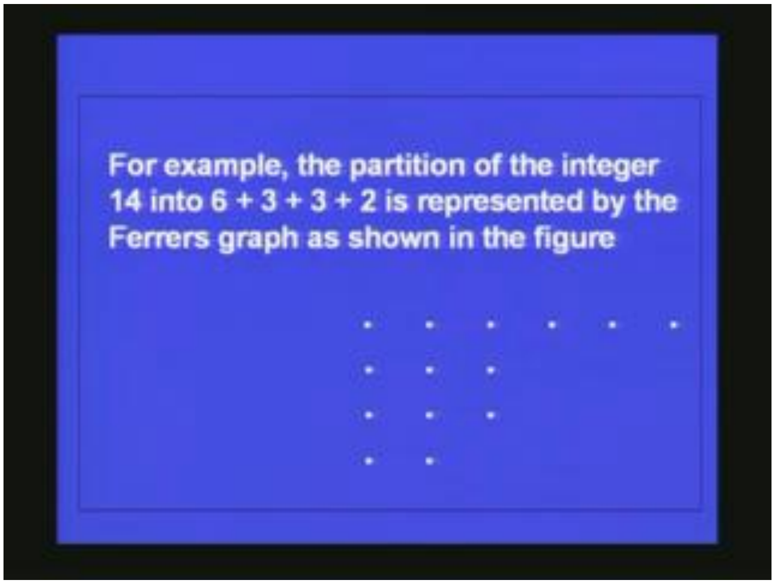

First is 6 then 3 3 and 2. Usually you arrange in such a way that the upper row is longer than the lower row or equal. This gives you some interesting results. That is, if you have a number of partitions, the number of partition is 4. The maximum value you are having is 6. So, if you interchange this and look at it in a slightly different way you can see that the number of partitions of a number in which you have the maximum number is 6 and it is equivalent to the number of partitions where the number is split into 6, because you have to just interchange the rows and columns. So some interesting properties can be derived from this Ferrers graph. Thus we have seen some enumerators as generating functions. We have seen the ordinary enumerators in the last lecture and today we have seen the exponential enumerator and this can be used for calculating the number of permutations of r objects out of n objects. And we have also seen how to distribute distinct objects into nondistinct cells and also nondistinct objects into nondistinct cell which is known as partition of integers.

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Next we shall see a very simple principle, this is known as the inclusion exclusion principle. Already we have informally seen this when we considered sets. Suppose you have a room in which there are 100 people, out of them say, 50 of them know Tamil and 40 of them know English and 20 of them know both English and Tamil.

What is the number of people who do not know either Tamil or English? You get the answer like this, 100 minus 50 plus 40 but this 20 you have counted twice, so you have to again add that 20. So from this you subtract 50 plus 40, but while doing that you have subtracted this 20 twice because this is the intersection. So you have to add that 20. So that will be 100 minus90 plus20 and so that will be 30. So the number of people who do not know either Tamil or English is 30. This is in essence the principle of inclusion and exclusion.

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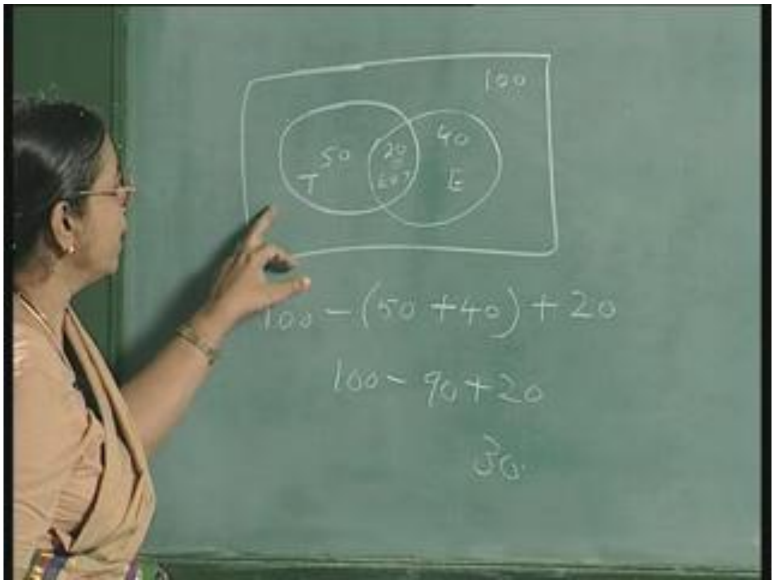

Let us formally see how we define this. Let A_1 be the subset of objects of S which have property P_1 and let A_2 be the subset of objects of S which have property P_2.

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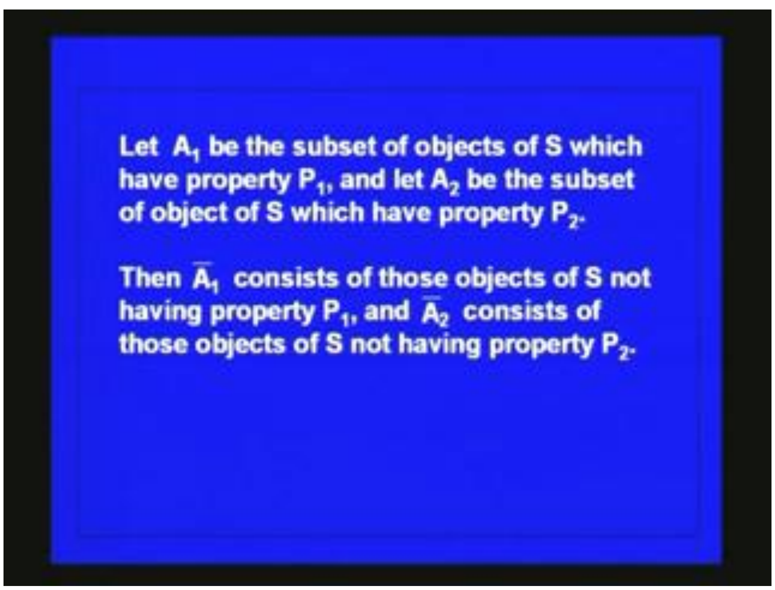

Then, A_1 bar consists of those objects of S not having property P_1 and A_2 bar consists of those objects of S not having property P_2. The objects of the set A_1 bar intersection A_2 bar are those having neither property P_1 nor property P_2. And that is given by this formula.

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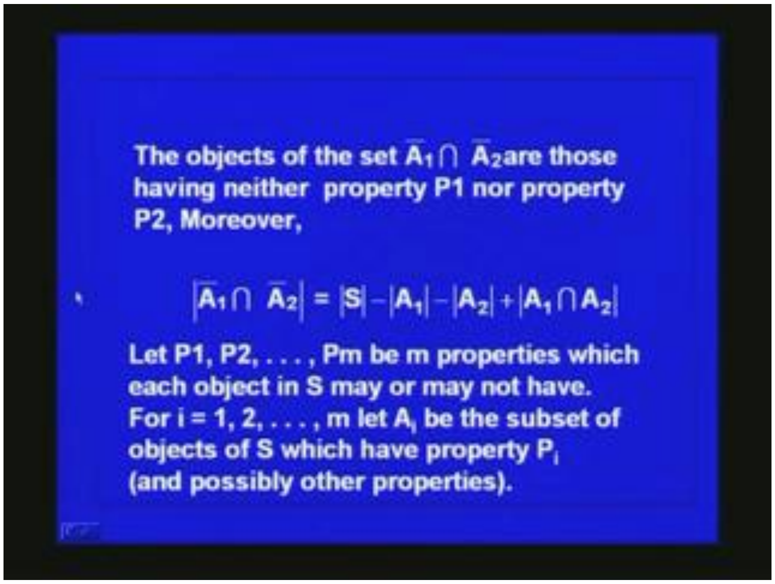

Number of elements in A_1 bar intersection A_2 bar is equal to the number of elements in S minus number of elements in A_1 minus number of elements in A_2. But since you have subtracted the intersection twice you have to add it, plus A_1 intersection A_2 bar. For example, here A_1 is the number of people who know Tamil and A_2 is the number of people who know English, then from the whole set you have to subtract the number of people who know Tamil minus the number of people who know English. But you have to add the number of people who know both English and Tamil. And that will give you the number of people who do not know either Tamil or English in that example.

You can generalize this; in this we have considered only two properties. We have generalized to m properties. Let P_1 P_2 and Pm be m properties which each object in S may or may not have. Then for i is equal to 1 to m, let A_i be the subset of objects of S which have property P_i. Then A_i intersection A_j is the subset of objects which have both the properties P_i and P_j and possibly others also.

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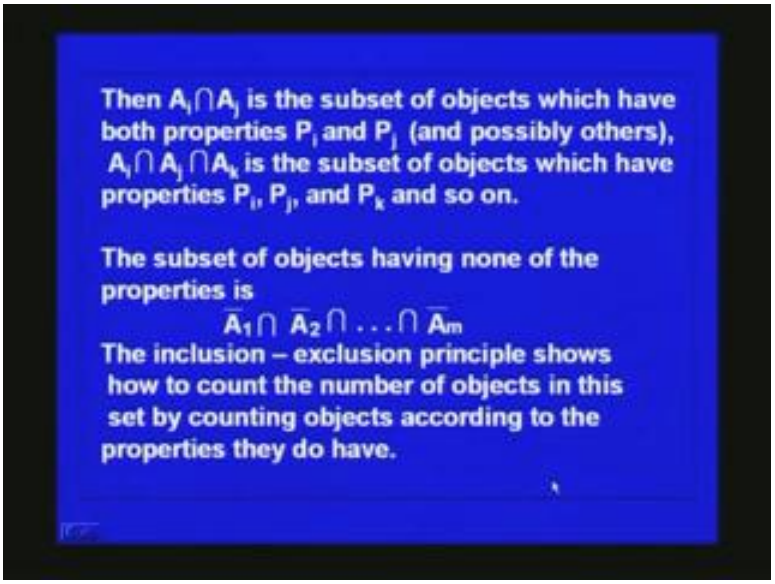

A_i intersection A_j intersection A_k is the subset of objects which have properties P_I, P_j and P_k and so on. Then the subset of objects having none of the properties is given by the formula A_1 bar intersection A_2 bar intersection A_n bar and what is the formula for this? That is what is given by the inclusion exclusion principle that shows how to count the number of objects in this set. By counting the objects according to the properties they do have. You want to count the number of objects which do not have either of the properties or any of the properties and how do you calculate it in terms of the number of elements having that property.

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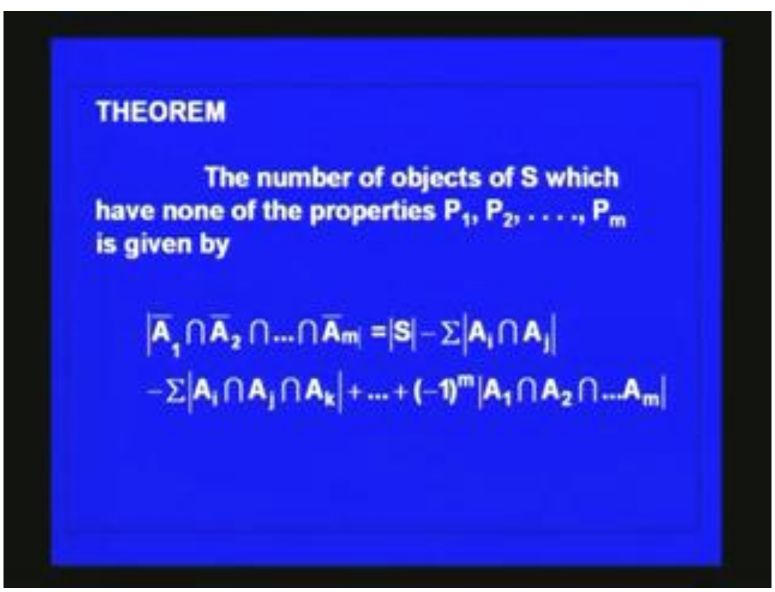

You know the value of A_1 you know the value of A_2 and so on. From that how will you calculate? The number of objects in S which have none of the properties P_1, P_2 and P_m is given by, A_1 bar intersection A_2 bar etc. The number of elements in this set is equal to the total number of elements minus, there is one term missing here, it will be like this, the number of elements A_1 intersection A_2 intersection Am bar equal to the number of elements in S minus sigma i A_i plus sigma i, j A ij having properties A_i and j that is you can say A_i intersection A_j minus sigma i, j, k. A_i intersection A_j intersection A_k and so on.

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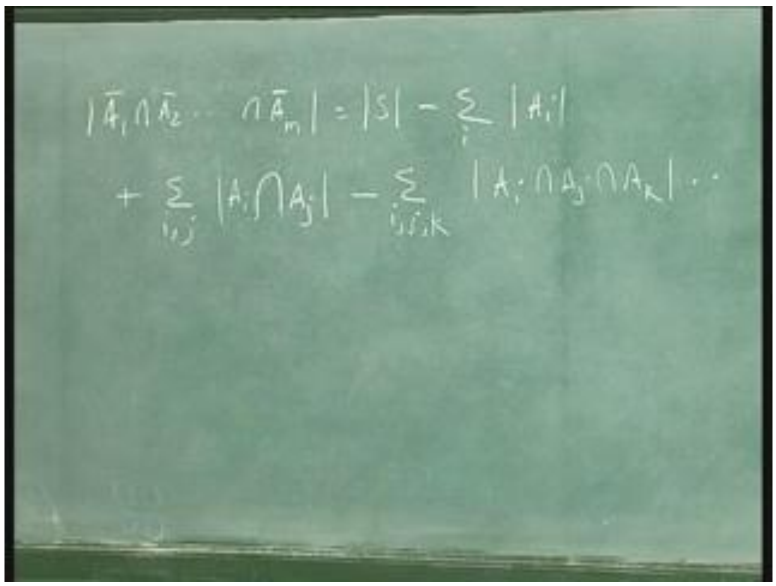

This term is missing here, so the general term will be minus 1 power m, A_1 intersection A_2 intersection A_m.

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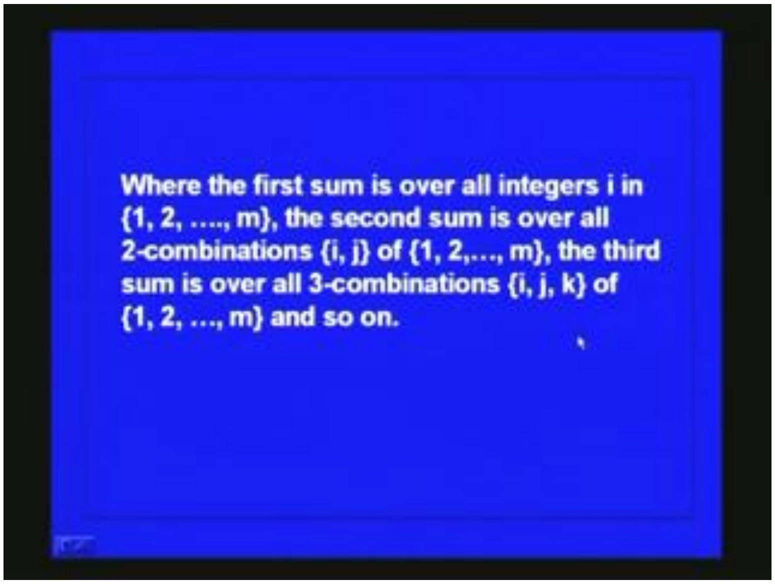

Here the first sum is over. All integers are i. Here you should take all integers ij such that i not equal to j, all two combinations from the set 1 to m, here you can take all the three combinations from 1 to m where the first sum is over all integers i in 1, 2, m, the second sum is over, all two combinations I, j of integers from 1, 2, m and the third one is over, all the three combinations of I, j, k from 1, 2, m and so on. So, this is plus then minus then plus then - and so on.

Suppose instead of that example, we have Tamil, English and Hindi speaking people in a room and the total number of people who do not know Tamil or English or Hindi that is equal to the total number of people in the room minus the number of people who know Tamil, the number of people who know English, the number of people who know Hindi plus the number of people who know Tamil and English plus the number of people who know English and Hindi plus the number of people who know Tamil and Hindi. In U. S. the number of people who know Tamil, English and Hindi. So, alternatively it is minus, plus, etc. Each sum is taken, here you are taking all the two combinations, here you are taking all three combinations and taking the sum.

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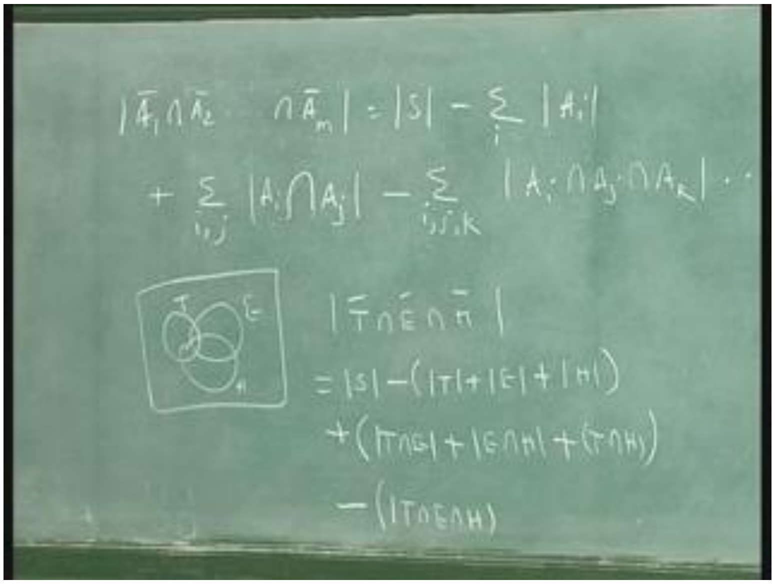

Let us consider one example making use of this, find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 5 or 6. You are considering numbers from 1 to 10,000 and numbers not divisible by 4, 5 or 6.

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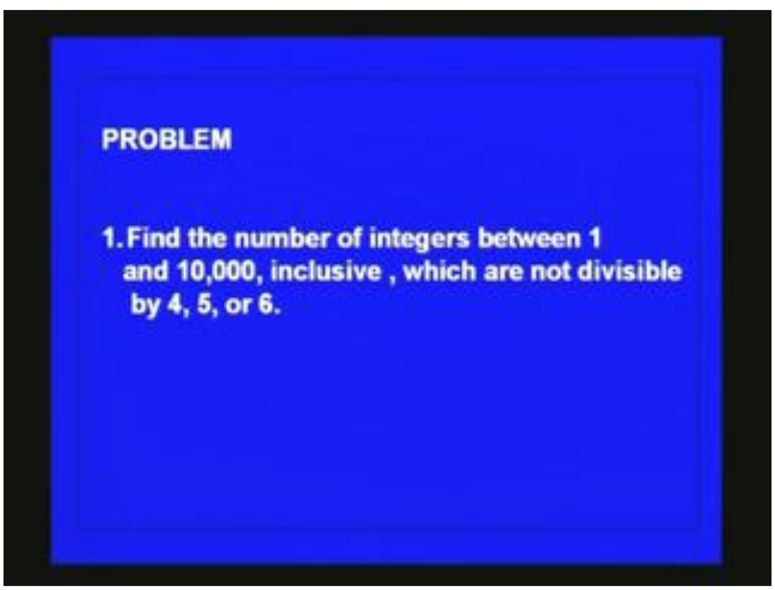

Let us consider the numbers between 1 and 10,000 divisible by 4. How many numbers in this are divisible by 4? We can very easily see that, if you divide this by 4, 2,500 of them are divisible by 4. How many of them are divisible by 5? Divide this by 5, you get 2,000. How many of them are divisible by 6? Divide this by 6, you will get 1,666. How many of them are divisible by 4 and 5? That is 4 and 5 that is 20. If you divide that by 20, it is

500. How many of them are divisible by 5 and 6? You have to consider the divisibility by 30 that will be how much? If you divide by 30 it will be 333.

How many of them are divisible by 4 and 6?
You must consider divisibility by 12 that will be 8 if you divide this by 12, 8 then 3. So how many of them are divisible by 4, 5 and 6? You have to consider divisibility by 60. Here you have to consider the divisibility by 12, here 30, here 20. If you divide by 60, the number will be 166.So the numbers not divisible by 4, 5 and 6 is from 10,000 we have to subtract the sum of 2,500 plus 2,000 plus 1,666 but add these three that is 500 plus 333 plus 833 and then again subtract 166.

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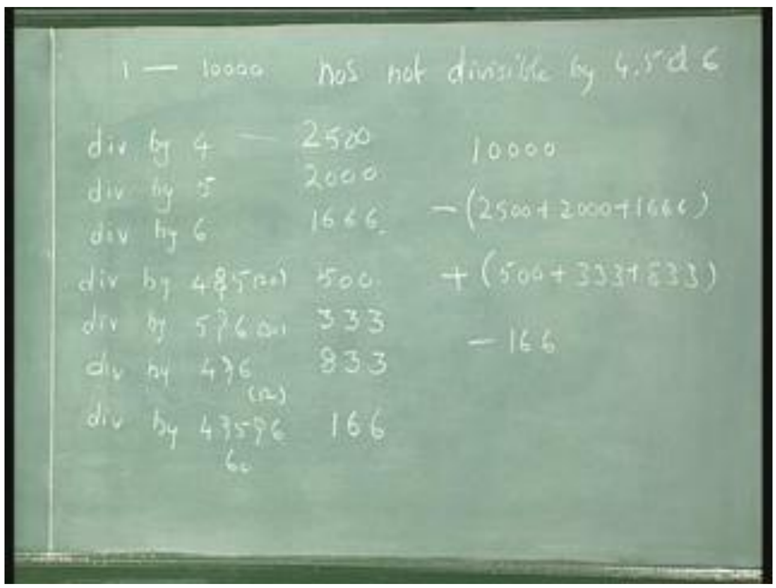

What will that be? 2,500, 2,000, 1,666 will add up to 6166 to which you have to also add 166, then 500, 333 will add up to 1666,you have to add 10,000 to this. So 11,666 minus 6,332 will give you 5334. The answer is 5,334.

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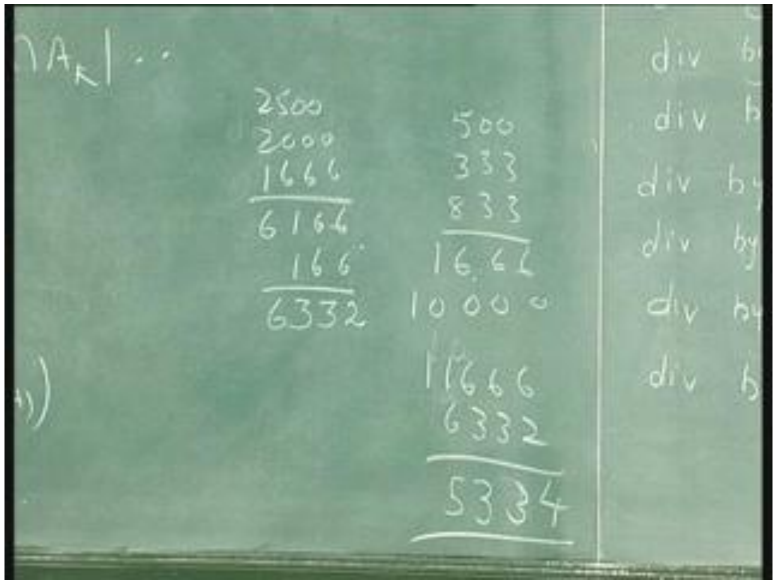

If you simplify this, the answer you get is 5,334. So, between 1 and 10,000, the numbers which are not divisible by 4 or 5 or 6 is equal to 5,334 which is five thousand three hundred and thirty four.

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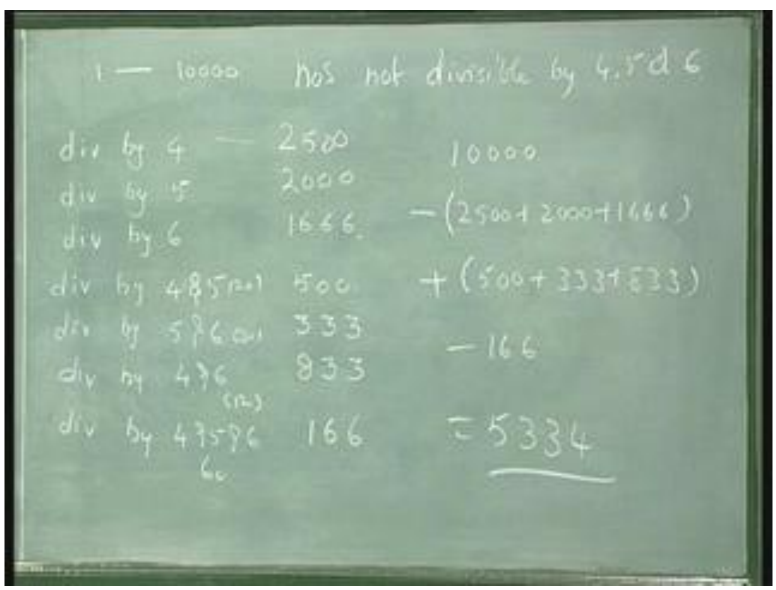

Let us consider one more problem, a bakery sells chocolate, cinnamon and plain doughnuts, chocolate doughnuts, cinnamon doughnuts and plain doughnuts. And, at a particular time, has 6 chocolate, 7 cinnamon and 3 plain. If a box contains 12 doughnuts, how many different boxes of doughnuts are possible? This is the question.

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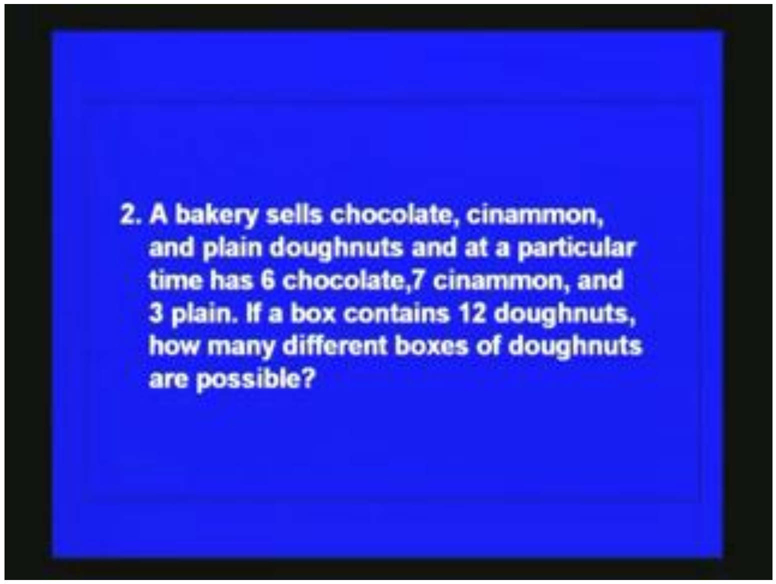

Let us see, how you can apply the principle of inclusive-exclusive for this. So, you are having 6 chocolate. So you are having three types of objects a, b, and c and in a, you are having 6 objects, in b you are having 7 objects and in c you are having 3 out of which you have to select 12. In how many ways can you do this? Now if you have unlimited number of them, suppose we are having unlimited number and then you want to select 12 out of them, that is with unlimited reputation it is given by the formula 3 plus 12 minus 1 by 12. That is without any limitation If you repeat, this is given by this formula, this is 14 by 12 is equal to 13 into 14 by 2, that is 91.

But, I can have only 6a’s, I cannot have more than 6a’s. So, suppose I have 7 a’s, I have to subtract those selections where I have 7 or more a’s. Suppose I select 7 a’s, the remaining 5 can be selected from a, b and c. So, greater than 6a’s, the number of search combination is, we have to select the remaining 5 from these three types of objects, so 3 plus 5 minus 1 by 5 and that is 7 by 5 and that is equal to 6 into 7 by 2 that is 21. If I have to also remove from this, where I can have more than 6b’s? So, more than 7b’s, suppose I have 8b’s, then such a selection is not possible. So from the total number of selection I have to subtract that. So if I look into that, if I have more than 7b’s, 8b’s or more, the remaining 4 can be selected from this. So that will be from all these three. So 3 plus 4 minus 1 by 4 and that will be 6 by 4 which is equal to 5 into 6 by 2 which is 15.

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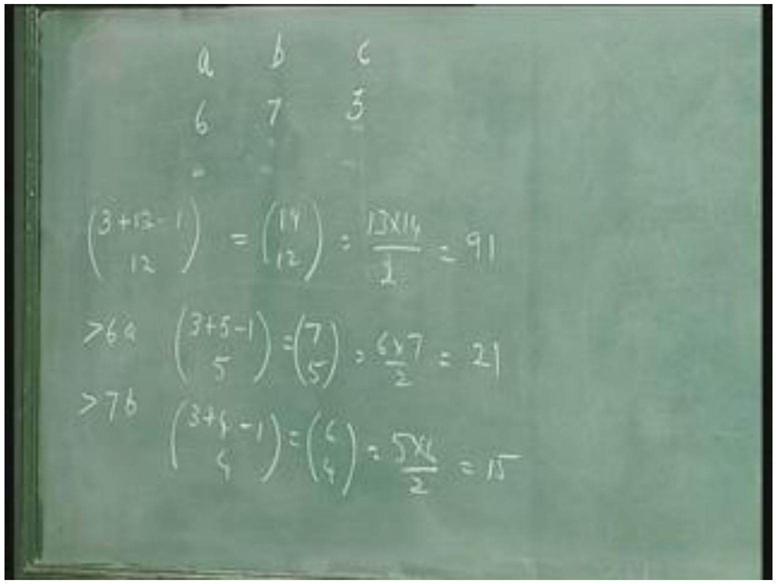

Similarly, if I select 4 or more c's, such a selection is not possible. So I have to subtract from this and then such a selection will amount to greater than 3c's, 4c's if I select then the remaining 8 can be selected from this. So that will be 3 plus 8 minus 1 by 8 that is equal to 10 by 8, this is equal to 9 into 10 by 2 is equal to 45. So, from 91 I have to subtract this, this and this. But I should add in that case using the principle of exclusion I have to also consider whether if you have selected more than 6a's, more than 7b or more than 3c, such a combination should be again added using the principle of inclusion and exclusion.

So if you select more than 6a and more than 7b, such a selection is not possible is it not? You have to select only 12 and 7 plus 8 would be in 15 which is not possible. Suppose I select more than 6a and more than 4c, what will happen? That is at least 7a's I should select more than 3c's, 7a's and 4c's, that is 11 elements I have already chosen. The remaining one I can select as a, b, or c. That can be done in three different ways.

Now if I select more than 7b and more than 3c, that is I have already selected 8b's and 4c's, 12 elements I cannot select any more a's. So this can be done only in one way. And there is no way you can select more than 6a, more than 7b and more than 3c this is 0. So, using the principle of inclusion exclusion the answer will be 91 minus 21 plus 15 plus 45 plus 3 plus 1 plus 0 minus 0 which should be 91 minus 81 plus 4 that is equal to 14. The answer is 14 and you get it like this.

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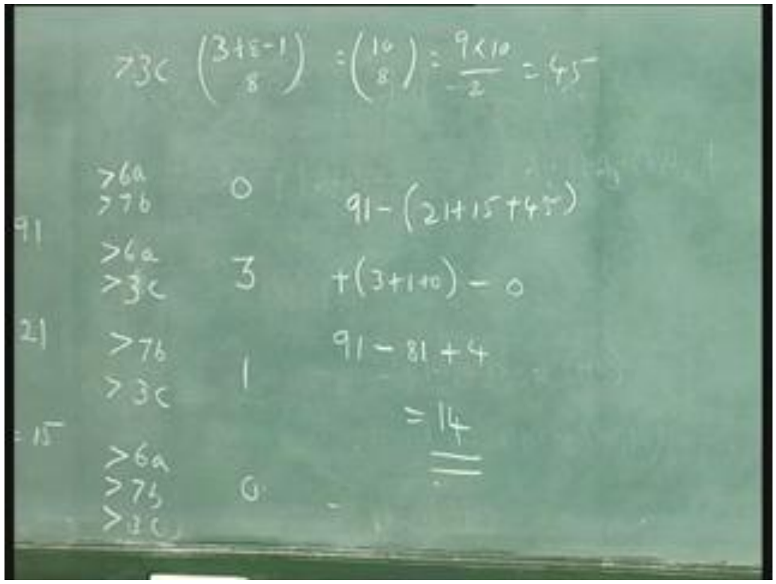

This problem you can also try using the idea of generating functions. I will leave it as an exercise for you to try this using the idea of generating functions. So in the last few lectures we have considered some principles from combinatorics, pigeon hole principle, permutation and combination, distribution of objects into cells, distinct and nondistinct objects, distinct and nondistinct cells, generating functions, principle of inclusion exclusion and so on. Making use of these generating functions also we can consider recurrence, relations and recursive functions which is our next topic.