Discrete Mathematical Structures
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Lecture - 3
Predicates & Quantifiers

In the last two lectures we saw about propositional logic. We saw what propositions were and about what are the logical connectives. The logical connectives which we consider were AND, OR, NOT implication and equivalence. And we also saw some identities about logical expressions. We also saw how to simplify logical expressions. We also saw what is meant by a tautology, a contradiction and a contingency. Towards the end of the last lecture I left you with a problem. We shall consider that problem. This will tell you how you can make use of logic and get some problems solved. Here is the problem which we considered earlier.

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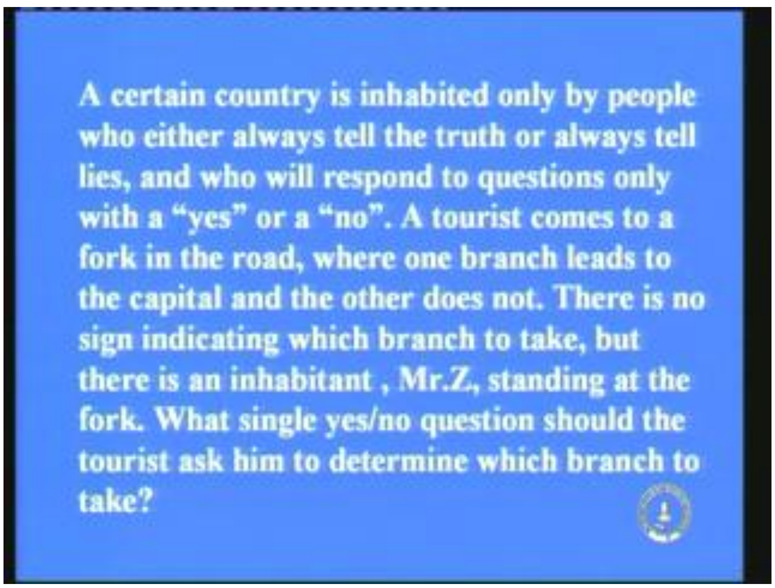

A certain country is inhabited only by people who either always tell the truth or always tell lies and who will respond to questions only with a yes or a no answer. A tourist comes to a fork in the road where one branch leads to the capital and the other does not. There is a sign indicating which branch to take, but there is an inhabitant Mr. Z standing at the fork. What single yes or no question should the tourist ask him to determine which branch to take? So the situation is like this; there is a fork but there is no sign board but Mr. Z is sitting over here and the tourist comes like this. Now he has to ask a single yes or no question so that if the answer is yes he must take the left road, this is the left road and this is the right road.

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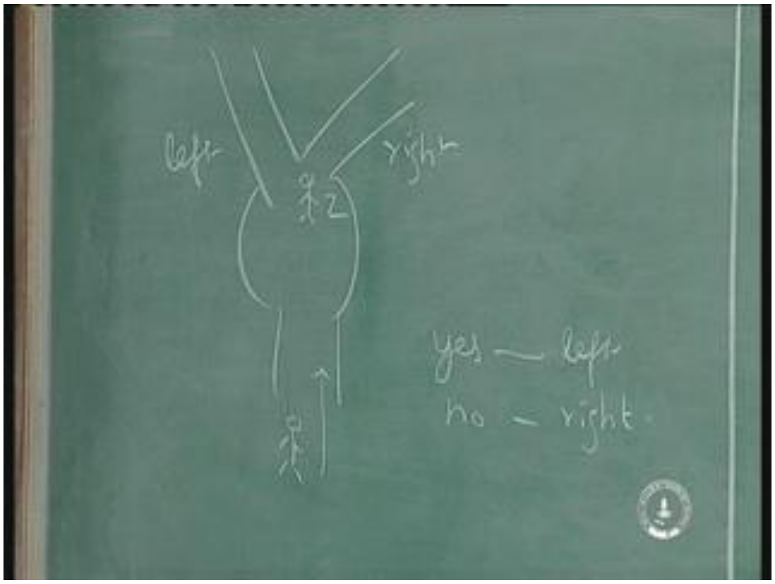

And if the answer is no he must take the right road. And there is a person sitting over there but this tourist does not know whether he is a truth teller or a liar. So there are four possibilities that arise like this; the left road may lead to capital or the right road may lead to capital. And the person may be a truth teller or a liar. Now, if the answer is yes he has to take the left road and if the answer is no he has to take the right road.

For all these possibilities this should happen. But the liar always lies and if the answer is yes he will say no and if the answer is no he will say yes. So the correct answer to the question the tourist has to ask is this, left, right, truth teller because the liar always inverts the answer, the correct answer should be like this. So the question will be something like this; is it true that you are a truth teller and the left road leads to capital or you are a liar and the right road leads to capital.

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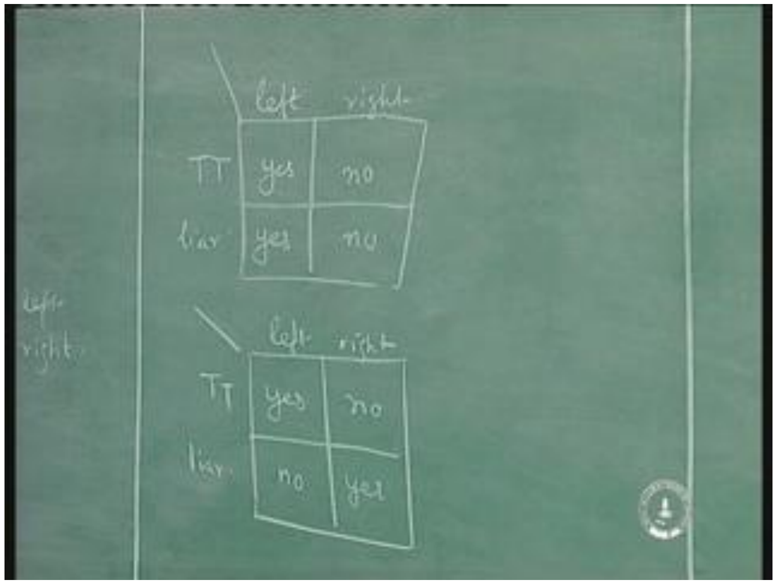

The combination of these two using OR because in either of these cases he will take the left row. So you can read it like this.

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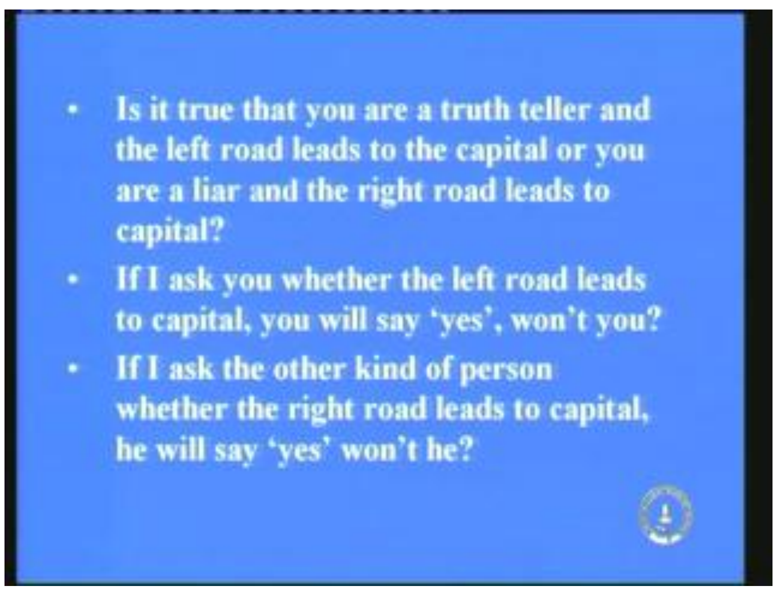

Is it true that you are a truth teller and the left road leads to the capital or you are a liar and the right road leads to capital? So look at this very carefully, if he is a truth teller and the left road leads to capital he will say yes and so the person can take the left road. And if he is a liar and right road leads to capital, the correct answer is yes but the liar will say no and so if it is a combination of liar and right the answer will be no and he will take the right road.

So like that you can look into the all the four possibilities and you will see that this is the question you have to ask. This is the single yes or no question by getting the answer yes he will take the left road and by getting the answer no he will take the right road. It is not that this is the unique answer, the question can be something like this also.

Is it true that if I ask you whether the left road leads to capital you will say yes, Will you not? So again you look at the four possibilities, if he is a truth teller and the left road leads to capital he will say yes and so in the next case also it will be yes, it is a combination of yes and yes and so he will say yes and you will take the left road.

But if he is a liar and the right road leads to capital, will you say yes? He has to say no but as he is a liar he will say yes. And a second negative, if I ask you whether the left road leads to capital whether you will say yes, he will say no, but because the second time negative he has to use he will say yes so, whenever the answer is yes the left road will be taken. So it is a sort of double negation, it is positive, some sort of an idea is there here. Another question which is equivalent or which will serve the same purpose is this, if I ask the other kind of person whether the right road leads to capital he will say yes will he not?

Here again think carefully and look into the four possibilities, you will realize that this is also a proper question, by getting the answer yes you will take the left road and no he will take the right road. So this is what we study about proposition; there are many interesting such puzzles where you make use of logic to solve a question like this.

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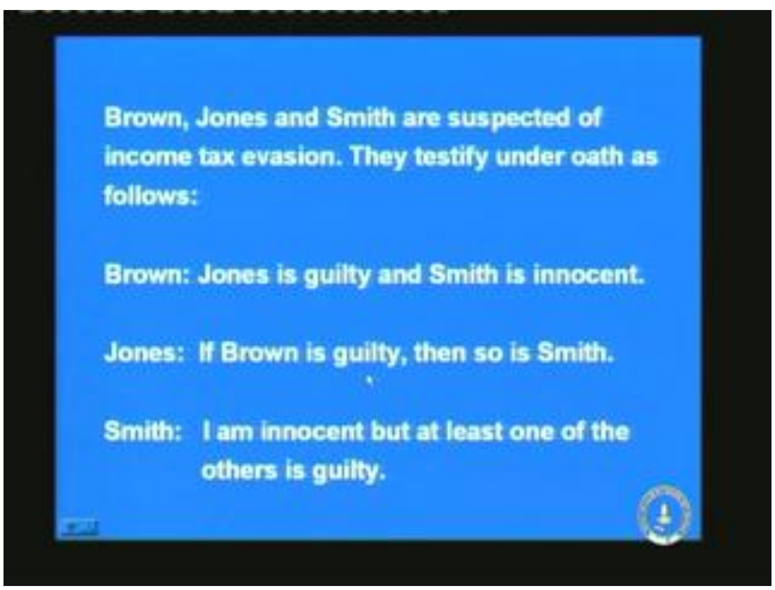

Let us consider this problem and I also mentioned this in the last lecture. Brown Jones and Smith are suspected of income tax evasion, they testify under oath as follows: Brown says Jones is guilty and Smith is innocent, Jones says if Brown is guilty then so is Smith, then Smith says I am innocent but at least one of the others is guilty. The question is, assuming everybody told the truth who is innocent and who is guilty? Assuming the innocent told the truth and the guilty lied, who is innocent and guilty?

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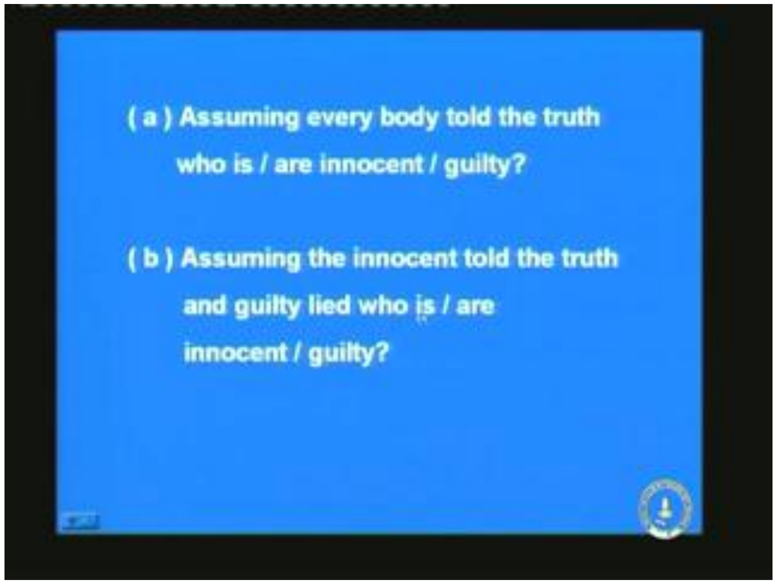

First part is easier as I told you in the last lecture. Now let us write the sentences in logical notation and try to get the answer. Suppose B stands for Brown is innocent, then NOT B will stand for Brown is guilty.

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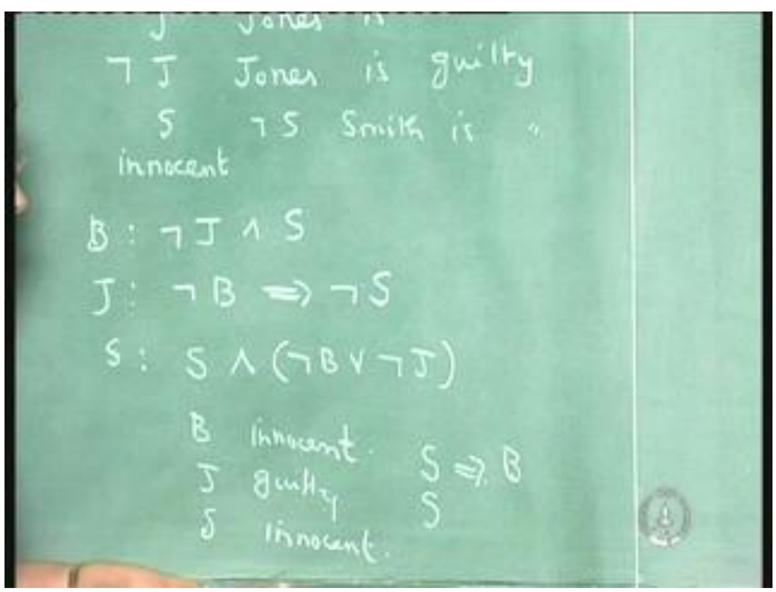

Similarly J stands for Jones is innocent and NOT J stands for Jones is guilty. Similarly S and NOT S, S will stand for Smith is innocent and NOT S will stand for Smith is guilty.

Now, what are the statements they are making? Brown is making the following statement: Jones is guilty and Smith is innocent, Jones says if Brown is guilty then so is Smith. So this should be written like this: Brown says NOT J AND S that is Jones is guilty and Smith is innocent, Jones says if Brown is guilty that is NOT B then Smith is also guilty that is Not B implies NOT S, then Smith Says I am innocent that is S and at least one of the others is guilty that is NOT B OR NOT J. So the statements they make can be transcribed in this manner.

Now suppose everybody tells the truth all these three statements should be true, in that case what is the answer you get? Brown, Jones, Smith, Brown says all three statements are true, Brown says NOT J AND S, so NOT J means Jones is guilty and Smith is innocent. And you take the second sentence that is true NOT B implies NOT S. What is the contrapositive of this? The contrapositive of that will be S implies B. P implies Q means the contrapositive is NOT Q implies NOT P. So the contrapositive of this is this that is true. That means if Smith is innocent then B is innocent, Brown is also innocent. We know that Smith is innocent, so from this we get Brown is innocent. So if all of them are telling the truth this is the answer you get. But the second portion of the question is if the innocent tell the truth and the guilty lied who is innocent and who is guilty? Now first let us take Brown, there are two possibilities: Brown may be innocent or Brown may be guilty. Consider this possibility Brown is innocent then he must be telling the truth.

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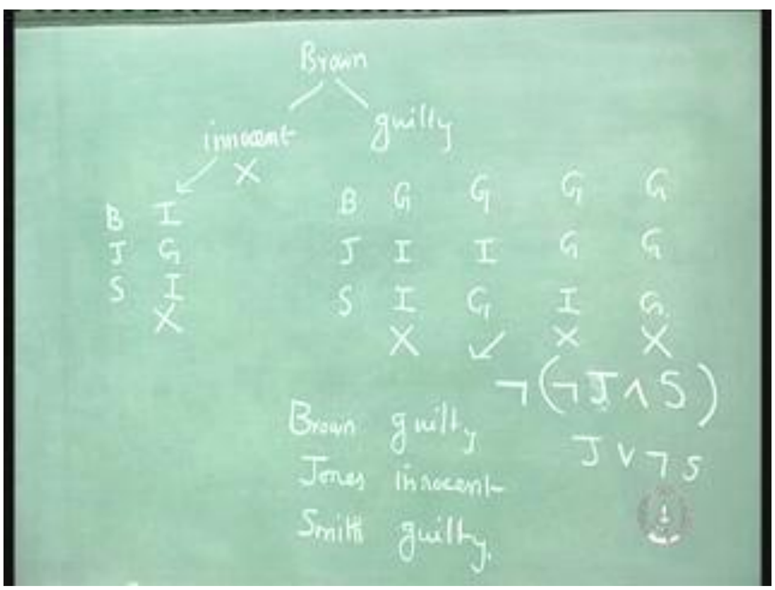

So in that case he must be telling the truth so the possibility is Brown is innocent, Jones is guilty and Smith is innocent because he says that Jones is guilty and Smith is innocent. But in this case what is the statement of Jones? If Brown is guilty then so is Smith or the contrapositive is if Smith is innocent then Brown is innocent and that is also true. So actually Jones is making a correct statement, if he is guilty he must be making a wrong statement so this possibility goes, so Brown cannot be innocent, this possibility is ruled out.

Now what are the other possibilities? Brown is guilty, so Brown is guilty but Jones and Smith they can be innocent, both can be innocent, or one can be innocent other can be guilty, one can be guilty, the other can be innocent. These are the other four possibilities.

Now let us take this sentence, this possibility that Jones is innocent. So he is telling the truth, what is he saying? If Brown is guilty then Smith is guilty but Smith is innocent. So he is making a wrong statement. So this possibility is ruled out. Now looking at this possibility Jones is saying the correct statement, if Brown is guilty then so is Smith and Smith is saying I am innocent and so on, he is making a wrong statement so this also works out correct.

So this looks like a possible answer. Now look at this again in these cases what is the statement of Jones? Jones says if Brown is guilty then so is Smith. He is lying but Smith is innocent so he must be telling the truth. Smith says I am innocent but at least one of the others is true. This looks okay, look at this first statement Brown is guilty so he must be saying a wrong statement. What is the statement of Brown? Jones is guilty and Smith is innocent. Now he is guilty so he must be telling the wrong statement, so you have to take the negation of that and that will be J OR NOT S.

That is, you have to rule out this possibility because here both of them are not correct, so this is wrong. Now what does this correspond to? This is again not a possibility because in this case Smith is guilty, so he must be lying and Jones is guilty he must be lying, so the statement that Brown is guilty then so is Smith is a truth statement which he is making but he has to lie, because he is guilty he has to lie but he is making a true statement, so this possibility is also ruled out. So the correct answer in this case is Brown is guilty, Jones is innocent and Smith is guilty. So, next we study about predicates and Quantifiers.

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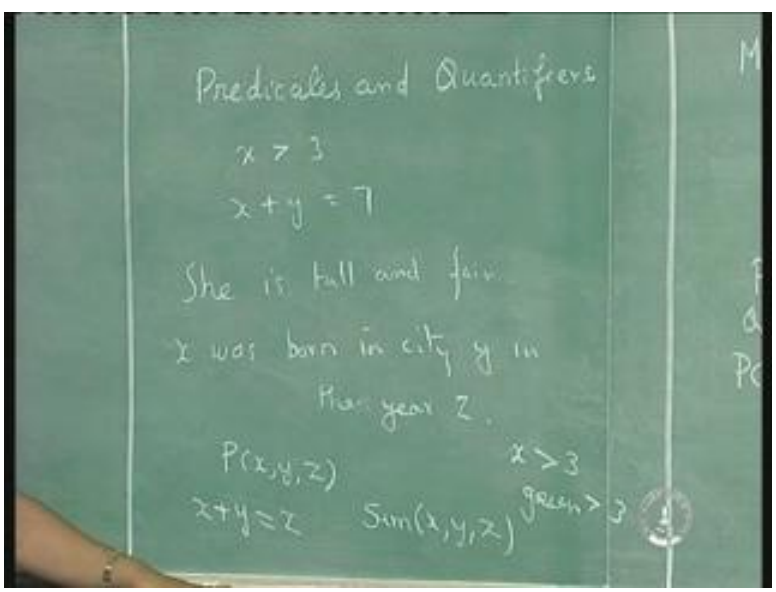

Now we already consider that a statement something like X greater than 3, x plus y is equal to 7. They are not propositions, they are assertions because the truth value you give them will depend upon what value you are going to give for the individual variables x and y.

In English language you may have statements like this; she is tall and fair, x was born in city y in the year z or he was born in the city y in the year z. We can use pronouns, pronouns will stand for individual variables. In this case we do not know whether it is true or false. It depends upon who is she and here who is S, what is the city y and which year is z and depending upon that the sentence will get the truth value or false. So they are called predicates and the value you give to the individual variables will give you the proper truth value to that. Now you usually denote the predicates like p(x, y, z). For example x plus y is equal to z, this you can represent as sum x, y, z.

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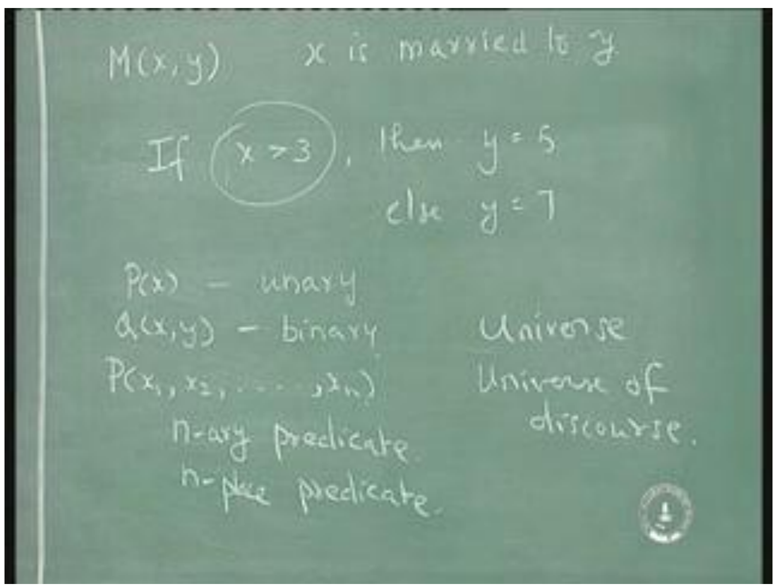

This stands for the predicate x plus y is equal to z or if you have m x, y this may stand for x is married to y. Again we do not have a unique value, the value will depend upon what value your going to give for individual variables x and y. In programs you frequently come across statements like, if x greater than 3 then say some y is equal to 5 else y is equal to 7. So you assign the value y to y if x is greater than 3 and otherwise you give the value 7.

Now this is a predicate here and when the control comes to this statement you evaluate at that particular instance x may be having a particular value and you give that value to x and find out whether statement becomes true or false. And if it is true you execute the then portion and if it is false you execute the else portion. Now in general you have the predicate of the form p(x) or q(x, y) etc this is a unary predicate. It has a single individual variable, this is a binary predicate.

In general you may have something like $p(x_1, x_2, x_n)$ and you may have n individual variables in a predicate, this is called a n-ary predicate or n-place predicate. Now you have to choose the value you give for x and y.

In this case x and y are to be taken from the set of human beings and in this case x and y can be real numbers and z can be a real number or x and y can be non-negative integers or integers and so on. They are chosen from a particular domain of values that is called universe or universe of discourse.

Now look at this; x was born in the city y in the year z. x takes the value from the set of human beings, y takes the value from the set of cities, z takes values from the set of years, so that is called the underlying universe. And in this case the sum of x, y, z represents the predicate x plus y is equal to z. So x, y, z may be taken from the set of integers or non-negative integers. Now in such cases you may have to specify the underlying universe because the statement which may be true for integers may not be true for real numbers or a statement which is true for real numbers may not be true for integers and so on. Therefore the underlying universe has to be specified. But sometimes like this you need not have to specify the underlying universe explicitly.

Look at this sentence: x was born in the city y in the year z. Obviously x has to be a human being, y has to be a city and z must be a year. You cannot have y as an integer and you cannot have z as a color and so on. And something like if you have the predicate x greater than 3, I cannot say x is green in color, green is greater than 3, it does not make any sense. So x has to be from the set of integers or real numbers or whatever it is.

So in some cases you need to specify the underlying universe explicitly and some times you need not have to spell it out explicitly, implicitly it will be understood. Now let us take a predicate x_1, x_2, again we can have predicate constants and predicate variables like you know sum of x, y, z is a predicate constant. It represents x plus y is equal to z.

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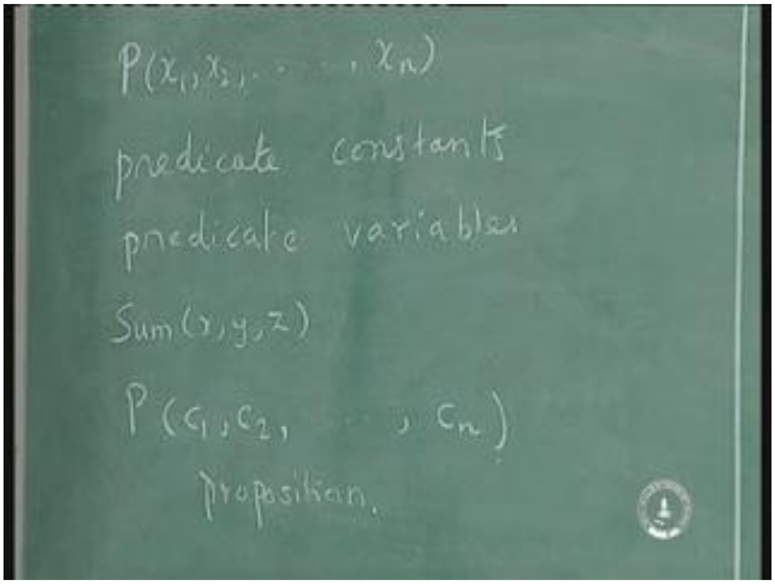

Now if you generally say p (x_1, x_2, x_n) is a variable you can assign any n place predicate to that and you also have expressions involving predicate variables and so on. Now when you assign a particular value for x_1, c_1 and a particular value c_2 and a particular value c_n for x_n, this predicate becomes a proposition. For example take this, x plus y is equal to z. If I assign the value x is equal to 2 and y is equal to 3 and z is equal to 5 it becomes 2 plus 3 is equal to 5 which is a proposition.

So if you assign particular values c_1, c_2, c_n to x_1, x_2, x_n then the predicate becomes a proposition and it takes a truth value true or false. Of course it has to take values from the underlying universe. Now a predicate x_1, x_2, x_n if this is true for all values c_1, c_2, c_n from the universe, you can call it as universe U and then you say that this predicate is valid in the Universe U, then you say p(x_1, x_2, x_n) is valid in U. If it is not true for all values of c_1, c_2, c_n but for some particular value it is true then you say it is satisfiable in U. If P is true for some c_1, c_2, c_n from U it may not be true for all but for at least one it is true, then you say that p is satisfiable in U.

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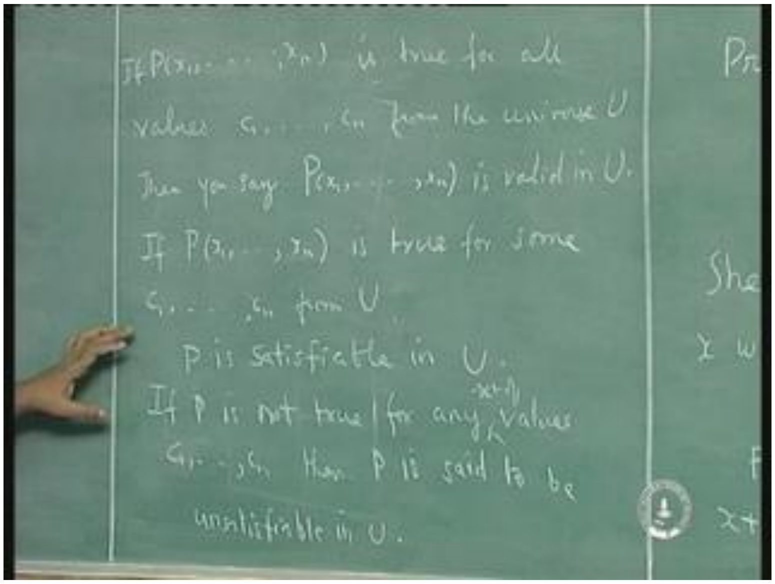

Now it may so happen that $p(x_1, x_2, x_n)$ is not true for any of the values you give for x_1 x_2, x_n. If p is not true for any set of values c_1, c_2, c_n then p is said to be unsatisfiable in U. So we talk about valid predicates, satisfiable predicates and unsatisfiable predicates and also we specify the Universe. Now you have a predicate say p(x, y) then you can make it a proposition by giving values for x and y. So give values for x and y and this becomes a proposition.

Now in this case you say that you are binding the variables x and y by giving values. There is another way you can bind the variables that is by making use of quantifiers. For example you have a predicate p(x), this is the unary predicate involving one single variable x. Now you can bind it using this quantifier for all of x. This is read as for all x or generally you read it as for all x. You can also read it as for every x, for any x, for each x, for arbitrary x.

Now when you use a quantifier the predicate is bound. Let us consider some examples making use of this. It becomes a proposition when you use quantifier. Let the underlying universe be the set of integers.

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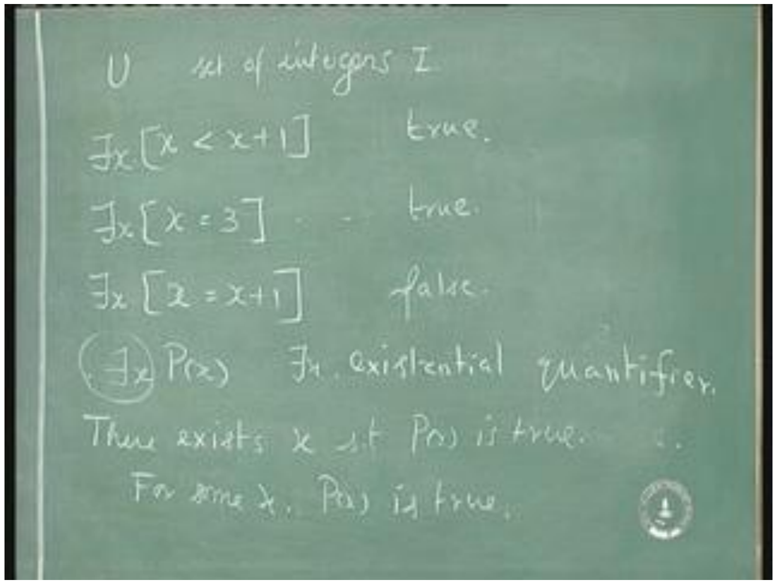

Look at these predicates x is less than x plus 1 and x is equal to 3, x is equal to x plus 1. Now to make it a proposition I can use a quantifier like this, look at this for all of x where x is less than x plus 1, this is a proposition. Is it true? This is always true. For any value you give for x, from the set of integers x will always be less than x plus 1 and so this is a proposition and it is always true.

And if you use for all of x here, for all of x where x is equal to 3 it is not correct, if you give value 4 to x then 4 equal to 3 is not correct. So it is true only you give the value 3 to x and for other values it is false. So saying that for all of x where x is equal to 3 where the underlying universe is the set of integers is not correct and this is a false statement. This is again a proposition and takes a value false. Now look at this one: x is equal to x plus 1, for all of x where x is equal to x plus 1.

Again for none of the values from the set of integers x is equal to x plus 1. So this is in fact not true for any values so saying that for all of x where x is equal to x plus 1 is not correct and it is a false statement so this takes the value false. So we see that using the quantifier for all of x you can make a predicate, a proposition and it takes the particular value true or false depending upon what universe you are specifying. Now the other quantifier is there exists x p(x), you can use the quantifier there exists x p(x). This is read like this there exists x such that p(x) is true. You can also read it as for some x p(x) is true.

Now let us again take the same three predicates and see what happens when you apply the quantifier there exists. This there exists x is called the existential quantifier. The earlier one which we considered for all of x is called universal quantifier. Now again let us take these three predicates and take the underlying universe as the set of integers. This is a predicate, if you use there exists x where x less than x plus 1 is it true or false? Obviously for every value of x, x will be less than x plus 1, so for some value also it is true. So this is a proposition which takes the value true, use there exists for this predicate, there exists x where x is equal to 3.

Again if you assign one value x is equal to 3 from the universe this will be true. So saying that for some x x is equal to 3 is correct. So this takes the value true and if you say there exists x where x is equal to x plus 1, it will not be true for any x is it not? x cannot be equal to x plus 1 for any integer. So there exists x where x is equal to x plus 1 is a false statement. So the point to remember is when we use a quantifier we are binding a variable and when you bind a variable a unary predicate becomes a proposition.

When it is a binary predicate involving two quantifiers using x and y you have to bind both of them to make it a proposition. So a predicate can be made into a proposition by binding the variables. There are two ways of doing that; one is by assigning values to individual variables or by using quantifiers. And mainly we use two quantifiers; one is for all x and another is for there exists x. There is one more quantifier which is denoted like this.

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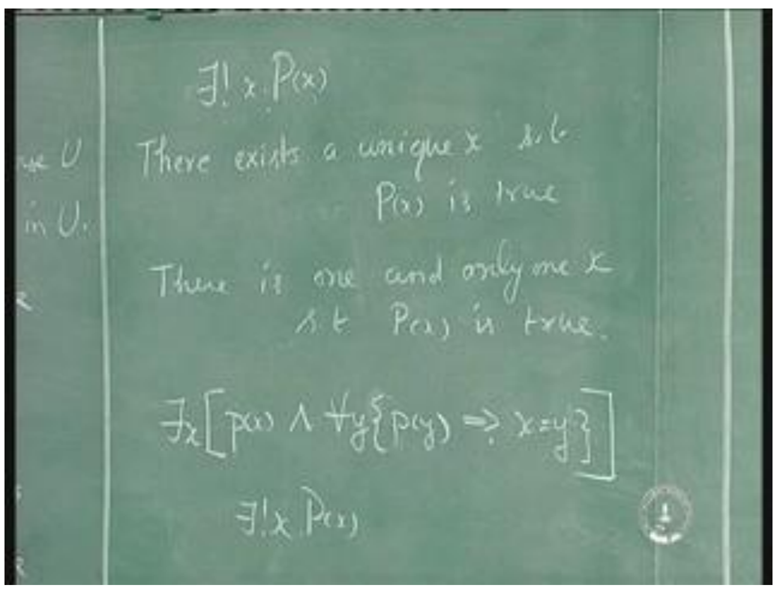

There exists a unique x such that p(x) is true. This is read as there exists a unique x such that p(x) is true or there is one and only one x such that p(x) is true. We use only for all and there exists but this is rarely used but anyway we learn about this also. The reason is you can express there exists a unique x in terms of for all and there exists which we shall see now.

Now again let us take the same three statements and use this quantifier there exists a unique x says there exists a unique x where x is less than x plus 1, is it true or false? Underlying set is the set of integers, if you take any value for x, x will be less than 1, it is not correct to say there is a unique value of x for which x is less than x plus 1. So this becomes a proposition which becomes the value false. And there is a unique x where x is equal to 3. For only one value of the variable x from the set of integer the assertion x is equal to 3 will be true that is correct. Only when you give the value 3 to x this will be true, if you give any other value it will be false, so there is a unique value for which this is true so this takes the value true.

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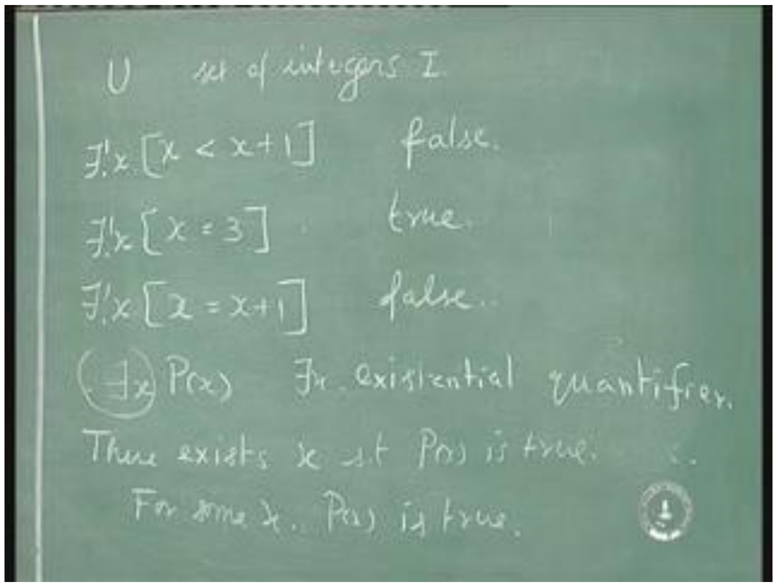

And then there is a unique x where x is equal to x plus 1. Again for none of the values this is true, so saying that there is a unique x for this is true is not correct, so this takes the value false.

Now I told you that this is not frequently used because you can express this in terms of for all of x and there exists x. So instead of writing there exists a p(x) you can write it this way there exists x p(x) and for all of y p(y) implies x is equal to y.

This is equivalent to saying there is a unique x such that p(x) that is why we do not use this frequently. But sometimes we also use it for conveniency, this is a lengthy expression and instead of that we can use that unique x in a convenient manner. Similarly there exists at most one x. This you can represent like this; there exists x for all y p(y) implies x is equal to y.

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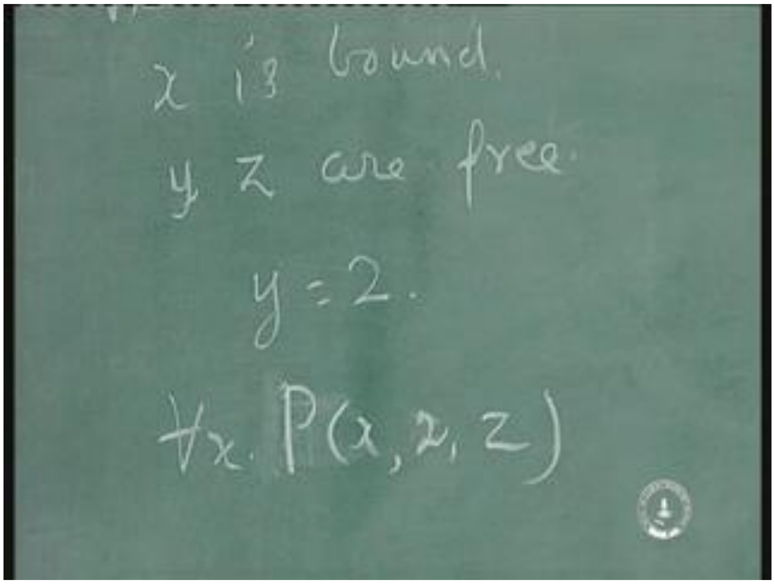

So if you have something like p(x, y, z) this is a predicate involving three individual variables x and y and z. If you use some quantifier like this for all of x p(x, y, z) then x is bound, x is bound here and y and z are free, they are called free variables. Again you can bind them by giving values or again using some other quantifier. So you can say y is equal to 2 and so you will get for all of x p(x, 2, z).

Now it is a unary predicate because this is bound, z is free and you have bound y by giving a value. If you say something like there exists z, for all of x p(x, 2, z) both x and z are bound and so this becomes a proposition and it will take a value true or false if you specify the underlying universe. Let us consider one more example like this. Now we have taken this x plus y plus z is equal to z sum. So sum x, y, z denotes this.

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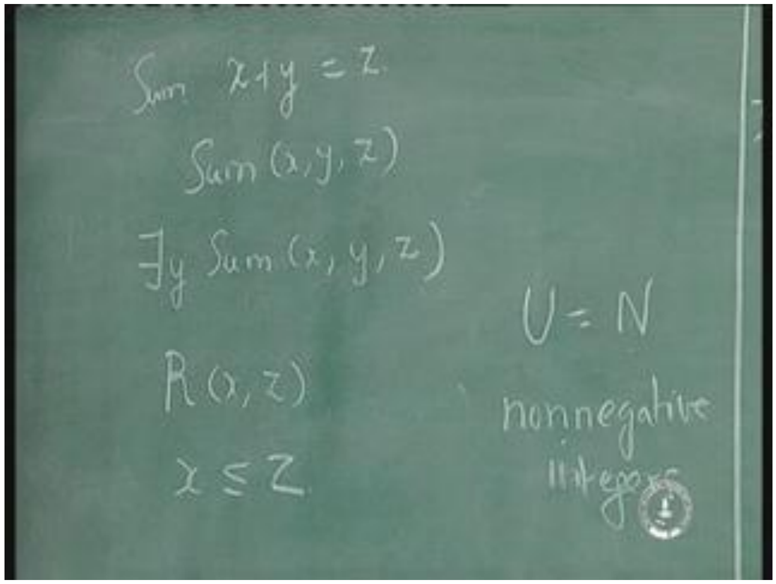

Now if I say there exists y sum x, y, z then y is bound and x and z are free. If you specify the underlying universe as the set of non-negative integers, N denotes non-negative integers, so x and y and z take values from the set of non-negative integers. Then what do you mean by saying that there exists sum of x, y, z that is there exists y such that x plus y is equal to z. If all of them are non-negative that means x has to be less than or equal to z. So this really stands for some predicate x, z which is x less than or equal to z if the underlying universe is the set of non-negative integers. Like that we have to analyze the statements.

Now how do you expand this or look into this. For example if the underlying universe consists of only three elements say 1, 2, 3 then for all of x p(x) should be true for every value here and so it represents p(1) AND p(2) AND p(3). There exists x p(x) this represents the compound statement p(1) OR p(2) OR p(3).

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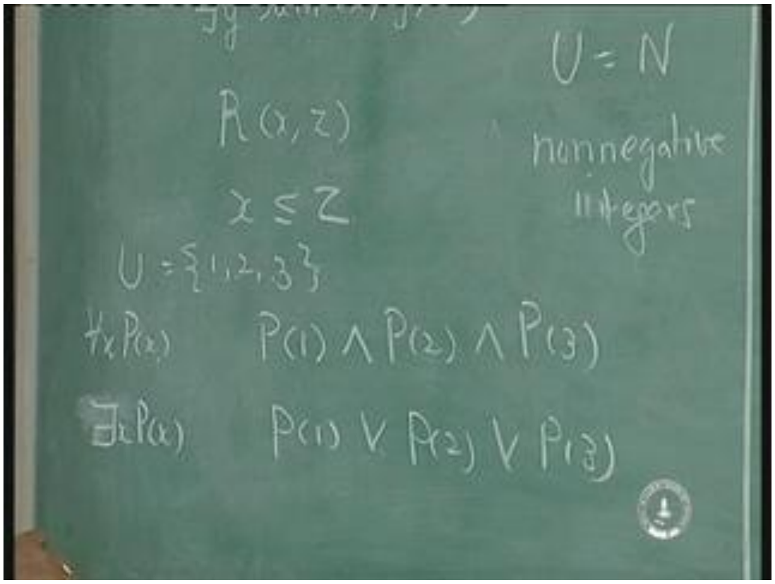

There exists a unique x p(x), how will you represent this? If it is true for 1 it should be false for 2 and 3, if it is true for 2 it should be false for 1 and 3, if it is true for 3 it should be false for 1 and 2. So you represent it as p(1) AND NOT(p(2)) AND NOT(p(3)) OR NOT(p(1)) AND p(2) AND NOT(p(3)) OR NOT(p(1)) AND NOT(p(2)) AND p(3).

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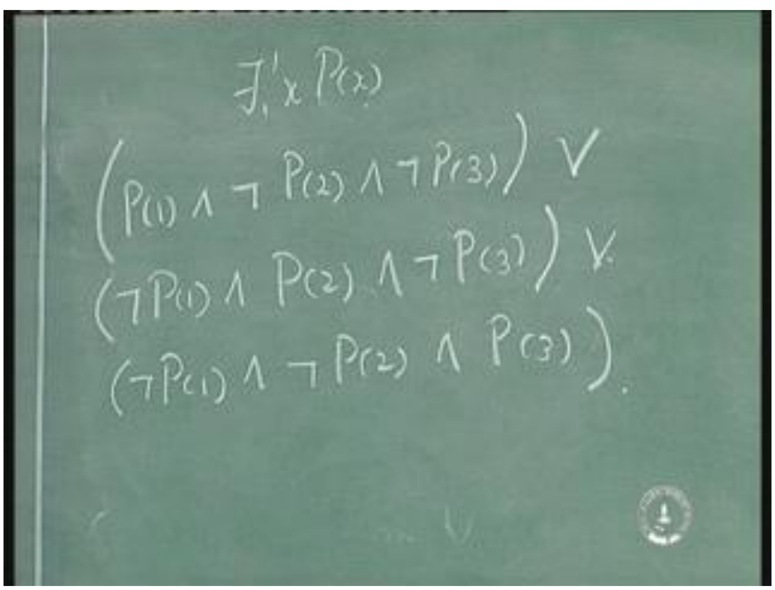

We can express like this if the universe is finite, in this case it consist of only three elements 1, 2 and 3. If it is infinite what shall we do? For example let the universe be the set of non-negative integers which is just 0, 1, 2 etc, then for all of x p(x) will be denoted by the infinite conjunction p(0) and p(1) and p(2) and like that. It is an infinite conjunction like this. There exists x p(x) is denoted by p(0), OR p(1) OR p(2) OR and so on which is an infinite disjunction.

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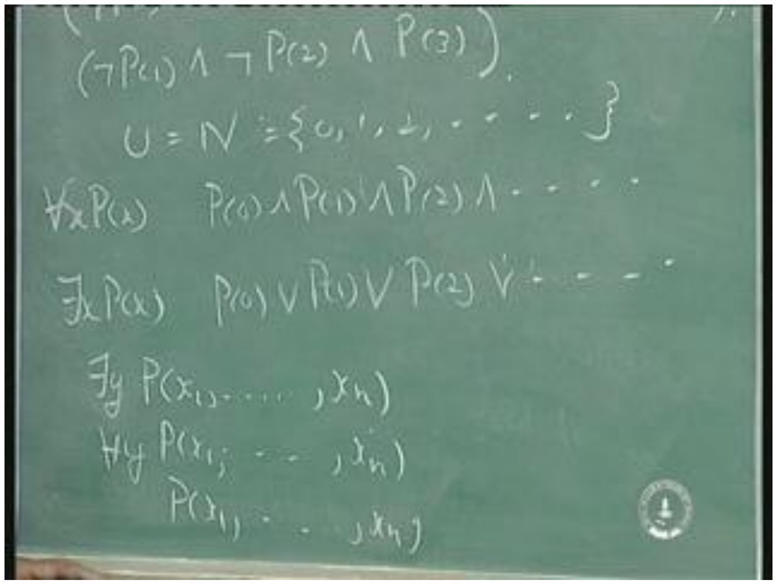

Now if you have a predicate involving individual variables p(x_1, x_2, x_n) and there is one more variable y and you say there exists y p(x_1, x_2, x_n) where the value of y is different from x_1, x_2, x_n then binding like this is not going to affect this portion, this does not involve y at all, if this does not involve y, whether you bind it with there exists y or for all of it even then it is not going to affect. These are all equivalent for all of y p(x_1, x_2, x_n) and without any binding of quantifiers also and they are all equivalent if this does not involve y. So we have to be careful about the scope of the quantifiers and which one it is going to bind and so on. Now look at these statements; for all of x for all of y p(x, y, p) is a binary predicate and you are having two variables x and y.

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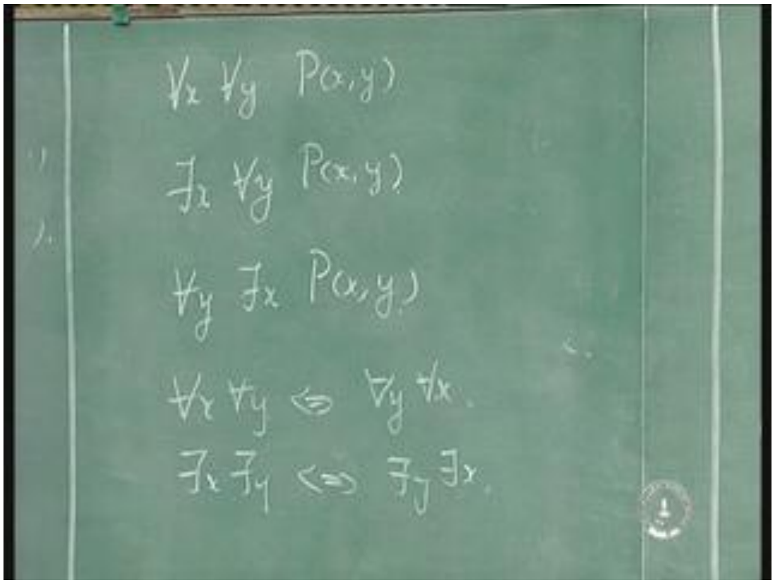

Look at this statement; for all of x for all of y p(x, y) how will you read this? You will read this as for all values of x and for all values of y p(x, y) is true. Now look at this, there exists y for all of y p(x, y) and for all of y there exists x p(x, y and x) is bound by the existential quantifier and y is bound by the universal quantifier. Are these two statements equivalent?

You are using the quantifiers in different orders, the order is exchanged, do they mean the same sentence or they are different? Can you interchange the order of binding? In this case you cannot change they represent different statements. Here you have to read it like this; there exist x such that for all values of y p(x, y) is true. Now the value of x is fixed independently or the value of y here. And look at this it says for any y there is x such that p(x, y) is true. Now here you take the value of y and the value of x depends on the value of y. So the meaning is different; you cannot interchange in this case because the meaning will change.

Let us look at some examples now, but you can always change for all of x for all of y into for all of y for all of x and there exists x and there exists y where you can change the order and this will not affect but here it will affect and the order will affect the statement. Now let us consider some example where changing the order of the quantifiers affects the meaning. Let the universe of discourse be the set of married persons, then when you say for all of x you say x is married to y, that means for

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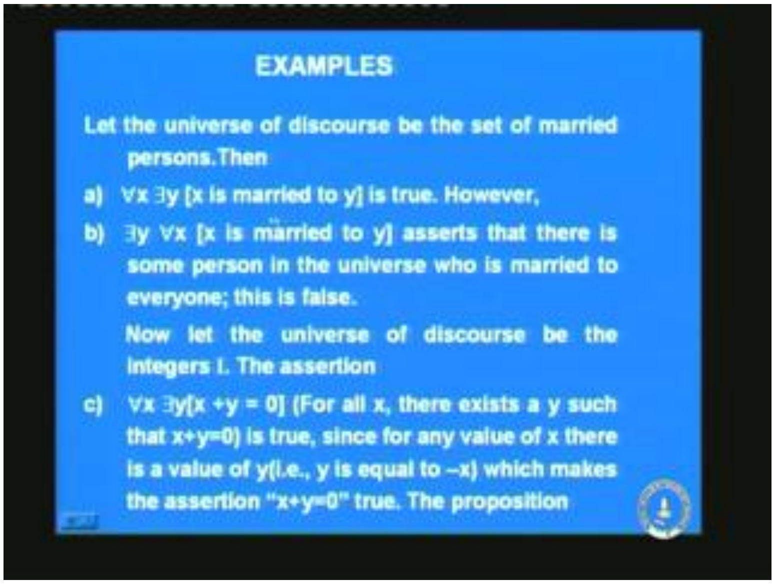

For any x there is a person y to whom x is married which is correct. So this is true. Whereas if you say there exist y for all of x, x is married to y that means there is a person y to whom everybody else including himself will be married. So that is a wrong statement, so changing the order affects the meaning completely so you have to be careful about the order in which you bind the variables and when use the quantifiers. Now let us take this set of integers as underlying universe.

Then look at this statement; for all of x there is a y such that x plus y is equal to 0, that is for all of x there exist y such that x plus y is equal to 0 and this is true because any value of x there is a value of y by taking y to be equal to minus x which makes the assertion true. Now look at this, there exists y for all of x, x plus y is equal to 0. This asserts that the value of y can be chosen independently of the value of x since no y exists which yields 0 when added to an arbitrary integer this is a false statement. So by interchanging the quantifiers once you get a true statement and another time you get a false statement.

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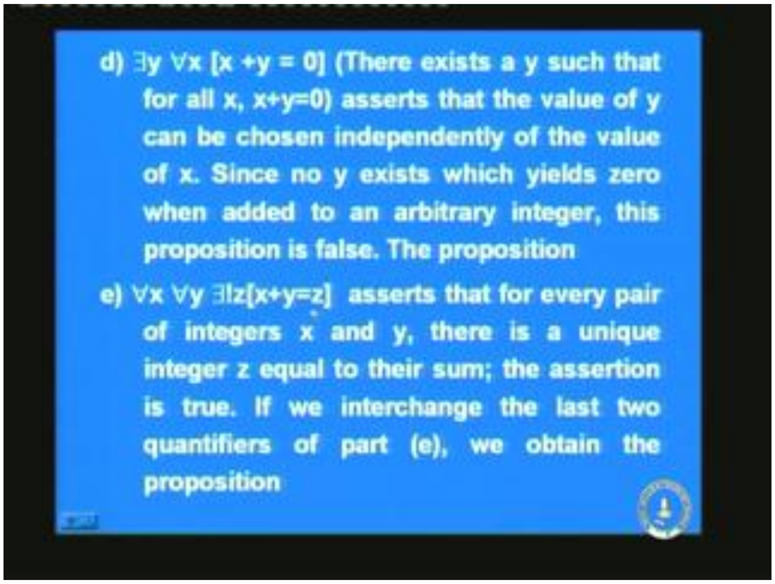

Look at this one, for all of x for all of y there is a unique z such that x plus y is equal to z. This is a true statement. You have to choose z to be the sum of x and y. But if you look at this statement for all of x there is a unique z such that for all of y x plus y is equal to z.

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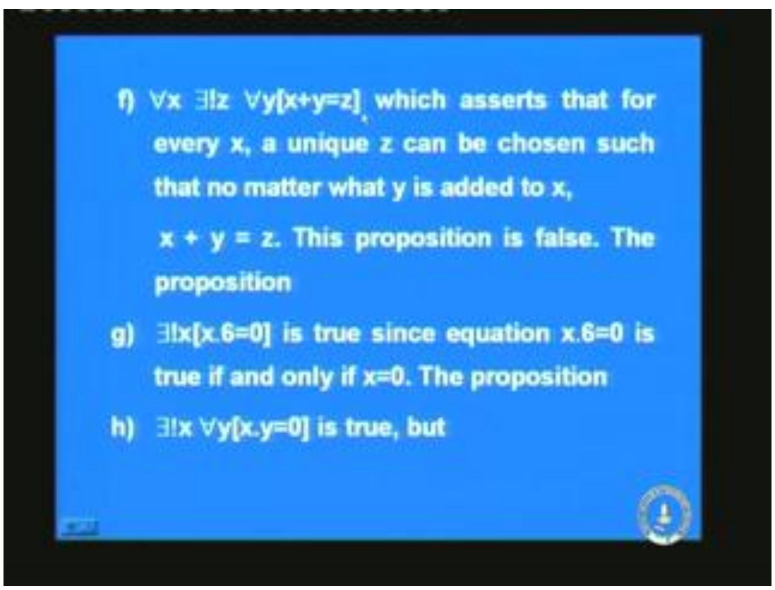

This is not a correct statement and so it is not it is a false statement. So here the value of z is chosen first and then you say whatever the value of y you choose x plus y will be z that is not correct. Look at some more statements, there is a unique x such that x into 6 is equal to 0, this is true because only if you give the value x is equal to 0 this will be true. If you give any other value this will be false. And look at this one, there is a unique x for all of y, x into y is equal to 0 this is true because only when you take x is equal to 0 for all y, x into y will be equal to 0, but if you interchange the order for all y there is a unique such that x into y is equal to 0 is false because if you take y to be 0 any value of x will satisfy that and so you cannot say that there is a unique x. So that is a false statement.

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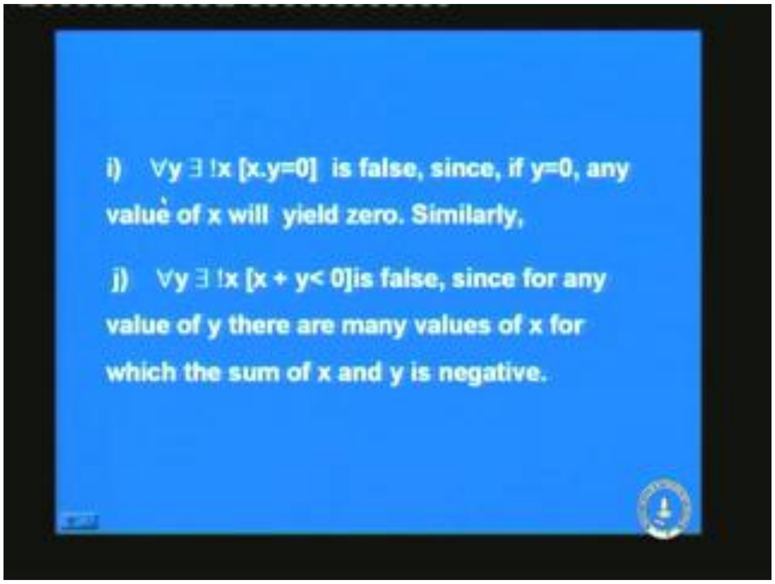

The last one, look at this, for all y there is a unique x such that x plus y is less than 0 is false because you have several ways of choosing x given a value of y so that the sum is less than 0.

For any value of y there are many values of x for which sum of x and y is negative so this is a false statement. So you have to be very careful when you quantify a variable or when you bind a variable using a quantifier and the order in which you bind the variable matters and it gives the meaning to the sentence. So in the next lecture we shall see how to convert English sentences into logical notation using quantifiers and vice versa.