Discrete Mathematical Structures
Dr. Kamala Krithivasan
Department of Computer Science and Engineering
Indian Institute of Technology, Madras
Lecture # 23
Equivalence Relations and Partitions

We have been considering about equivalence relation. An equivalence relation is a binary relation which is reflexive, symmetric and transitive. A binary relation R on a set A if it satisfies these three conditions it is called an equivalence relation. And we have seen some examples of equivalence relation.

One is the mod k set of integers, take the set of integers and the relation mod k. That is two integers a and b are related if a minus b is equal to some n times k or a and b leave the same remainder when divided by the integer k. This is called mod k relation and we have seen that it is an equivalence relation because it satisfies the three properties reflexive, symmetric and transitive. And it divides the set of integers into k equivalence classes.

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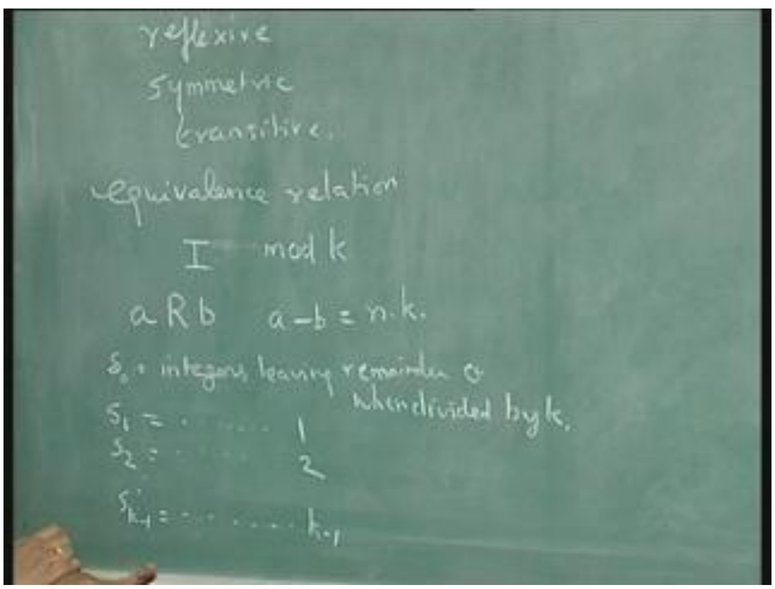

The set S_0 is integers leaving remainder 0 when divided by k. S_1 is the integers leaving remainder 1 when divided by k, S_2 is the set of integers leaving remainder 2 when divided by k and $S_{k \text{ minus } 1}$ is the set of integers leaving remainder k minus 1 when divided by k. And we find that if you take any two of them they are disjoint, there is no element which is present in both of them. These are called classes of the equivalence relation and we find that here there are k equivalence classes S_0 S_1 up to $S_{k \text{ minus } 1}$. And each class is separate from the other that is the classes are all disjoint. And the union of them will make the whole set I, the union of all these equivalence classes will make the whole set I and that is what we learnt in this theorem.

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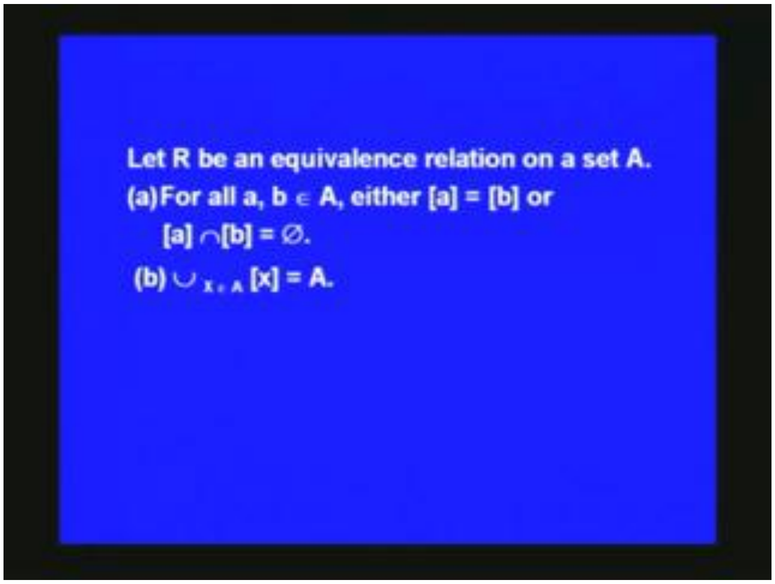

Let R be an equivalence relation on a set A then for all a, b belonging to A either the equivalence class of a is the same as the equivalence class of b or equivalence class of a and equivalence class of b are disjoint that is what we meant by equivalence class of a intersection equivalence class of b is phi, they are disjoint. And the union of all such equivalence classes makes the whole set A that is given by the second part. From this we immediately get this result.

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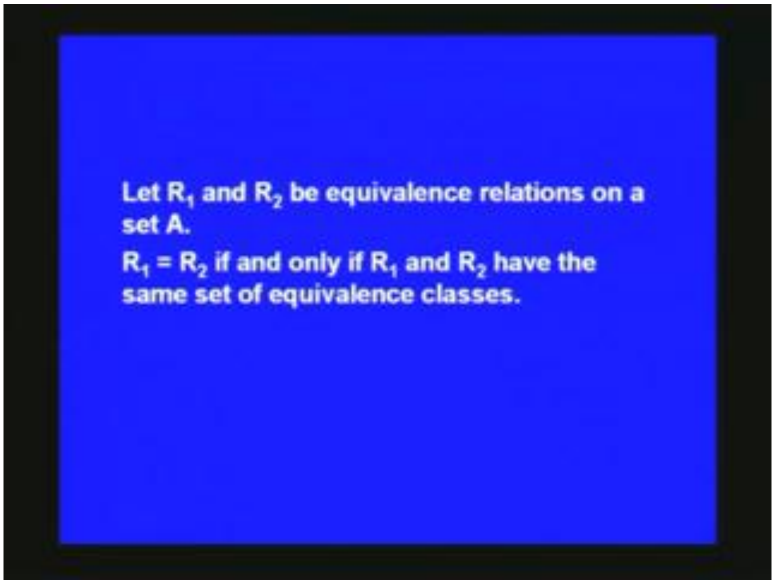

Let R_1 and R_2 be equivalence relations on a set A then R_1 is equal to R_2 if and only if R_1 and R_2 have the same set of equivalence classes. Suppose I have a set A consisting of a, b, c, d, e, f then I have a equivalence relation on this, equivalence relation you know that is represented by a complete diagraph so suppose I have this c, d, e, f I have this equivalence relation R_1 on a represented like this. This divides a into equivalence classes a and b be in one equivalence class, c and d in one equivalence class and e and f in another equivalence class.

Suppose I define another equivalence relation R_2 if it divides a, b, c, d, e, f into different equivalence classes this is e different equivalence. Suppose I have a, b, c in one class and d, e in one class and f in one class if I draw the diagram there will be a complete diagraph with a, b, c there will be a complete diagraph with d and e and there will be a complete diagraph with f, they are different this is not the same as this.

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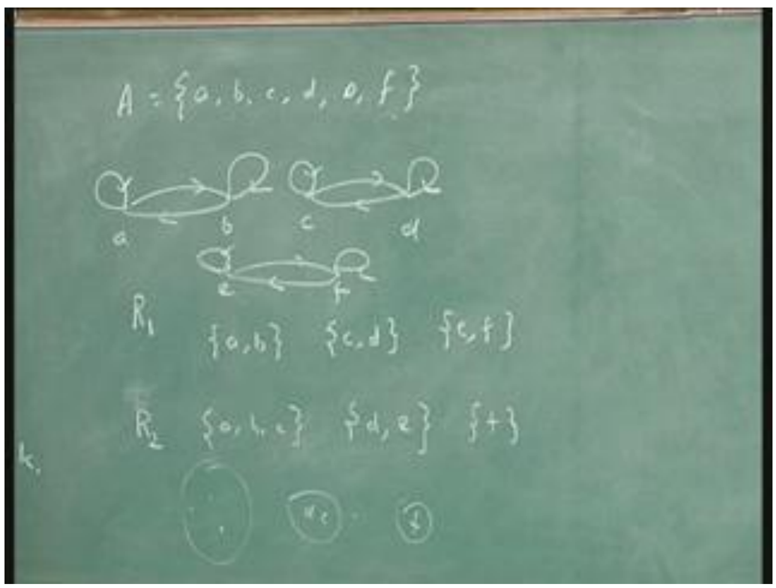

When will R_2 be is equal to R_1?
Only when you have the same diagraph for R_2 also only then the relations are the same. And in that case a and b will be in one equivalence class, c and d will be in one equivalence class, e and f will be in one equivalence class. So R_1 and R_2 divide a into the same equivalence classes. R_2 also will give the same equivalence classes. Only in that case you can say that R_1 and R_2 are equivalent so that is what is meant by this slide. R_1 is equal to R_2 if and only if R_1 and R_2 have the same set of equivalence classes. I have explained this as an example, this is true for any set and any equivalence relations R_1 and R_2 on A.

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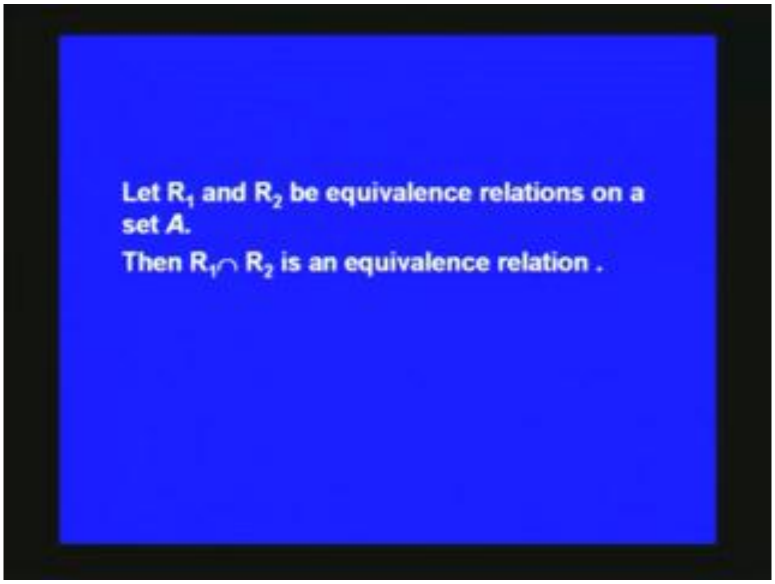

Now, what do you mean by the intersection of equivalence relations R_1 and R_2?
We know that a relation is essentially a set. It is a set of ordered pairs; a binary relation is a set of ordered pairs defined on a set A cross A. So you define R_1 and you define R_2 as a set of intersection you can define R_1 intersection R_2. So R_1 is an equivalence relation on A, R_2 is an equivalence relation on A and what can you say about R_1 intersection R_2? This is also an equivalence relation on A. Let us take an example, the previous one, this is R_1 the set A is a, b, c, d, e, f and the set consists of six elements a, b, c, d, e, f and R1 is a equivalence relation defined like this.

Let R_2 be defined like this a, b, c, d then e, f, so how many equivalence classes you have here? You have a in one equivalence class, b and c in one equivalence class, d in one equivalence class, e and f in one equivalence class.

Now what can you say about R_1 intersection R_2?
It will have all the arcs which are present both in R_1 and R_2. So you will find that e, f these arcs are present in both so they are present in the intersection but the self loops are present in both the relation R_1 and R_2 so they are present here. But the arcs between a and b are present in R_1 but they are not present in R_2 so it will not be present in the intersection. Similarly, the arcs between c and d are the ordered pairs cddc they are present in R_1 but they are not present in R_2 so they will not be present in the intersection. Similarly, bccb the ordered pairs bccb are present in R_2 but not in R_1 so they will not be present here.

Now look at this relation, is this an equivalence relation? This is an equivalence relation because it is reflexive, it is symmetric and it is transitive.

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So how can you say that it is reflexive, it is symmetric and it is transitive?
Now take R_1 and R_2 both will have the self loops because R_1 is reflexive both are equivalence relation. Please remember that R_1 and R_2 are equivalence relation on the set A. So both are reflexive and the self loops will be present in both the things so in the intersection also they will be present therefore the reflexive property is okay.

What about symmetry property?
If you have an arc in one direction you should also have an arc in the other direction. If such a thing is present in R_1 and such a thing is present in R_2 it will be present in R_1 one intersection R_2 two also. For example here between e and f such an arc is present a pair of arcs are present that will be present in the intersection also. But if it is present in R_1 and if it is not present in R_2 or if it is present in R_2 and not present in R_1 then it will not be present in the intersection.

Symmetric means between any two pair of nodes either there should be no arcs or if you have one arc in one direction you should also have an arc in the other direction. So that condition will also be satisfied. So symmetry property is also satisfied.

What about transitivity?
Now, if you have a, b and b, c in R_1 and in R_2 also, so if you have a, b and b, c in R_1 and you also have a, b and b, c in R_2 then it will be present in R_1 intersection R_2. But R_1 is an equivalence relation so it is transitive, so a, c will be present in R_1. Similarly because a, b and b, c are present in R_2 and R_2 is transitive this will be present in R_2 also. So such an arc will be present in R_1 intersection R_2 so the transitive property is also satisfied.

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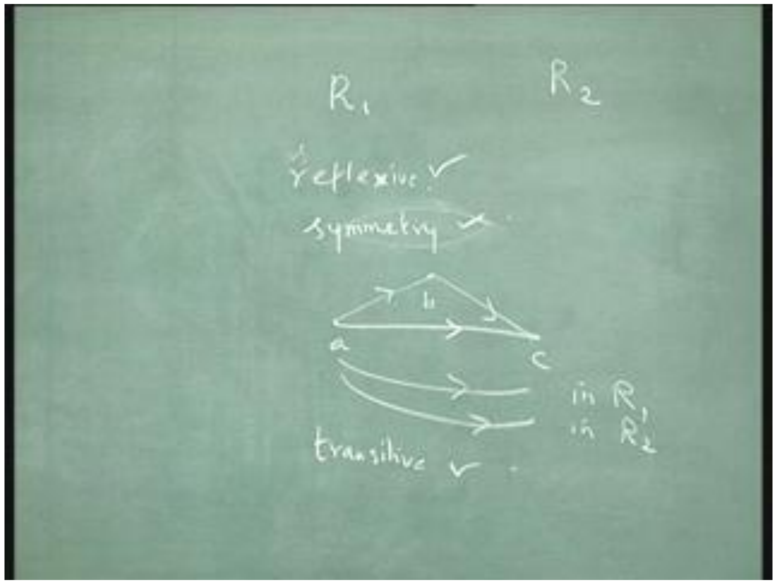

Since all these properties are satisfied R_1 intersection R_2 is an equivalence relation. So this is an equivalence relation.

For example, we have considered R_1 like this then R_2 like this on the set A having six elements. R_1 intersection R_2 consists of these elements, the equivalence classes here are a will be in one equivalence class, b alone will be in one equivalence class, c alone will be in one equivalence class, d will be in one equivalence class, e and f will be in one equivalence class. These are the equivalence classes corresponding to R_1 intersection R_2. But we cannot say like this for the union.

Suppose R_1 is an equivalence relation and R_2 is an equivalence relation can you say R_1 union R_2 is an equivalence relation? We cannot say that, R_1 union R_2 need not be an equivalence relation. And R_1 an equivalence relation on a, b, c, d defined like this, a, b, c, d by the diagraph like this. Here a, b are in one equivalence class and c, d are in one equivalence class. Let R_2 be an equivalence relation defined like this. Here, a is one equivalence class, b and c are in one equivalence class, d is in one equivalence class.

Now what can you say about R_1 union R_2?
R_1 union R_2 will have all the arcs present in this or this. It will have self loops at all nodes, it will have this because it is in R_1, it will have this because it is in R_2, it will have this because it is in R_1 this is R_1 union R_2. The reflexive property is not affected, R_1 union R_2 will be still reflexive. The symmetric property is also not affected R_1 union R_2 is still symmetric. But R_1 union R_2 is not transitive, you have an arc from a to b and you have an arc from b to c but you do not have an arc from a to c. That is, you have the ordered pair a, b you also have the ordered pair b, c but you do not have the ordered pair a, c is not present. So this is not transitive so this is not an equivalence relation.

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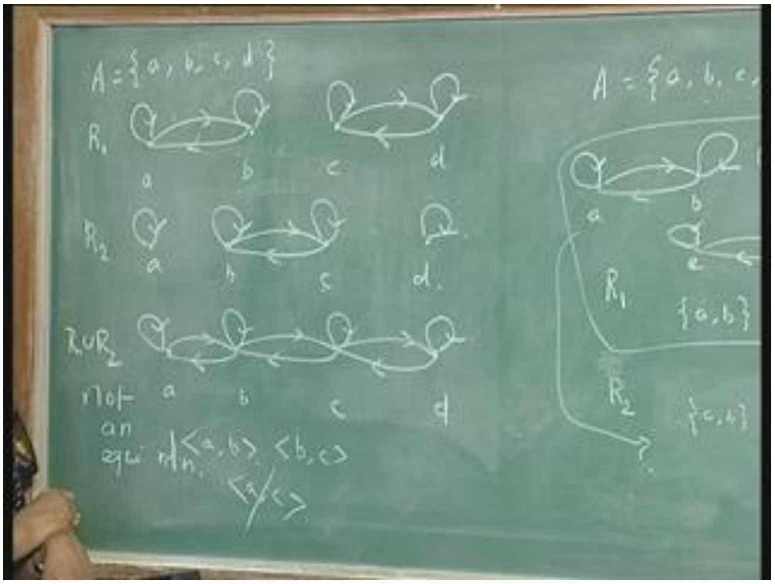

But if you take the transitive closure of that then that will be an equivalence relation. Only the transitive property gets affected so if you take the transitive closure you will get an equivalence relation.

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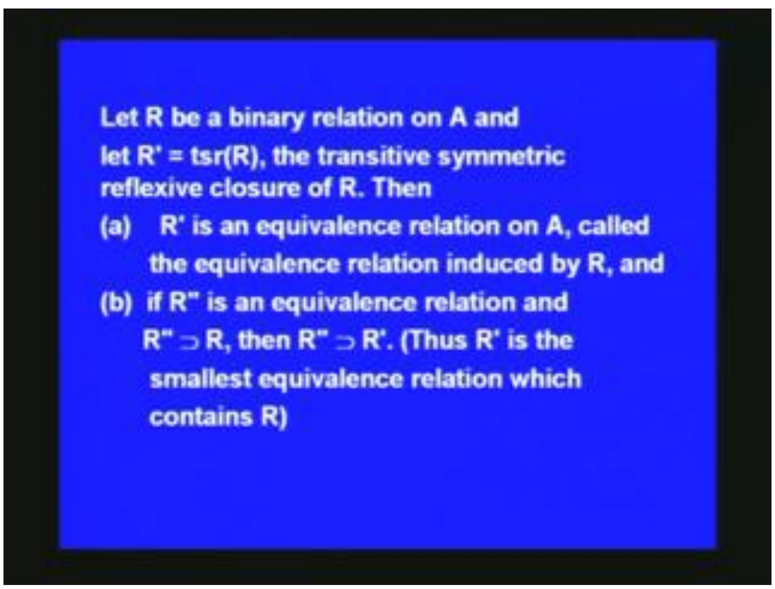

Let R be a binary relation on a set A and first take the reflexive then the symmetric then the transitive closure of R then the resulting relation is an equivalence relation. R dash is the transitive symmetric reflexive closure of R, then R dash is an equivalence relation on A called the equivalence relation induced by R and R_2 dash if R_2 dash is an equivalence relation and if R_2 dash contains R then R_2 dash contains R dash. Thus R dash is the smallest equivalence relation which contains R.

So for example take this;
Take a set A with two elements a and b like this, this is not reflexive because you are not having an element, it is not symmetric also. Now first take the reflexive closure of that and that will be adding a self loop here, then take the symmetric closure that is you will be adding another one. Obviously it is transitive so while taking the transitive closure you do not have any problem, this will be the same as this that will give you an equivalence relation.

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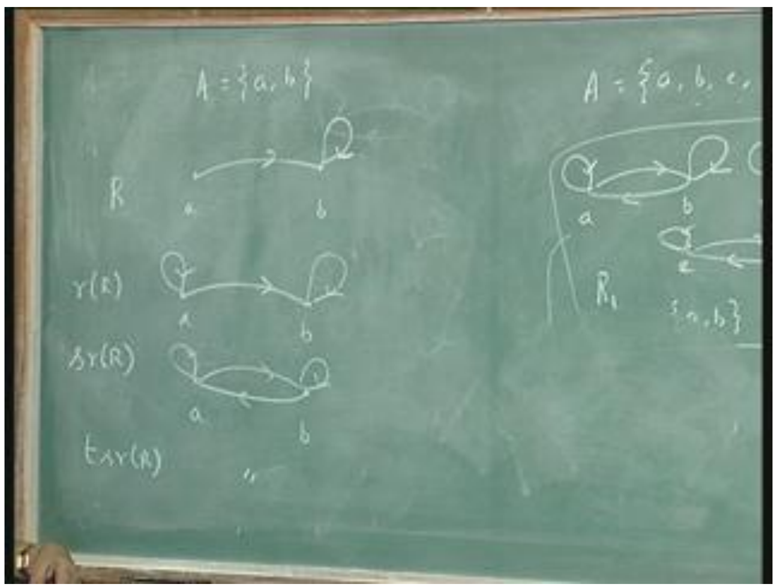

So when you first take the reflexive closure the reflexive property is satisfied. And then when you take the symmetric closure you get the symmetric property but the reflexive property will not be affected by that. Then when you take the transitive closure the reflexive and the symmetric property will be retained and you also additionally get the transitive closure.

Now if you take any other relation R_2 dash containing equivalence relation R_2 dash containing R it will also contain this. Here we have taken only two elements, suppose I take some more elements, suppose I have four elements a, b, c, d and something like this R is this a, b, c, d then when I take the reflexive, symmetric and then transitive closure you get the equivalence relation a b c d like this, I will add one more element e it is like this e.

Suppose I have an equivalence relation R on a set with five elements like this then when I take the transitive symmetric reflexive closure of R I get this. Consider relation R_2 where you have a b e and c d like this, this is an equivalence relation because it is represented by a directed graph which has got components that are complete diagraphs. Here there are three equivalence classes a b is in one equivalence class, c d is in one equivalence class, e is in one equivalence class. here there are two equivalence classes a b e are in one equivalence class, c d is in one equivalence class.

Now you can very easily see that R_2 contains this, this contains this. Obviously R_2 contains R, R is present here also so R_2 will contain this. If an equivalence relation contains this, it will also contain the symmetric. This is the smallest equivalence relation containing R. This is another equivalence relation which contains R. You see a b is present there, the self loop is present there, c d is present there so this also contains R but it is a bigger equivalence relation. It will always contain the transitive symmetric reflexive closure of R which is the smallest equivalence relation containing R, that is what this result says.

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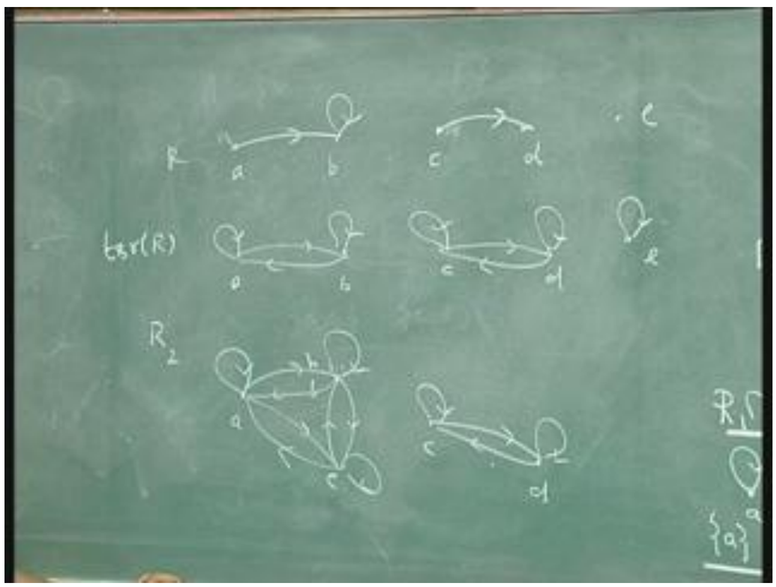

Now we shall see the correspondence between partitions and equivalence relation.

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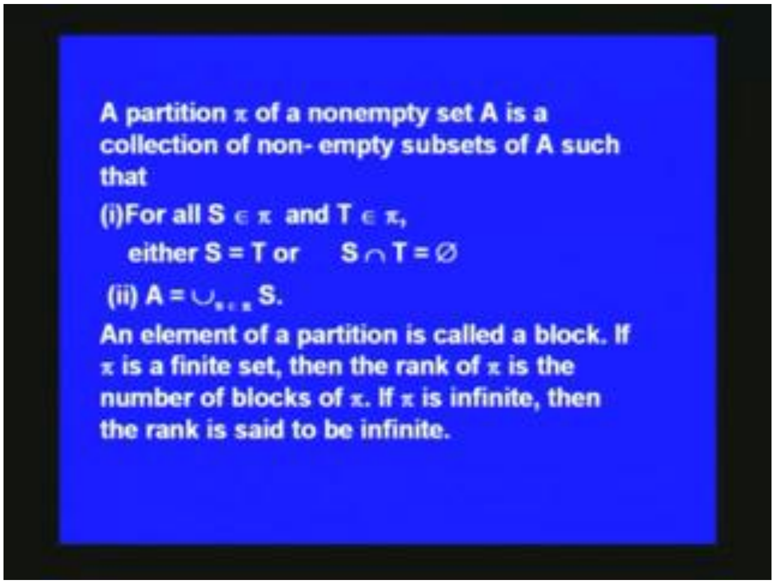

What is a partition?

Before going into the formal definition take some examples. Let me have a set A having elements a, b, c, d, e, f, g, h then I divide the elements into blocks. Suppose I can put a, b in one block c, d, e in one block, f and g in one block, h in one block like that we can put. The eight elements are divided into four blocks. You are dividing the eight elements into blocks that is called a partition, this is a finite set.

If you take the infinite set of examples of integers you can divide them into three blocks having negative integers in one block, positive integers in one block, 0 in one block. So you can divide like that into three blocks. Or if you take the set of non negative integers then you can divide them into two blocks one having the even non negative integers and another having odd non negative integers. You can partition into odd and even numbers.

That is you can divide it into blocks so that one element is present in only one block and it will not be present in two blocks but every element will be present in one block. So a partition pi of a nonempty set A is a collection of nonempty subsets of A such that for all S belonging to pi and T belonging to pi either S is equal to T or S intersection T is equal to pi, A is equal to union of S belongs to pi, union of S belonging to pi S is A. That is, you are putting the elements of a set a into blocks and separate blocks are disjoint and one element will belong to one and only one block.

An element of a partition is called a block. If pi is a finite set then the rank of pi is the number of blocks of pi. For example, the rank of this is number of blocks 1 2 3 4 there are 4 blocks so the rank is 4 or the index is 4, you can use the word index also for this, here the rank is 4 which is the number of blocks, you can also say the index is 4.

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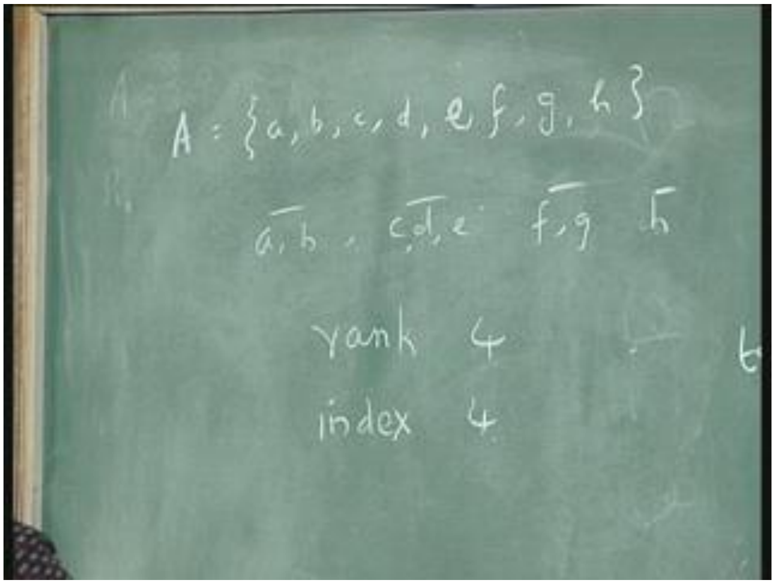

If phi is an is a partition then the rank on a infinite set, for example the set of non negative integers you are dividing it into odd and even integers so the rank is 2, there are two equivalence classes or two blocks in the partition. I should say there are two blocks in the partition and so the rank is 2 but you may also have infinite number of blocks you can put every non negative integer in one block and in that case the rank will be infinite, the number of blocks will be infinite. So if you take the set of non negative integers 0, 1, 2, 3 up to this each one you can put in one block like that then the rank is infinite.

If you put all the even in one block and all the odd integers in one block you are having only two blocks then you say that the rank here is 2. But if it is a finite set obviously the rank has to be finite only.

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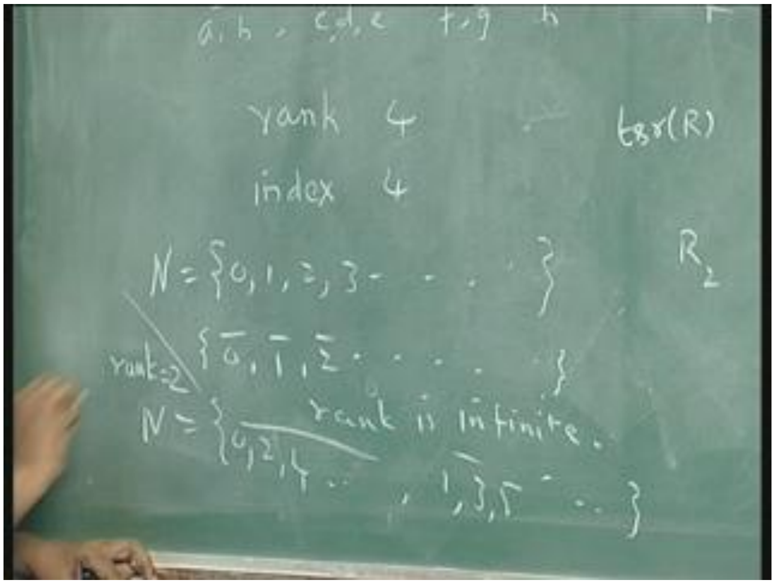

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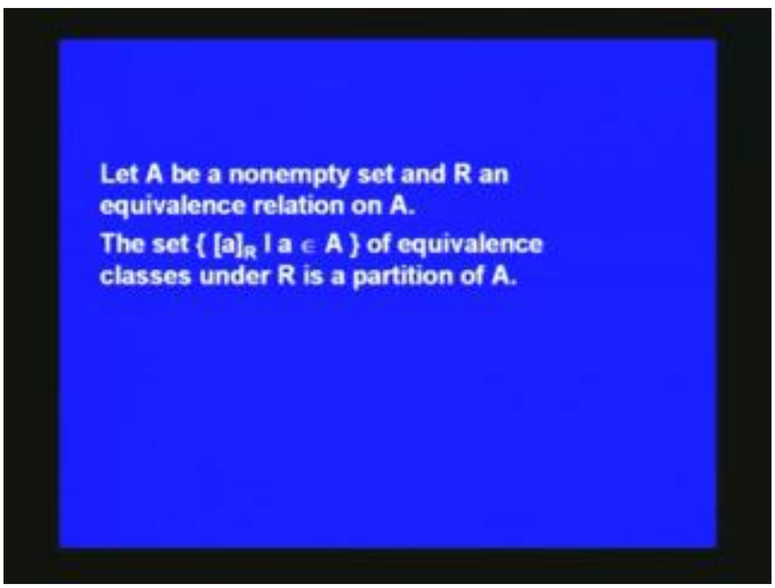

Now, we see the correspondence between the equivalence relation and the partition. Let A be a nonempty set and R an equivalence relation on A then the collection of the equivalence classes a_R a belongs to A of equivalence classes under R is a partition of A. In all these examples we have seen that, again take this example. The underlying set here is a b c d, the equivalence relation R is represented like this so the equivalence classes are a and b will belong to one equivalence class, c and d will belong to one equivalence class. Actually this corresponds to the partition where a and b will be in one block and c and d will be in one block. So this equivalence relation induces a partition on A, that is this partition a and b in one block c and d in one block. Each equivalence class will be one block. So you say that an equivalence relation induces a partition on A.

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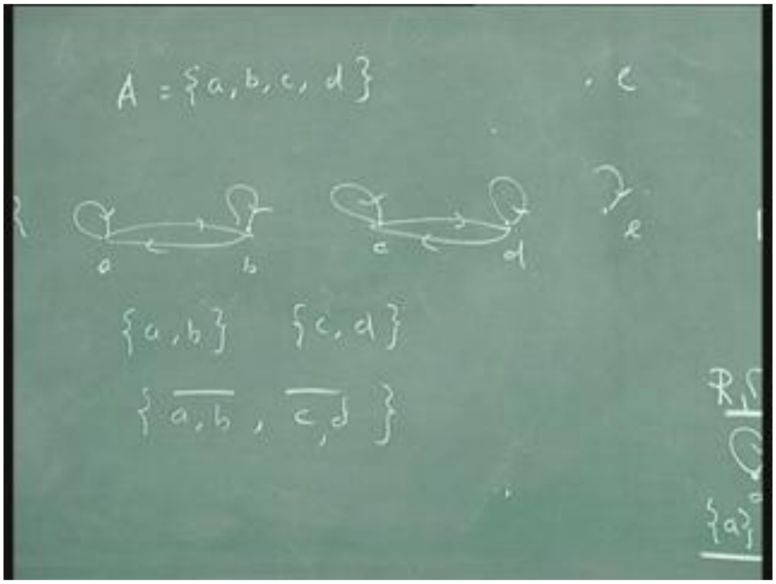

This is what we have seen.

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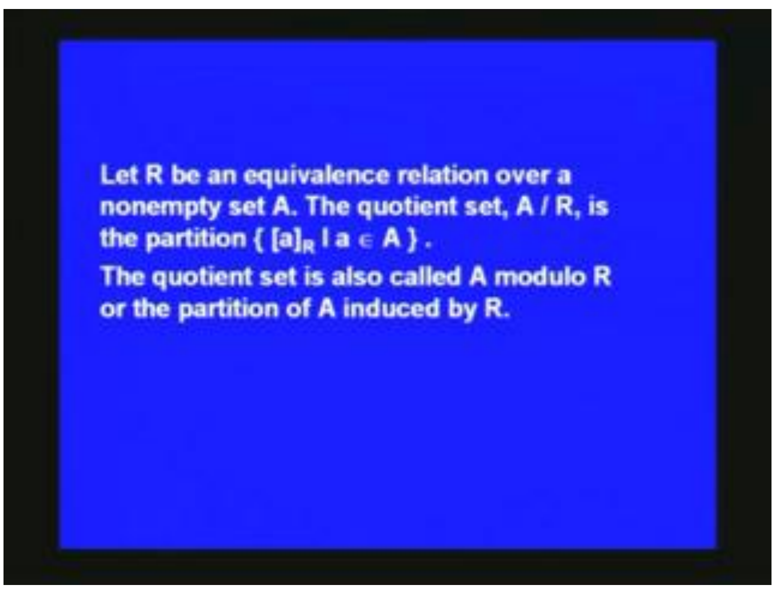

Let R be an equivalence relation over a nonempty set A. the quotient set a_R is the partition this is called the quotient set, this is called A modulo R the partition of A induced by R.

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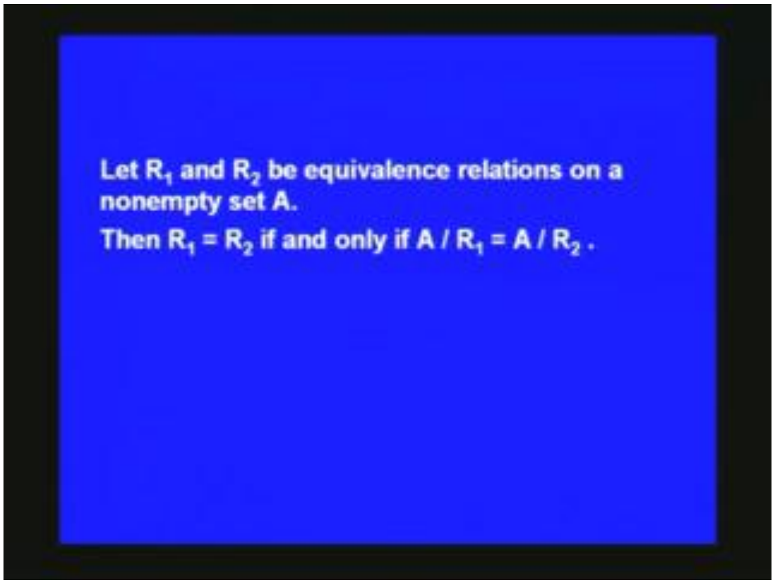

Now, immediately you can see, let R_1 and R_2 be the equivalence relation on a nonempty set A. then we have seen that R_1 is equal to R_2 only if they have the same set of equivalence classes, we have seen this result earlier. R_1 and R_2 be equivalence relation on a set A. R_1 is equal to R_2 if and only if R_1 and R_2 have the same set of equivalence classes, this we have already seen. So if R_1 and R_2 are equivalence relation and R_1 is equal to R_2 they have the same set of equivalence relation so they will induce the same partition on the underlying set.

A by R_1 is equal to A by R_2 means they will induce the same partition or the underlying set will be divided into equivalence classes in the same way in both the cases. That is the same partition is induced in both cases that is what is meant by this result.

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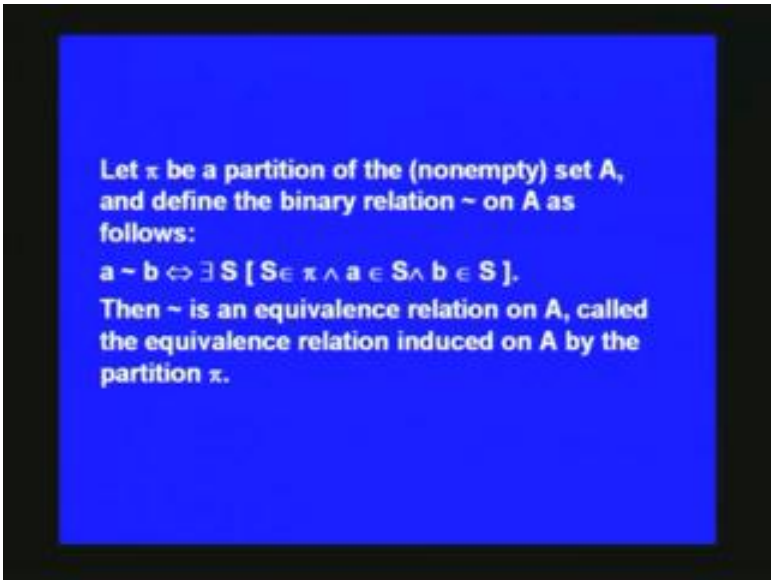

So we have seen that an equivalence relation induces a partition on the underlying set. Conversely you can have a partition and the partition can induce an equivalence relation on a set A, the idea is very similar.

So look at this set, you are having a set A having eight elements and there is a partition of that. there are four blocks a, b belongs to one block; c, d, e belongs to one block; f, g belongs to one block and h alone belongs to one block, this induces an equivalence relation on the set A where a and b will be equivalent so you have a, b the equivalence relation can be represented by this diagraph, f and g are equivalent so you will have like this and c, d, e are equivalent and h alone is one equivalence class. So this partition induces this equivalence relation on the set A. And formally if we define this you say a is related to b if a and b belong to the same block.

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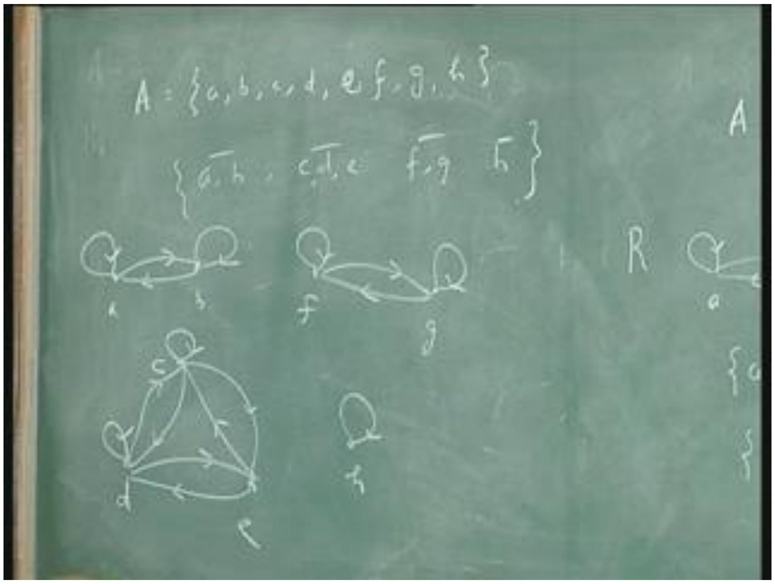

Formal notation is defined like this; let pi be a partition of the nonempty set A and define the binary relation equivalent which is this relation on A as follows, a is related to b is equivalent to saying that there is a block S such that S is a block of pi and a and b belong to the same block. This is an equivalence relation because reflexive property will be satisfied a will be related to a because it is in the same block.

And if a and b are in the same block b and a in the same block so symmetric property will be satisfied. When you say a and b are in the same block b and a are in the same block. Then transitive property is also satisfied because if a and b are in one block and b and c are in one block that is a, b, c all of them will be in the same block so that a and c will be in the same block. So the transitive property is also satisfied.

So since this relation satisfies all the three properties reflexive, symmetric and transitive you find that this is an equivalence relation. And you call this equivalence relation as a equivalence relation induced on A by the partition pi. So this is called an equivalence relation on A called the equivalence relation induced on A by the partition pi. So we see that an equivalence relation induces a partition on the set A and a partition on the set A induces an equivalence relation on A. Let pi be a partition of a set A and R an equivalence relation over A. Then pi induces R if and only if R induces pi, obviously that is true.

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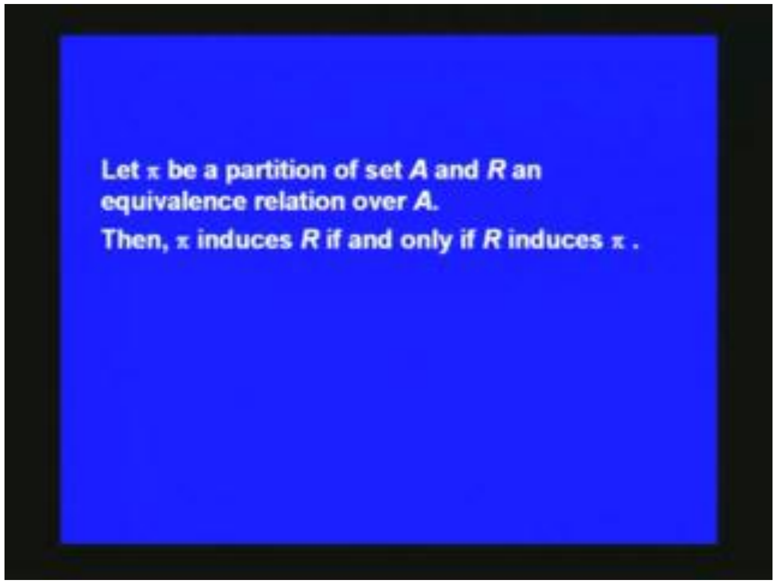

Look at this, this is the set A this is a partition pi and this is the equivalence relation R. P_i induces this equivalence relation R and you can see that R induces this partition pi on A so pi induces R if and only if R induces pi. This is the correspondence between equivalence relations and partitions.

Next we shall see what is meant by a refinement of a partition and also what is meant by product of two partitions and sum of two partitions.

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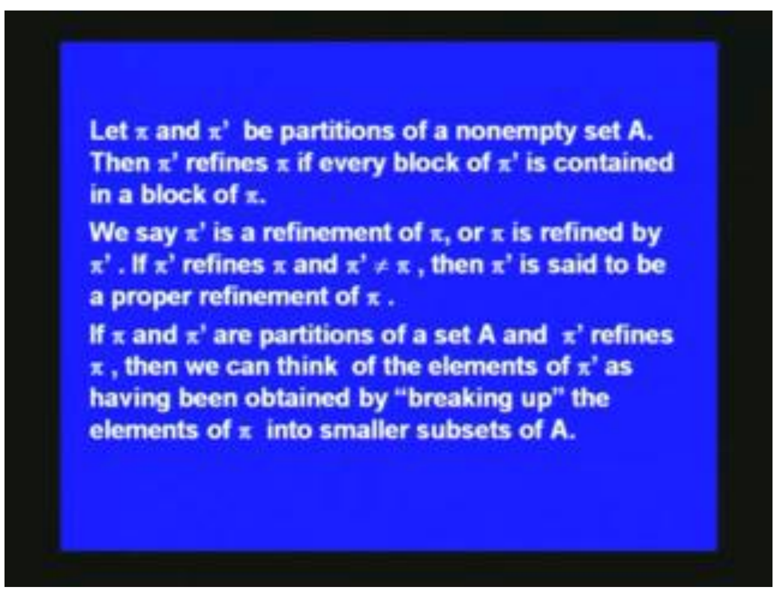

Let pi and pi dash be partitions on a nonempty set A then pi dash refines pi. If every block of pi dash is contained in a block of pi we say pi dash is a refinement of pi or pi is refined by pi dash. If pi dash refines pi and pi dash is not equal to pi then pi dash is said to be a proper refinement of pi.

Let us see what is meant by that. Take an example, A is the set say a, b, c, d, e, f then pi is a partition on A where a, b are in one block c, d, e, f are in one block. Pi dash is another partition where a, b is in one block c, d is in one block e, f is in one block.

Now look at pi and pi dash. There are three blocks here there are two blocks here. Every block of pi dash is contained in a block of pi a, b, this is contained here this is contained here this is also contained here. If this condition is satisfied you say that pi dash is a refinement of pi or pi is refined by pi dash, that is, in a sense what is meant by pi and pi dash?

Pi and pi dash are partitions of A and pi dash refines pi then we can think of the elements of pi dash as having been obtained by breaking up the elements of pi into smaller subsets of A. So here this is broken up into smaller elements so you are dividing into two parts so you are breaking it up. So this is called the refinement of all and pi is said to be refined by pi dash. Of course pi is its own refinement if they are not equal one is called the proper refinement of the other, pi dash is a proper refinement of pi.

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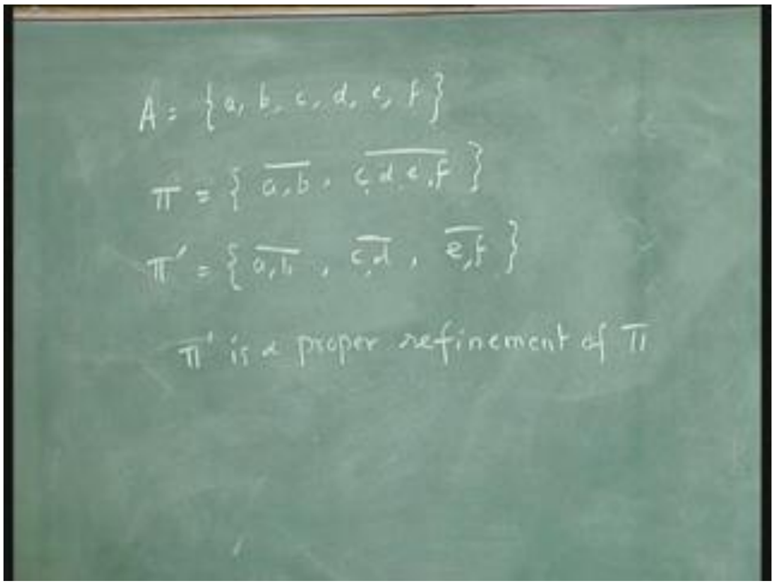

Now, obviously a partition induces an equivalence relation. So pi will induce an equivalence relation on A, pi dash will also induce an equivalence relation on A. What is the connection between these two equivalence relations?
Let us see that.

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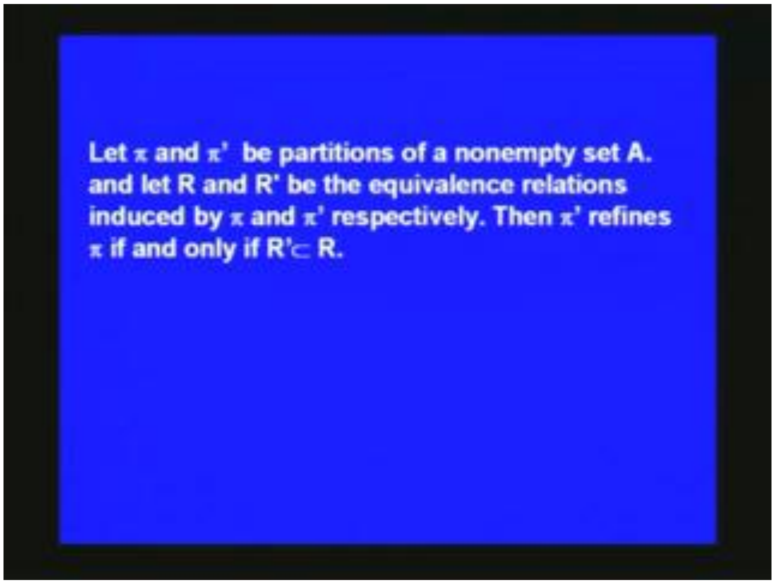

Let pi and pi dash be partitions of a nonempty set A. And let R and R dash be the equivalence relations induced by pi and pi dash respectively. Then pi dash refines pi if and only if R dash is contained in R. Obviously see here, pi induces an equivalence relation where a, b are connected by, it is a complete diagraph and c, d, e, f are connected in all possible manners whereas pi dash you will have a b like this but this is R, R is induced by pi so c, d, e, f and all pairs of arcs are present. This is the equivalence relation represented by a diagraph R which is induced by pi.

Now pi dash also induces an equivalence relation. It will not have all these pairs but it will have these elements, pi dash induces R dash which will have these elements. Obviously whatever is present in R dash is present in R also so R dash will be contained in R. So you can very easily see that pi dash refines pi if and only if R dash is contained in R.

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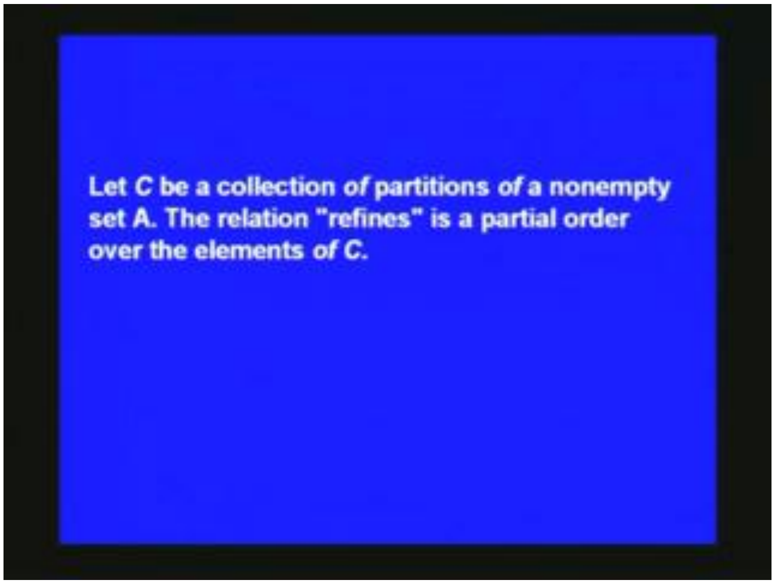

Let C be a collection of the partitions of a nonempty set A, then the relation refines is a partial order over the elements of the set C. Again I will take just three elements in a set A and let us see what the partitions are.

Take a set A having three elements a, b, c, what are the partitions? Pi_1 is a partition having all the three elements, pi_2 is a partition having each one in a separate block, pi_3 will be a partition having a and b in one block and c in a separate block, pi_4 will be a in one block b and c in one block, pi_5 will be a and c in one block b in one block.

Now which one refines this?
Pi_2 will refine this, this, this and this pi_2 will refine every one of that and pi_3 will refine this, pi_4 will refine this, pi_5 will refine this, this is baking this like this. But none of pi_3 will refine pi_4 or pi_5, pi_4 will not refine pi_3 it is not breaking up like that so this will induce a partial order like this. Pi_2 will be the lowest one or the least element, pi_3, pi_4 and pi_5 and pi_1 is refined by everything. So this is a partial order it is a poset diagram or a Hasse diagram pi_2 refines pi_3, pi_2 refines pi_4, pi_2 refines pi_5 and by transitivity pi_2 refines pi_1 and pi_3, pi_4, pi_5 refine pi_1 but pi_3 and pi_4 there is no connection pi_4 and pi_5 there is no connection and so on. So it is represented by a partial order like this.

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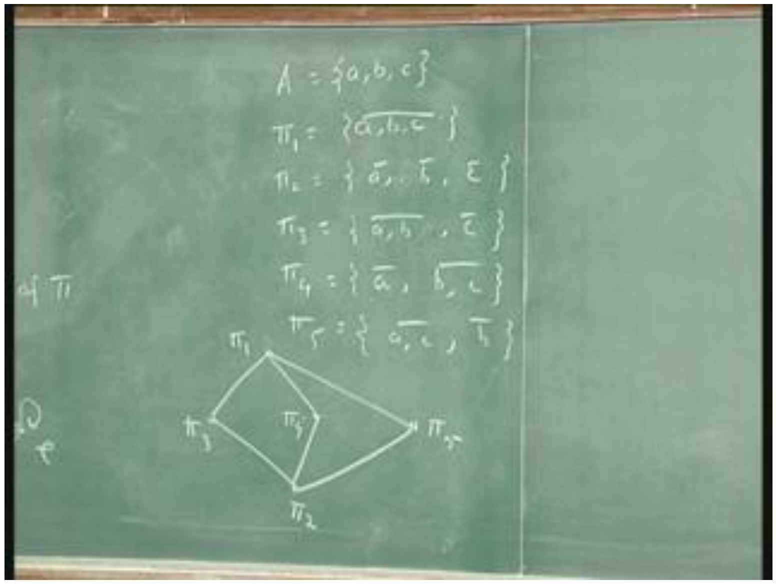

Next we shall see what is meant by the product of two partitions.

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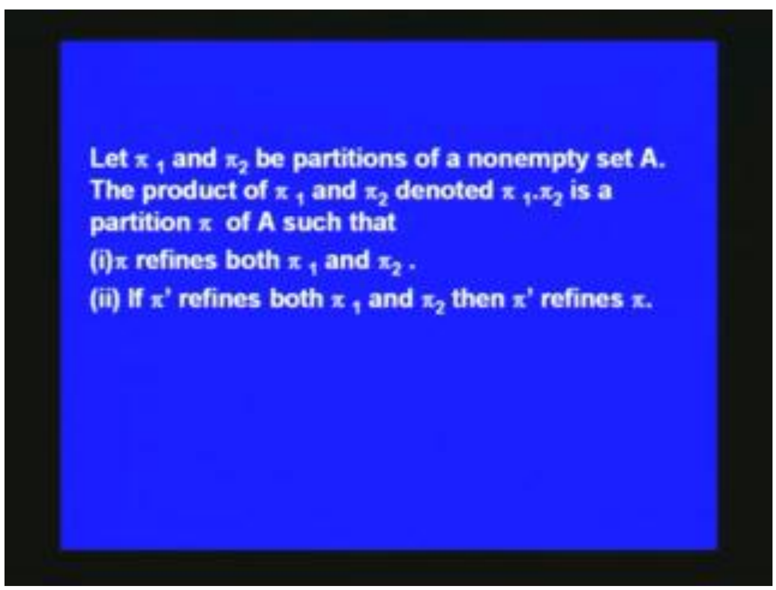

Let pi_1 and pi_2 be partitions of a nonempty set A. the product of pi_1, pi_2 denoted by pi_1. Pi_2 is a partition pi of A such that pi refines both pi_1 and pi_2. If pi dash refines both pi_1 and pi_2 then pi dash refines pi. So let us see what is meant by the product of a partition. Let a b c d be a set, this is a set and pi a partition like this a, b in one block c, d, e in one block and f in one block, this is pi_1. And pi_2 is, a, b in one block and c, d in one block and e, f in one block, so this is an example.

What do you mean by the product of pi$_1$ and pi$_2$?
Pi$_1$ we have represented like this, sometimes you represent like this the product of pi$_1$ and pi$_2$.

What is this?
This is a partition pi such that pi should refine both pi$_1$ and pi$_2$ and any other partition which refines both pi$_1$ and pi$_2$ should refine pi also, so it is obtained like this. So consider this in this example; a, b is a block here a, b is also a block so that will be retained as it is. Now, c, d is one block, c and d are in the same block here so c, d again will be a block here. But e, f is in one block here, e and f are in different blocks here so you cannot have e, f in one block e will be in a separate block f will be in a separate block. Here you have to split c, d, e will be split as c, d and e and e, f here will be split as e and f.

Now you see that this one refines pi$_1$. Each block is contained in a block of this, this also refines pi$_2$. So you see that pi refines both pi$_1$ and pi$_2$ and if you take say pi dash is equal to a, b, c, d, e, f this is some other partition you see that pi dash refines pi$_1$ and pi dash also refines pi$_2$ you see that pi dash also refines pi so that is the condition. The product of two partitions pi$_1$ and pi$_2$ which is denoted as pi refines both pi$_1$ and pi$_2$. And if pi dash refines both pi$_1$ and pi$_2$ then pi dash refines pi.

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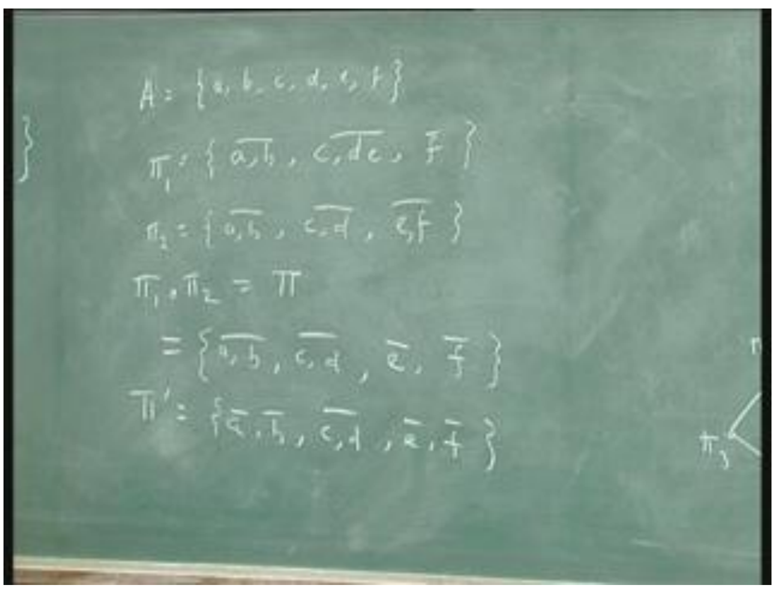

Now what is the connection between this and the equivalence class is represented by them?
Let R$_1$ and R$_2$ be the equivalence relations induced by the partitions pi$_1$ and pi$_2$ of a nonempty set A. Then the relation R is equal to R$_1$ intersection R$_2$ induces a product partition pi of pi$_1$ and pi$_2$. So, if this induces the equivalence relation that equivalence relation will have three components, what are the components here? Corresponding to pi$_1$ if you have R$_1$ then R$_1$ will have a, b the equivalence relation can be represented like this

then c, d, e and f and R_2 will be represented by a, b, c, d, e, f and pi induces R that will be a, b, c, d and e and f.

You can see that the intersection of these two is this. See, this is present here, this is present here, it is present here and c, d this is present here, this is also present here it is present here and the other elements which are not present in this are removed and you have a self loop at e and f. So the intersection of R_1 R_2 is R which is the equivalence relation induced by the product of pi_1 and pi_2 induced by pi.

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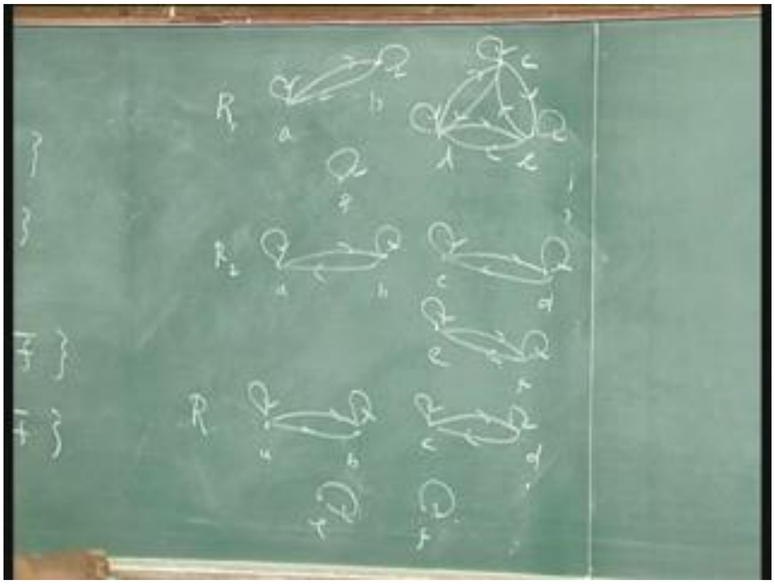

Now we will see what is meant by the sum of two partitions?

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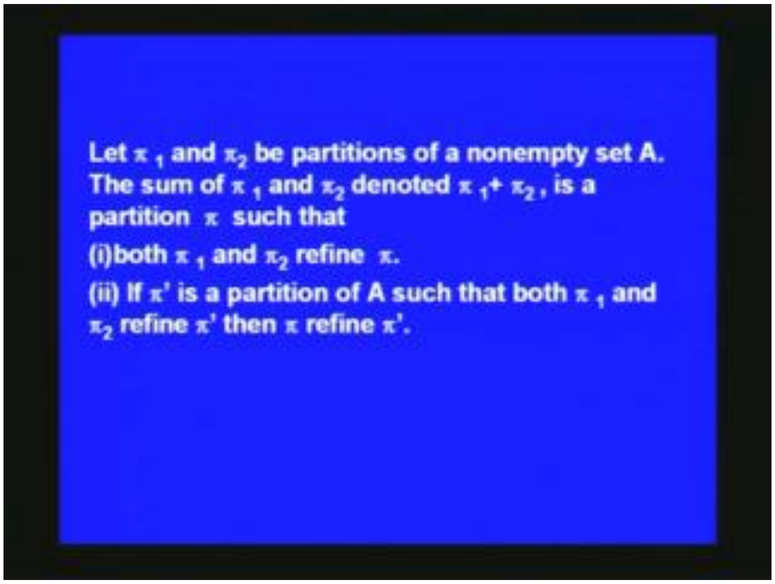

Let pi$_1$ and pi$_2$ be partitions of a nonempty set A, then the sum of pi$_1$ and pi$_2$ denoted by pi$_1$ plus pi$_2$ is a partition pi such that both pi$_1$ and pi$_2$ refine pi if pi dash is a partition of A such that both pi$_1$ and pi$_2$ refine pi dash then pi refines pi dash.

Take the same example here, A is a set like this pi$_1$ and pi$_2$ are partitions like this. The sum of these two pi$_1$ pi$_2$ is a partition pi such that both pi$_1$ and pi$_2$ will refine pi. So here a, b is a block here a, b is a block here so that will be in one block. Then c, d is in one block here but c, d is in block which also contains e so you have to include that also here. And here e and f are in the same block so you have to include that also here. So this is the sum of pi$_1$ and pi$_2$ a, b will be in one block and c, d, e, f will be in another block.

Now you see that pi$_1$ refines this pi$_2$ also refines this. If I take another one say the whole set a, b, c, d, e, f in one block pi$_1$ and pi$_2$ refine this but you note that pi also refine this. So if there is another partition which is refined by pi$_1$ and pi$_2$ it will be refined by this sum also. So that is the definition of this.

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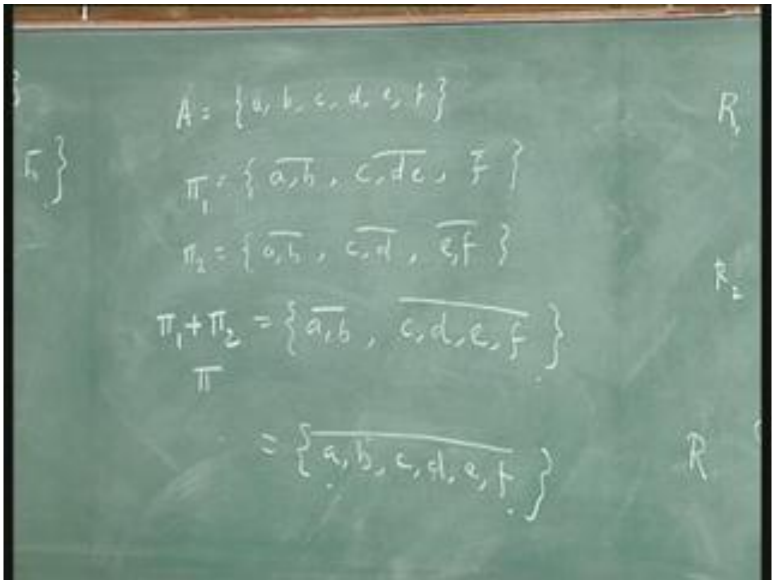

And what is the connection between these equivalence relations?

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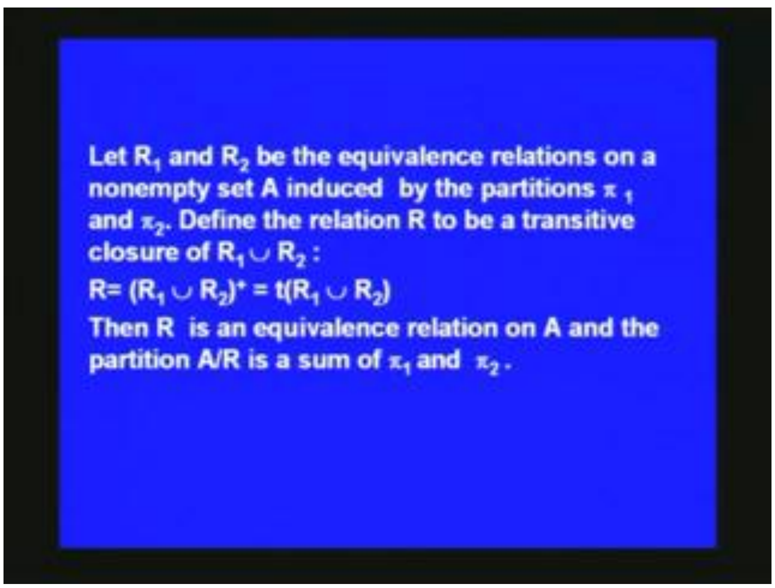

Let R_1 and R_2 be the equivalence relation on a nonempty set A induced by the partitions pi_1 and pi_2. Now, we have seen that R_1 and R_2 are equivalence relations but R_1 union R_2 need not be an equivalence relation. Now, pi_1 will induce an equivalence relation R_1, pi_2 will induce an equivalence relation R_2 we have seen what is that in this example R_1 is this and R_2 is this. Now pi induces another equivalence relation but you see that R_1 union R_2 need not be an equivalence relation.

But what is the equivalence relation induced by pi? That is R and in this case it will be a, b and c, d, e, f in one block and the self loops all over like this. This is the equivalence relation R induced by pi and it is the transitive closure of R_1 union R_2. If you take the transitive closure of R_1 union R_2 that is the equivalence relation induced by the sum partition. R_1 union R_2 plus is equal to t of R_1 union R_2. Then R is an equivalence relation on A and the partition A by R is the sum of pi_1 and pi_2.

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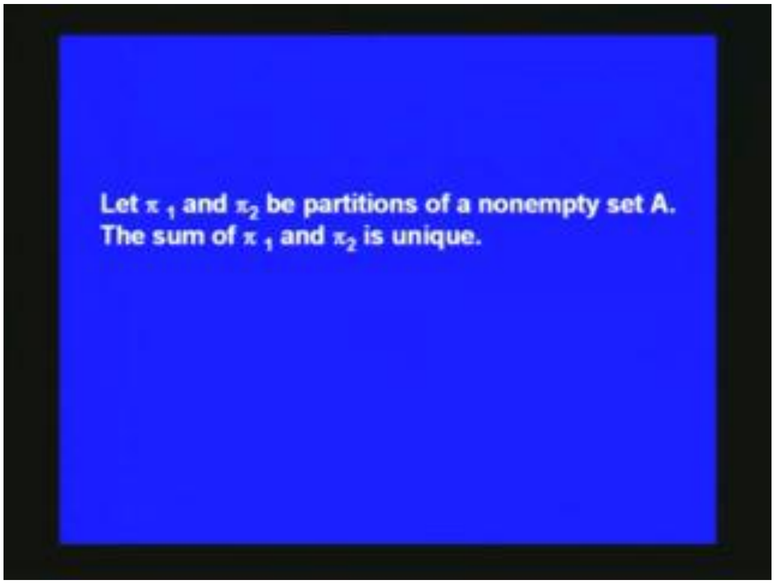

Now if you have two partitions pi_1 and pi_2 the sum is unique you cannot have two different sums why?
If I have pi dash as the sum of pi_1 I have two partitions pi_1 and pi_2 the sum of pi_1 and pi_2 is unique why? If I have two of them say pi of two sums say pi dash and pi_2 dash this will refine this by definition and this will refine this. That means pi dash is equal to pi you cannot have two different sums they will be equal. And similarly, the product of two partitions if you have two partitions pi_1 and pi_2 the product of the partition is denoted by this you can have only one product. If pi dash and pi_2 dash are two products like that pi dash will refine pi_2 dash and pi_2 dash will refine pi dash by the definition and so they will be equal so the product is also unique.

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So we have seen the equivalence relation in detail. This has got a lot of application in several fields of Computer Science. And we have also seen the connection between partitions and equivalence relations and what is meant by the product of partitions, sum of partitions and how they induce new equivalence relations on the underlying set.