Discrete Mathematical Structures
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Lecture - 18
Special Properties of Relations

Today we shall consider some special properties of list. Let us take binary relation. Let R is a binary relation on A. That is A cross A the domain and the co domain are A. then R is reflexive, if R is reflexive, if x is related to x for all x belonging to A. R is irreflexive if x is not related to x for every x in A. R is symmetric if x related to y implies y related to x for every x y belonging to A. R is antisymmetric if x related to y and y related to x together imply x equal to y for every x belonging to A. R is transitive if y related x, z together imply x related to z for every x, y, z belonging to A. R is asymmetric if x related to y implies y0 related to x.

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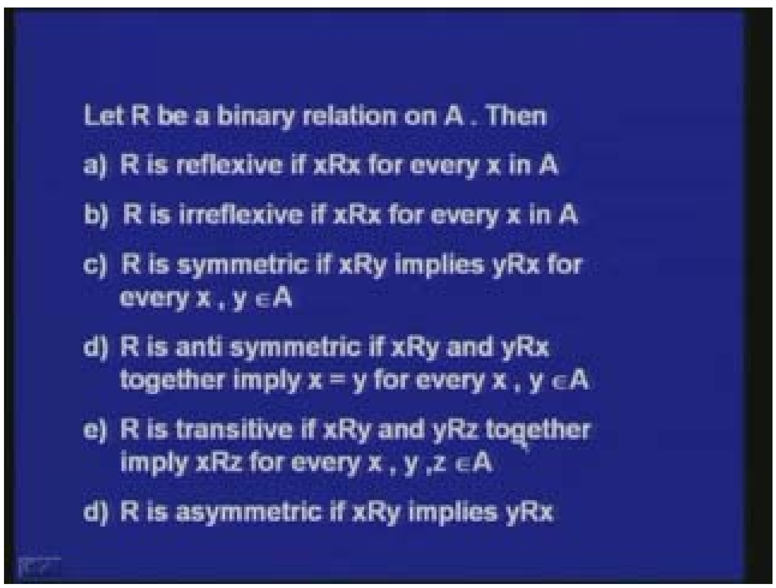

So we study six properties. What are the six properties I will again repeat them. Reflexive means for all x x will be related to x. Irreflexive means for no x x will be related to x. And symmetric means x is related to y means y will be related to x. Antisymmetric means if x is related to y and y related to x if both these hold that means x is equal to y. Transitive would be x is related to y and y related to z means x will be related to z. Asymmetric would mean if x is related to y implies y0 related to x.

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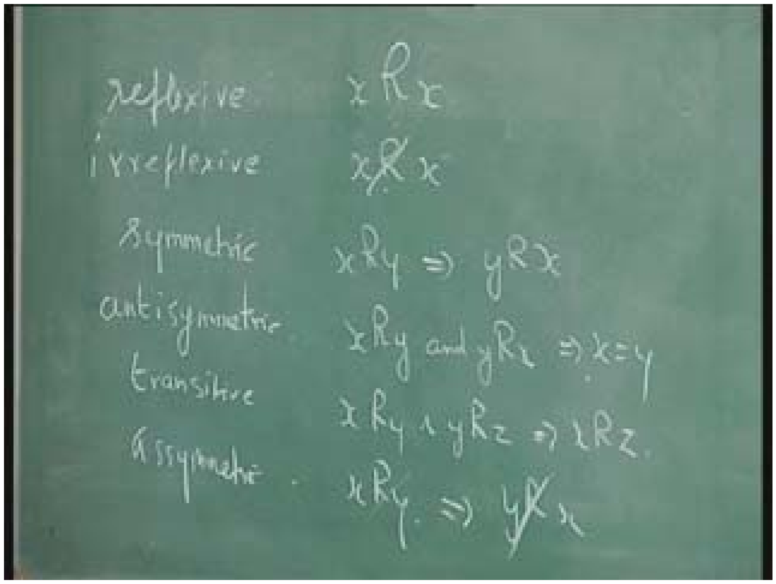

In the case of graphs reflexive means when you represent a binary relation as a directive graph reflexive means there will be a self loop on every node. Irreflexive means there will not be a self loop on any node, no node should contain a self loop. Symmetric would mean between any two nodes if there is no arc that is okay but if there is an arc like this in one direction there should be an arc in the other direction. No arcs between them, is also okay. Antisymmetric is, if there are two nodes there are no arcs between them it is okay but if there is an arc like this you should not have an arc like this.

Asymmetric means if you have an arc like this you should not have an arc like this. Not only that antisymmetric you can have self loops but asymmetric you should not have self loops. Transitive would mean if there is an arc between two nodes x and y and there is an arc between y and z that is x is related to y and y is related to z then there will be an arc between x and z, x should be related to z.

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Now let us take some examples and see whether they are symmetric, reflexive, etc. For example take the set of natural numbers and the equality relation on them. Then x is related to x for every x in n. take the set of natural numbers or non negative numbers and the equality relation if you take any x will be related to x that is a reflexive relation. It is also a symmetric relation because you do not have any arcs between anything, x is related y means x and y should be the same you cannot have for different x, y an arc belonging to them, it will not be irreflexive. And less than relation if you take that is not a symmetric relation A less than B means B will not be less than A so that is not a example of a symmetric relation. But A less than B means B will not be less than A so it is an example of a antisymmetric relation. And if A less than or equal to B it is also an example of antisymmetric relation. And less than is also a transitive relation because A less than B and B less than C means A will be less than C. So that is an example of a transitive relation.

Let us consider some relations and see what properties they have, take the equality relation. Now I will not take many elements just three elements I will take. The equality relation if you have three elements say a, b, c the equality relation on a b c will be represented like this.

What are the properties it satisfies? Let me write down all the properties:
Its properties are reflexive, irreflexive, symmetric, antisymmetric asymmetric, transitive of which how many are satisfied? In a relation there are only three elements here a, b, c what are the properties this relation has, because there is a loop at every node it is reflexive and it is not irreflexive. We will slightly deviate, suppose I have something like this a, b, c and I have self loop here but not self loop here what about this relation is it reflexive? This is not reflexive because you are not having a self loop at every node, it is not irreflexive because irreflexive means you should not have self loop at any node but here at a, you are having a self loop so this is not irreflexive also. So a relation can be

neither reflexive nor irreflexive, it can be neither of them, so that is possible. It is a symmetric relation? If a is related to b then b should be related to a, that is symmetric definition. But if there are no arcs between any two nodes that is okay still it will be a symmetric relation. If a is related to b then b should related to a. if a is not related to b for a different case then that is okay. So this is symmetric.

Now is it antisymmetric? If a is related to b and b is related to a then a equals b, that is what you are having since if a is related a and so on. That is you should not have something like this in the antisymmetric relation, if there are no arcs that is okay. So it antisymmetric also and is it asymmetric? Asymmetric, if a is related to b b should not be related to a, that is, you should not have self loop anywhere. So it is not asymmetric. Is it transitive? What is the condition for transitivity? If a is related to b and b is related to c then a should be related to c. Here you have only self loops no other nodes are related so it is transitive also. So equality relation is reflexive, symmetric and transitive, that you must know. This is just for three elements we have drawn, for simplicity sake you can define for non negative integers and so on.

Consider some more examples; take the universal relation on three elements. Again for convenience sake just taking three elements give examples where we have, this is the universal relation on three elements. What can you say about this relation, is it reflexive? You are having self loop at every node so it is reflexive. Is it irreflexive? No because irreflexive means you should not have self loop at any node. Is it symmetric? Whenever there is an arc from one node to another there should be a reverse arc so it is symmetric. Is it antisymmetric? Antisymmetric means you should not have something like this between two different nodes. If you have x related y and y related x then x should be equal to 1, so it is not antisymmetric. Is it asymmetric? No, because asymmetric you should not have like this nor you can self loops so it is not asymmetric. Is it transitive? Whenever you have a related to b and b related to c you are having a related to c and so on, it is transitive. This is about the universal relation on three elements.

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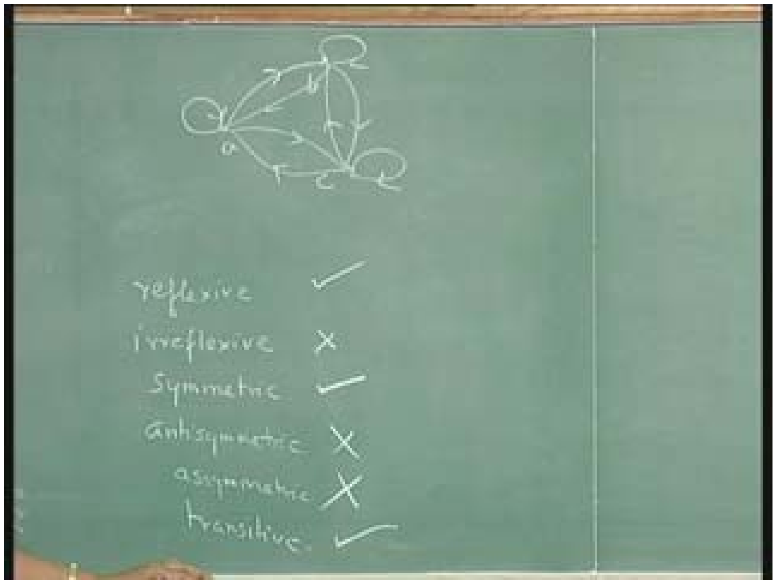

Let us take some more examples; look at this a, b, c you have this relation. Is it a reflexive relation? A reflexive relation should have self loop at every node, here you are having only at this node you are not having self loops here so it is not reflexive. Is it irreflexive? Irreflexive means you should not have self loop at any node, here you having self loop at this node so it not irreflexive. So totally a relation can be neither reflexive nor irreflexive, this is an example of that. Is it symmetric? Either there are no arcs between two nodes or if there is an arc directed in one way there should be arc directed in the reverse direction, so it is symmetric. Is it antisymmetric? No because if b is related to c and c is related to b that should imply b is equal to c, here b and c are different so it not antisymmetric, it is not asymmetric because a is related to b, b should not related to a so it is not asymmetric. Is it transitive? It is not transitive also because b is related to c, c is related to b that means combining these two you should have a self loop at this node if it is transitive, but you are not having self loop at that node so it is not transitive also. So this is an example which is symmetric but none of the other properties hold.

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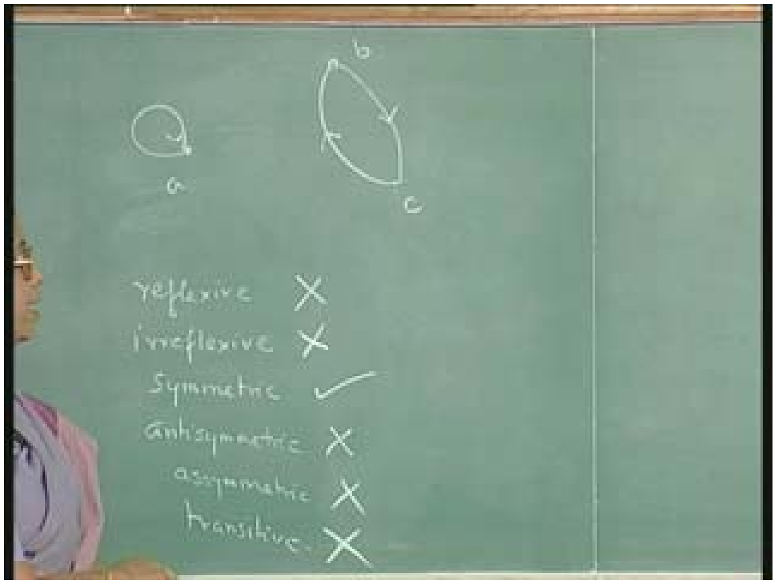

Look at the empty relation; no arcs, no relation, no ordered pairs, is it reflexive? It is not because reflexive means you should have self arc so it is not reflexive. Is it irreflexive? Yes it is irreflexive because you are not having self loop at any node. Is it symmetric? Symmetric means either you should have both the arcs between two nodes or no arcs. So if x is related to y then y should be related to x but x is not even related to y that is okay it is symmetric.

Similarly, is it antisymmetric? If x is related to y and y is related to x it should imply x is equal to y but no arcs between two elements is also okay and it is antisymmetric. Similarly, if x is related to y y should not be related to x. But there is no such thing also this is asymmetric. And it is transitive also because if x is related to y and y related to z x should be related to x no arcs between any of them so it will be transitive, all these conditions are okay to a empty relation.

Take some more examples, one more example we shall take. Consider this a, b, c consider this. What can you say about this, it is a reflexive relation? No self loops so it is not reflexive. Is it irreflexive? There is no self loop at any node so it is irreflexive. Is it symmetric? In symmetric means a related to b would imply b related to a that is not satisfied this is not symmetric. Is it antisymmetric? If a is related to b and b related to a together imply a is equal to b. it is antisymmetric because you are having one arc the other way around it is not there so it is antisymmetric. Is it asymmetric? Yes a is related to b but b is not related a so it is asymmetric also. Is it transitive? If a is related to b and b is related to c you should have an arc from a to c that is not there so this is not transitive. So the properties of this relation is like this.

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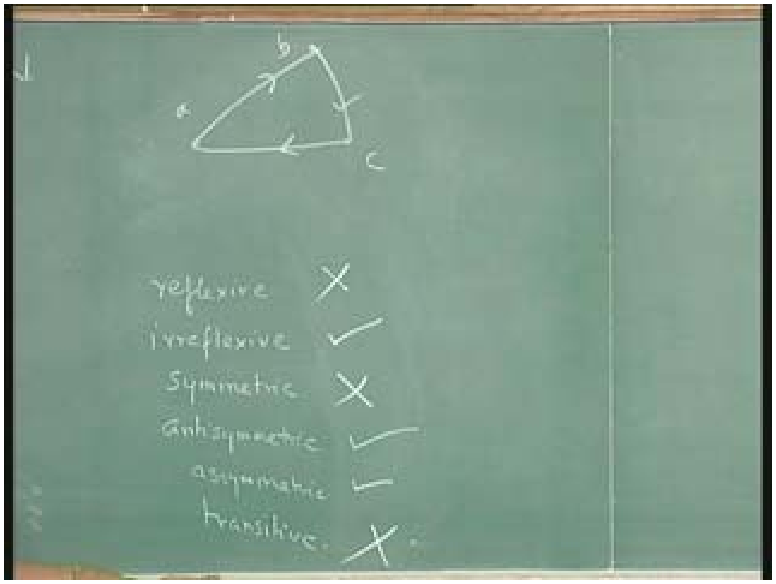

Now, what about the less than relation on non negative integers? a is less than b if a is less than b where a and b are non negative integers what can you say about this relationship? Is it reflexive? a is less than a is not correct so it is not reflexive. Is it irreflexive? Yes a is not related to a for any a so it is irreflexive. Is it symmetric? If it is symmetric a less than b would imply b less than a that is not there so it is not symmetric. Is it antisymmetric? Yes it is antisymmetric a less b and b less than a that is not at all true so this is antisymmetric. a less than b means b is not less than a so it is asymmetric also. Is it transitive? If a is less than b and b is less than c a will be less than c so it is transitive. So these properties hold for less than relation.

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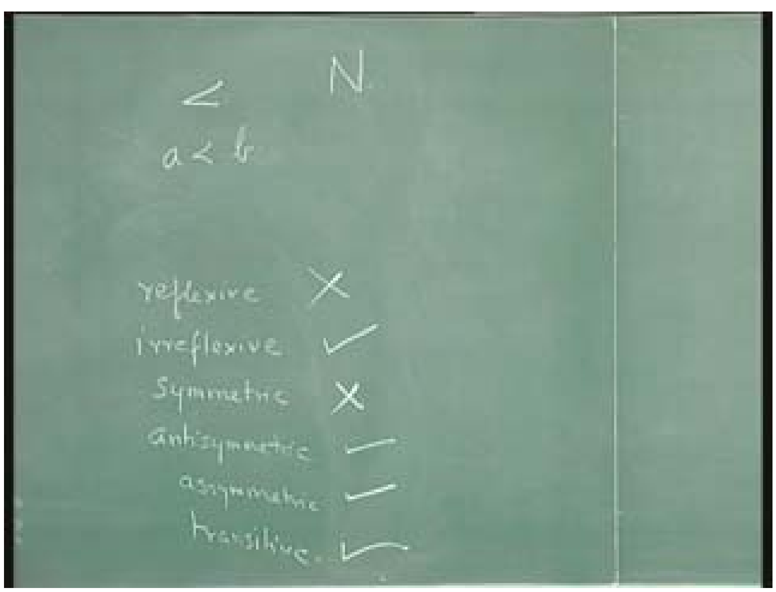

Let us consider less than or equal to relation. Instead of less than you consider less than or equal to relation on the set of non negative integers. a less than or equal to b, then is a less than or equal a? Yes a is equal to a so a is less than or equal to a so it is reflexive. It is not irreflexive, if a is less than or equal to b we will see take 4and 6 then 4 is less than is equal to 6. Can you say 6 is less than or equal to 4? You cannot say, so this is not symmetric. Is it antisymmetric? What can you say about this? You will not have like this but you can self loops so it is antisymmetric. Is it asymmetric? Asymmetric means you should not have self loops also, 4 less than or equal to 4 will hold but then if a is related to b b should not be related to a so it is not asymmetric also. The different between antisymmetric and asymmetric is, antisymmetric you can self loops, asymmetric you cannot have self loops. Transitive, if a is less than or equal to b and b is less than or equal to c will a be less than or equal to c? Yes 4 less than or equal to 6 or 6 less than or equal to 6 it will hold the transitive property.

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So it will be reflexive, antisymmetric and transitive. So these are some properties of relations and they are used in many places sometimes you may want to have a relation which has got special features you may want a transitive relation to start with and you may want a symmetric relation to start with and so on. So these properties play an important role in several fields. Then we shall later define what is meant by reflexive closure, transitive closure and the important role they play in many fields. We will see that later.

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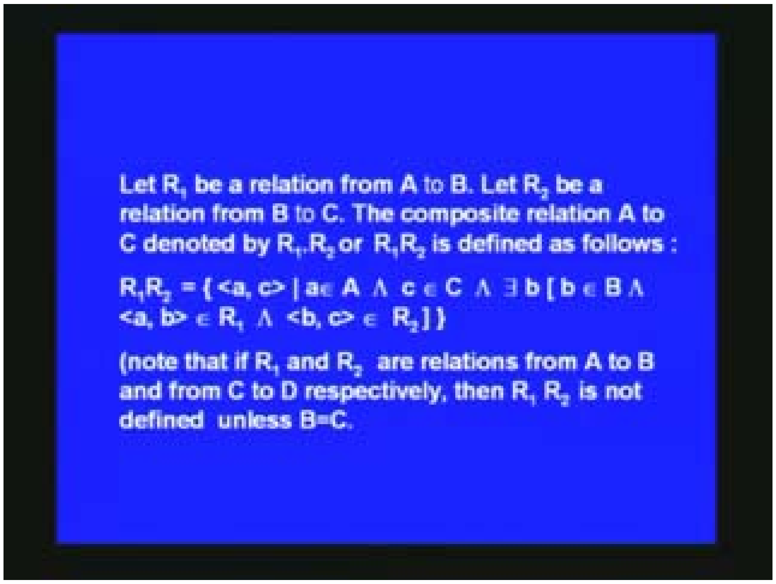

Now, let us next consider composition of relations. What you mean by the composition of relations? Let R1 be a relation from A to B and R_2 be a relation from B to C. the composite relation A to C denoted by R_1. R_2 or R_1 R_2 is defined as follows; R_1 R_2 is a, c where a belongs A and c belongs to C and there exists b belongs to B such that a, b belongs to R_1 and b c belongs to R_2.

Note that if R_1 and R_2 are relations from A to B and from C to D then R_1 and R_2 is not defined unless B is equal to C. So what do you mean by composition of relation? Now, R_1 is a relation from A to B, there are two sets A, B some elements are there, some elements are there. So R_1 represents some arcs like this it is the ordered pair where the first element belongs to a second element belongs to b something like this. R_2 is a relation from the set B to a set C something like this. So this represents the ordered pairs b, c where b belongs B and c belongs C. Then how do you define the composite relation R_1 and R_2? If you have an arc from here to here this is an element of R_1 R_2. And similarly, you have an arc from here to here and arc from here to here so this is an element of R_1 and R_2.

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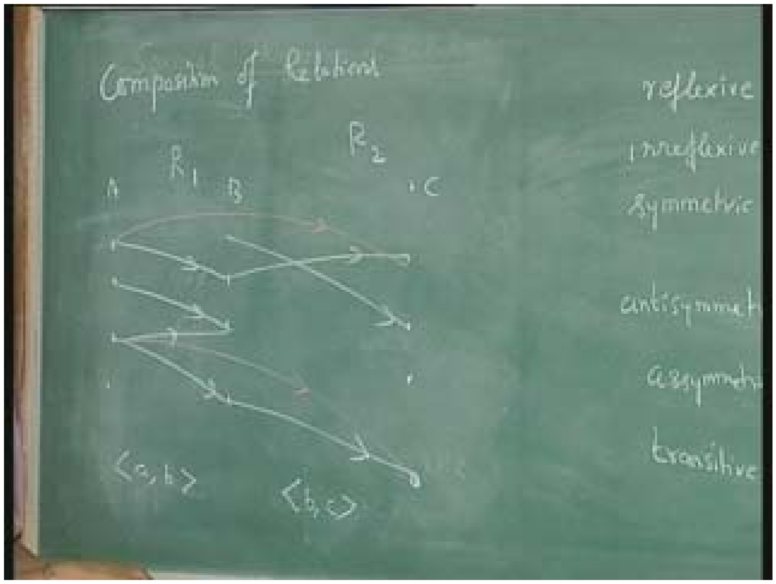

Let us see the definition once again. Let R_1 be a relation from A to B and R_2 be a relation form B to C the composite relation A to C denoted by a R_1. R_2 or R_1 R_2 is defined like this. It represents ordered pairs a, c such that a belongs A and c belongs C and there exists b in B such that a, b belongs to R_1 b, c belongs to R_2. See, if it is A to B and R_2 is from something else to something else and the co domain of R_1 should be the same as the domain of R_2 if they are different it is not well defined. So with composition of relation we have several theorems, we shall see what they are.

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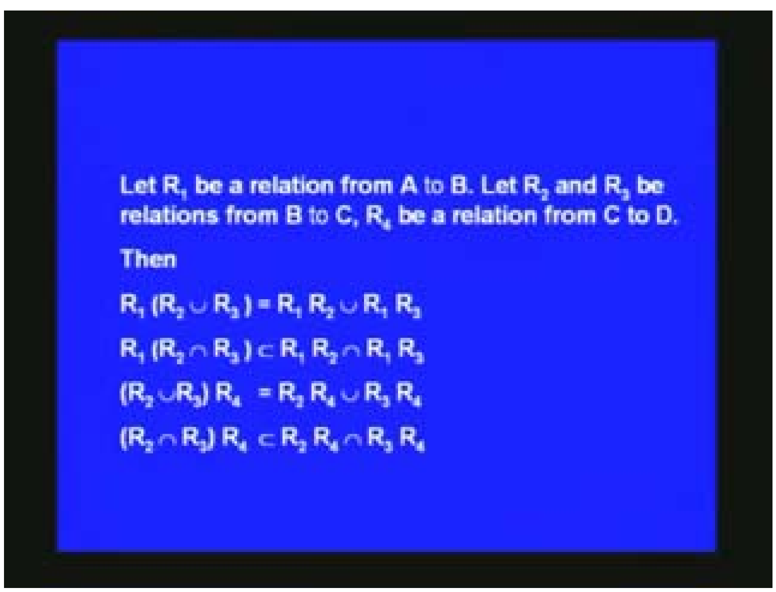

Let R_1 be a relation from A to B and R_2 and R_3 are relations form B to C, R_4 is a relation from C to D then you have the following results R_1 cross R_2 union R_3 is equal to R_4, R_2 union R_2 R_3 R_1 combined with composition with R_2 intersection R_3 is contained in R_1 R_2 intersection R_1 R_3 and the other results. R_2 union R_3 cross R_4 gives you R_4 and R_3 R_4 union. And R_2 intersection R_3 combined with R_4 is contained in R_2 R_4 intersection R_3 R_4.

Let us take the first result and prove the others can be proved in a similar manner. The first one is R_1 R_2 union R_3 is equal to R_1 R_2 union R_1 R_3 how do you prove that? What do you mean by some element x, y or let me take it as a, b otherwise a, c belongs to R_1 R_2 union R_3. Take an ordered pair b, c belonging to R_1 R_2 R_3. This is equivalent to saying a belongs to A, c belongs to C and there exists b such that a, b belongs to R_1 and b, c belongs to R_2 union R_3. And of course b belongs to B there exists b, b belongs to B and a, b belongs to R_1 and b, c belongs to R_1 union R_3.

Now this is equivalent to saying a belongs to A, c belongs to C and there exists b b belongs to B and a b belongs to R_1 and b c belongs to R_2 union R_3 you can write like this b, c belongs to R2 or b, c belongs to R_3. This means a belongs to A, c belongs C and there exists b, b belongs to B and AND distributes over R so you can write this as a, b belongs to R_1 and b, c belongs to R_2 or a, b belongs to R_1 and b, c belongs to R_3. And that is equivalent to saying a belongs to A, c belongs to C etc, and there exists b, b belongs to B and this one means a, b belongs to R_1 and b, c belongs to R_2.

So if you bring it in there exists b, b belongs B and a, b belongs to R_1 and b, c belongs to R_2 or there exists b, b belongs to B and a, b belongs to R_1 and b, c belongs to R_3 which would mean there exists a belongs to A, c belongs C and there exists b such that b belongs to B and a, b belongs to R_1 and like this and b, c belongs to R_2 or there exists b, b belongs to B and a, b belongs to R_1 and b, c belongs to R_3. So for this one you can say and this is equivalent to saying a belongs to A and that conditions is always there c belongs to C and there exists b, b belongs to B and so a, b belongs to R_1 and so on that would mean a, c belongs to R_1 R_2 or a, c belongs to R_1 R_3 that is equivalent to saying a, c belongs to R_1 R_2 union R_1 R_3. So a, c belongs to R_1 cross R_2 union R_3 is equivalent to saying a, c belongs to R_1R_2 union R1 R_3. So the equivalence can be proved, the other three results can be proved in a similar manner.

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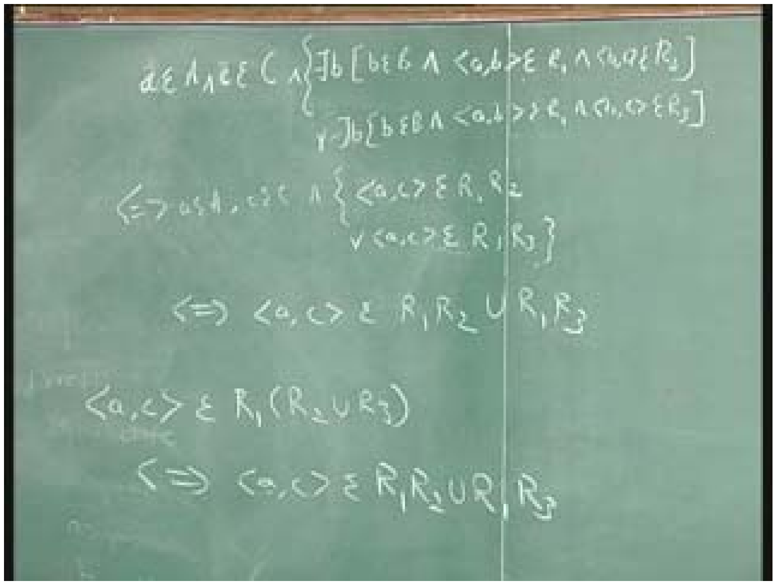

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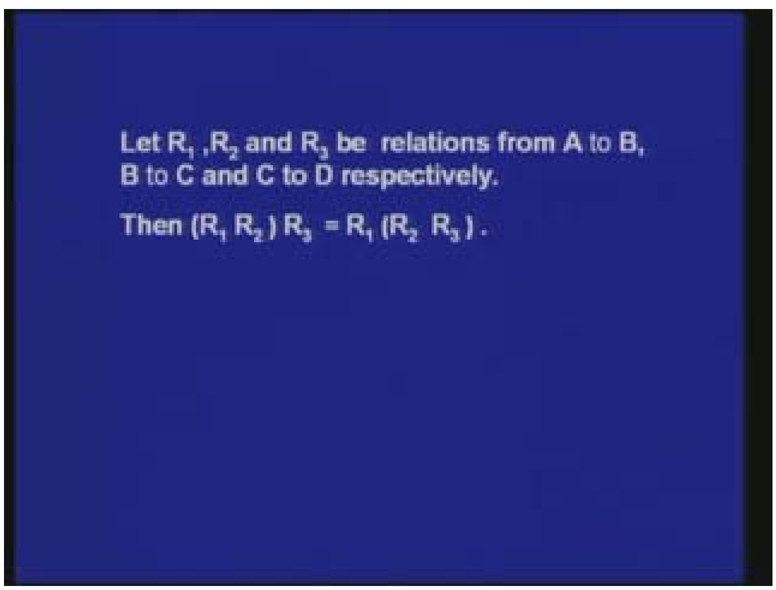

The composition of relation is associative that is R_1 R_2 R_3 is equal to R_1 cross R_2 R_3. Let R_1 and R_2 and R_3 be relations from A to B, B to C and C to D then R_1 R_2 combined with R_3 is equal to R_1 combined with R_2 R_3. You can figuratively write like this: say suppose I have A, B, C, D R_1 is a relation from A to B, R_2 is a relation from B to C, R_3 is a relation from C to D. So let us denote by some ordered pairs like this: R_1 may have some ordered pairs like this, R_2 may have something like this, R_3 may have something like this.

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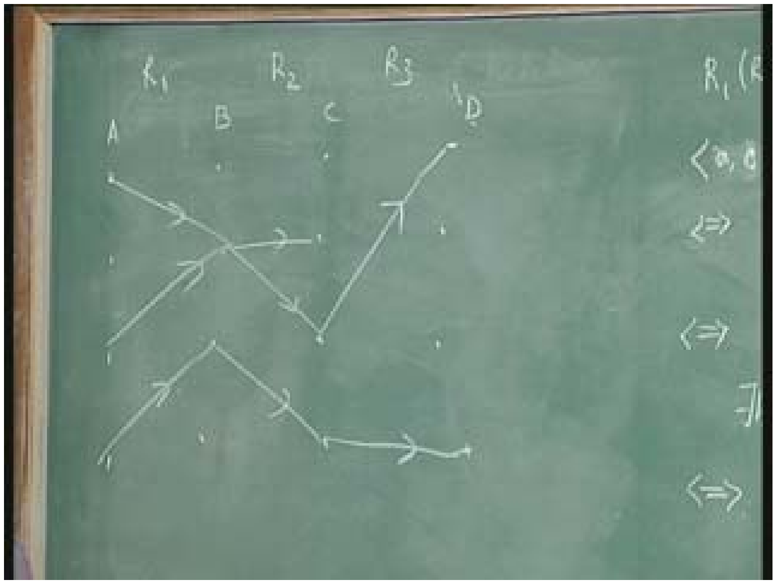

Now, what does R_1 R_2 represent? R_1 R_2 will be, if there is an arc from here to here and here to here R_1 R_2 have an arc like this that is from here to here and here to here so R_1 and R_2 will have an arc like this. from here to here and here to here you have so R_1 and R_2 will also have this and R_1 has this R_2 has this so R_1 will have something like this and this is in R_1 this in R_2 so R_1 and R_2 will have an arc like this. What about R_2 R_3? If you have from here to here and here to here R2 will have something like this, R_2 R_3 will have an arc like this it will also have an arc like this. Now R_1 R_2 has 1, 2, 3, 4 arcs 5 arcs R_3 has 2 arcs like this.

If you try to combine, there is an arc from here to here and if you combine with this you will get an arc from here this is an ordered pair belonging to R_1 R_2 cross R_3 and similarly you have an arc from here to here so combining with this an element from R_3 you have an arc an ordered pair belonging to R_1 R_2 combining with R_3. Now, suppose take R_1 and R_2 R_3 what are the elements in R_2 R_3? R_2 R_3 has only two arcs one this and one this and R_1 has one arc from A to this and let us go back R_1 R_2 also have an arc from here to here and R_3 has from here to here so there will be 1 arc from here to here in R_1 R_2 cross R_3 because R_1 R_2 has this arc and then R_3 has this.

Now look at R_2 R_3 it has got only 2 arcs one this and this. R_1 has this arc this arc this arc. Now if there is an arc like this and a blue arc representing R_2 R_3 you see that this is there in R_1 R_2 R_3. Similarly, if you have a blue arc like this in R_1 R_2 R_3 one arc from R_1 to R_2 like this you have an arc like this in R_1 cross R_2 R_3. Similarly, there is a blue arc here and there is an arc in R_1 so you have an element R_1 R_2 R_3. So either way you take they are equivalent so composition of relation is associative. You can also write like this; what do you mean by saying that a, d belongs to R_1 R_2 cross R_3 that is equivalent to saying there exists c such that a, c belongs to R_1 R_2 and c, d belongs to R_3. Now expanding this,

this is equivalent to saying there exists c and this one you can write as there exists b where b belongs to B of course, a, b belongs to R_1 and b, c belongs to R_2 and here you have c, d, this is one part c, d belongs to R_3.

Now note that b does not occur here so you can take there exist be outside, the quantifier can be taken outside because it does not occur here. So you can write it as there exists c there exists b, a, b belonging to R_1 and b c belongs to R_2 and c, d belongs to R_3. Now you can always interchange the order there exists x, there exists so you can change for all x for all y you can interchange this we have earlier seen. But we cannot change if you have one for all x and there exists the meaning will differ.

But here both are there exists so I can keep there exists b here and take c inside there exists c b, c belongs to R_2 and c, d belongs to R_3 and that is equivalent to saying that there exists b a, b belongs to R_1 and because b, c belongs to R_2 and c, d belongs to R_3 would mean b, d belongs to R_2 R_3. And this is equivalent to saying that a, d belongs to R_1 R_2 R_3. So if a pair a, d belongs to R_1 R_2 cross R_3 it also belongs to R_1 cross R_2 R_3 and vice versa if it belongs to this it will belong to that. So the composition of relation is an associative operation.

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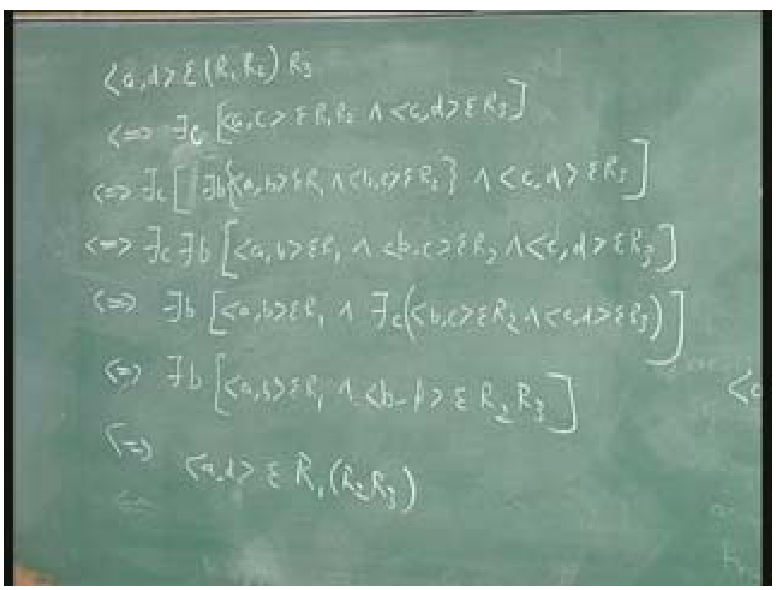

Now because of this associative property what can you say about R squared or something like that when R is a relation on A cross both the domain and the co domain are same. So R is a binary relation on A cross A.

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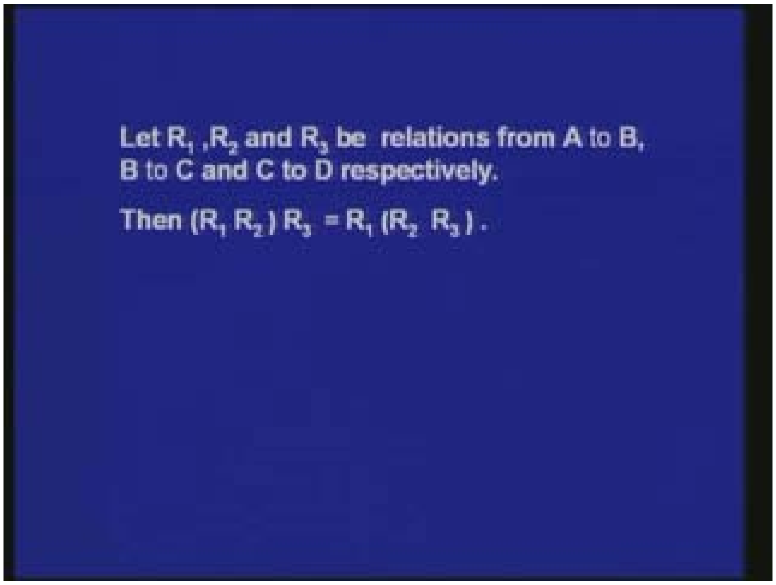

Then what can you say about how do you define R squared or cubed etc?

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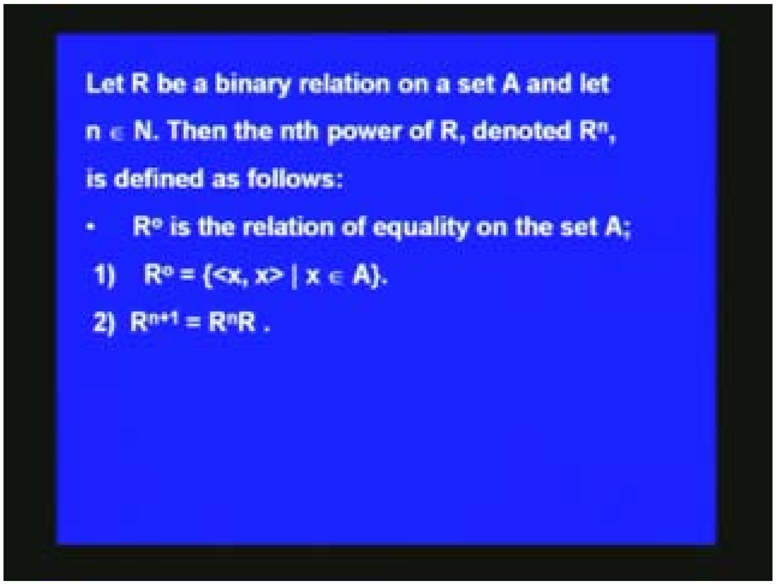

Let R be a binary relation on set a A and let n belonging to N, N is a natural number or a non negative integer. Then the nth power of R denoted by R power n is defined as follows: R power 0 is the relation of equality on the set A, R power 0 is x, x where x belongs to A. R power n plus 1 is equal to R power n cross R. So it like this; you can talk about R power 0 that is the equality relation x belongs to A. R power 1 is just R this is defined on A cross A. What can you say about R squared? R squared is defined as R cross R, because of the associativity without any problem we can write that. Similarly, R cube is R squared cross R, without any problem you can write it as R, R, R either you can put the bracket this way or this way does not matter so without ambiguity you can write it as R. R. R or R cube. And proceeding like this R power n is R cross R cross R n times that is R power n minus 1 cross R. R power n plus 1 of course is R power n cross R.

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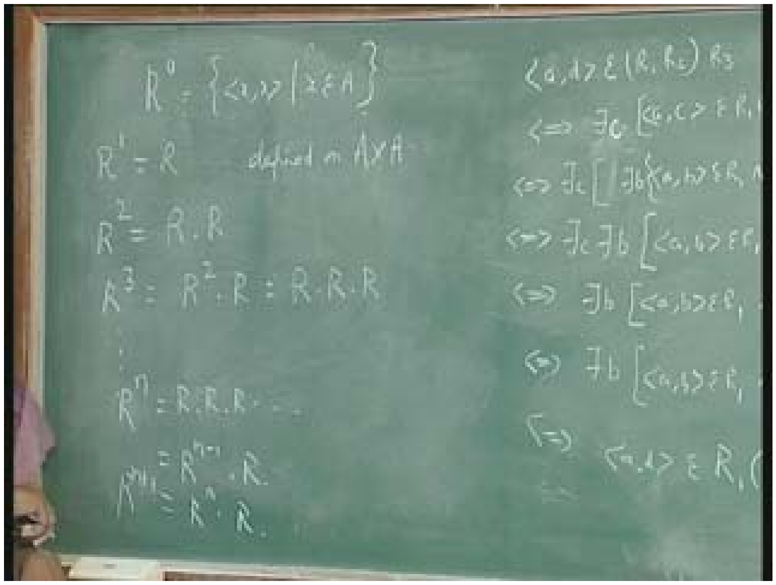

This is the way you define the composition of relation or the power of a relation.

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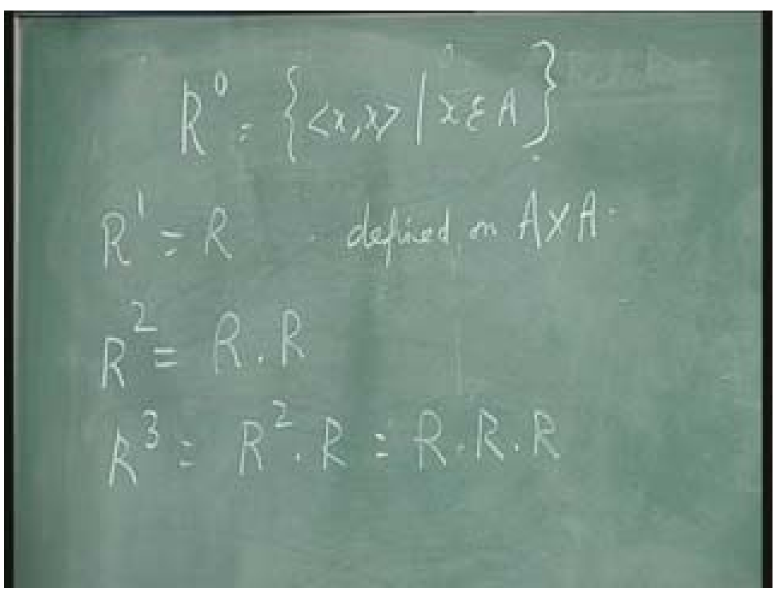

Let me take one or two examples to see what you mean by this. Take this example, let the relation be defined on three elements a, b, c and you have this relation, this is R. What is R power 0? R power 0 is the equality relation on a so it will be represented by self loops like this R power 0 is represented by this graph, R power 1 is just R and it is represented like this. What can you say about R squared? a, b, c. Now if there is a path of length two this one that will be replaced by a single line in R power 2. So there is a to b and b to c and that it will be combined as a to c, there is an arc from b to c and then c to a so there will an arc from b to a and there is an arc c to a and a to b so there will be an arc from c to b. So R squared in this case is represented by this one.

Now what can you say about R cubed? R cubed is R squared cross R, in R squared you have a to c, in R you have c to a so combining that you will have self loop at node a which is this. Similarly, there is an arc from b to a in R squared and a to b in R so there will be a self loop at b in R cubed or you can look at it in another way if there is a path of length three that will be replaced by a single arc. Here, there is a path of length three from a to a so that is replaced by a single arc which is a self loop. Similarly, there is a path of length three at this node so that will be replaced by this self loop at b and similarly you have a path of length three from c to c in arc so that will be replaced by a single arc that is a self loop at c.

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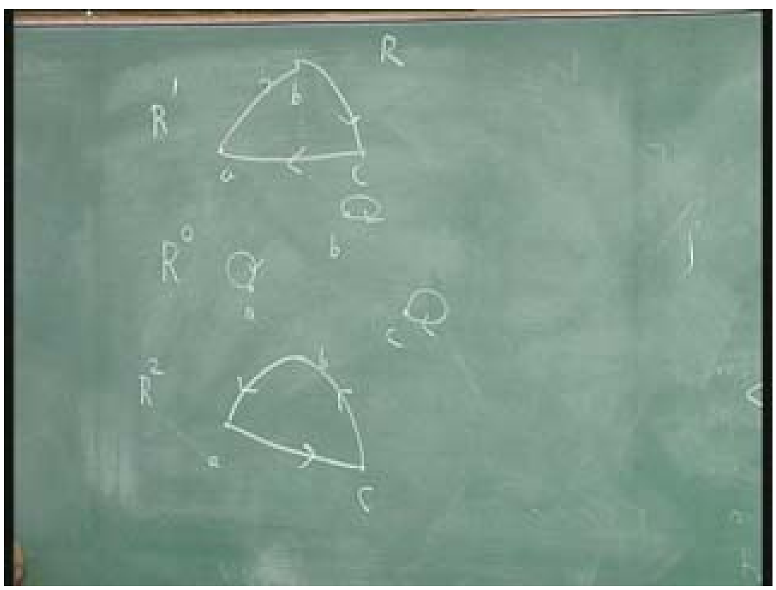

Now what can you say about R power 4? R power 4 will be R cubed cross R a, b, c that is R cubed has a self loop along with this it will make an arc like this, R cubed have a self loop at b along with this arc there will be an arc like this, R cubed has a self loop at c and that along with this arc will be like this. So you find that R is equal to R power 4 and you can also see that R power 0 is equal to R cube that is the equality relation self loops at every node, R power 1 is equal to R power 4 and R squared will be R cross R that is equal to R power 4 cross R so R squared is R power 5, R power 6 again will be equal to this, R power 7 will be equal to this, R power 8 is equal to this and so on. So in this case where you have only three elements and the graph is like this you see that, this is the particular example you find that, this is R power 0, this is R square, this is R cube then R power 4 again becomes equal to R power 1 that is R and so on.

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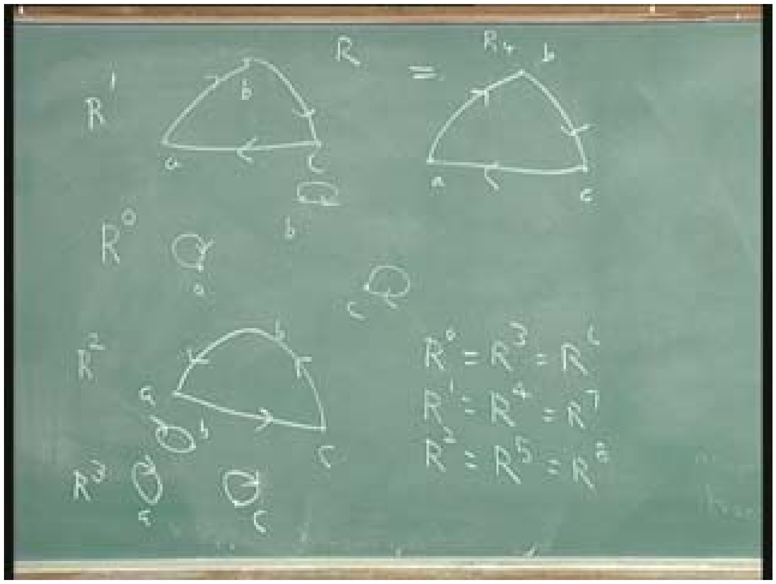

We have this very simple relation.

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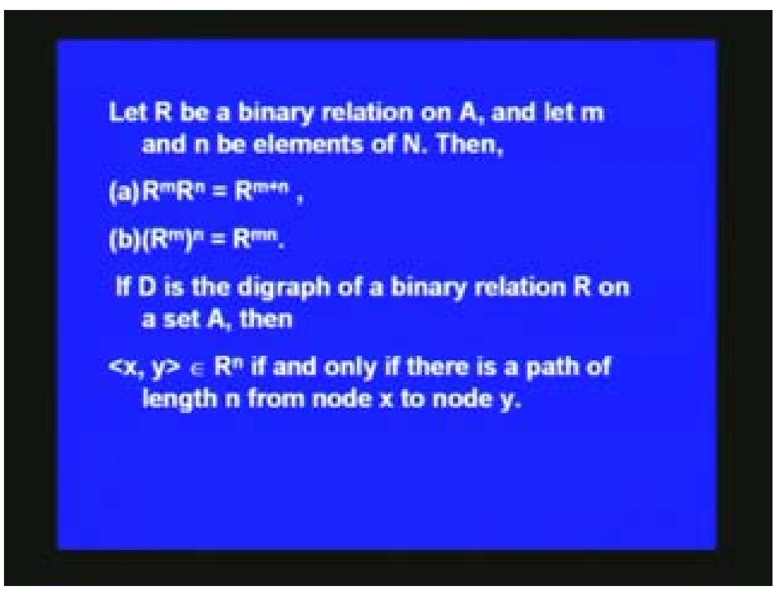

This is very easy, this can be proved by induction in a very easy manner. Let R be a binary relation on A and let m and n be elements of N power m cross R power n is equal to R power m plus n. R power m times R power n is R power m, n. This also we can see every easily. You can use induction to prove that if D is a diagraph of a binary relation then x, y belongs to r power n if and only if there exist a path of length n from node x to node y. So if you have something, if there is a path of length not a single arc a path of length n from x to y this will be in R power n and from y to z say there is a path of length m and this will belong to R power m. So in R power n cross R power m you have to combine this and there will be a path from x to z of length n plus m, you can see that this path is of length n and this path is of length m so there is a path of length n plus m to this that means that path belongs to R power n plus m.

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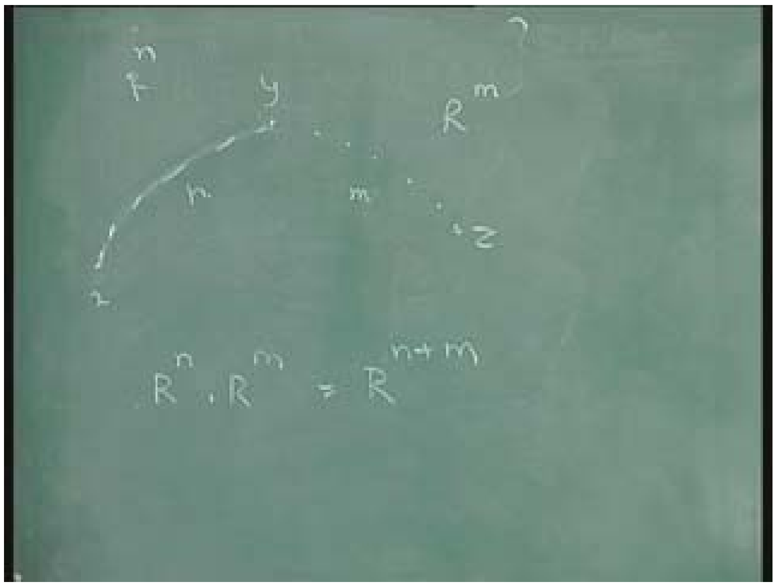

So R power n plus m in a theoretical manner you can use induction and say that R power n cross R power 0 is R power n, R power n cross R is R power n plus 1 and so on. So R power m cross n is equal to R power m, n how do you prove that? So in R power m you have a path from x to y of length m now R power m cross n is R power m cross R power m cross R power m like that n times. That is it is equal to R power m plus m is equal to m like that n times is R power n, m. You have n paths of length m like that and so on.

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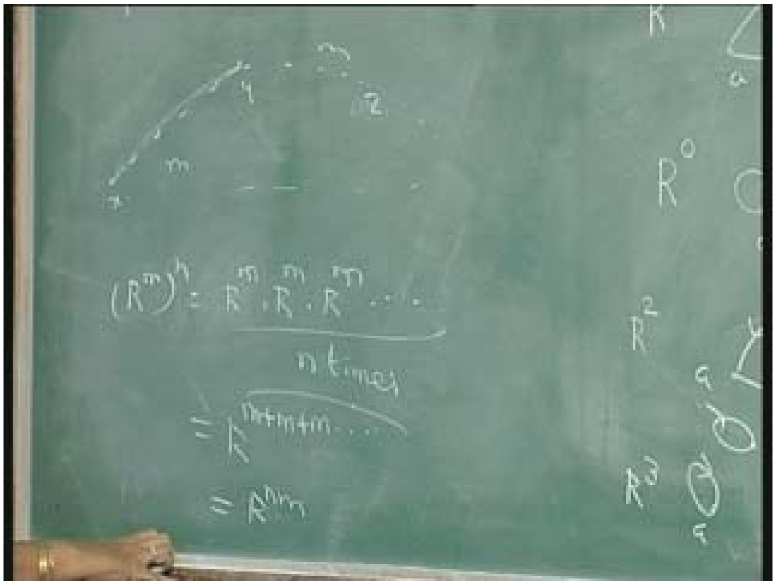

So in essence it amounts to say, like in a ordinary integer you have a power m cross a power n is equal to a power m is equal to n and a power m cross n is equal to if a is an integer say this will hold or if a is any real number for that matter m and n are integers then you have these results. A similar result holds for the power of R also. You have this, now we have seen that R power m cross n is R power n, m R power n, m or m, n the same. And we can also see that if x, y is a pair which belongs to R, n if and only if there is a path of length n from a node x to a node y, we can consider some more examples of how to represent a composition of relations and the power of a particular relation R and what other properties it has.