Discrete Mathematical Structures
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Lecture # 21
Order Relations

Today we shall consider about order relation. We have been considering binary relations on a set A. So we consider binary relations on a set A. A transitive relation, a binary relation that is transitive defined on a set A is called an order relation if it helps to compare two elements of the set. Though we may not be able to compare all pairs of elements some pairs of elements we may be able to compare and such a relation a transitory relation on a set A which helps us to compare elements of the set is called an order relation.

We shall consider several order relations like the partial order, the quasi order, the linear order, well order etc. First let us see what is meant by a partial order. A binary relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive.

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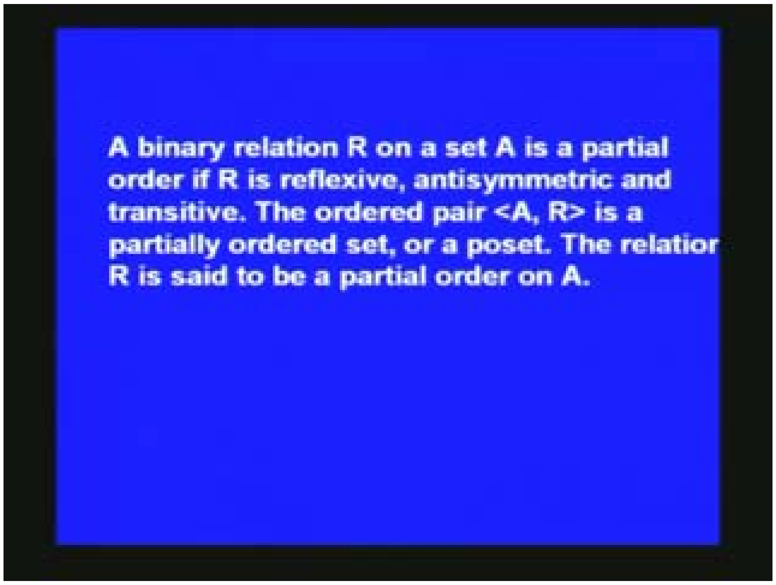

So, a partial order is a relation a binary relation R if it is reflexive, antisymmetric and transitive if it has got all these three properties. The ordered pair A, R is a partially ordered set or a poset. So we see that a binary relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive. The ordered pair A and the relation R on A is a partially ordered set or called a poset. The relation R is said to be a partial order on A. Let us consider some examples of partial order.

If you take some set say B and the power set of B, it is a set of all subsets of B then this contained relation is a partial order because you can compare two elements. For example, if is equal to a, b, c, d then a, b is one subset and a, b, d is another subset. This is contained in this and such a relation is called a partial order. It is reflexive because a, b is contained in a, b.

Let us see the three properties; it is reflexive, it is antisymmetric because if you take two subsets S_1 contained in S_2 and S_2 contained in S_1 that would mean S_1 is equal to S_2 which is the definition of antisymmetric. And it will also satisfy the transitive property because if S_1 is contained in S_2 is contained in S_3 obviously S_1 will be contained in S_3, this is the definition for transitivity. So, this relation is reflexive, it is antisymmetric and transitive and so this is a partial order.

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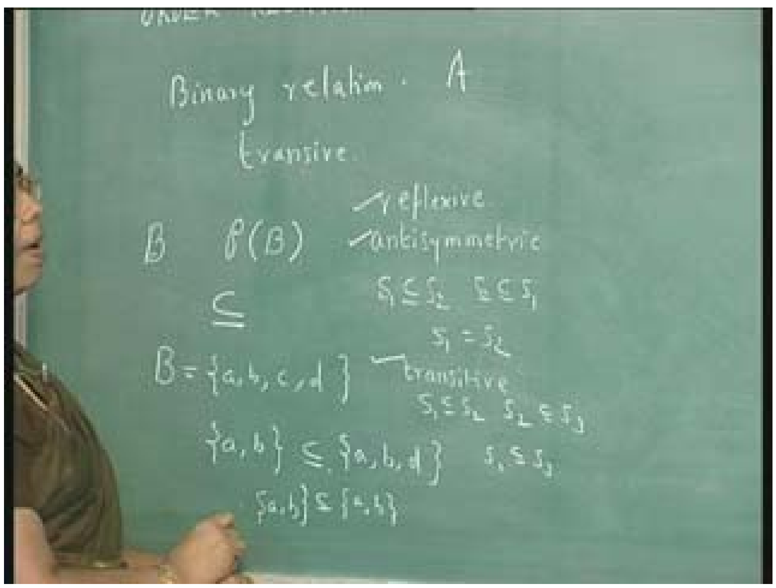

And the set A R that is here the power set of a set B and the relation is contained in this is a poset. This example is a poset. You call it as a partially ordered set or a poset. Another example is, take the set of non negative integers and the less than or equal to relation. Here, again if you see if a is a non negative integer a will be less than or equal to a and so the reflexive property is satisfied, it is reflexive. And you can see that it is antisymmetric if a is less than or is equal to b and b is less than or is equal to a this would imply a is equal to b so the antisymmetry property is also satisfied. And obviously if a is less than or is equal to b and b is less than or is equal to c then from this you can conclude that a is less than or is equal to c so the transitive property is also satisfied. So this relation is reflexive, antisymmetric and transitive. So the less than or equal to relation on the set of non negative integers and you can also take set of integers same thing will hold. So this satisfies all these properties and therefore it is called a poset or a partially ordered set.

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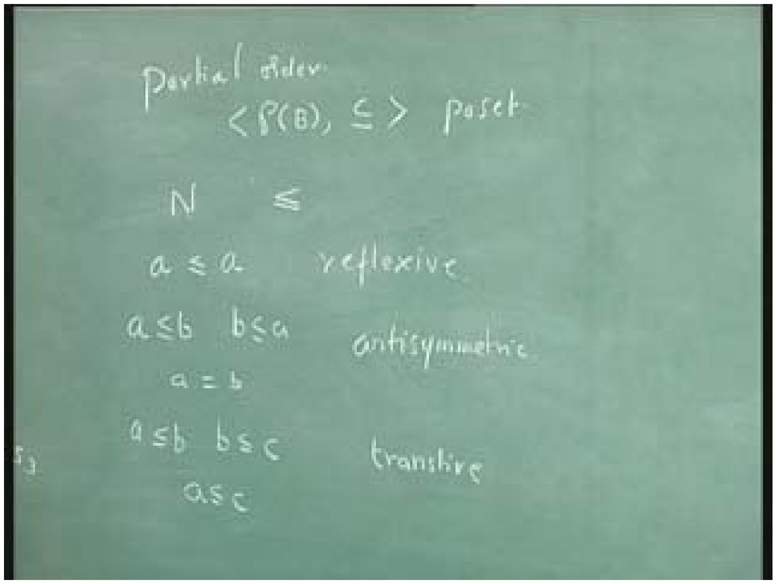

Now, let me take a simple example of a set of a relation. Take a set A a, b, c, d and a relation on the set represented like this a b c d like this. So this relation has ordered pairs.

What are the ordered pairs here?
A, b and a, c and b, d and a, d then a a, b b, c c and d d. So these are the ordered pairs in the relation represented by this diagram.

Now is it a partial order?
You can see that you have self loops at every node so the reflexive property is satisfied. And antisymmetric means between two nodes you can have no arcs or arc in only one direction, you should not have in two directions so antisymmetry property is also satisfied.

And is it a transitive relation?
Here you find that from a to b, b to d you have so you also have a to d so the transitive property is also satisfied and this is a partial order.

Now, when we want to represent a partial order as a diagram we usually do not write everything on the graph. As a diagram we want to represent a partial order we will not write everything or we will not draw all the arcs we will write only the minimum number of arcs. For example, we know that a partial order is reflexive so there will be self loop at every node and it is not necessary to explicitly draw the self loop. So usually a partial order is represented by means of R dash, if R is a partial order diagram only will represent R dash such that R dash star is equal to R.

A relation R dash whose reflexive transitive closure will be R will be represented on the diagram that is called a hasse diagram or a poset diagram. In this case it will be represented like this: a, a to b, b to d and a to c this is all and you do not mark the arrows also here, the arrows are supposed to be directed like this from below to above directions are like this. So, if you take the reflexive closure of this you will have the self loops and also if you take the transitive closure because there is an arc from a to b and b to d there will be an arc from a to d. So the whole thing is represented by means of a smaller relation R dash and that too without the directed arc and we assume that the arcs are directed from below to above and this smaller relation reflexive transitive closure is R. This is easier to draw, in this case there are only four elements in the set so it may not make much of a difference but in other cases it may make a difference. So, it is easier to draw a diagram like this rather than the whole diagram with self loops and arcs and so on.

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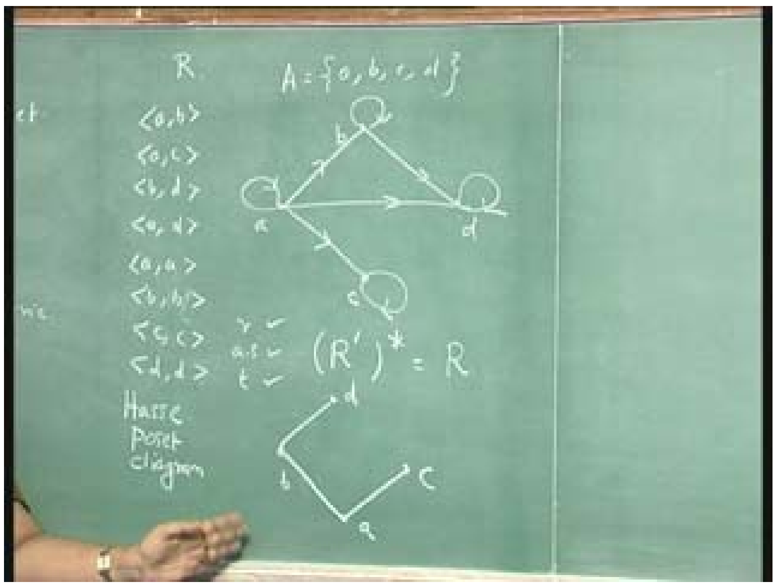

Another example of that is, take the set of integers the set A you take from 1, 2, 3 up to 12. And a is related to b, the relation is defined like this a is related to b if a divides b. You can very easily verify that this is a partial order. It is reflexive and it will be antisymmetric and it will be transitive. Each number divides itself, if a divides b and b divides a that means a is equal to b if a divides b and b divides c then a will be dividing c so all the three properties such as reflexive, antisymmetric and transitive are satisfied so this is a partial order on the set A containing integers from 1 to 12.

Now how will you represent it as a Hasse diagram or a poset diagram?
Now 1 will be below that, so 1 divides 2 it divides 3 and it divides 5 it divides 7 it divides 11 this is 2, 3, 5 we have to mark six more elements 1 2 3 4 and 2 divides 4 so 4 will be like this. Here, there is no arc between 1 and 4 but by transitivity 1 also divides 4. Then 4 divides 8, 1 2 3 4 5 6 so 2 and 3 divides 6 so 6 will be somewhere here, 2 divides 6 and 3 divides 6 so it will be like this. Then what about 1 2 3 4 5 6 7 8 9, 3 divides 9 so 9 will be marked like this, then 10 2 and 5 divide 10 so 10 you can mark somewhere here like this and then 2 divides and 5 divides you can mark at the same level this level also you can mark. Then 11 is there 12 3 and 4 and 6 also divides 12 so 4 and 6 divide 12 so 12 will be marked like this 4 divides 12 and 6 divides 12 and so on. This is the poset diagram or the

hasse diagram for representing the partial order which contains the set A of integers from 1 to 12 and the underlying relation is; a related to b if a divides b.

Therefore, you can easily see that 1 divides 2 and 1 divides 8 also or may be 2 divides 8 but you do not have a direct arc from 2 to 8 you have an arc from 2 to 4 then 4 to 8 these arcs are directed from below to above. So like this and like this the arcs are directed. You do not have a direct arc from 1 to 9 but 3 to 9. Similarly, 12 is divided by 3, 2, 6, 4. But you only draw the lines from 4 to 12 and 6 to 12. So this is the way you draw a poset diagram or a Hasse diagram.

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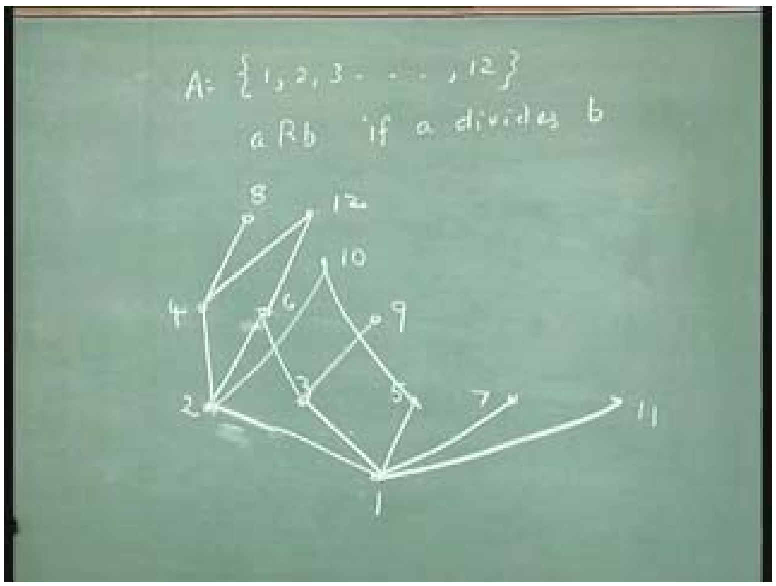

Now, if you do not want to have the reflexive property you define what is known as a quasi order.

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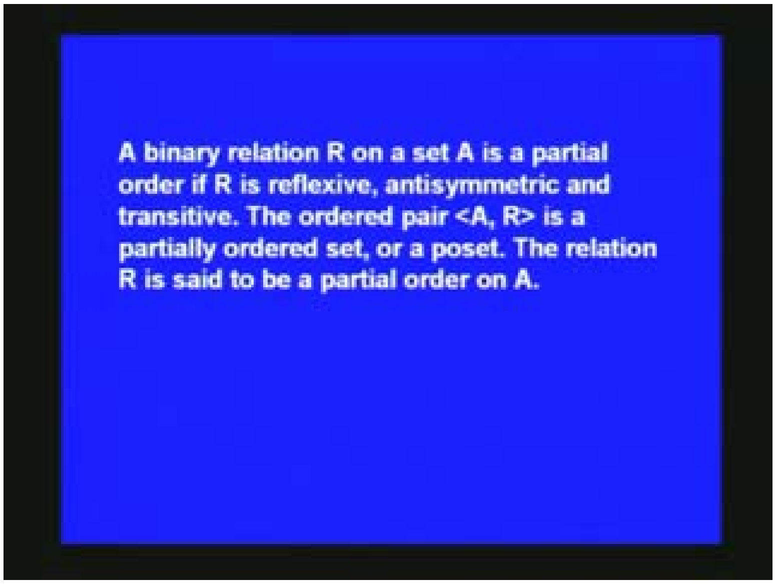

Let R be a binary relation on a set A, R is a quasi order if R is transitive and irreflexive.

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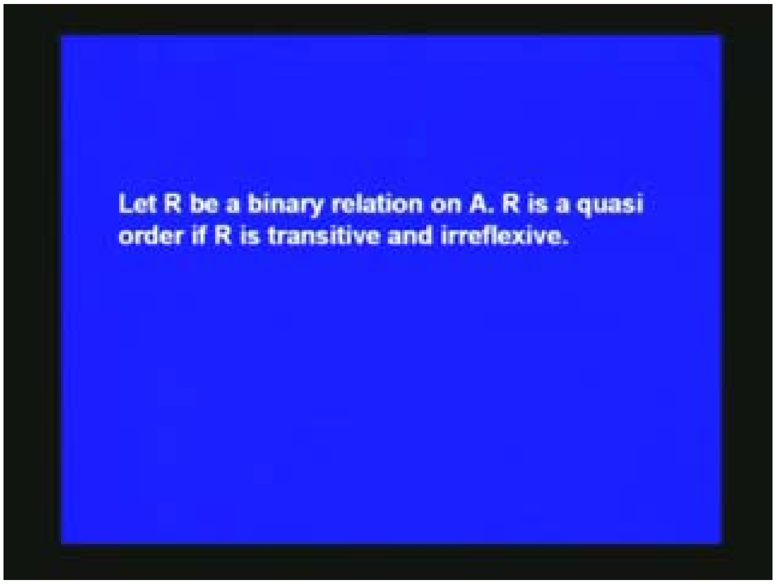

So partial order, we were considering partial order it has to satisfy the three properties reflexive, antisymmetric and transitive. A quasi order is almost similar to that. You have the irreflexive instead of reflexive that is no self loops and this property also you will have and you have the transitive property. We need not have to specify that antisymetry property explicitly because these two together will imply antisymetry property. Let us see how it is.

What is the definition of antisymetry property?
If x is related to y and y related to x this means x is equal to y this is the way antisymetry property is defined. But you will see that if you have irreflexive property and transitive property this will be always false, this will always be false. And we know that in an implication if the premise is false the implication is true this is what we have seen in large.

And why is this always false?
If you have x related to y and y related to x then the set has transitive property this from. You can conclude that x is related to x. But if it is irreflexive for any x x is not related to x if it is irreflexive so this is not correct and so this is always false. If this is true x will be related to x because of transitivity which we know is not correct. So the premise of this implication is always false and because the premise is always false the whole statement is true and so the antisymetry property will automatically hold. So these two conditions together will imply this. So a quasi order is a binary relation on a set A which is irreflexive, antisymmetric and transitive.

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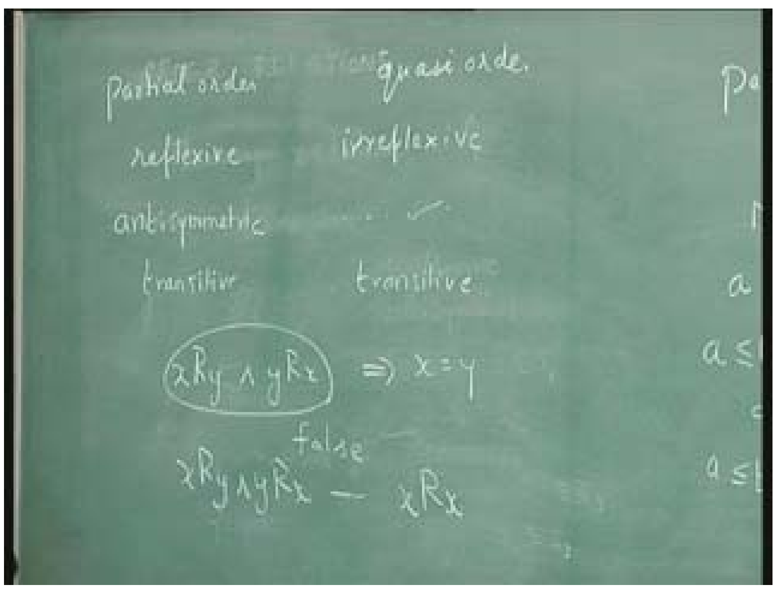

Examples are; if you consider a set B and the power set of the set B and use proper containment instead of containment the relation is the proper containment and in this case reflexive property will not hold. If it is containment reflexive property will hold. If it is proper containment reflexive property will not hold so it will be irreflexive. The other two conditions antisymetry and transitive will automatically hold. And if you take the set of non negative integers or the set of integers and consider the relation less than, we have seen that less than or equal to as a partial order if you consider the relation less than then it will be a quasi order because it is not reflexive it is irreflexive and less than is antisymmetric and transitive. So this is an example of a quasi order.

Some other examples of quasi order are; if you take a set of courses in curriculum and c_1 related to c_2 course one is related to course two if c_1 is a prerequisite for c_2 you consider the set of courses in some curriculum and you define the relation R such that c_1 is related to c_2 if c_1 is a prerequisite for c_2, this is irreflexive, a course cannot be a prerequisite for itself and it will be antisymmetric because if c_1 is a prerequistive to c_2 obviously that c_2 cannot be a prerequistive so that sort of a premise will always be false in the antisymmetric condition.

What about transitive if property?
If c_1 is a prerequisite for c_2 and c_2 is a prerequisite for c_3 then c_1 is obviously a prerequisite for course c_3 also so this is a transitive relation, it is irreflexive and transitive and so it is a quasi order.

Another example is; you want to sort of allocate the tasks in a big way. I mean you have number of tasks you have to find out which one should be completed before other and so on. For such a thing in operations research you must use what is known as a PERT diagram. And there it tells you which jobs can start after which job only after this job is completed the other job can start and so on. So in such a thing you define a relation like x is related to y if x has to be completed before y starts then such a relation will again be irreflexive and transitive and so it is an example of a partial order.

You come across such examples in several things, in practical use, in several fields of life though you may not realize that you are really using a partial order or something like that, technically you may not use that word but in several aspects of practical life in several fields in computer science you use this idea. So, with this relation we have the following results.

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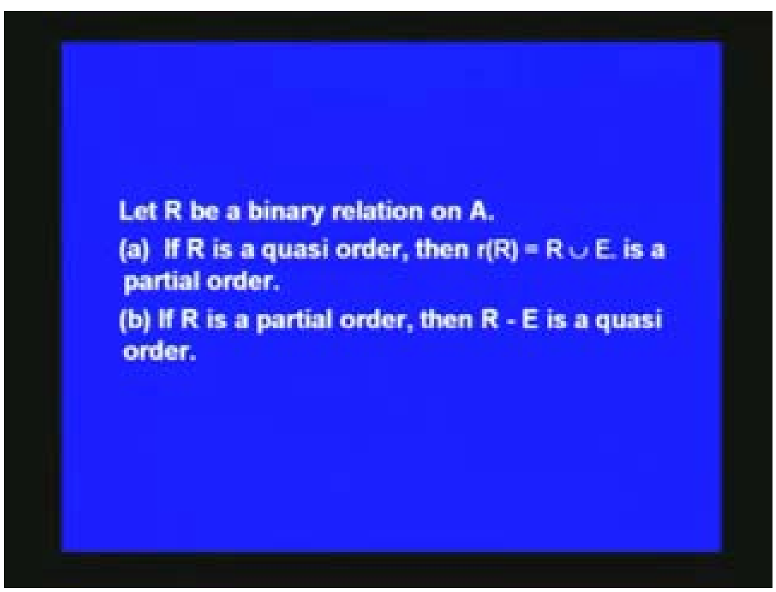

Let R be a binary relation on A then if it is a quasi order it is irreflexive. So if you add E to that, if the reflexive closure you take and add E it will be a partial order obviously. Then if R is a partial order it will be reflexive and if you take away the E if you remove all the self loops then R minus E will be irreflexive and so it will be a quasi order. So from a quasi order you can very easily get a partial order by adding E and from a partial order you can get a quasi order by removing the self loops or by removing the elements of the equality relation E.

Now, these two things we have considered sometimes some more conditions may be there. See, for example, in the case of that divides relation in a partial order you may be able to compare elements but you may not be able to compare all the elements.

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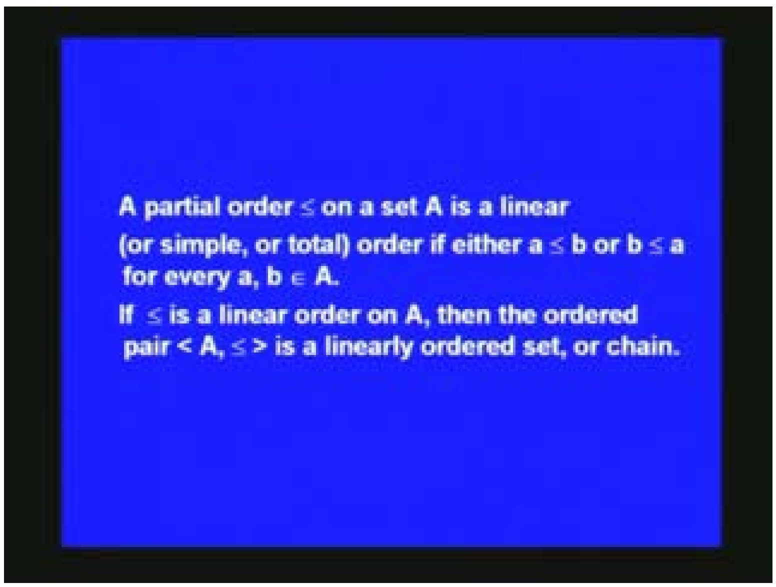

Take for example this one which we considered. You know that 2 is related to 4 you know that 4 is related to 12 and so on 4 relates to 12 some elements you are able compare. But what can you say about 7 and 9? There is no relationship, neither 7 is related to 9 or 9 is related to 7 there is no relationship between them that is why this is called a partial order. You cannot compare every pair of element. So only in some pairs that relation exists. Now, if in a partial order you are able to compare every pair of elements then it is called a linear order.

When you are comparing two elements, a partial order is usually denoted by this set A and the relation is you can use the less than or equal to symbol but it is slightly different from the less than or equal to symbol, usually it is denoted like this, the underlined set A is denoted like this, the relation is denoted like this, usually when you write it is slightly different from the less than or equal to. But if you are not able to use this symbol you can use less than or equal to also slightly different from the less than or equal to symbol.

Now, if you take any element you may be able to say a is less than or equal to b or b is less than or equal to a. but in the partial order it may not be possible to do like this for every pair of elements. But if you are able to do for every pair of elements you say that a will be less than or equal to b or b will be less than or equal to a then that is called a linear order and you say it is a chain.

A partial order on a set A is linear sometimes you call it is as a simple order and sometimes you call it as a total order if either a is less than or equal to b or b is less than or equal to a for every a b belonging to A. That is if you take any two elements you will be able to compare them. If that is a linear order on A then the ordered pair A in that partial order relation is a linearly ordered set it is called a linearly ordered set, this is called a linearly ordered set.

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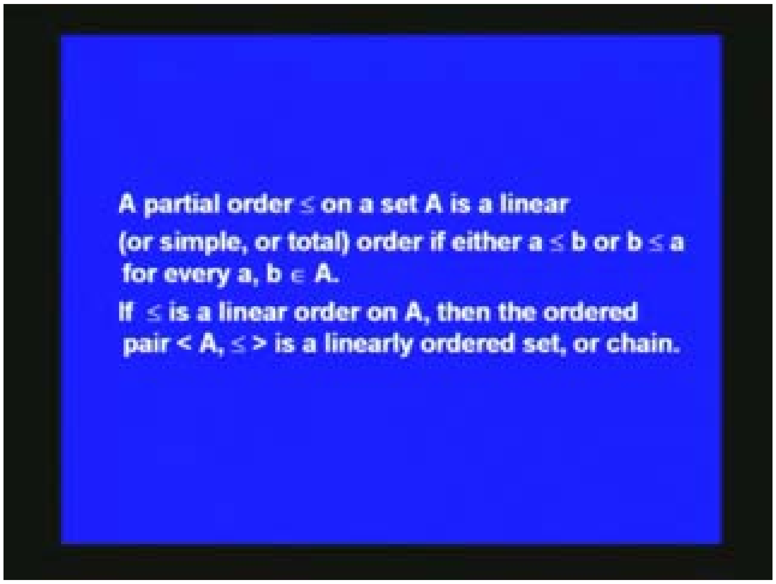

Now, take the set of integers and the less than or equal to relation. You can take the less than or equal to relation. You can write like this: 0 minus 1 minus 2 minus 3 1 2 etc. You can compare any two of them. Any two elements if you take you will be able to compare. if you take the set of non negative integers and the less than or equal to relation it will be like this: 0 1 2 3 it looks like a chain, the poset diagram will look like a chain that is why we use this name chain, this is again a linear order. But if you take the set B and all subsets of B for example take B to be a, b, c, d and take all subsets of B and the contained relation this is not a linear order because for example take a, b and take b, c you cannot say this is contained in this or nor this contained in this you cannot compare them so this is not a linear order.

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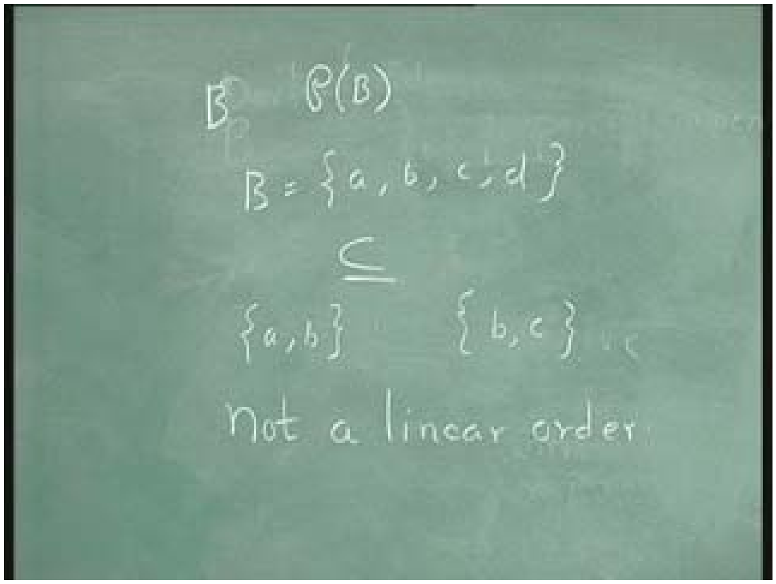

Now let us see what a well order is.

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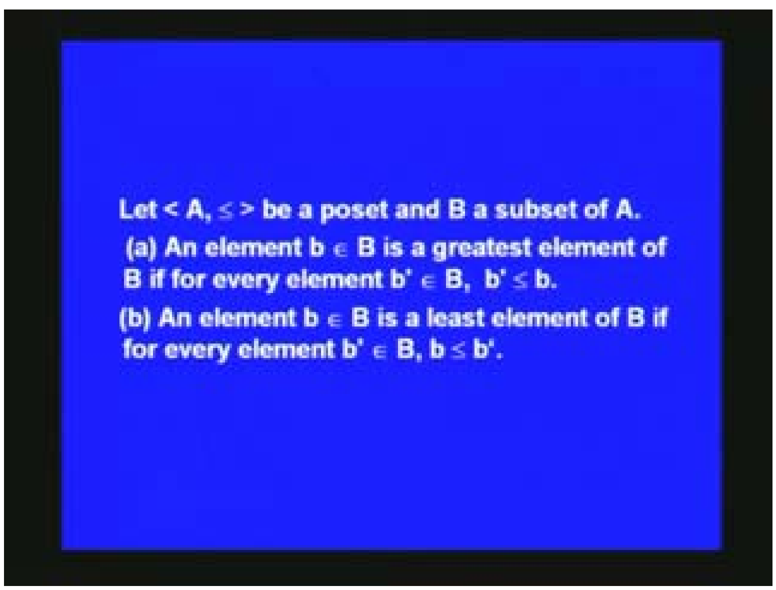

And before going to well order we will see what is meant by a greatest element and what is the least element of a set. Let A and this relation be a poset you have a partial order A and this is like this you have a partial order. I will read it as less than or equal to though it is slightly different from the less than or equal to function. I will read it as a less than or equal to, A less than or equal to be a poset and B a subset of A. Then an element of B is a greatest element of B if for every element b dash belonging to B b dash is less than or equal to b. An element b belonging to B is the least element of B if for every element b dash belonging to B b is less than or equal to b dash this is the definition. If I take an example it should be clearer to you.

Take the set A as the power set of a set having two elements. so this has got and the contained relation a, b is one set the empty set is one, a alone is one subset b alone is one subset. The poset diagram is defined like this. This is contained in this and this is contained in this obviously this is contained in this because of the transitivity.

Now, if you look that the whole set a, b this is the greatest element because every other element is less than that. And this is the smallest element here or the least element this is the greatest element and this is the least element. But I do not consider any all of them suppose I consider only this and this. If I consider only this and this, among this you find that this is the greatest element because everything else is less than that and this is the least element because this is less than every other element.

Now, if I consider this, a set consisting of only this and this, a subset consisting of a subset of A, I am considering B the definition is with respect to a subset B of A. So the subset has only these two elements, if you consider this they cannot be compared. This is not less than this or this is less than this. And this subset has no greatest elements because you cannot define an element with which you can compare these two and say that these two are less than that. So, if you take a subset having these two alone this has no greatest element neither it has got least element it does not have a least element either. So a greatest element may exist a least element may exist or may not exist. So, after having defined let us see what it is and if the element exist if the greatest element or the least element exist it is unique.

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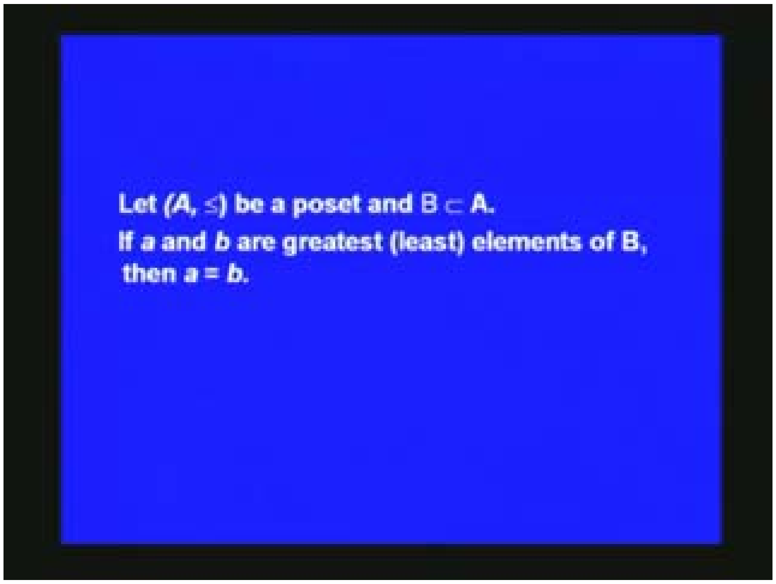

Let A less than or equal to be a poset and B is contained in A B is a subset of A and if A and B are the greatest elements of B suppose you are having two elements A and B as the greatest element then A should be equal to B. Or in other words, the greatest element if it exists is unique, why? Suppose I have two elements A and B both of them are greatest elements. Then by the definition of greatest elements a will be less than is equal to b because b is the greatest element and by the definition of greatest elements because a is the greatest element b will be less than or is equal to a in which case from this you can conclude that a is equal to b. So the greatest element need not exist but if it exists it is unique. Similarly, the least element if it exists is unique.

Now having defined the greatest element and the least element we will see what happens. A binary relation R on A is a well order. If R is a linear order and every nonempty subset of A has a least element. The ordered pair A R is called a well order well ordered set and R is a well ordering of A.

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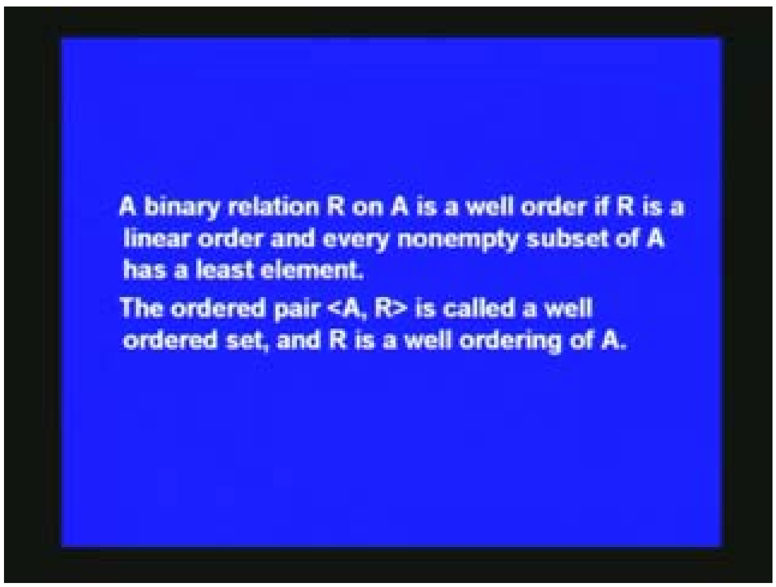

So, first we defined partial order quasi order then we defined linear order and in this linear order if you put the restriction that every nonempty subset has a least element then it is called a well order. Let us see some examples.

Consider the set of non negative integers and the less than or equal to relation. You see it can b represented by a poset diagram like this 0 1 2 3 etc. This is a well order because if you take any subset that will have a least element. Instead of that you take the set of integers and the less than or equal to relation the poset diagram will look like this it is not complete it is like this 1 2 minus 1 minus 2 like that. Now, if you take a subset say all set of negative integers that has no least element, you cannot define the least element for that. So this is not a well order.

Sometimes it may be really necessary to have a well order if you have a linear order it may not be sufficient we may require a well order. a slightly different version of that we have studied in proving program correctness which we considered earlier after logic portion we considered that and there we defined what is known as a well founded set that was slightly different and the definition is slightly different from well ordered set but there also the condition was that every subset should have a least element. So, you can see that N less than or equal to, if you take the set of non negative integers and less than or equal to relation that is a well ordered set.

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Now, we have considered set operations on sigma stars taking sigma star and you find some lexicographic ordering and so on. Before going into that we saw that i less than or equal to is not a well order. But you can induce a well order on that, induce a well order on it or into it, how do you do it?
Here two integers a and b are compared like this a less than or is equal to b the comparison is like this. Now if they define the relation a less than or equal to b in this manner such that a less than or equal to b if mod of a is less than mod of, the numerical value is less than b if or if mod of a is equal to mod of b then a is less than b.

If you define the relation like this you can define a well order into the set of integers, it is like this. For example, if you do that 0 you will order the elements in this way: minus 1 and 1 will be less than or equal to minus 2 and 2 because if the modulus the value of that is less than the value of this then this will come before that so they will be ordered like this. And within them minus x and plus x the minus x will come because of this condition minus x will come before plus x. So minus 3 plus 3 like that you can order them so you can say this is the first element, this is the second element, this is the third element, fourth, fifth, sixth, seventh like that you can say.

And if you define like this you can compare any two of them. If you take again you can compare any of them and also if you take any subset that will have a least element in this case. But you must remember that the relation we are defining is not less than or equal to on the set of integer but it is a relation R which is defined this way: a is related to b if the value of a is less than the value of b, numerical value of a is less than the numerical value of b and if they are the same a is less than b. So if you define this relation it becomes a well order but not under less than or equal to.

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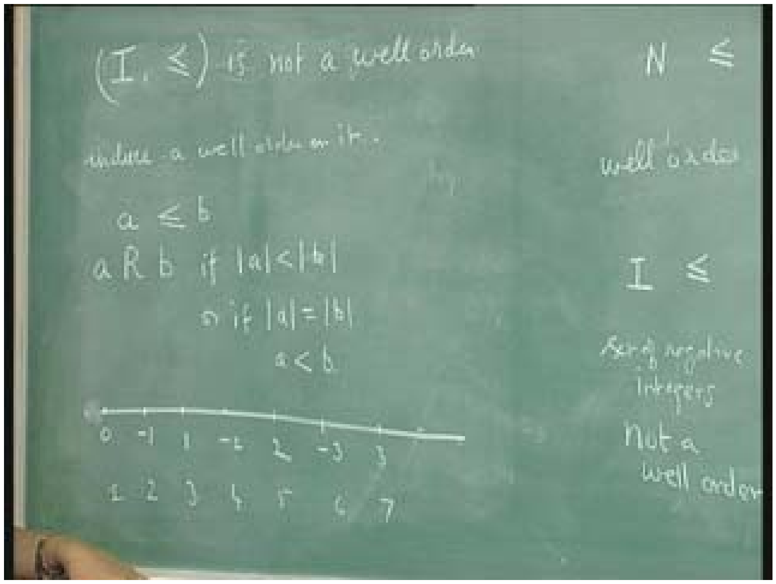

The next is we consider some operations on sigma star. What is sigma? Sigma star we have seen earlier. Sigma is an alphabet and sigma star denotes the set of all strings over sigma where lambda or epsilon denotes the empty string R, lambda or epsilon denotes empty string of length 0. We have seen all these earlier such as what is a length of a string and so on. On sigma star you can define two types of relation one is called the lexicographic ordering and another is called the well ordering or canonical ordering. Let us see what that is.

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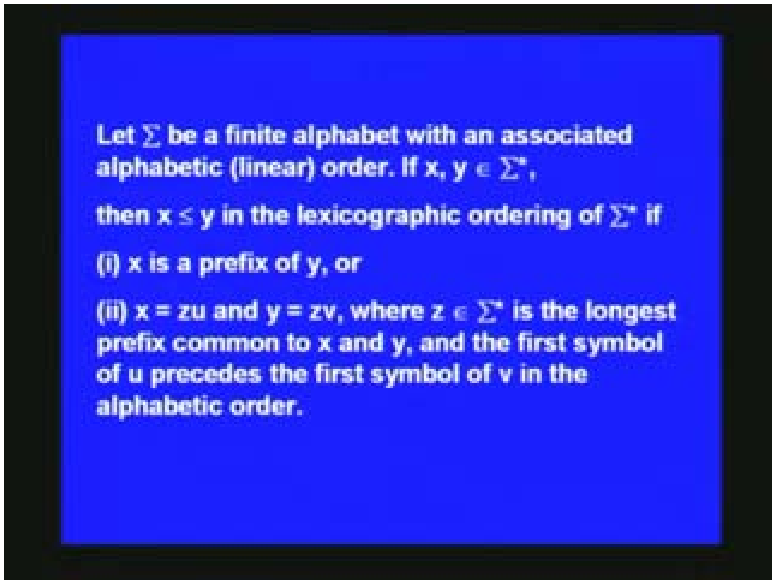

Let sigma be a finite alphabet with an associated alphabetic order, what do you mean by that?

Sigma is a, b, c here a comes before b and b comes before c that is the associated order on it. So generally you write a, b, c or you can write a_1 a_2 a_3 which one is first that you should mention clearly. And one point we have to note that all this linear order and well order if you define on a finite set then any linear order will be a well order because it will always have a least element. A linear order on a finite set will be a finite chain like this so it will always be a well order you will have the least element but only on infinite set that problem will come. So let sigma be a finite alphabet with an associated alphabetic linear order if x belongs to sigmastar then x is less than or equal to y in the lexicographic ordering of sigmastar

If x is a prefix of y or x is equal to zu and y is equal to zv where z is a string in sigmastar and that is the longest prefix common to x and y and the first symbol of u precedes the first symbol of v in the alphabetic order. It is lexicographic ordering is the same as the order followed in a dictionary. So if you have two strings x, y when do you say that x is less than or is equal to y? x comes before y in the dictionary. That is possible when x is a prefix of y that is suppose I have a_1, a_2, a_n and another string a_1, a_2, a_n and an plus 1 something like that am this will come before this in the dictionary so this x will come before y, this is x and this is y. Otherwise x is of the form say some a_1, a_2, a_i, a_i plus 1 etc a_n, y is of the form a_1, a_1 is the same as this and up to a_i it is the same then $_{bi}$ plus 1 and so on. And this comes before this in the order, a comes before b like that. In that case you say x will come before y in the dictionary or x is less than or is equal to y. This is the alphabetic ordering or the lexicographic ordering.

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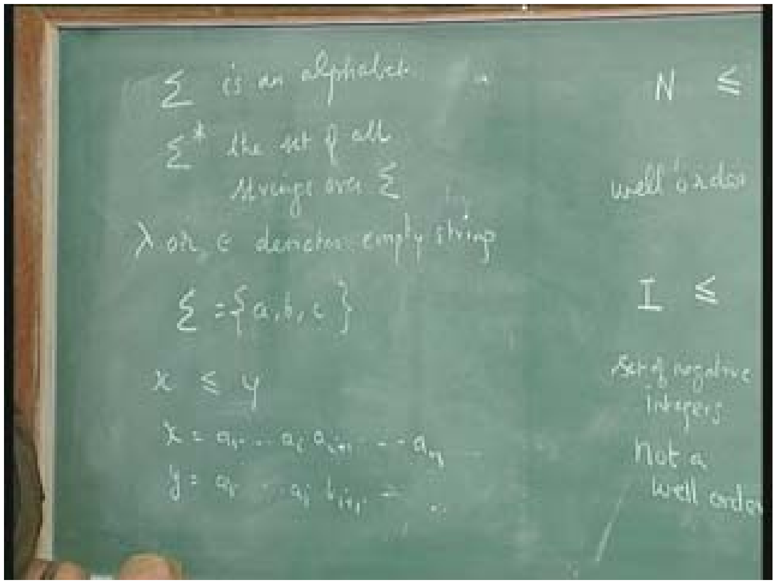

Let us take some strings and compare. For example; take a, bc, aabc, aaba which will come before this?
This will come before this. If you want arrange this a will come before these and among these aaba will come before aabc and bc will come afterwards. This is called the lexicographic ordering or a linear order. This is a linear order because if you take any two strings you can always say that this string should come before the other string so it is an example of a linear order.

For example, take sigma to be a, b, c and then we have considered some strings, is it a well order? It is a linear order, this is a linear order, is it a well order? It is not a well order, why?
You cannot find a least element in a subset. Well order means if you take a subset of that it should have a least element. For example, consider this set b, ab, aab, aaab like that consider the set. That is a power n b n greater than or is equal to 0. Now you can see that if you take ab this will come before b and if you take aab that will come before this and if you take aaab that will come before that. But what is the least element, which one is the least element here? This is less than this, this is less than that, what is the least element? There is no least element in this set so this is not a well order. In order to compensate for that, you define a well order on the set of alphabetic order.

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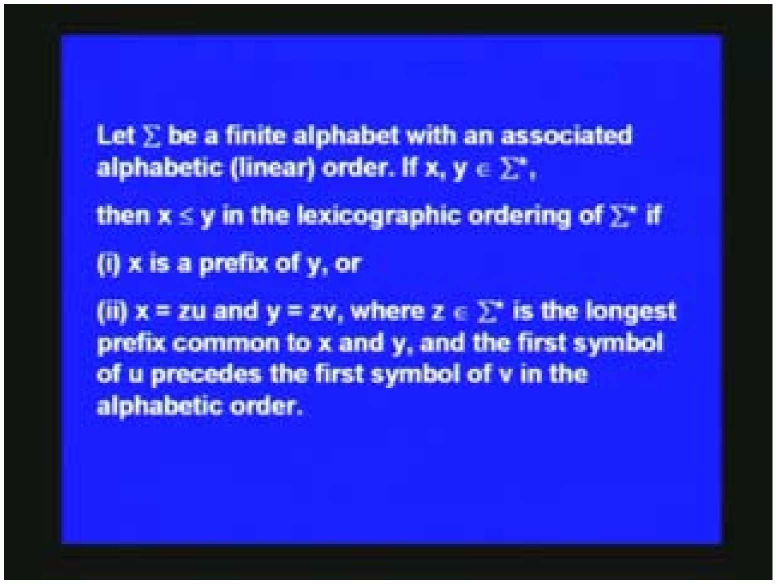

Let sigma be a finite alphabet with an associated alphabetic order and let the mod of x denote the length of x then x is less than or is equal to y in the standard ordering of sigmastar. If the length of x is less than or is equal to y or if the length f x is equal to length of y then x precedes y in the alphabetic ordering or lexicographic ordering of sigmastar. If you define this way again you get a linear order and not only that you can talk about the least element in this order. For example; take sigma to be a, b, c again a comes before b and b comes before c.

How will you talk about the strings?
Here you can arrange them in the order, first the empty string will come then strings of length 1 will come which is a, b, c then strings of length 2 will come and among that they will be arranged in the alphabetic order. So strings of length if you arrange aa, ab, ac they will be in this order ba, bb, bc, ca, cb, cc and so on. If you take ab, abb and abac this will come before this because the length of this is 3 and the length of this is 4 even though in the alphabetic order this will come before this but in the well order this will come before this because the length of this is less than the length of this.

And if you take two strings of the same length abc and aca something like that, which one will come first?
This will come before that because in the alphabetical ordering this will come before this. So, if you compare two strings the smaller string will come before the lengthier string in this order and if they are of same length then the string which will appear in the alphabetical order first will come before the string which will appear later. Now in this case consider the set b, ab, aab, and so on. This will be the least element. This is of length 1 and the rest of them are length more than that so that will be the least element.

Since you can arrange them into one corresponding to the set of non negative integer 0 1 2 3 4 5 and so on so you can arrange them in this order you can talk about the ith string in this enumeration. When you can do this it is always a well order so you can always talk about it, any subset you take you can always talk about the least element of the set, so this is a well order. And this is very important because in many cases you may want to have an enumeration of the strings over the alphabet sigma. I will continue with this enumeration.

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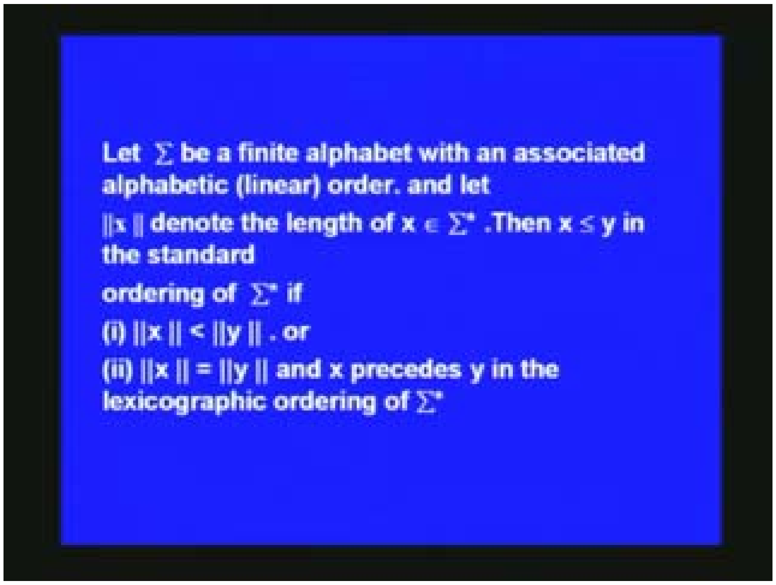

Suppose I have a string xa what will be the successor of xa?
xb will be the successor of, x for xa what is the successor what is the next string? It will be xb. For xb what will be the successor in this well order? It will be xc. For xc what is the successor? It will be ya where y is the successor of x. And similarly if you take the predecessor for xb the predecessor will be xa, for xc the predecessor will be xb x is some string and for the string xa the predecessor will be yc where y is the predecessor of x. This is a way you define. You can arrange the strings of length 3 also here. We have arranged the first strings of length 0 then 1 then 2. If you want to arrange strings of length 3 it will be in this order aaa, aab, aac, aba, abb, abc and so on. Then it is aca, aca, acb, acc and so o.

So you find that the successor of xa is xb, the successor of xb is xc and the successor of say xa will be yc where the successor of xc will be ya and y is the successor of this aa for aa ab is the successor this is what we have seen. So, from this example we can very easily see the successor of xa is xb, the successor of xb is xc, but the successor of xc is ya where y is the successor of x, ab is the successor of aa. And similarly, you can that the predecessor of xb is xa, the predecessor of xc is xb and the predecessor of xa is yc where y is the predecessor of x, this is the way you define. So you can arrange the strings in this order and you can have a 1 to n mapping with the set of non negative integers. You can talk about the ith string in the element supposing in this well order over the alphabet a, b, c.

If I ask about what is the 47th string in the enumeration you can always say you can find out and tell what the string is and so on. And this sort of an idea is very useful in proving some very difficult theorems in automata theory about decidability. So we shall see more about this ordering in the next lecture.