Discrete Mathematical Structures

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Lecture-2

Propositional Logic

(contd.)

Let us continue with what we learnt in propositional logic in the last lecture. First we shall review what we learnt in the last lecture. We saw what is meant by a proposition and taking propositional variables P and Q PQ.

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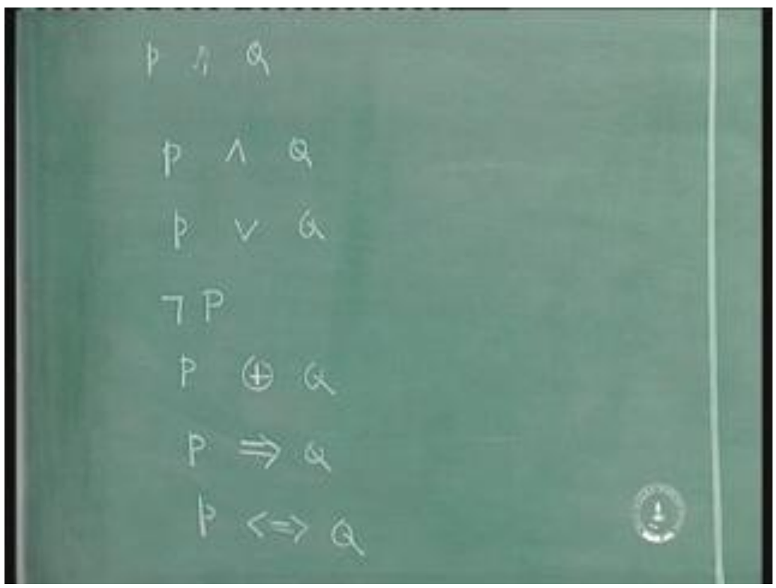

We consider logical operators AND, OR, unary operator NOT, and Exclusive OR operator, implication and equivalence. To recall what we studied earlier P and Q is true only when P is true and Q is true, P or Q is true only when either P is true or Q is true or both of them are true. It is false only when both P and Q are false considering NOT P. NOT P is true only when P is false and NOT P is false when P is true.

We consider the Exclusive OR operator like this: P exclusive Or Q. This compound expression is true if one of P and Q is true and the other is false. It will be false when both of them are true or when both of them are false. We have seen how to express P implies Q in different ways. We can also say if P then Q, Q if P, P only if Q and so on.

This is true if P is false or Q is true. That is in three cases it will be true when P false Q true, P false Q false, P true Q true. It is false only when P is true and Q is false. P is equivalent to Q if P and Q both takes the same value, that is if P is falser Q is false this will take the value 1, if P is true and Q is true also this will take the value 2, but if one of them is true and the other is false this will take the value false or 0.

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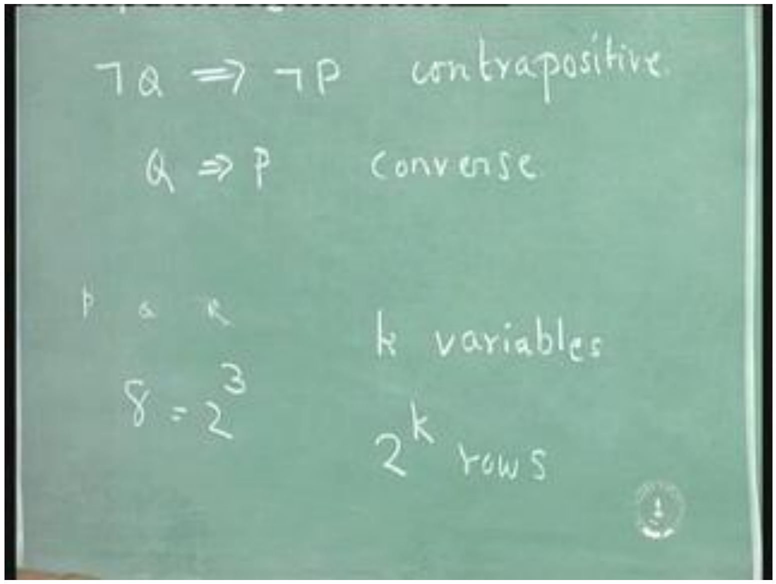

We have also seen what is meant by a contrapositive. For P implies Q the contrapositive is NOT Q implies NOT P, this is called the contrapositive. And Q implies P is called the converse of P implies Q. We also saw how to draw two tables for propositional forms or well formed formulae of propositional logic.

First we will have columns for each one of the variable and there will be a row corresponding to each assignment of values for the variables. Suppose there are three variables each can be true or false. So there are totally 8 equal to 2 power cube possible assignments and there will be 8 rows in the table for any propositional form involving P, Q and R. In general if you have k variables there will be first k columns for each one of the variables and there will be 2 power k rows, each one standing for one assignment of the truth values for the variables. We have also seen what is meant by a tautology.

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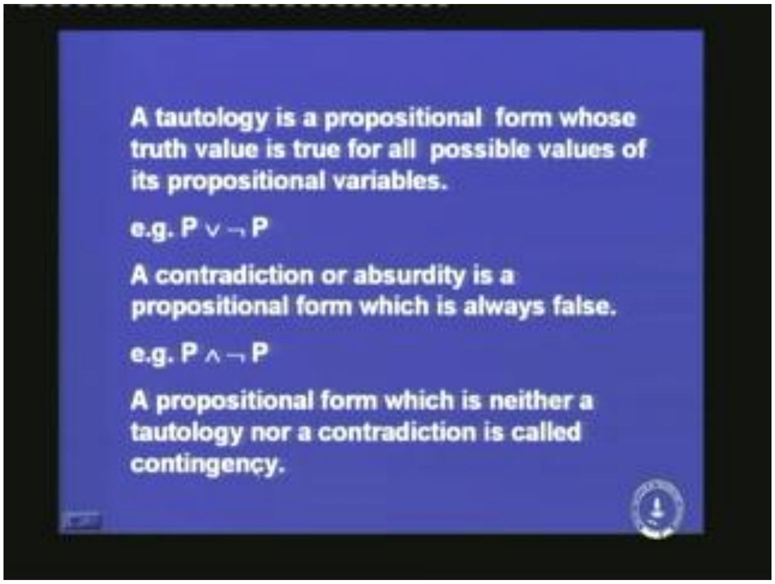

A tautology is a propositional form whose truth value is true for all possible values of its propositional variables example P OR NOT P. A contradiction or absurdity is a propositional form which is always false example P AND NOT P. A propositional form which is neither a tautology nor a contradiction is called contingency.

Now, in some cases it may not be necessary to have all the 2 power k rows for a truth table. For example, if you want to show that a propositional form is a contingency it is enough if you show one row were the resultant expression takes the value 1 and another row were the resultant expression takes the value 0, this shows that for some assignments it will take the value 1 and for some other assignment it will take the value 0 and so it is a contingency. We need not have to write all the 2 power k rows. And similarly in some cases you may have a simplified truth table.

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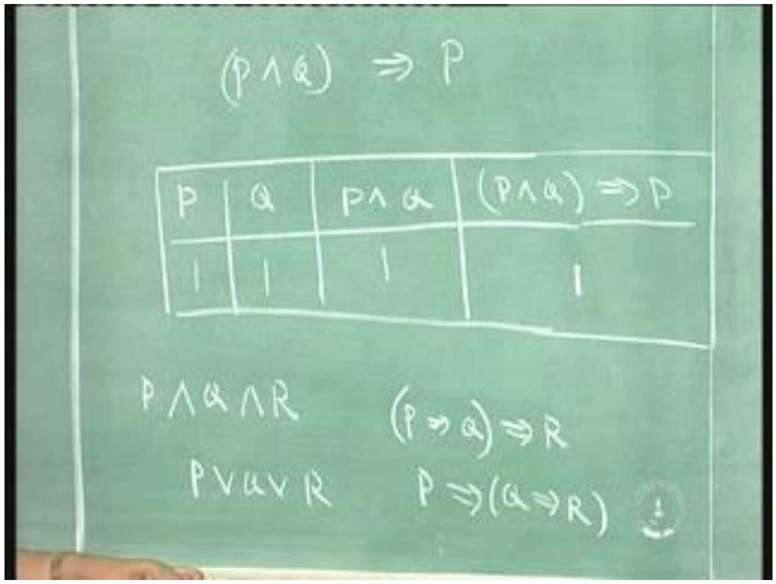

For example I want to show that P AND Q implies P, this is a tautology. So actually I should write 4 rows to show that this is a tautology and in the last column I should show that everything is 1 1 1 1. But it is not necessary to write all the 4 rows because when will this implication be false? This implication will be false when this is true and this is false. So when will this be true? When both P and Q are true. So it is enough if I write only 1 row for this. When P and Q will be true? When P is true and Q is true. So it is enough if we consider this row alone. So in this P and Q is true and obviously P and Q implies P because both the antecedent and the consequence are true, this implication will be true. So we find that it is not always necessary to write all the rows. In some cases it is enough if we write few rows which are necessary to show what we want. We have also seen some logical identities, let us recall what we have seen.

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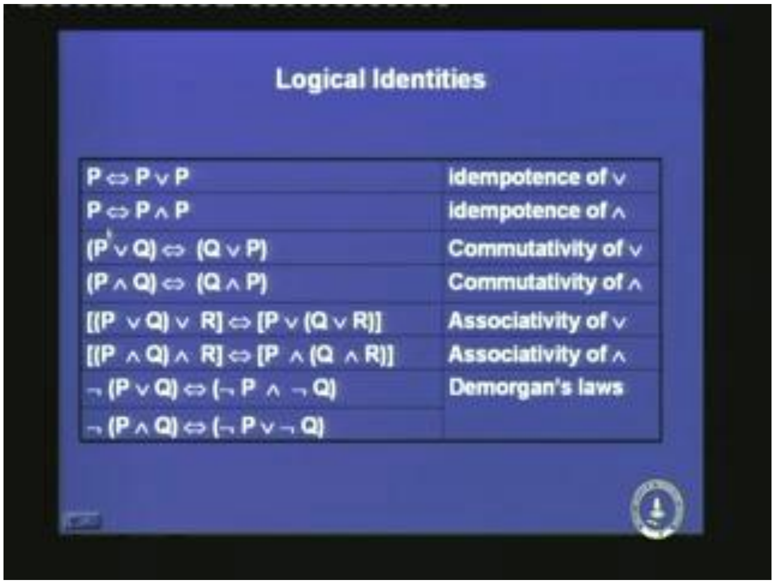

This we have seen in the last lecture. Some of them are very clear. As I told you AND is associative, OR is associative. So you can write something P AND Q AND R without any ambiguity. Similarly you can also write P OR Q OR R without any ambiguity. But whenever you are in doubt you must use parenthesis so that the expression is unambiguous. (Refer Slide Time: 09:49 min) But if you take P implies Q implies R you cannot write like this because this is ambiguous. Do you mean P implies Q implies R or do you mean P implies Q implies R? Implication operator is not associative and we have to use proper parenthesis to represent what we mean.

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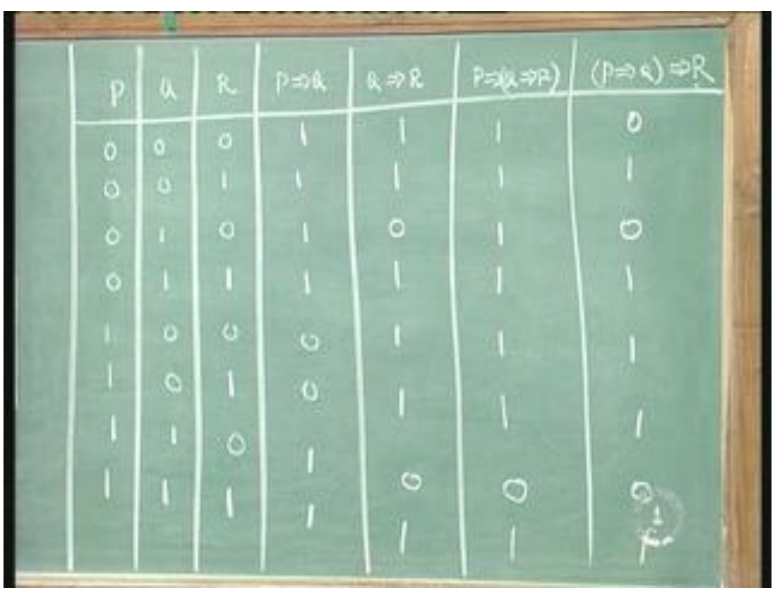

See the tables for these two expressions: P Q R, P implies Q, Q implies R, P implies Q implies R, P implies Q implies R. Let us draw the truth table for this and see what happens. So there will be 8 rows giving different values for P, Q and R. We know that 1 stands for truth and 0 stands for false. When is P implies Q true? In these cases it will be true and in these two cases it will be false. When will Q implies R be true or when will it be false? When Q is true and R is false it will be false in other cases it will be true. So writing down the expression or the truth value in this column you find that when Q is true and R is false this is false but when Q is false or when R is true this will be true.

Now look at this column and this column, let us fill the truth value for these two columns. When can this be true? It will be true when P is false or Q implies R is true, it will be false when P is true and Q implies R is false. So taking that, you see that whenever P is false this will be true and what about Q implies R? Q implies R is false here and P is true. So at this point it will take the value false. Here both the antecedent and the consequence are true in these three rows so this implication will be true.

Now let us go to the last column P implies Q implies R that will be false if this is true and this is false, it will be true if this is false or this is true. So whenever R is true the compound expression will be true. So looking at this, it will be true here, it will be true here and it will be true here. Look at this: P implies Q is true but R is false so this will be false, here P implies Q is true and also R is true so that is 1, here again P implies Q is true but R is false so this will be false and here P implies Q is false and R is also false so this will be true and here P implies Q is true and R is false so this will be false. So you can see that the last two columns are not the same. So it very much depends upon how you interpret this value that is whether your going to put the parenthesis here (Refer Slide Time: 09:49 min) or parenthesis here, the meaning becomes entirely different. So implication is not associative and you have to be careful when you write an expression of the form P implies Q implies R.

You have to put parenthesis in a proper manner. We have seen some logical identities in the last lecture.

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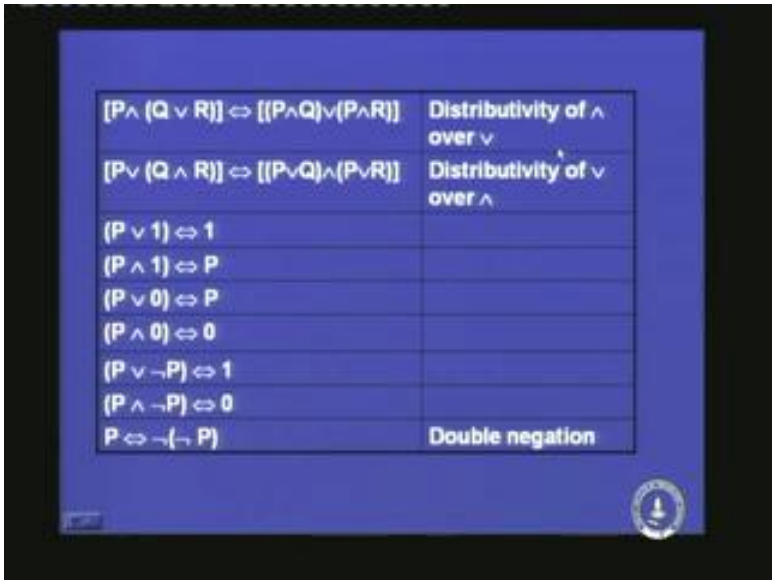

The idempotence of OR and AND are all tautologies, then commutativity of OR and commutativity of AND, associativity of OR and Demorgan's laws, distributive laws, then laws involving one of the operands as true or false, one stands for true and other false, double negatation, NOT of NOT of P is P and implication. Let us see what this means.

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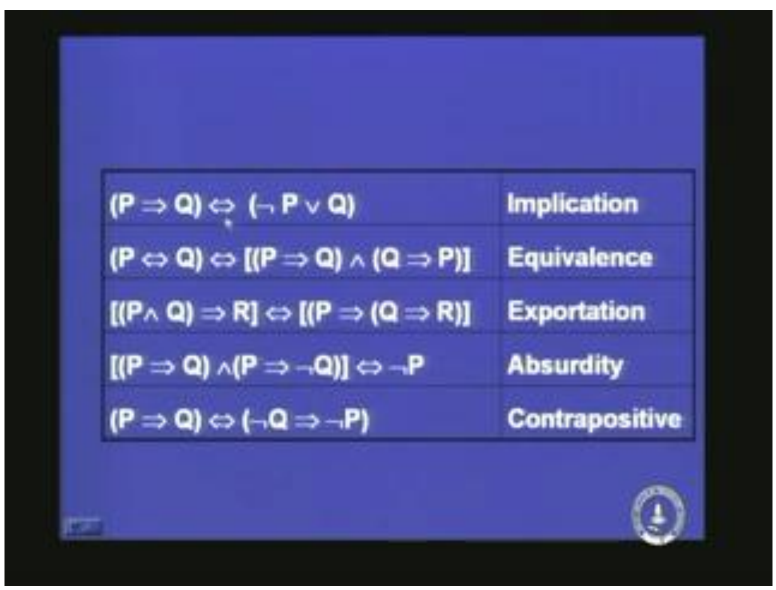

P implies Q is equivalent to NOT P OR Q. Let us construct the truth table for this and see what it is.

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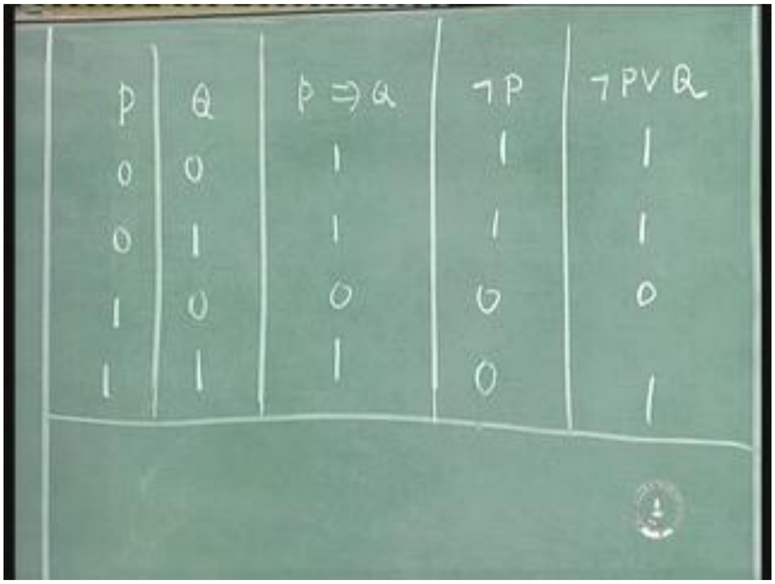

P Q, P implies Q, NOT P, NOT P OR (Q), so giving all four possible values for P and Q let us fill this column. P implies Q is false in this case and in the other three cases it is true we know this. Now what about NOT P? NOT P is true when P is false and it is false when P is true. So the last column is the orring of this two.

When is the orring of this false? When both of them are false and in this case both of them are false so it will be false. In the other three cases at least one of them is true so this will be true. Now look at the third column and the fifth column. You find that they are identical. So P implies Q is equivalent to saying NOT P OR Q. So we can use these equivalences for simplifying propositional forms or if you look at it as Boolean algebra or Boolean expressions and replace one expression by an equivalent 1.

So you want to simplify something and if you have P implies Q, you can replace it by NOT P OR Q. And the next one is, P is equivalent to Q is equivalent to P implies Q AND Q implies P. (Refer Slide Time: 15:45 min) For this also we can draw the truth table and see that they are equivalent. Similarly for every one of these things we can draw the truth table and see that whatever you have on the left side, this is equivalent to this. If you draw the truth table the two columns will be identical. Similarly P AND Q implies R is equivalent to saying P implies Q implies R and this is called Exportation. We can draw the truth table for this also having 8 rows.

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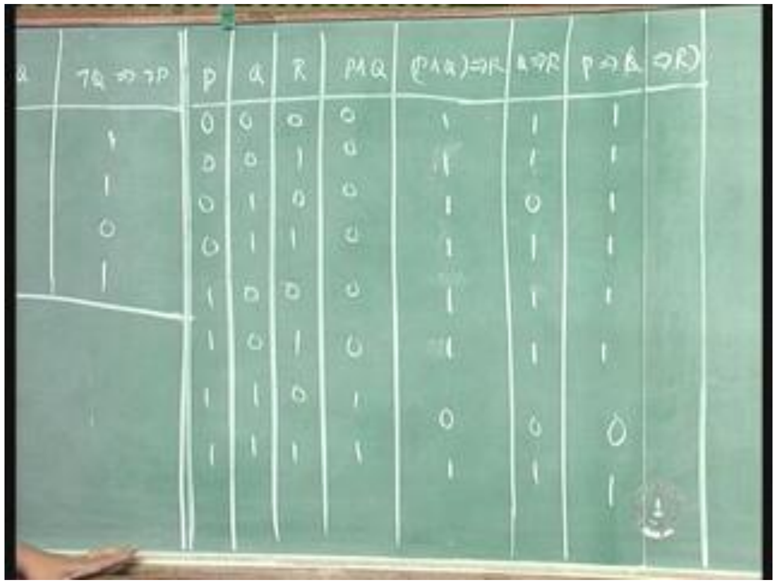

If you draw the truth table some expressions or identities you are considering. The exportation rule says that P AND Q implies R is equivalent to P implies Q implies R. Let us draw the truth table and see. You can see that the truth table for this is like this. There are 3 variables so there will be 8 rows having all the 8 possible values or 8 possible assignments. Then P AND Q is true only when P is true and Q is true so this is the value for P AND Q. And P AND Q implies R will be false only when this is true and R is false in this case. So only in this case it will be false otherwise it will be true.

Let us consider the value for Q implies R, again it will be false when Q is true and R is false that is in these 2 rows alone it will be false rest of them will be true. And taking the last one P implies Q implies R will be false only when P is true and Q implies R is true.

So comparing this and this you will realize that this is false only when this is true and this is false that is in this case. So if you look at this column or this column you see that they are identical and that is what this logical identity says: P AND Q implies R is equivalent to P implies Q implies R. Then we have rule for absurdity that is P AND Q and P implies Q and P implies NOT Q and equivalent to NOT P.

This is the one which we will use for proving something by contradiction and this is called proof by contradiction and this is the law which will use for proving such theorems. This we have to see contrapositive P implies Q is equivalent to NOT Q implies NOT P. If you look at the table giving all the four possible assignments to P and Q you will get four rows.

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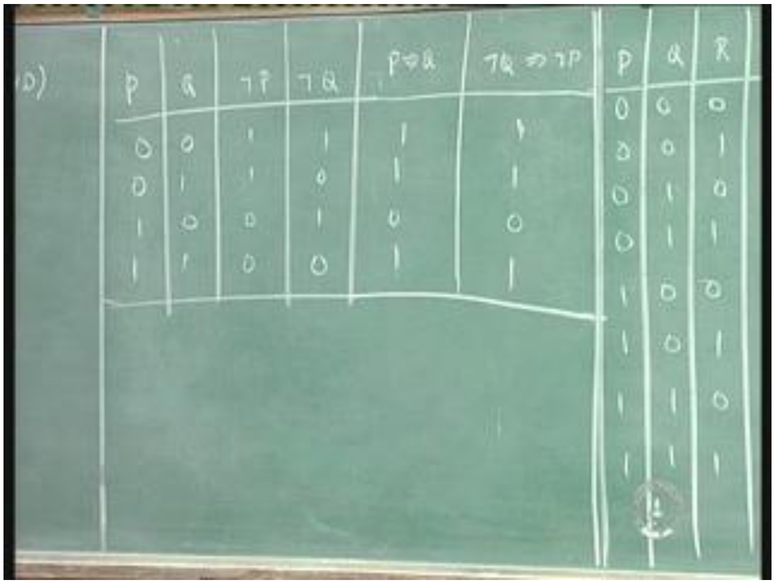

When P is true Q will be false and so on and this is the column for NOT P and this is the column for NOT Q, this is the truth value for P implies Q, this we already know, for NOT Q implies NOT Q that will be false only when Q is true and NOT P is false. That is in this case NOT Q is true NOT P is false only in this case it will be false otherwise it will be true. And so if you look at the last two columns you see that they are identical.

So whenever an expression involving implication P implies is true, it is equivalent to say saying NOT Q implies NOT P. Just for English sentence we consider “if I fall into the lake I get wet” if I am not wet means I have not fallen into the lake. That is the contrapositive of it. So like this we consider these rules and these rules can be made use of to simplify logical expressions or well formed formulae of propositional logic. Let us take an example of an expression and see how to simplify it.

Let us consider this expression, we will make use of these identities which we considered earlier and simplify this expression. This is A implies B OR A implies D implies B OR D, this is the expression. Here A B D are propositional variables.

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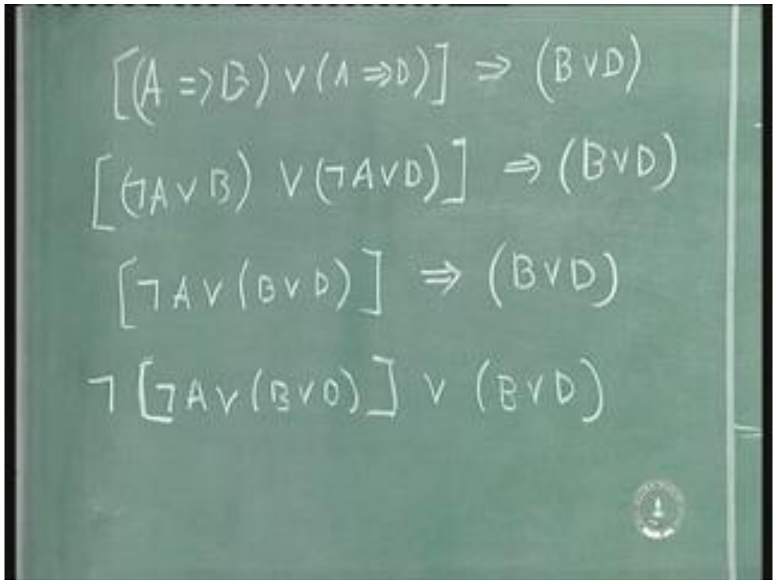

Now let us see how to simplify this. First we shall change the implication into OR. We know that A implies B is equivalent to saying NOT A OR B and similarly A implies D is equivalent to saying NOT A OR D. So you can reduce this expression to this. Now NOT A is common in both these cases, so you can take it out and write it as NOT A OR B OR D. Now there is a implication involved here so again we can make use of the result that P implies Q is equivalent to NOT P OR Q and write this as NOT of NOT A OR B OR D OR B OR D.

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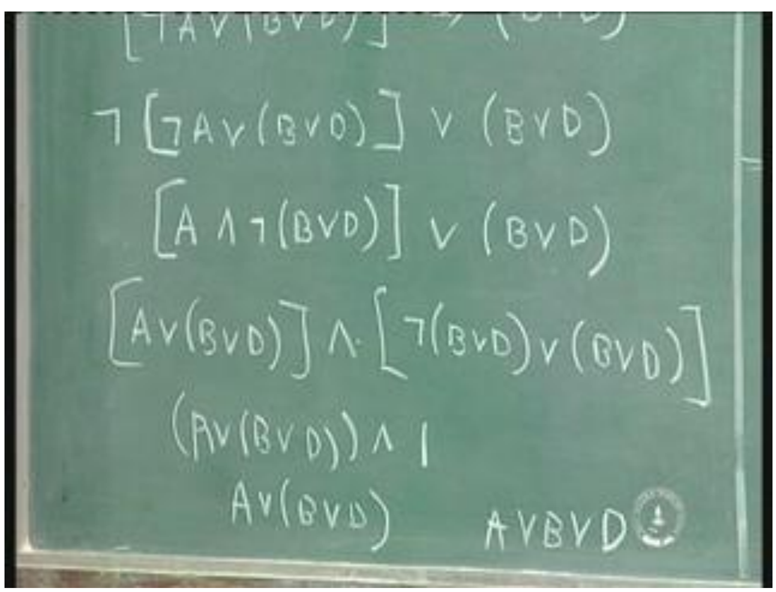

Now when you have a NOT out and you want to bring it inside we have to use what are called Demorgan's laws which we have already seen. These are called Demorgan's laws. So this will become A AND NOT B OR D, this OR becomes AND when you bring the NOT inside and NOT of NOT of A will become A. Now using the distributive law this will become A OR B OR D AND NOT B OR D OR B OR D.

Now we know that if you have NOT P or P that reduces to 1 and it is always true and it is a tautology. So this will become A OR B OR D AND 1 that is true and that is nothing but A OR B OR D. Now because of the associative property of OR you can write it as A OR B OR D without ambiguity. We can remove the parenthesis here which is not necessary because of the associativity of OR. Like that by using these identities we can simplify logical expressions.

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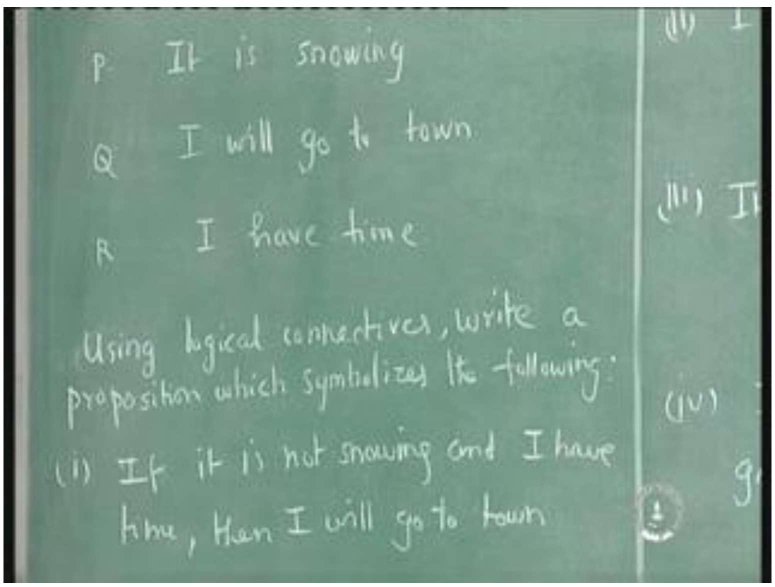

Let us consider some sentences in English and try to convert them into logical notations and also try to write down English sentences from the logical notations. Let us take an example: Now let us consider this P stands for the sentence it is snowing and Q stands for I will go to town, R stands for I have time. Now using logical connectives write a proposition which symbolizes the following.

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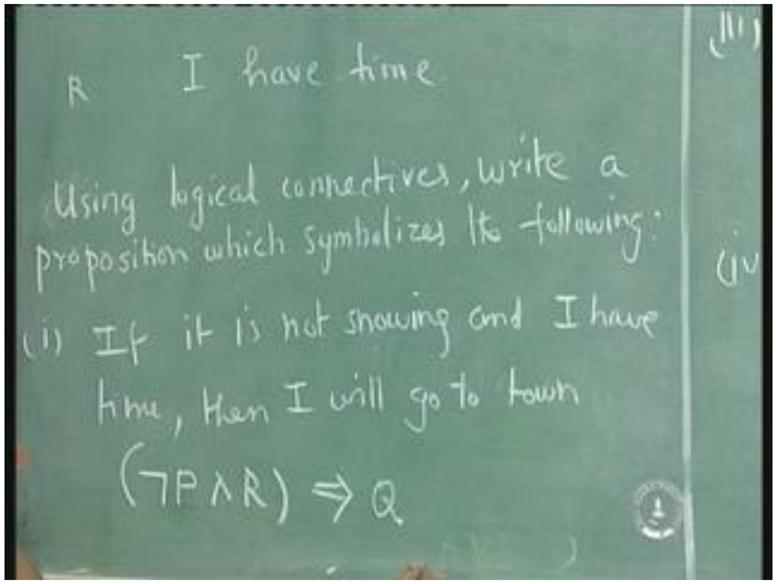

First we have, if it is not snowing and I have time then I will go to town. How can you write this in logical notation "if it is not snowing?" For if it is snowing, the logical notation is P, so it is not snowing means NOT P. And the logical notation for I have time is AND R. If this is so, then the logical notation for I will go to town is implies Q. So this sentence can be written logically in this form as NOT P AND R implies Q. The next one is I will go to town only if I have time. So for this P implies Q can also be read as P only if Q. So the logical notation for I will go to town only if I have time will be Q implies R.

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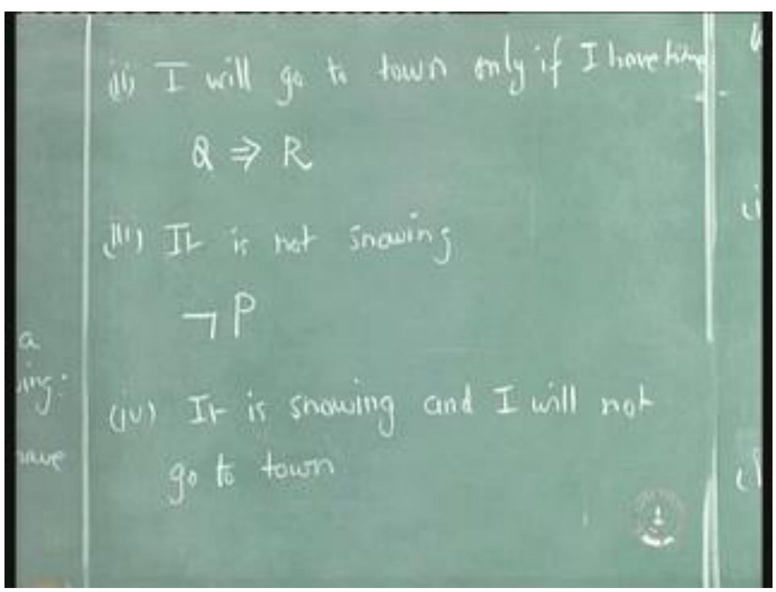

The third sentence is a very simple sentence: it is not snowing, here P stands for it is snowing, so NOT P will stand for it is not snowing. Next you have, it is snowing and I will not go to town. What is the logical expression for this? For snowing it is P and for I will not go to town is NOT Q.

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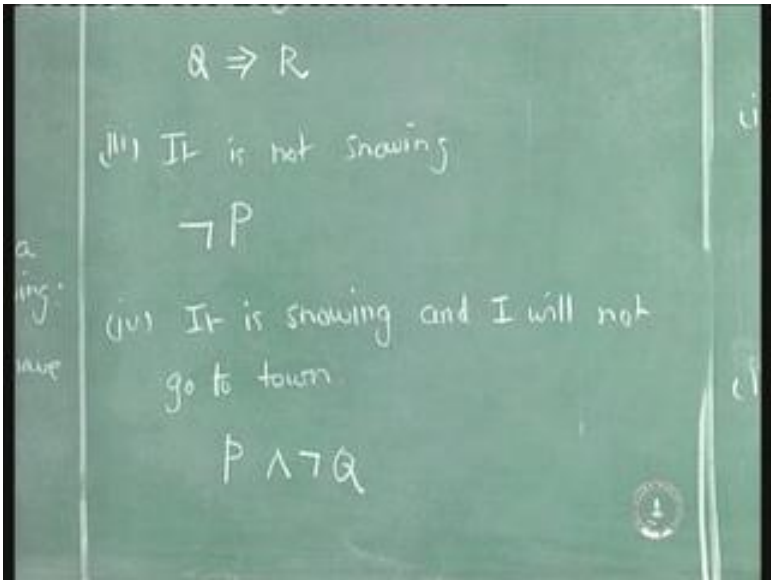

Q is I will go to town, so NOT Q is I will not go to town. Like that these English sentences can be written in logical notation. And if you have a proposition how will you write it in English? How will you interpret properly and write it in English? Now write a sentence in English corresponding to each of the following propositions: Q is equivalent to R AND NOT P.

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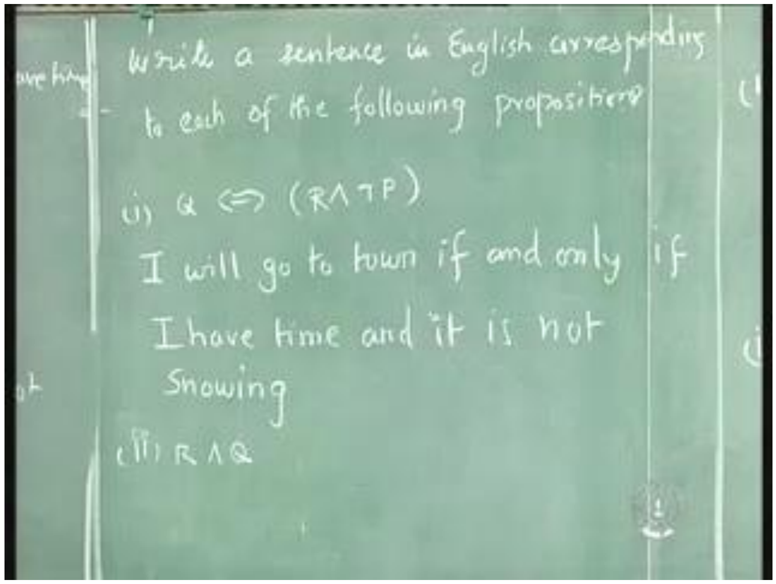

How can we write this in English? This can be read as if and only if. And what does Q stands for? Q stands for I will go to town. So this can be written in the form I will go to town if and only if. What is the condition other side R AND NOT P? What is R? R is I have time and P is it is snowing. So if I have time and it is not snowing, so this proposition Q is equivalent to R AND NOT P. If you want to write in English it takes this form: I will go to town if and only if I have time and it is not snowing.

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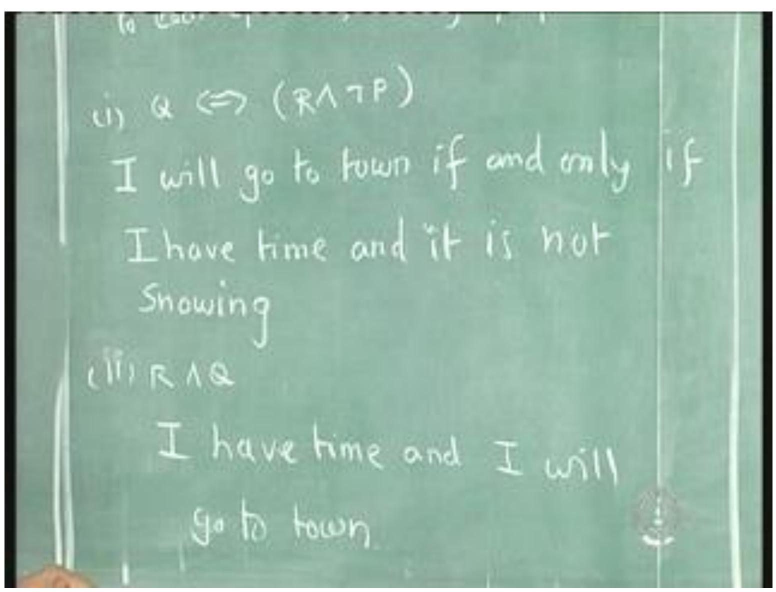

Now how will you transcribe this proposition into English? How will you write a sentence which is equivalent to this R AND Q? R stands for I have time and Q stands for I will go to town so this can be written in the form I have time and I will go to town.

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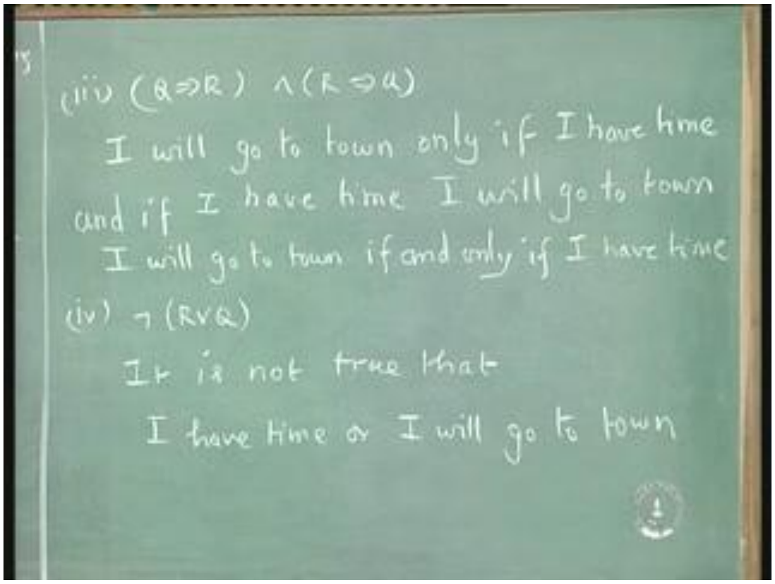

Then next one is Q implies R and R implies Q. How will you transcribe this in English? Q stands for I will go to town and R stands for I have time, so Q implies R can be written in the form I will go to town only if I have time then R implies Q must be written as, if I have time I will go to town or equivalently Q implies R AND R implies Q is equivalent saying Q is equivalent to R. So this can also be written in the form I will go to town if and only if I have time. The last is NOT R OR Q, again R stands for I have time and Q stands for I will go to town, can be written in the form it is not true that I have time or I will go to town. So like this you can transform English sentences into logical notations and if you have logical notations you can write down the corresponding English sentences.

You have to be careful when you use inclusive OR or exclusive OR but usually either or would refer to exclusive OR and other wise it will be inclusive OR but you have to be careful about this. Let us consider some more logical identities; they are all tautologies involving implications. And later on we shall see that they are also called rules of inference.

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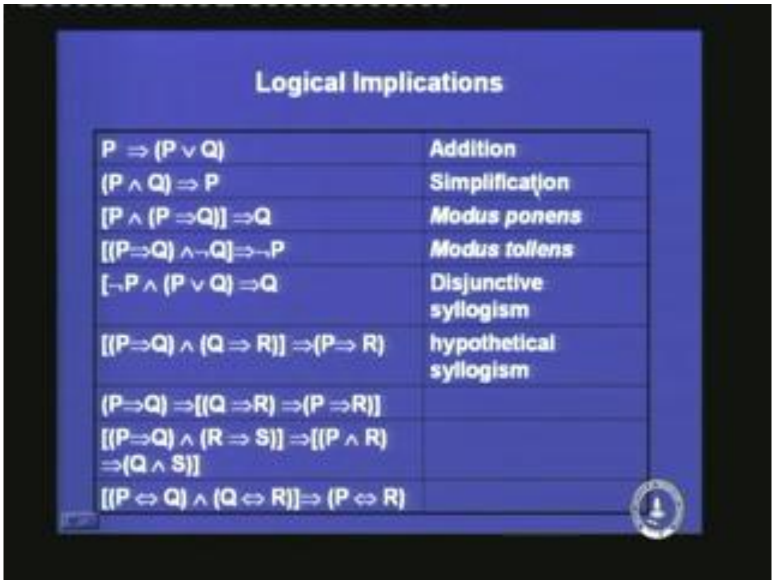

So these are the logical implications which are tautologies. In addition P implies P OR Q, see when you have P by adding something the value is not altered so from P you can conclude P OR Q. If you look at it as a pool of inference from P you will be able to conclude P OR Q.

And simplification is the second rule P AND Q implies P if you have P and Q then from that you can conclude P because P and Q will be true only when P is true and Q is true and so you can conclude P from that if you look at it as a rule of inference. Then you have this P AND P implies Q implies Q this is called Modus ponens.

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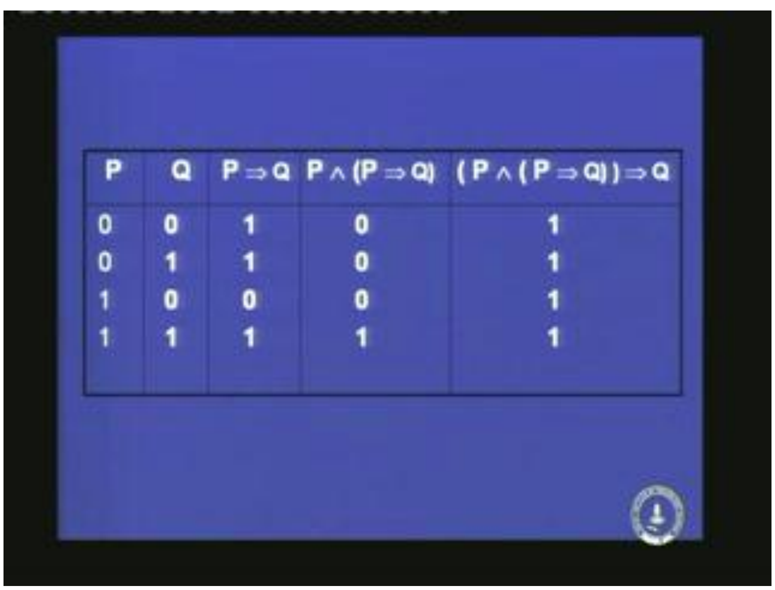

Let us draw the truth table and see how it looks. Considering the truth table for Modus ponens we have four possible assignments for P and Q so you have four rows and we have written down the four possible values. For P implies Q this is the truth value and we also know this. Now P AND Q, P AND P implies Q will be true only when P is true and P implies Q is also true. So you find that the truth value for this is, in these three cases it takes the value 0 and in this case it is 1.

The last column stands for the entire statement P implies P implies Q, P AND P implies Q implies Q. This will be true if the antecedent is false or the consequence is true. So in these three rows the antecedent is false so the compound statement will be true. In the last case, when the antecedent is true the consequence is also true so this compound statement is again true.

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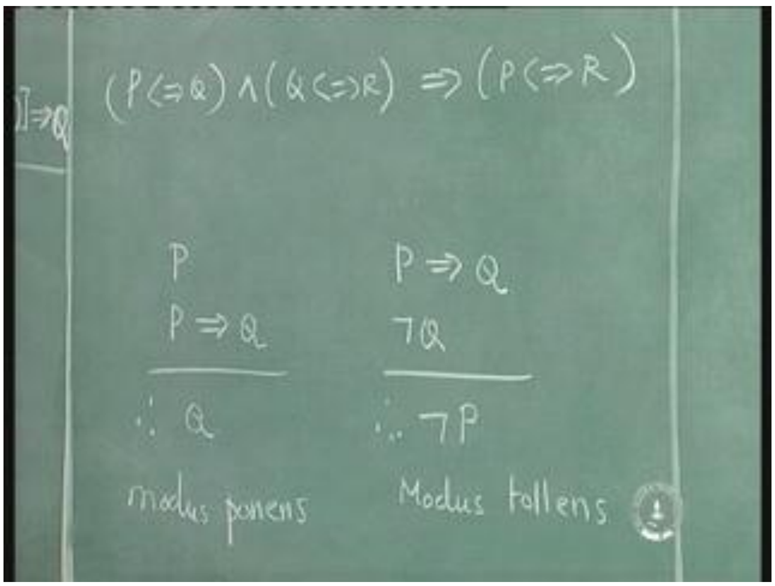

So if you look at the last column you always have 1, it is a tautology. This is called Modus ponens and when you write it as a rule of inference you write like this, P implies Q and from this you conclude Q.

This is what is meant by Modus ponens. Similarly the next implication is Modus tollens, P AND Q AND NOT Q implies NOT P. (Refer Slide Time: 37:41 min) We can draw the truth table for this also in a similar manner but when we write it as a rule of inference it means if you have P and Q that is P implies Q and NOT Q, from this you can conclude NOT P, this rule therefore NOT P is called Modus tollens whereas this is called Modus ponnens. And we have some more rules NOT P AND P OR Q implies Q, this is called Disjunctive syllogism. And P implies Q AND Q implies R implies P implies R rule is called hypothetical syllogism. We will come across this again when we study rules of inference.

This rule states that if you have P implies Q implies Q implies R implies P implies R is anding of two things and this is equivalence. P implies Q and R implies S implies P AND R implies Q AND S and this should be equivalence. P is equivalent to Q AND Q is equivalent to R would imply P is equivalent to R, that last rule is if P is equivalent to Q AND Q is equivalent to R this would imply P is equivalent to R. We will make use of this in logical inference when you come to that. Now before that I will leave you with a problem and may be I shall give you the solution in the next lecture. But let us see what the problem is.

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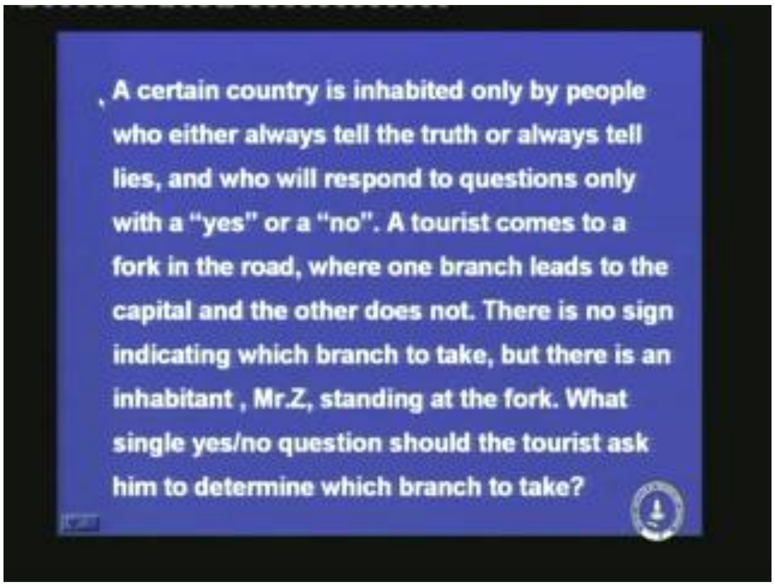

A certain country is inhabited only by people who either always tell the truth or always tell lies, that is either the person will be a truth teller or a liar and who will respond to questions only with a yes or a no answer.

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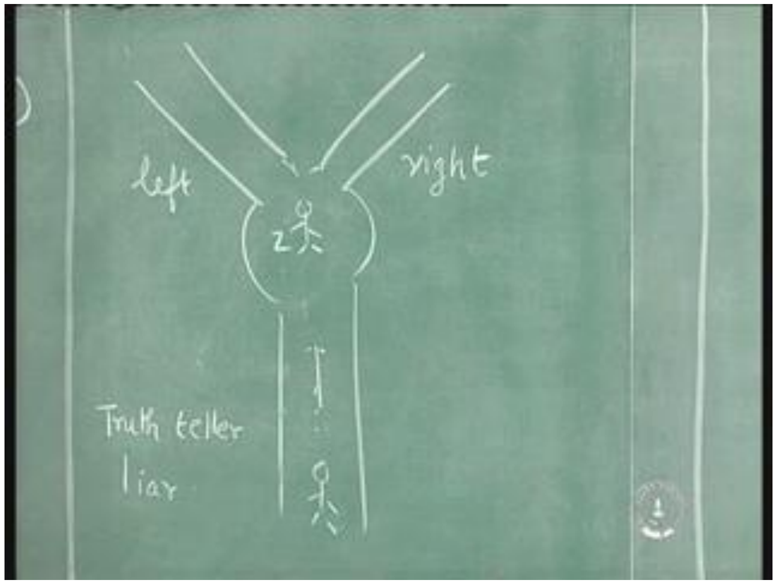

A tourist comes to a fork in the road so this is the situation. You have a fork, the tourist is approaching this place. A tourist comes to a fork in the road where one branch leads to the capital and the other does not. There is no sign indicating which branch to take but there is an inhabitant Mr.Z standing at the fork. What single yes or no question should the tourist ask him to determine which branch to take. So this tourist is approaching this fork, he sees a person sitting here, this person may be a truth teller or he may be a liar.

A truth teller always tells the truth, a liar always lies. And either the left road leads to the capital or right road leads to the capital. This tourist wants to find out; there is no board here so he wants to ask this question. But this person Z the inhabitant there may be a truth teller or may be a liar. This person does not know, the tourist does not know whether he is a truth teller or a liar and he will respond only with an yes or no answer, he will not say anything more.

In that situation this tourist should ask only one question, a single question for which the answer will be a yes or no and by listening to that answer like this. If it is yes he will take the left road which is the correct road, if it is no he will take the right road which is the correct road. So what is the single yes or no question he should ask? So you must look at the four possibilities.

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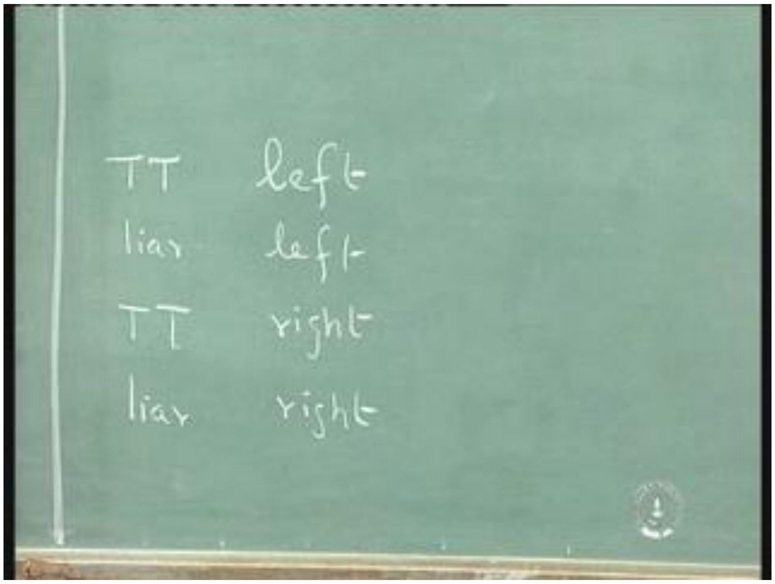

The person may be truth teller and the left road may lead to capital, the person may be a liar and the left road may lead to capital, the person may be a truth teller and the right road may lead to capital, the person may be a liar and right road may lead to capital. So there are four possibilities and the single yes or no question should take care of all these things. Let us consider some more similar problems.

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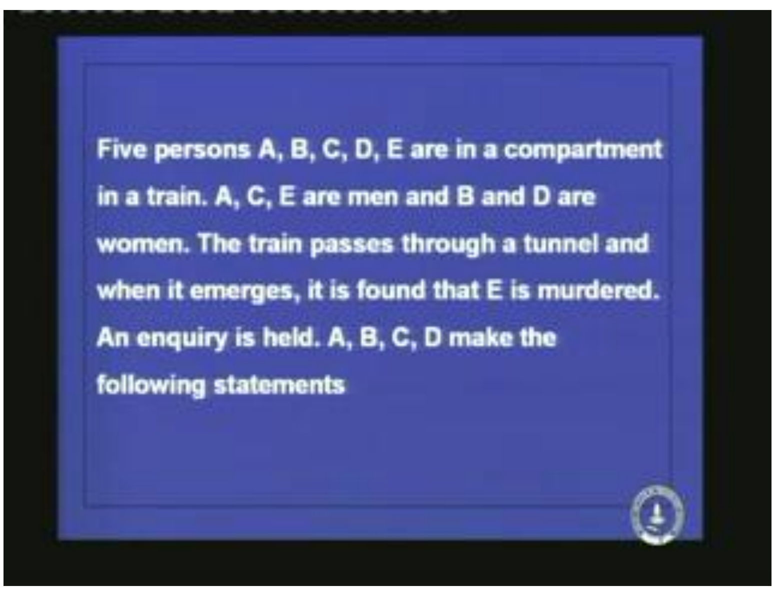

Five persons A, B, C, D, E are in a compartment in a train. A, C, E are men and B, D are women. The train passes through a tunnel and when it emerges it is found that E is murdered and an enquiry is held.

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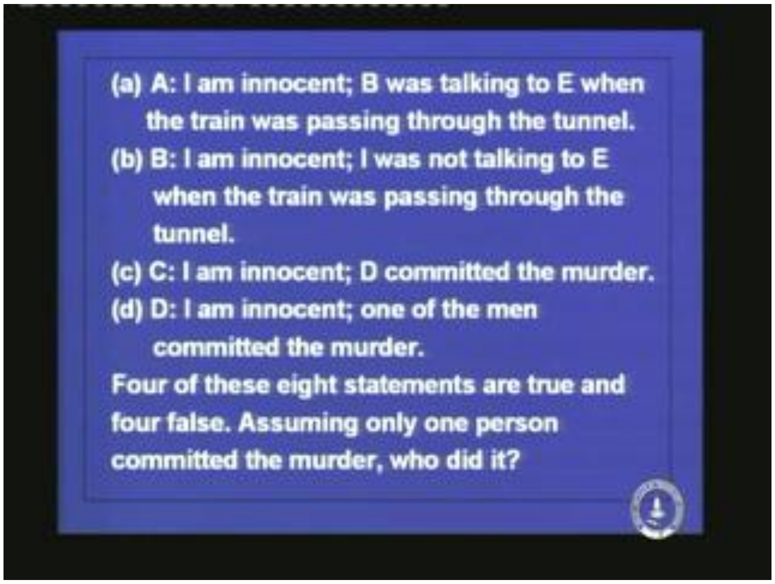

And A, B, C, D makes the following statements: A says I am innocent B was talking to E when the train was passing through the tunnel and B says I am innocent I was not talking to E when the train was passing through the tunnel, C says I am innocent D committed the murder, D says I am innocent and one of the men committed the murder.

You must remember that A and C are men and B and D are women and each one is making two statements. Four of these eight statements are true and four are false. You have to assume that four of these eight statements are true and four of them are false. Assuming only one person committed the murder. From these statements you must find out who committed the murder. And each one is making two statements and therefore there are eight statements out of which four are true and four are false.

Now from these arguments or from these statements how will you find out who has committed the murder? Look at the first four statements each one makes: A says I am innocent and B says I am innocent and C says I am innocent and D also says that I am innocent. Now only one person committed the murder, so out of A, B, C, D three of them must be saying the truth and one is lying.

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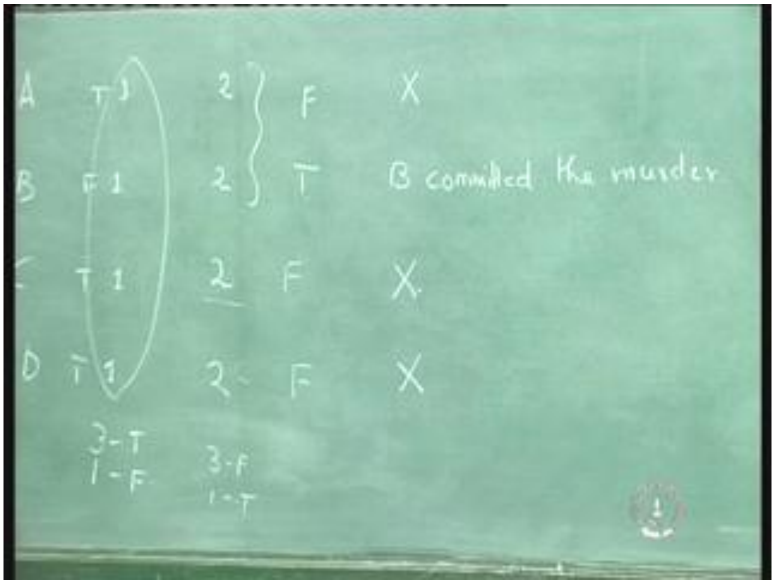

So there are eight statements: A is making statement one and two, B is making statement one and two and C is making statement one and two and D is making statement one and two. Out of these all four statements are I am innocent out of which three of them are true one is false. So out of the four, three are true and one is false. So, in the remaining four, three of them must be false and one must be true because it is given that four of them are true and four of them are false. And look at the second statement of A and B what is that? B was talking to E when the train was passing through the tunnel and B says I was not talking to E when the train was passing through the tunnel. One is the negation of the other, if what A says is true then what B says is not true and if what B says is true then what A says is not true. So out of these two, one is false and the other is true.

Either of them may be false the other may be true. So it amounts to saying that this statement is false and this also is false. So the second statement of C and D are false. What is the second statement of C? D committed the murder and that is false. So D did not commit the murder. What does C say, the second statement of D is one of the men committed the murder, so that is also false that means A and C did not commit the murder. So who committed the murder? B committed the murder.

In that case the first statement of A is true, the first statement C is true, this is true, this is false, this is true, this is false, this is true, this is true and these two are false and one of them is true and other is false. So four statements are true and four statements are false. So this satisfies the condition and so we should come to the conclusion that B has committed the murder. We can look at similar problems like this. As an extension of the example which I mentioned just a few minutes back, let us consider one more problem, this is much more difficult than that problem.

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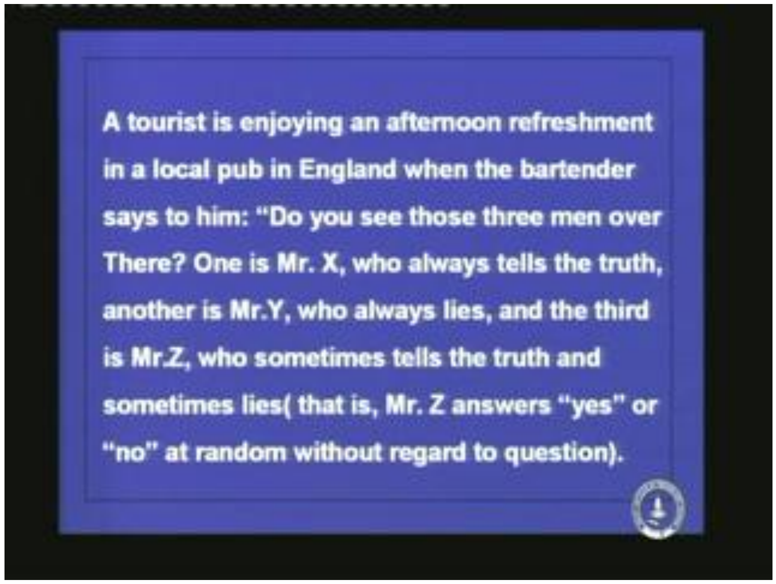

A tourist is enjoying afternoon refreshment in a local pub in England when the bartender says to him: Do you see those three men over there? One is Mr. X who always tells the truth, another is Mr.Y who always lies and the third is Mr.Z who sometimes tells the truth and sometimes lies, that is Mr.Z answers yes or no at random without regard to the question. You may ask them three yes or no questions always indicating which man should answer.

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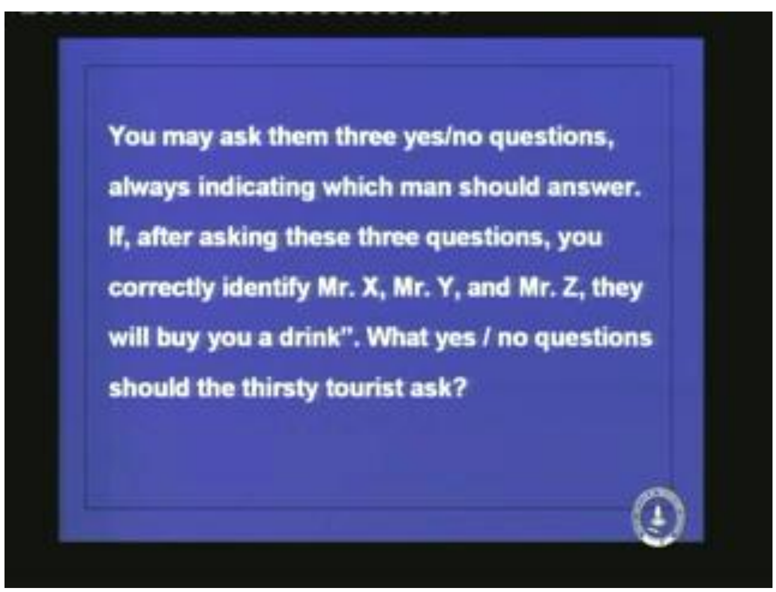

If after asking these three questions, you correctly identify who are Mr.X, Mr. Y and Mr. Z, they will buy you a drink. What yes or no questions should the thirsty tourist ask? The problem is like this: A person is sitting here, the tourist is sitting here, the waiter comes and he points out to three people standing out there: 1, 2, 3 they are standing in a row and the waiter tells the tourist, look at those three people: one is Mr. X who is a truth teller he always tells the truth another is Mr. Y who is a liar who always says yes who always lies, third person is Z sometimes he lies and sometimes he tells the truth.

Now this tourist can ask these three people three questions. Each time he can ask the first question and point out who should answer the question and then depending upon the answer he can ask the second question. So again he can ask the second question and point out whom should answer the question and similarly again he can ask the third question. After getting the answer for all the three questions the answer should be only in the form of yes or no.

So if it is correct X will say the correct answer, for Y if it is yes he will say no, if it is no he will say yes, for Z without looking into that he will randomly say yes or randomly say no. So what are the three questions he should ask so that at the end he is able to find out who is Mr. X , Mr. Y and who is Mr. Z. Actually this is not a very easy problem, this is slightly difficult. But the first question is important. The first question should be asked in such a way that you eliminate Z.

Actually it is easier to deal with a person who always tells the truth or who always lies. It is difficult to deal with the person like Z who sometimes lies and sometimes tells the truth. So here we have to find out who is Z and eliminate him. And after eliminating Z the second and the third question can be conveniently asked. Think about the answer for this problem and shall lead you to another similar problem.

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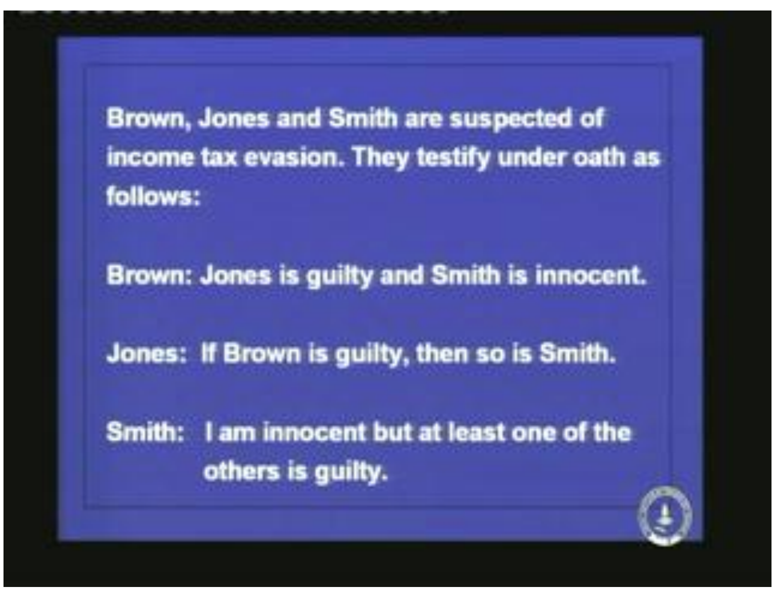

Brown Jones and Smith are suspected of income tax evasion. They testify under oath as follows: Brown says Jones is guilty and Smith is innocent, Jones says if Brown is guilty then so is Smith, Smith says I am innocent but at least one of the others is guilty. Assuming everybody told the truth who is or who are all innocent and who are all guilty?

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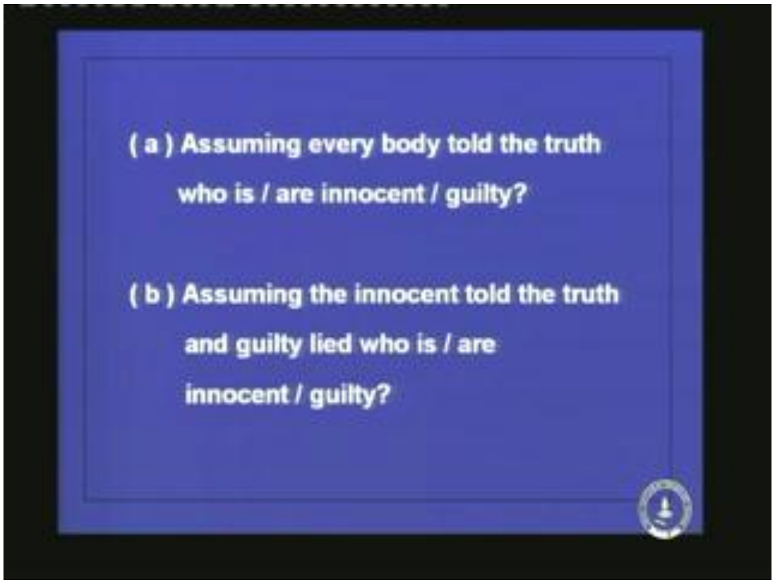

Assuming the innocent told the truth and the guilty lied who is and who is innocent? Look at the statements they make, we have to transfer all of them into logical notations. But the first portion is easy, assuming everybody tells the truth from the first statement you can infer that Jones is guilty and Smith is innocent and if Brown is guilty then so is Smith.

So the contrapositive of that will be if Smith is not guilty Brown is not guilty and so Brown is innocent. So the answer to the first portion is Brown is innocent, Jones is guilty and Smith is innocent. The second portion is slightly more involved, you can try the second portion also. So in the next lecture we shall consider predicate calculus and use of quantifiers such as existential quantifiers and inversal quantifiers and further concepts in logic.