**Discrete Mathematical Structures**
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**Lecture - 30**
**Generating Functions**

In the last two lectures we saw about Permutations and Combinations. P(n, r) denotes the number of Permutations of r objects out of n objects and C(n, r) denotes the number of Combinations of r objects out of n objects. And we have seen some results about this where repetitions are allowed and where different types of arrangements are considered and so on. We also saw that how to distribute distinct and non distinct objects into distinct cells and what is the connection between them and Permutations and Combinations. And today we shall look at the same thing Combinations and Permutations but from a different point of view making use of a tool called Generating Functions.

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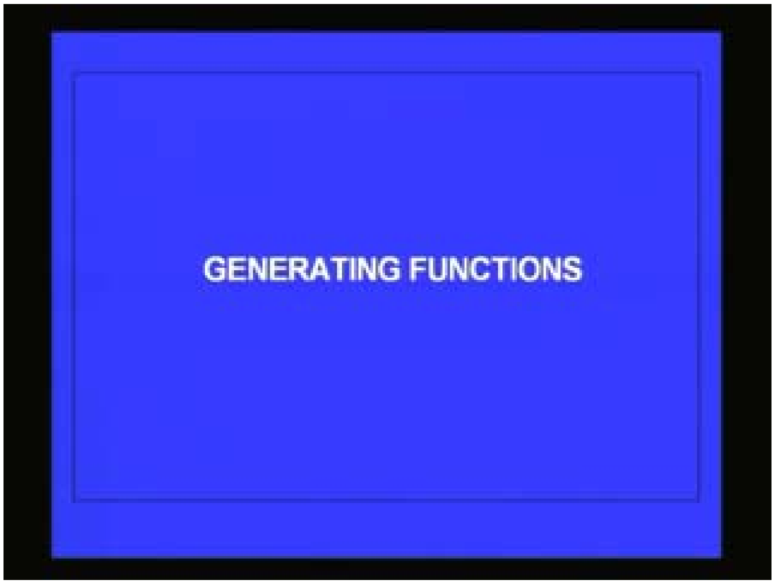


Here also we will see about Combinations and Permutations. But we are going to get the answers or the formulae using some other method, the concept of a generating function. So we shall consider generating function, we shall see two types of generating functions the ordinary enumerator and also the exp1ntial enumerator.

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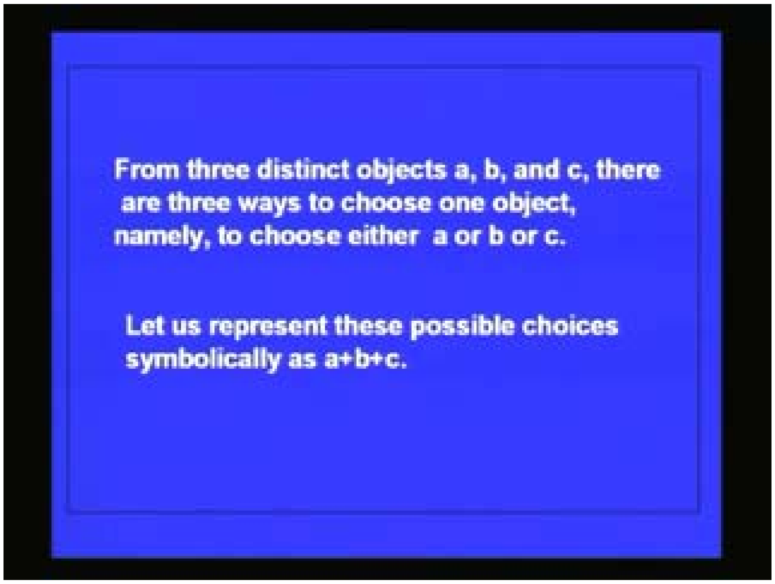


Now, from three distinct objects a, b and c there are three ways to choose one object namely you choose either a or b or c. Let us represent these possible choices symbolically as a plus b plus c. So if you have three objects a, b, c and you want to select 1 of them you can choose a or you can choose b or you can choose c that you can represent as a ba plus b plus ca or b or c.

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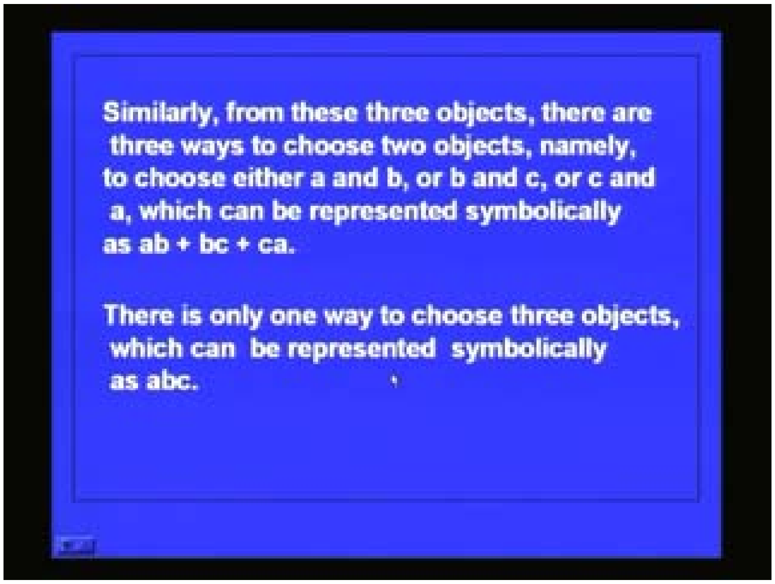


Similarly, from these three objects there are three ways to choose two objects namely you can choose two objects either a and b or b and c or c and a. And this can be represented symbolically as ab, a and b, b and c, c and a that is a and b or b and c or c and a. And if you want to choose all the three objects then there is only one way of doing it and that can be symbolically represented as abc.

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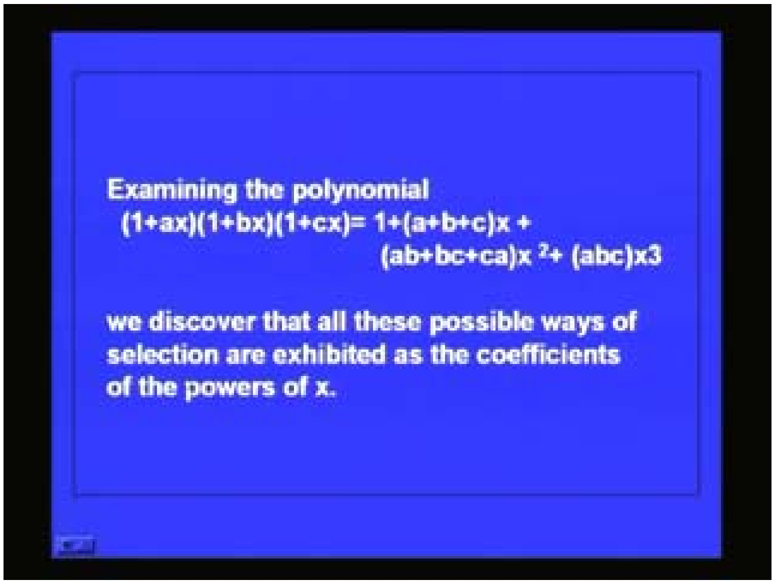


Now if you look at this polynomial (1 plus ax) (1 plus bx) (1 plus cx) if you expand this formula this is equal to 1 plus (a plus b plus c)x plus (ab plus bc plus ca)x squared (abc) x cubed. Now the coefficient of x power 1, x is x power 1 represents the number of or the possible ways of choosing one object out of the coefficient of x squared is (ab plus bc plus ca) and that represents the possible ways of choosing two objects out of the three objects that is (ab, bc or ca). The coefficient of x cubed represents the possible way of choosing all the three objects abc that is abc. Now we discover that all these possible ways of selection or exhibited as the coefficients of the powers of x.

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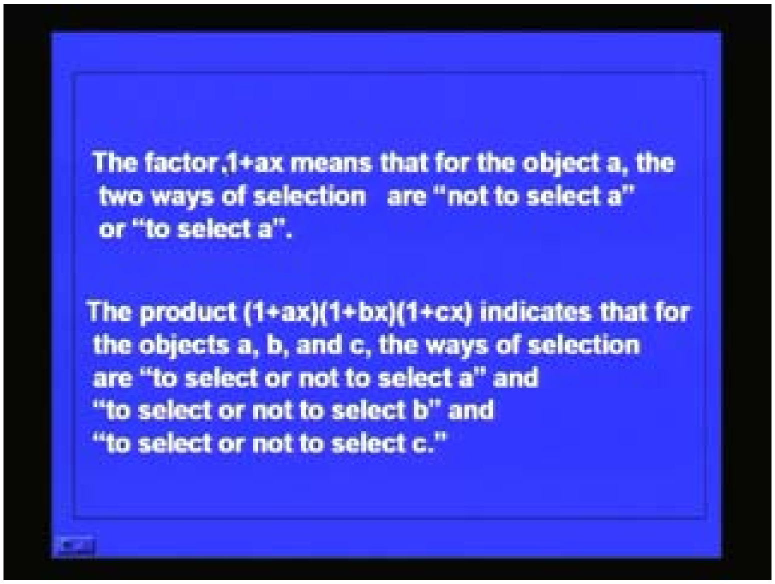


Now how do you interpret this? The factor (1 plus ax) means that for the object a there are two ways of selection, one is not to select a, that is one way of doing it and another one is to select a. So that is represented as 1 means not selecting a and the other is selecting a, ax so it is represented like this.

Now if you look at the left hand side of the previous equation this portion this is the product like this, this means that for the object a, b and c the ways of selection are to select or not to select a that is represented by this factor. And you can select or not select b and that is represented by this factor. You can select or not select c and that is represented like this. So together, this gives you the number of ways of selecting one object or two objects or three objects out of the three objects a, b, c and the coefficient of x power r and the right hand side gives you the possible ways of selecting r objects out of these three objects.

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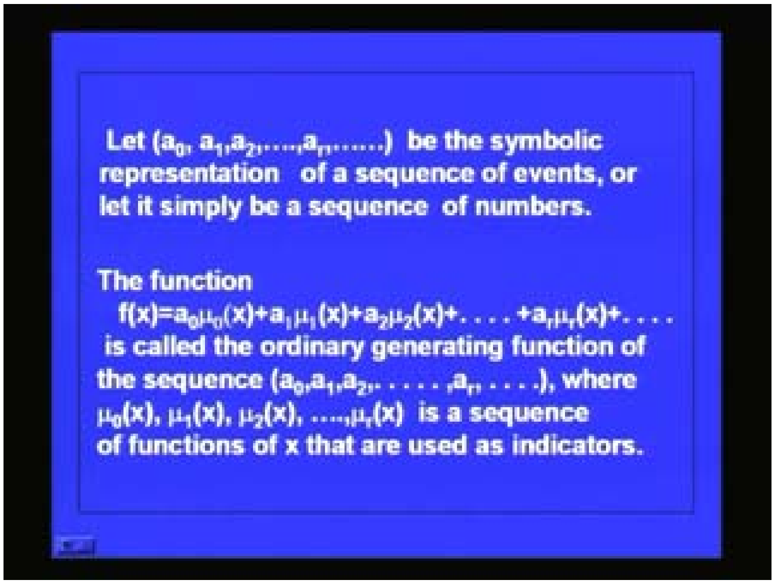


In general you can see that if you have a sequence ($a_0$ $a_1$ $a_2$ $a_r$) be the symbolic representation of a sequence of events or let it simply be a sequence of numbers, then the function f(x) is equal to $a_0mu_0(x)$ plus $a_1mu_1(x)$ plus $a_2mu_2(x)$ etc $a_rmu_r(x)$ is called the ordinary generating function of the sequence {$a_0$ $a_1$ $a_2$ $a_r$} where $mu_0(x)$, $mu_1(x)$, $mu_2(x)$, $mu_r(x)$ is a sequence of functions of (x) that are used as indicators. This is the sequence {$a_0$ $a_1$ $a_2$ $a_r$} and you are using the indicator functions this $mu_0(x)$, $mu_1(x)$, $mu_2(x)$ are called indicator functions.

Using these sequence of functions as indicator functions if you represent f(x) like this that is called the ordinary generating function of the sequence $a_0$mu(x) $a_1$mu(x) and so on. If you look back here the sequence is 1(a plus b plus c) (ab bc ca) (ab plus bc plus ca) and (abc) and if you use 1x, x squared, x cube etc as indicator functions this is the ordinary generating function.

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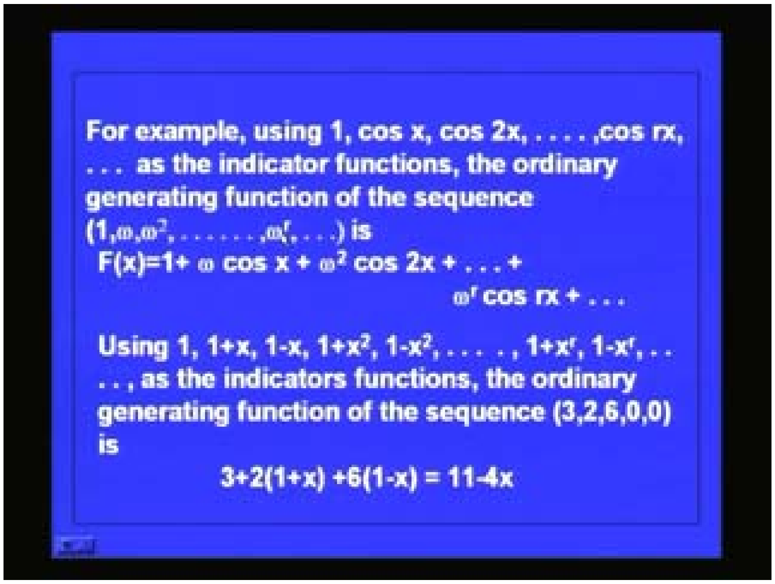


For example, if you use indicator functions 1 cosx cos 2x etc as indicator functions then the ordinary generating function of the sequence 1 omega omega squared omega cubed etc is given by this expression f of x is equal to 1 suppose here the function is 1 then omega second here you have cos x then you have omega squared third here is cos 2x so you have cos 2x here and so on.

Therefore, for omega r you will have cos r objects. So the expression is 1 plus omega cos x omega squared cos 2x etc omega power r cos rx. This is the ordinary generating function for the sequence 1 omega omega squared etc using the sequence 1 cos x cos 2x cos rx as indicator functions. Now you can choose the indicator functions in any way but you have to choose them in the proper manner that a particular sequence can be represented in a unique manner or one expression uniquely represents one particular sequence.

Two sequences should not give rise to a same expression or the function. Let us see using 1 (1 plus x), 1 minus x, (1 plus x squared), 1 minus x squared etc as indicator functions. Suppose (1 plus x squared) r 1 minus x squared r as indicator functions and suppose you take this sequence 3, 2, 6, 0, 0 as a sequence what will be the ordinary generating function? So 3 into 1, 2 into (1 plus x), 6 into 1 minus x, 0 into (1 plus x squared), 0 into 1 minus x squared that is 3 plus 2 into (1 plus x) plus 6 into 1 minus x. If you simplify this will reduce to 3 plus 2 plus 6 is equal to 11, 2x minus 6x will give minus 4x. So the ordinary generating function becomes is equal to 11 minus 4 x for this particular sequence using these as indicator functions.

Now this expression should not represent another sequence for the same set of indicator functions. In that case it is not advisable to use them as indicator functions. Look at this sequence; 1, 3, 7, 0, 0 and 1, 2, 6, 1, 1 with the same set of indicator functions what will you get as the generating function? It is 1 into 1 then 3 into (1 plus x), 7 into 1 minus x, 0

into (1 plus x squared), 0 into 1 minus x squared and so on. Let us see what you will get here?
You get 1 plus 3 into (1 plus x) plus 7 into 1 minus x this will reduce to 1 plus 3 plus 7 is equal to 11 and 3x minus 7x will give you minus 4x. The same 11 minus 4x you are getting. And it is representing this particular sequence as well as this sequence.

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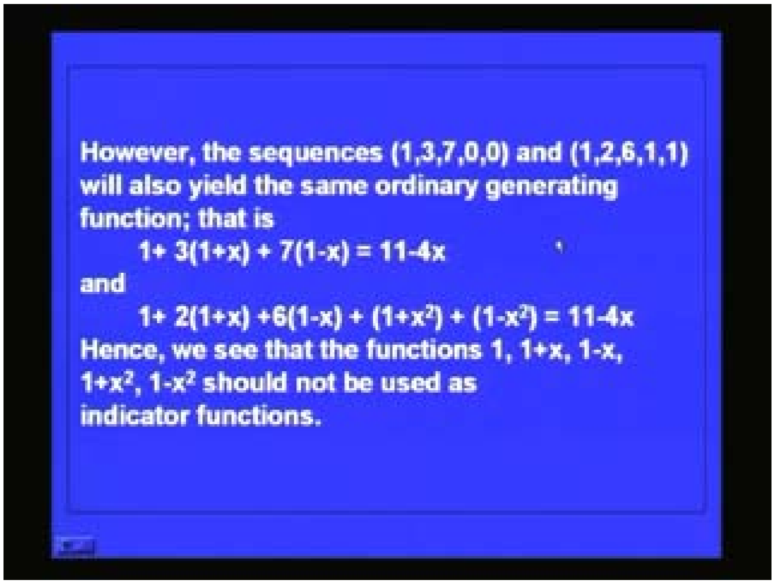


It also represents another sequence. Suppose you take the sequence as 1, 2, 6, 1, 1 let us see what the ordinary generating function is for the same set of indicator functions. You have 1 into 1 2 into (1 plus x), 6 into 1 minus x, 1 into (1 plus x) squared 1 into 1 minus x squared. If you simplify this you realize that you get 1 plus 2 plus 6 plus 1 plus 1 will give you 11 this x squared will cancel with this x squared and as far as the coefficient of x is concerned you have 2x minus 6x squared which will give you minus 4x. So, again you see that you are getting the same ordinary generating function. And it is representing 3 sequences which are not correct. So these functions you cannot use as the indicator functions. The indicator functions should be such that for each sequence the expression you get is unique or each expression uniquely represents as a sequence. Now, the easy way out is you choose 1x, x squared, x cube, x power 4, x power r etc.

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That is this sequence is taken as the set of indicator functions. In that case the coefficient of x power r gives you the value ar, the rth element in the sequence which you want to represent and there is only one way of doing it so there is no problem. So usually 1x, x squared, x cubed etc are taken as indicator functions. We shall restrict our consideration to only those sequences. Now, let us see how the generating function can represent a Combination.

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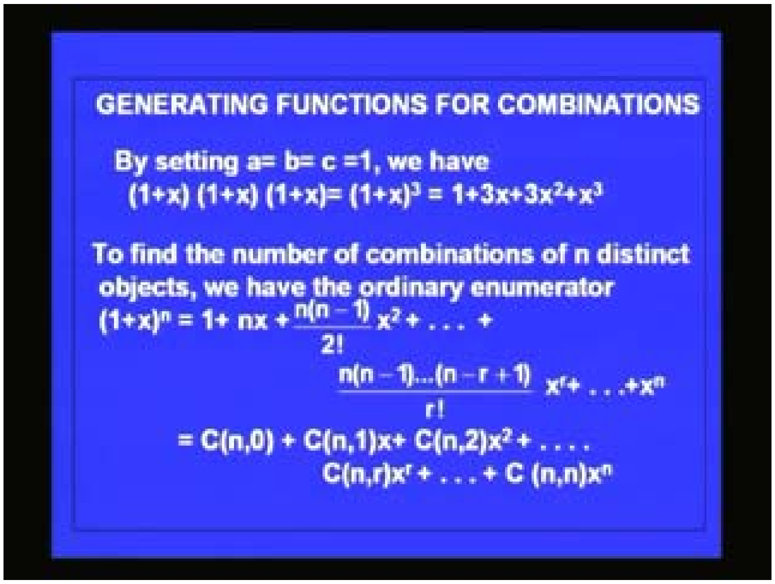


We have earlier seen how (1 plus ax) (1 plus bx) (1 plus cx) represents the way of choosing $a_0$, choosing a, choosing $b_0$, choosing b, choosing $c_0$, choosing c etc. Now put a is equal to b is equal to 1 in that expression then you will get (1 plus x) (1 plus x) (1 plus x) that is (1 plus x cubed) equal to (1 plus 3x plus 3x squared) plus x cubed.

What is this 3?
The coefficient represents the number of ways of not choosing any object out of three objects, this gives you the number of ways of choosing one object out of three objects, this gives you the number of ways of choosing two objects out of three objects and the coefficient of this gives you the way of choosing all the three objects. So the coefficient of x power r gives you the number of ways of choosing r objects out of n objects.

If you generalize this to find the number of combinations of a distinct object we have the ordinary enumerator ordinary generating functions, this is a binomial function binomial expression (1 plus x) power n is 1 plus nx plus n into n minus 1 divided by 2 factorial into x squared etc. Or you can represent this as C(n, 0) C(n, 1)x plus C(n, 2) squared and so on. The coefficient of x power 0 or the constant term here is the number of ways of using no objects out of n objects. The coefficient of x power 1 or x is given by this, this is the number of ways of choosing one object out of n object. The coefficient of x squared gives you the number of ways of choosing two objects out of n objects and the coefficient of x power r gives you the number of ways of choosing r objects out of n objects. Of course the last term, the number of ways of choosing all the n objects out of n object is 1 C(n, n) is 1 as we know.

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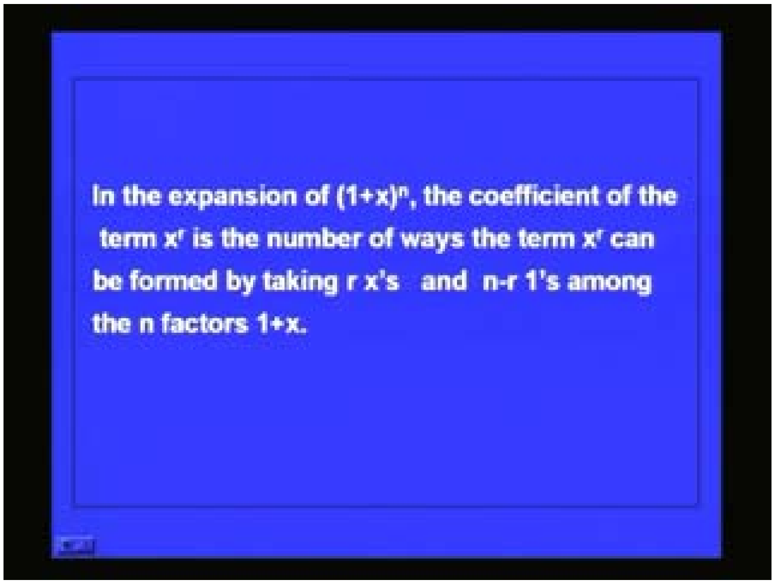


In the expansion of (1 plus x) power n the coefficient of the term x power r is the number of ways of the term x power can be formed by taking r x’s and n minus r 1’s among the n factors 1 plus x. That is, you are selecting r object and leaving out n minus r objects out of n objects.

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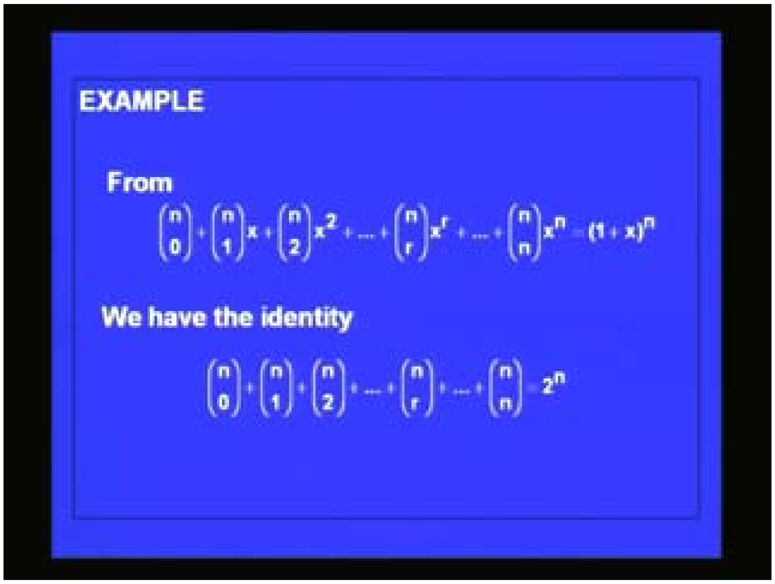


So using this generating function concept you can prove certain identities. For example, you have this; this is the expression you have seen earlier. Instead of writing C(n, r) we are writing it as (n, r). This also represents the number of ways of r objects out of n objects.

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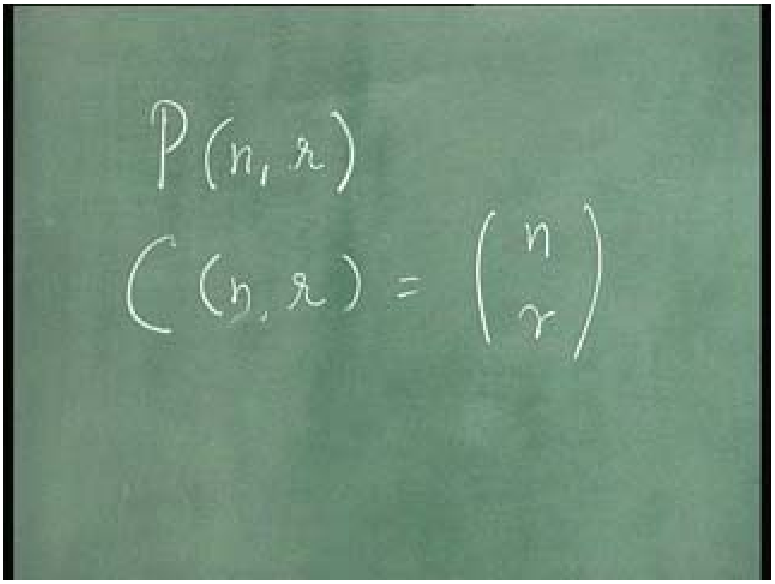


So C(n, r) can also be written in this form that is what we are making use of here for simplicity. Now this is the binomial expansion C(n, 0), C(n, 1)x plus C(n, 2)x squared etc is equal to (1 plus x) squared n. Now in this equation you put x is equal to 1. If you put x is equal to 1 here this becomes 2 power n and here if you put x in every place you get this, the left hand side becomes this.

What is this?
This is the number of ways of choosing zero objects or one object or two objects or three objects or n objects out of n objects. This is the number of ways of choosing zero object, this is the number of ways of choosing one objects, number of ways of choosing two objects and so on.

You can very easily see that from a combinatorial point of view you have n objects and for each object you have two possibilities either you can select or not select. So, for n objects there will be 2 power n possibilities. For each object you have two possibilities for n object the possibility will be 2 power n that is what you have in the right hand side. So the number of ways of choosing zero object, one object, two objects or n objects out of n objects that is a left hand side is seen to be is equal to 2 power n. Thus, from combinatorial point of view also we are able to see this. And we can get this from the binomial expression also using the area of generating function. This is what we have seen by setting x is equal to 1 we got that.

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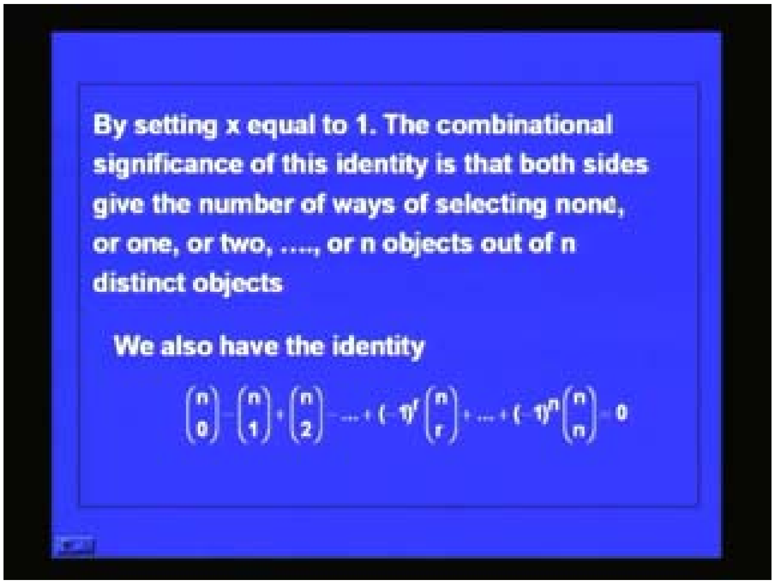


Setting x is equal to minus 1 in the previous one this will become 0 and alternate terms will become negative, this will be negative, this will be positive, this will be negative and so on.

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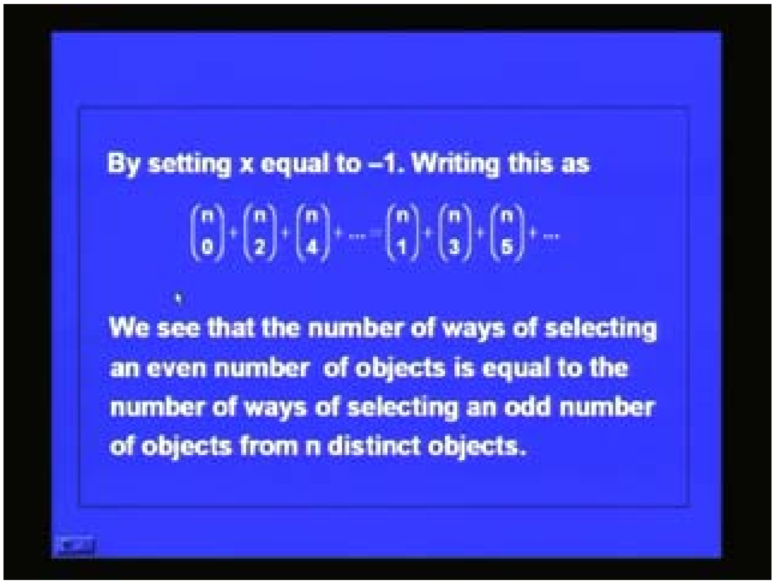


So you will get (n 0) minus (n 1) plus (n 2) minus 1 power r n r minus 1 power n n n is equal to 0 right hand side will become 0. So if you take all the negative terms to the right hand side this will be (n 0) (n 2) (n 4) and (n 1) (n 3) (n 5) C(n, 2) C(n, 4) etc.

What does the left hand side represent?
The left hand side represents the number of ways of selecting zero objects out of n objects or two objects out of n object or four objects out of n objects and so on.

What does the right hand side represent?
It represents the number of ways of one object out of n objects three objects out of n objects five objects out of n objects. So, the left hand side represents the number of ways of selecting an even number of objects from n objects. The right hand side represents the number of ways of selecting an odd number of objects from n objects. We see that the number of ways of selecting an even number of objects is equal to the number of ways of selecting an odd number of objects and n distinct object. Left hand side is equal to the right hand side so the number of ways of selecting an even number of object from n object is equal to the number of ways of selecting an odd number of objects from n object. This also we are able to get from the generating function idea. Now look at this, this is an identity.

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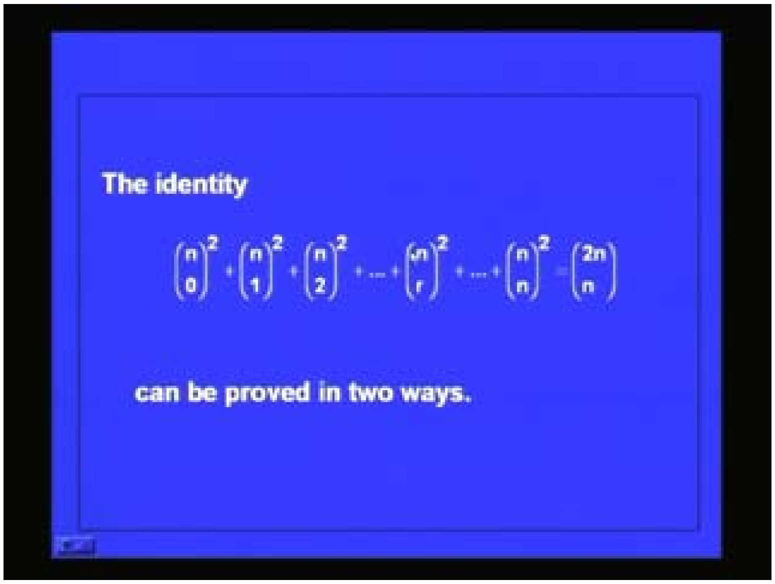


How can you prove this identity?
C(n, 0) squared (n 1) squared (n 2) squared etc C(n, 2) squared C(n, r) squared etc is equal to C(2, n) n this can be proved in two ways let us see how we can prove this. Look at the ordinary enumerator (1 plus x) power n that is (n 0) plus (n 1) x plus (n 2) x squared plus (n n) x power n.

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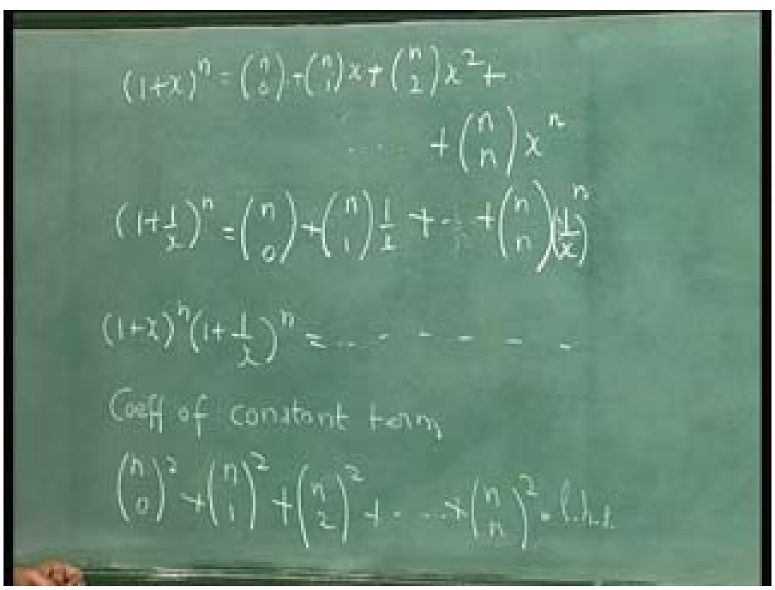


Look at (1 plus n) squared n that is equal to (n 0) plus (n 1) into 1 divided by x plus etc plus (n n) 1 divided by x power n. Now if you multiply these two you consider (1 plus x) power n into 1 divided by x power n where you have to multiply this whole thing by this.

Now, what is the coefficient of x power 0 or the constant term here?
The right hand side will be some expression.

What is the coefficient of x power 0 or the constant term?
This and this will give you the constant term, that is (n 0) the whole squared and this when you multiply with this will give you a constant term that is (n 1) the whole squared, this and the coefficient of 1 squared will give you the constant term so that is (n 2) squared and so on. And lastly when you multiply this by this you get a constant term and that is (n n) thus you get the left hand side of the expression which we want to put. So we want to prove this is equal to this so by considering this and this and multiplying out you get the coefficient of x power 0 or the constant term as this, which is the left hand side.

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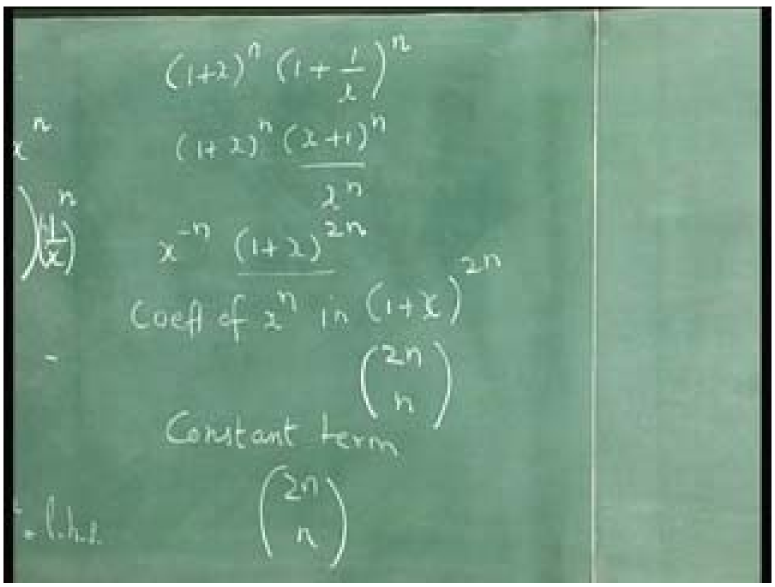


Now, how do you get the right hand side?
You have (1 plus x) power n this is what you are considering, (1 plus x) to the power of n, now this 1 you can write as (1 plus x) power n this x if you take it is x plus 1 power n divided by x power n or x power minus n this is (1 plus x) power 2n. In this expression if you take the coefficient of x power n when you multiply it by x power minus n you get a constant term. So what is the coefficient of x power n in (1 plus x) power 2n the coefficient of x power n in this is 2n power n when you multiply this by x power minus n you get the constant term. So if you look at it in a different way the constant here is 2n, instead of saying coefficient of the constant term I could have said constant term itself. The constant term is this and the constant term becomes equal to 2n n. So you find that you look at it in two different ways the expression (1 plus x) to the power of n into 1 plus 1 divided by x to the power of n then in one way you get the left hand side and in another way we can get the right hand side and they are equal. This is one way of looking at it.

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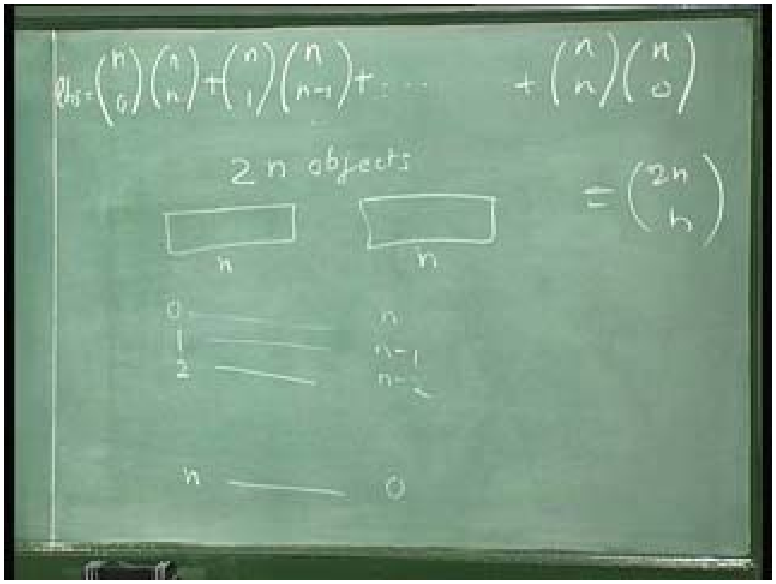


You can also prove it by combinatorial argument.
Now, the left hand side is n 0 whole squared but you know that n C r is equal to n C(n minus r) the number of ways of choosing r objects out of n objects is the number of ways of leaving out (n minus r) objects out of n objects. So this identity we know, so making use of that the left hand side will be is equal to n 0 and n n plus n 1 n n minus 1 and so on plus n n n cubed.

By a combinatorial argument what does this represent?
You have two n objects and you separate them into two sets of n objects. Now I want to choose n objects out of 2n objects, I can choose 0 from here and n from here which is given by this expression or I can choose 1 from this set n minus 1 from this set and that is given by this, I can choose 2 from this set and n minus 2 from this and that will be this.
In the next term and the last I can choose all the n from this and 0 from the second set, so that is given by this. So in essence it amounts to the number of ways of choosing n objects out of 2n objects and we know that that is given by 2n. So, by a combinatorial argument also you can prove this is equal to 2(n n).

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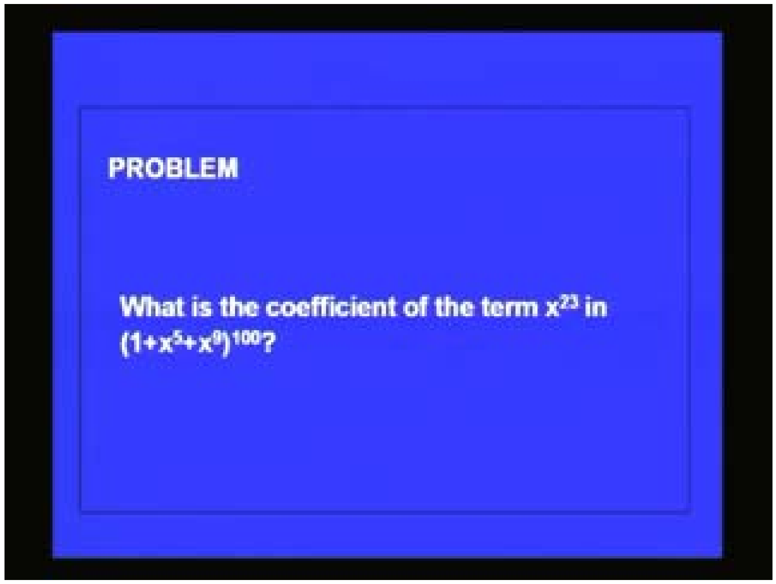


Now, let us consider a small problem, what is the coefficient of the term x power 23 in (1 plus x) power 5 plus x power 9 to the power of 100.

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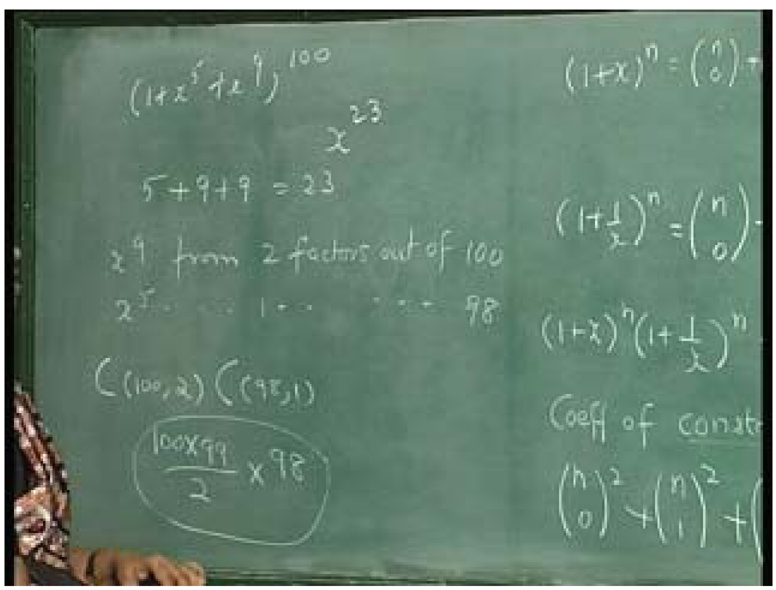


You have (1 plus x) power 5 plus x power 9 to the power of 100. You want to find the coefficient of x power 23 how can you find this out?

Now how can you get 23 by having 5s 9s etc?
There is only one way of doing it. You can have 5 plus 9 plus 9 this will give you 23. Otherwise if you have four 5s then you will have 3 which is not possible. If you have two

5s then also you will be left with 19 which you cannot account for, three 5s then you will be left with 8 which is not accountable. So there is only one way of splitting 23 into 5s and 9s and that can be done in only one way. So if you have hundred factors like this, out of the hundred factors you have to select 2 factors or x power 9 from 2 factors and x power 5 from 1 factor then only you will get x power 23.

So you have to select x power 9 from 2 factors out of 100 and you have to select x power 5 from 1 factor that is out of hundred out of the remaining 98. So it will be C 100, 2 from 2 factors you are selecting x power 9. After doing that you will be left out with 98 factors from which you have to select 1 x power 5. That can be done in C 98, 1. From the rest of the factors you have to select only 1. So this will give you the answer 100 into 99 divided by 2 into 98. You can multiply and get the answer. This is the coefficient of x power 23 in this expression. When you allow repetitions what happens? You are having three objects abc but you allow repetition now.

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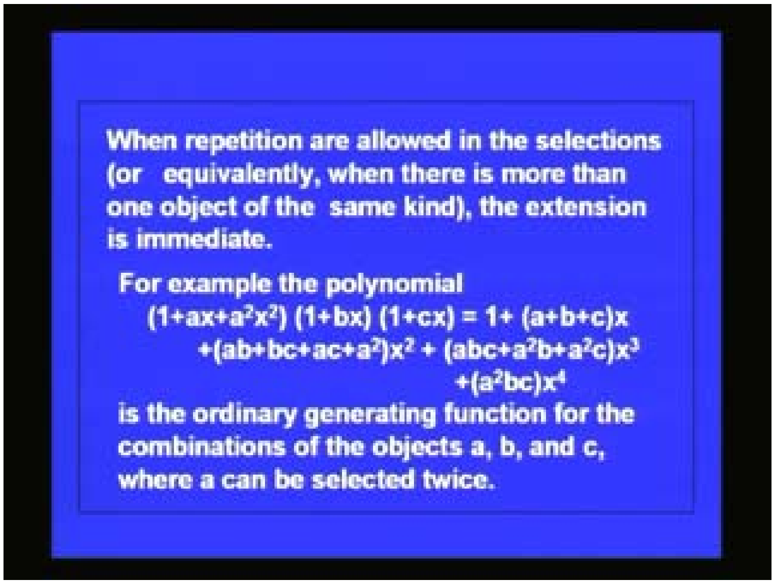


For a, you can choose it two times, either once or twice you choose and b and c you can choose only once. In that case how does the expression become or how does the generating function look like? When repetitions are allowed in the selections or equivalently when there is more than one object of the same kind or you can look at it like this you are having 2 red balls, 1 blue ball and 1 green ball from which you have to select 1, 2, 3 balls. The extension is immediate, you can look at it like this: 1 plus ax plus a squared x squared that is the object a need not be selected or it can be selected once or it can be selected twice. The object b can be not selected or selected once, the object c can be not selected or selected once.

Now if you expand this you get this which means if you want to select one object out of the three objects where a can be selected twice,

see when you select only once it does not matter whether you can select a twice or not, the coefficients become only a plus b. The coefficient tells you the number of ways of selecting one object out of three objects a, b, c where you allow a to be selected twice.
But when you look at the coefficient of x square the possibilities are like this: either you can choose a and b or you can choose b and c or you can choose a and c or you can select a twice. So a squared also is in this term.

Similarly, when you want to select three objects or select it twice is you can select a, b, c or you can choose a squared twice and b once or a squared twice and c once. And if you want to select 4 of them a has to be selected twice b and c also have to be selected. So this is the ordinary generating function for the Combinations of the objects a, b, c where a can be selected twice or you are allowing a to be selected twice. You must note the difference between this and the polynomial 1 plus ax into 1 plus c.

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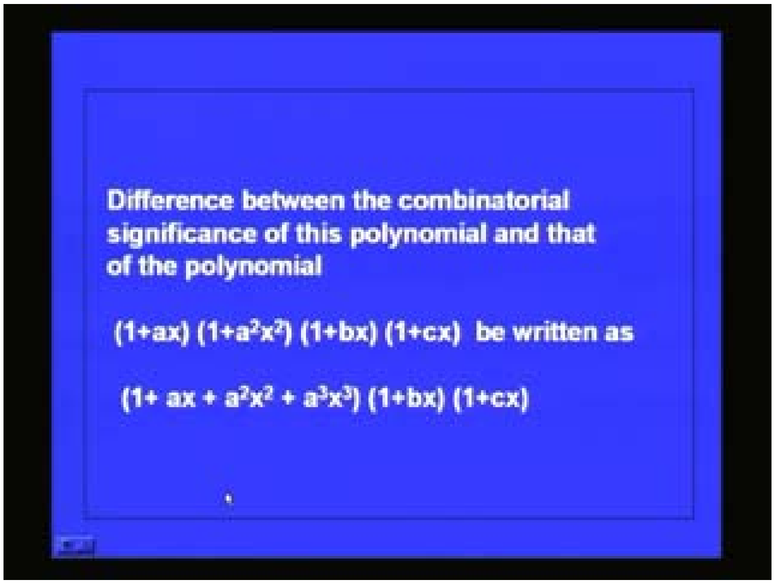


We have considered like this 1 plus ax plus a squared x squared for the selection of a. It is not the same as 1 plus ax into 1 squared x plus ax because in this case the expression will become like this, this is different. Here, you are allowing a to be selected once, twice, thrice and so on so that is not the same thing.

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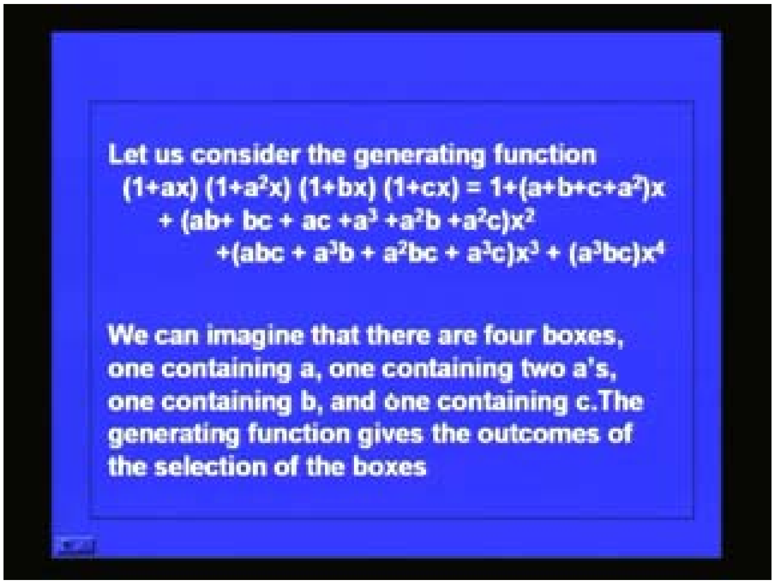


Therefore, if you look at this expression 1 plus ax into 1 plus a squared x1 plus bx into 1 plus cx etc if you expand it will be like this, what is this represent? It represents the following thing: We can imagine that there are four boxes one containing a, one containing two a's, one containing b and one containing c. The generating function gives the outcomes of the selections of these boxes. So you are having four boxes.

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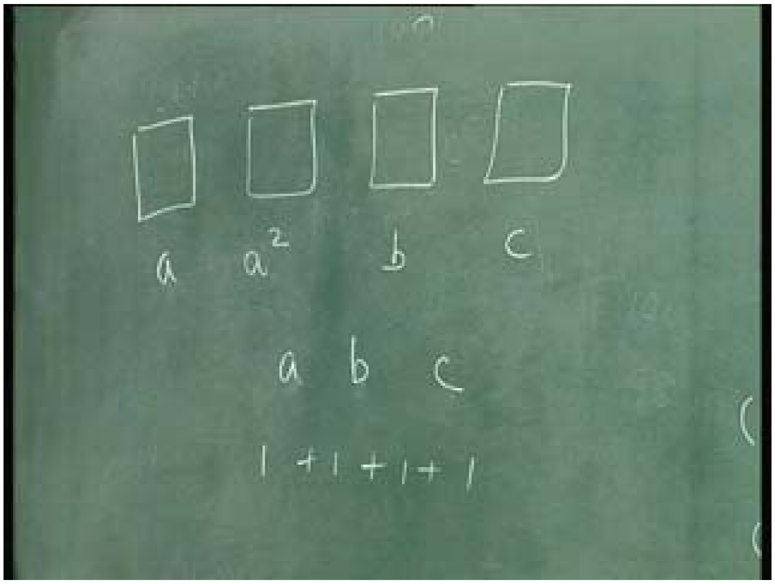


This box contains one a, this box contains two a's, this box contains b, this box contains c. If you select one of them what will be the outcome?

It could be a or it could be b or it could be c or it could be a squared. If you select two boxes then if you choose the box containing a and b you will get ab, you choose the box containing b and c you will get this, you choose the box containing a and c you will get this but if you choose the box containing a squared 1, another box you can choose so a squared and a will give you a cubed, a squared and b will give you a squared b, a squared and c will give you this.

Similarly, for the coefficient of x cubed and x power 4 so this represents that there are four boxes one containing a and one containing 2 a's one containing b and one containing c and this generating function gives the outcome of the selection of the boxes. When you select one box what happens and when you select two boxes what are the things inside that and so on. This is different from what we considered earlier where you have three objects abc one object is allowed to be selected more than 1 that is, a is allowed to be selected twice.

Now instead of putting abc if you put them as one the ordinary enumerator for the Combinations of the object abc where a can be selected twice is given by this instead of a you are just considering the number of ways only. So it is (1 plus x) plus x squared into (1 plus x) into (1 plus x), for b it is (1 plus x), for c it is (1 plus x), for a it is (1 plus x plus x squared) if you expand this you will get this. So, the number of ways of choosing one object out of three objects abc where a you can select twice is 3.

So if you have three objects abc the number of ways of selecting one of them is you can choose a, you can choose b, you can choose ab that is one, you can choose bc that is one you can choose ca that is one and a can be selected twice. If it is selected twice it gives you 1. So totally it is 4 and that 4 represents the coefficient of x squared like this. Like this you can get the coefficient of x cubed and x power 4 also.

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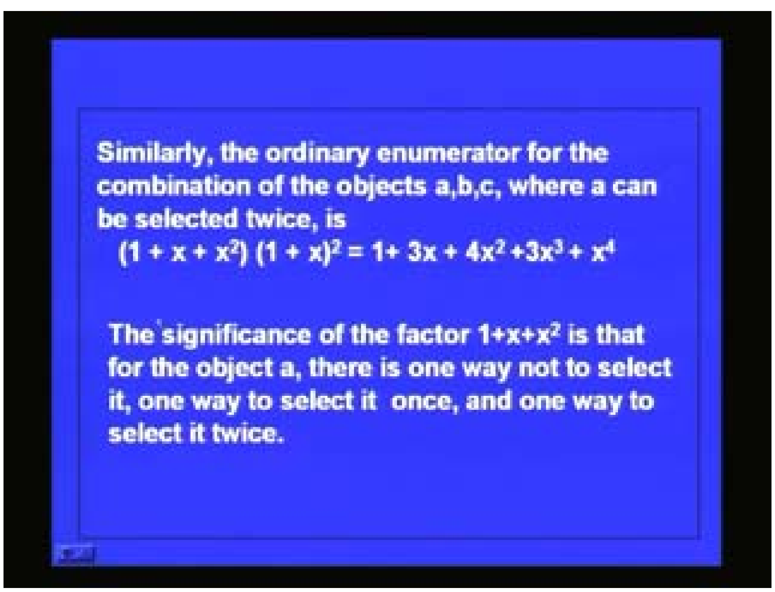

The significance of the factor (1 plus x plus x squared) is that the object a there is one way not to select it one way to select it once and there is one way to select it twice, that is what it meant by this expression (1 plus x plus x squared). Given two each of p kinds of objects and one each of q additional kinds of object in how many ways can r objects be selected? This is the problem, let us see how we tackle that.

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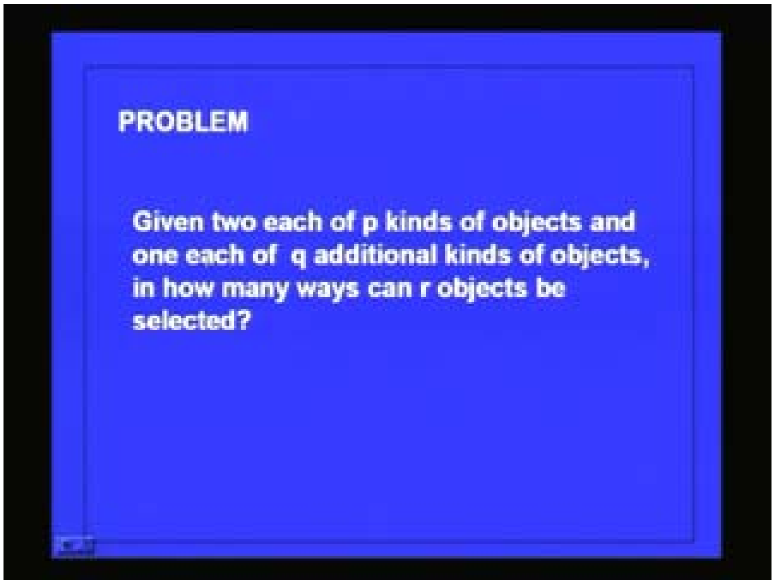


So you have p objects, they can be selected twice. Given two each of p kinds of objects so if you take one of them you can select it once or you can select it twice. This is the enumerator for the p kinds of objects.

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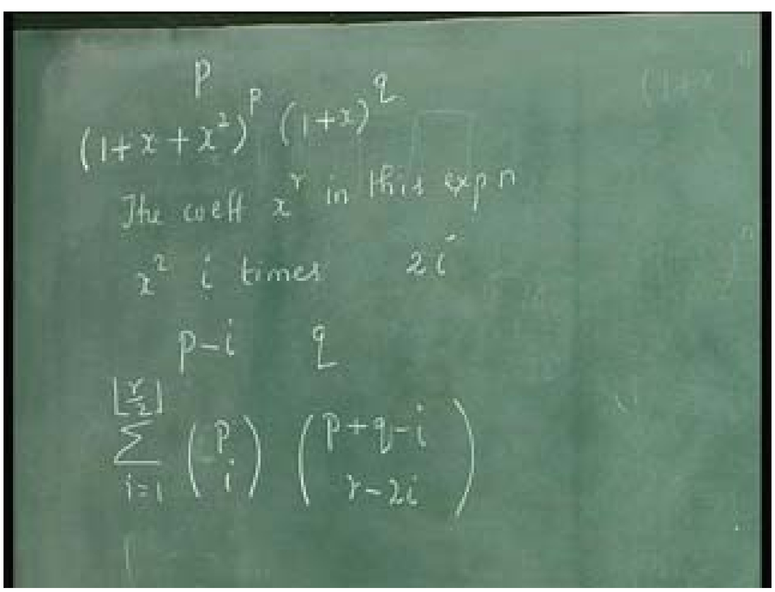

And you are having q additional kinds of objects, you have only one of them so either you can choose them or not choose them. So this is the enumerator for the expression. And generally see, first object you are having two of them, second object you are having two of them like that p of them, then q of them you are having only one. So the first object you need not select or you can select only one of them or you can select two of them.

Like that for all the p objects so the enumerator for that is this the ordinary enumerator. Then the remaining q objects you are having only one of them so either you can select it or not select it. And totally you are expected to select r objects out of the p plus q out of 2p plus, in that case what is the number? The answer is the coefficient of x power r in this expression.

Now, what is the coefficient of x power r in this?
Now you can select x squared i times then that will give you a power of 2I, then there will be how many factors remaining here?
There will be p minus i factors remaining here and already there are q factors. Out of these factors and these factors you have to select the remaining coefficient, x power r is the remaining coefficient you have to choose. So the idea is if you say it is sigma i is equal to 1 to r divided by 2. From this p factor we have to select pi. And from the remaining p minus i plus q factors that is p minus q plus factor you have to select and this r if this is even it is okay otherwise you have to select the integral part of that, this will give you the answer.

What does that mean?
It means that suppose r is 13, 13 means you cannot select x squared, 13 divided by 2 is 6.05. You cannot select 6 1 divided by 2 times so you have to select 6 times x squared maximum you can select is 6 times and 1 you have to choose from here. So from i is equal to i factors you are selecting here and that will account for 2 power i and the rest of them you have to select from the p plus q minus i factors. So, this will give you the answer for this problem.

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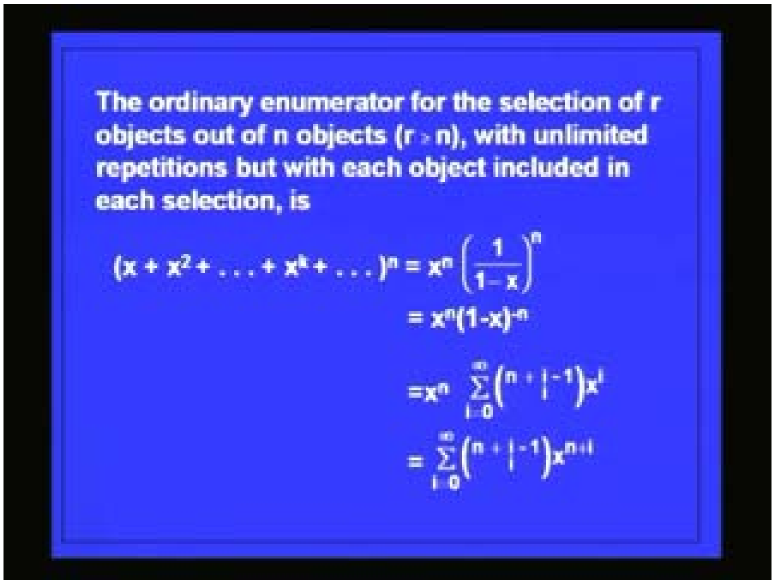


Now, the ordinary enumerator for the selection of r objects out of n objects where r is greater than or is equal to n with unlimited repetitions but with each object included in each selection is this. Now before that each object need not be included so you are selecting r objects out of n objects, what is the expression for this?
This we know is C(n plus r minus 1). So we are selecting r objects out of n repetition and the expression for that is C(n plus r minus 1) or r C(n plus r) both are equal, how do you get this?
You get (1 plus x plus x squared) etc. For one object it can be selected 1 need not be selected at all but for one object the enumerator is this. For n objects the enumerator is this and this is nothing but 1 divided by 1 minus x whole power n or is equal to 1 minus x to the power of minus n. In the binomial expansion this will expand as 1 plus n x plus n into n plus 1 by 2 factorial x squared and so on. The coefficient of x power r be n into n plus 1 etc n plus r minus 1 divided by r factorial x power r, if you expand using binomial theorem this will be the expression. Now the coefficient of x power r will give you the number of ways of selecting r objects out of n objects with unlimited expression and this is this expression and this is nothing but this.

We have earlier seen this by different arguments. Now, if you have to select each one of the n object at least once then what is the expression?
We will consider that, the ordinary enumerator for the selection of r objects out of n objects, now r has to be greater than or is equal to n because each one of them you are selecting at least once with unlimited repetitions but with each object is included in each selection that is given by this expression x plus x squared plus x power 4 etc to the power of n. For each object the possibility is it can be selected once, twice, etc.

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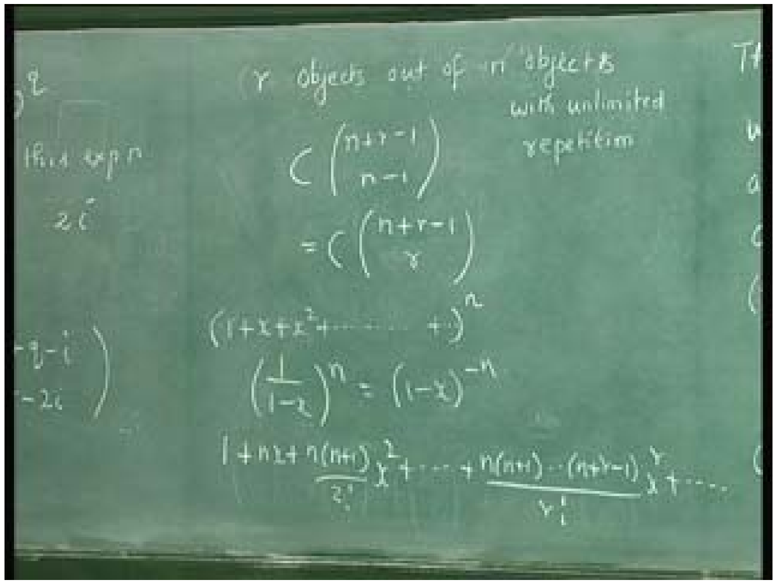


Please note that the constant term which was there in the earlier case is not here because you have to select each object at least once.

So the ordinary enumerator is given by this expression and if you take out the x power n this is equal to x power n into 1 divided by 1 minus x power n to the power of minus n. And this expression if you simplify this if you expand this portion and simplify x power n is like this, the expression for 1 minus x power minus n using binomial expression is sigma i is equal to 0 to infinity n plus i minus 1 i's that is C n plus i minus 1 i x power i. This is the way you expand this 1 minus x to the power of minus n. So this will be sigma i is equal to 0 to infinity n plus i minus 1 1 C that is the combination C n plus i minus 1 i x power n plus 1.

Now put (x plus i) with unlimited repetition where each object is selected at least once. So if you put (n plus i) is equal to r this previous equation here it becomes r so instead of saying i is equal to 0 to infinity you can say r is equal to n to infinity. So, this is x power r then instead of saying i is equal to 0 to infinity now it will be r is equal to n to infinity.

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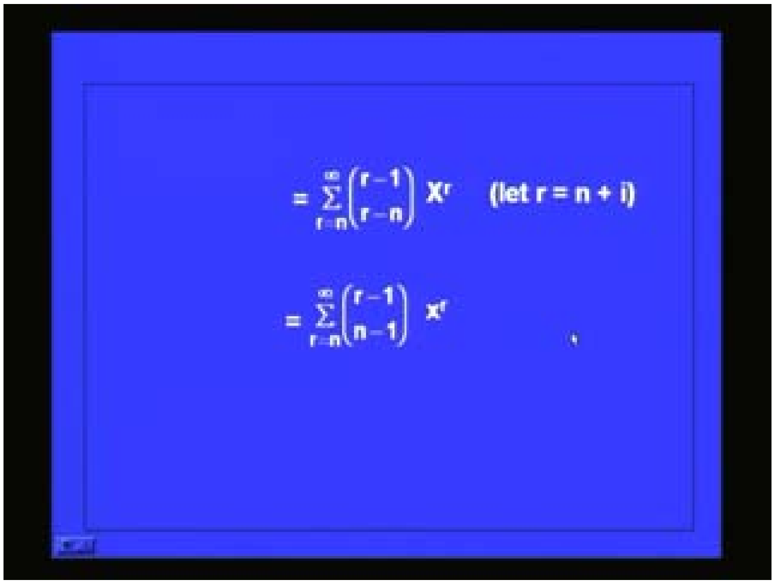


So the expression for that is r minus 1 rn r minus n, n plus i is r so it will be r minus 1I, r is equal to n plus i so i will be (r minus n). So this is rewriting that previous expression. It will be is equal to C r minus 1 r minus n x power r where the summation ranges from r is equal to infinity. Now the coefficient of x power r which is (r minus 1 r minus n) you can also rewrite this in this way because C(n, r) is equal to C n n minus r you can rewrite this expression like this. The coefficient gives you the number of ways of selecting r objects out of n objects where each object is selected at least once. We have seen this expression also in the case of distributing r objects into n distinct cells where each cell should contain at least one object and so on.

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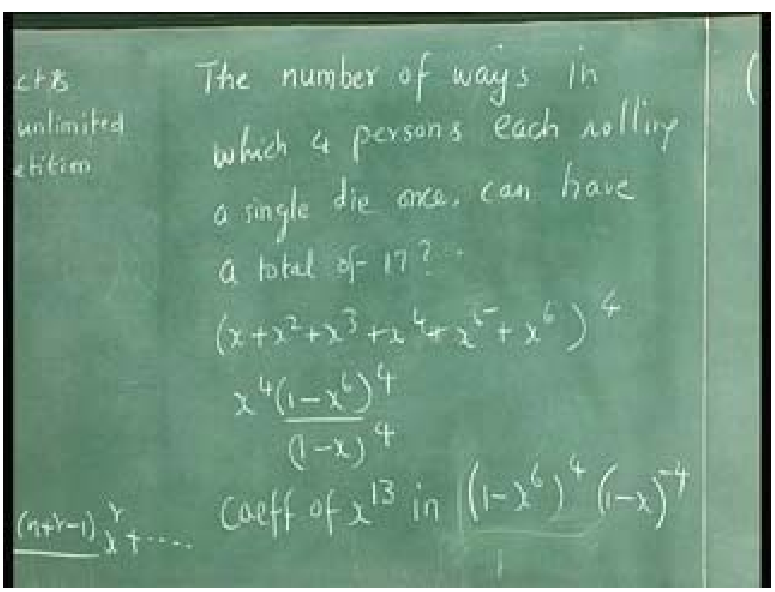

As an application let us see a simple problem. What is the number or ways in which 4 persons each rolling a single die once can have a total of 17. You are having 4 people each person is rolling a die. And the outcome of each die will be between 1 and 6. And what are the number of ways you can have a total of 17 from that. So for one person it could be 1 or 2 or 3 or 4 or 5 or 6 so the outcome can be like this.

You are having 4 persons so ordinary enumerator for that will be (x plus x squared plus x cubed plus x power 4 plus x power 6) to the power of 4. And you have to find the coefficient of x power 17 in this. So we have to find the coefficient of x power 17 in (x plus x squared plus x power 6) power 4.

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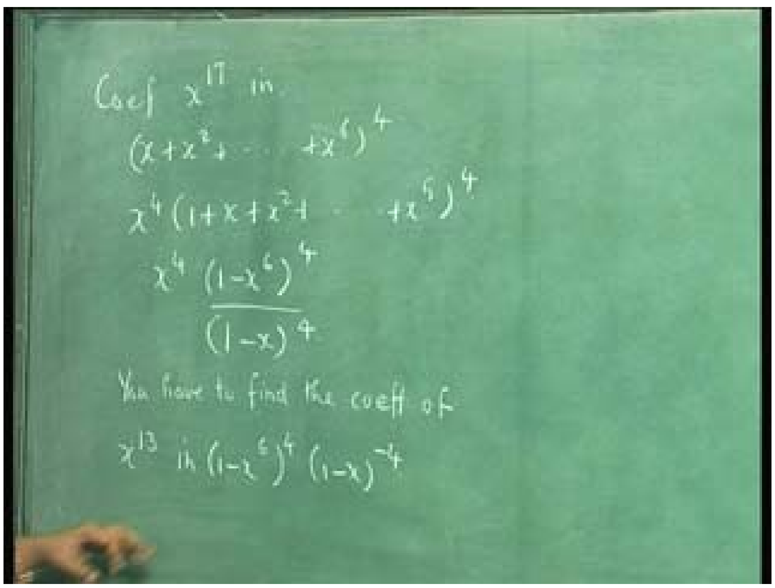


This we can write as x power 4 into (1 plus x plus x squared) plus x power 5 to the power of 4 and this is the geometric series so this you can write as x 1 to the power of 4. So in a sense you have to find the coefficient of x power 13 plus 4 you are accounting for here in 1 minus 6 to the power of 4 to 1 minus 6 to the power of minus 4. Let us see what is the coefficient, 1 minus 6 to the power of 4 is 1 minus 4x power 6 plus 6x power 5 minus and 1 minus x power minus 4 is 1 plus 4x plus 4 into 5 divided by 2 factorial x squared plus 4 into 5 into 6 divided by 3 factorial x cubed x squared. So, in the product we have to find the coefficient of x power 13.

How can you find x power 13?
You can multiply this by x power 13 here the term consisting of x power 13 here and you can multiply this by the term having x power 7 here and you can multiply this term by this term by this term.

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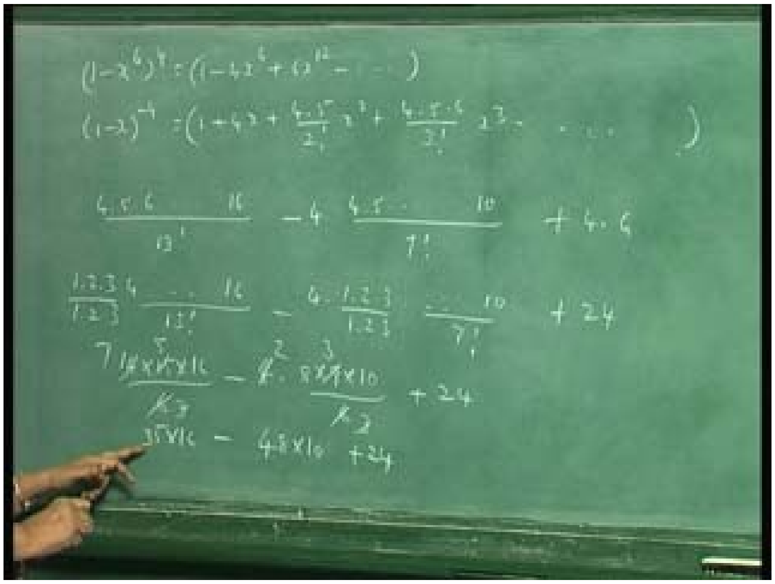

Therefore, in three ways you can get x power 13 so let us add those coefficients. So if you multiply this one by the coefficient of x power 13 here what will be the coefficient of x power 13 in this expansion? That will be 4 into 5 into 6 up to 16 divided by 13 factorial that is we are considering the coefficient of x power 13 then minus the product of this term with the coefficient of x power 7 here, x power 6 and x power 7 will give you x power 13. So that is 4 times 4 into 5 into 6 up to 10 divided by 7 factorial then if you multiply x power 12 divided by x you get x power 13 so these two will give rise to another x power 13 that is 4 into 6.

Thus, if you simplify this you can multiply and divide by 1 into 2 into 3 4 into 16 divided by 13 factorial minus four times 1 into 2 into 3 divided by 1 into 2 into 3 up to 10 divided by 7 factorial plus 24 and this will be 14 into 15 into 16 divided by 6 13 factorial will cancel off again here up to 7 will cancel off so this will give rise to 4 into 8 into 9 into 10 divided by 6 plus 24 and this is 7 3 5 35 into 16 minus 48 into 10 plus 24 and this will be 16 into 5 is equal to 80, 45, 560 minus 480 plus 24 plus 80 plus 24 is equal to 104. So there are 104 ways in which you can get 17 by rolling 4 dice. So we have considered some examples here and we have seen how the concept of a generating function and ordinary enumerator can be used to solve some problems in combinations, the number of ways of selecting.

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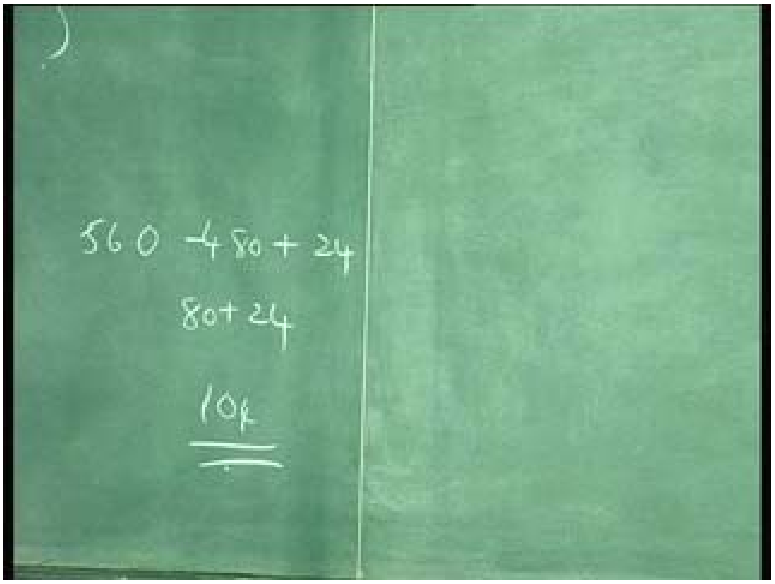


But when we consider the number of ways of Permutations or arrangement there is a slight problem. The very big advantage in using an ordinary enumerator is that the binomial coefficients C(n, r) gives you the number of ways of r object out of n objects. So you have a very simple closed form expression whereas the same will not be true in the case of Permutations. So in the case of Permutations what sort of an enumerator we should have so that we get a decent closed form expression.

So, in the next lecture we will consider generating functions for expressing the Permutations, the number of ways you can permute r objects out of n objects.