Discrete Mathematical Structures
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Lecture # 15
Graphs (contd.)

So we were considering graphs and we were considering undirected graphs. So we know what is meant by node or vertex and edges. We also saw how to represent a graph by means of adjacency matrix. Now, when do we say that two graphs are isomorphic, isomorphic graphs. In fact if in any structure two items have equivalent properties then we say we consider them in an equivalent manner. For example, we consider two triangles to be congruent, these two triangles are congruent because the sides are equal and also the angles are equal. And this triangle and this triangle are set to be similar because the sides are proportional and the angles are equal. They have some similar structure.

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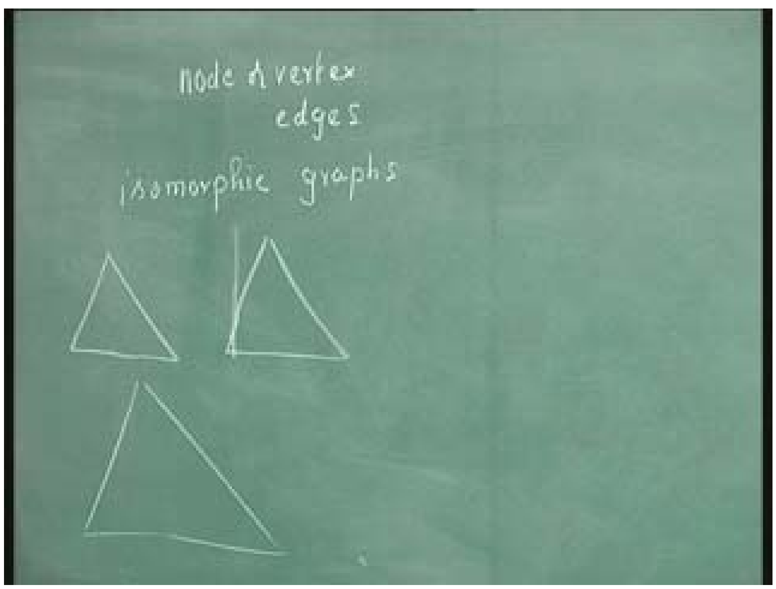

Similarly, we also saw that a graph mainly brings out the incident relationship between the nodes. The vertices are some entity and the graph brings out the connection between them. It is immaterial whether we draw it is a curved line or as a straight line and so on. This is what we have seen earlier. So in such a representation when do you say that two graphs are isomorphic? Two graphs are isomorphic if there is a one to one mapping between the vertices which preserves the incident relationship.

Look at these two graphs; these two graphs are isomorphic they have eight vertices and you can see that the incident relationship is preserved. Look at these two graphs; you can identify vertex e with vertex v_5 and vertex v_4 with d and v_3 with c, v_1 with a and v_2 with b. So some what you can write like this a corresponds to v_1, b corresponds to v_2, c corresponds to v_3, d corresponds to v_4 and e corresponds to v_5. So there is an edge between v_4 and v_5 there is an edge 1 between d and e that incident relationship is preserved. e_1 corresponds to 1, e_2 corresponds to 2 and the 2 is between a and d and e_2 is between v_1 and v_4 you can see the correspondence. 3 is between c and d here e_3 is between v_1 and v_3.

e_3 corresponds to 4 and e_4 corresponds to 3 c d, e_3 corresponds to v_1 and it connects v_1 and v_3, v_1 corresponds to a and c. So e_4 corresponds to e_3 which connects v_1 and v_3 corresponds to 4 and e_4 which corresponds to v_4 and v_3 corresponds to 3. Between a and b there is edge five which corresponds to e_5 between v_1 v_2. And between v_2 and v_3 or between b and c you have edge six which corresponds to e_6. So here you will find that e_1 corresponds to 1, e_2 corresponds to 2, e_3 e_4 corresponds e_5 corresponds to 5, e_6 corresponds to 6 and the incidence relationship is preserved.

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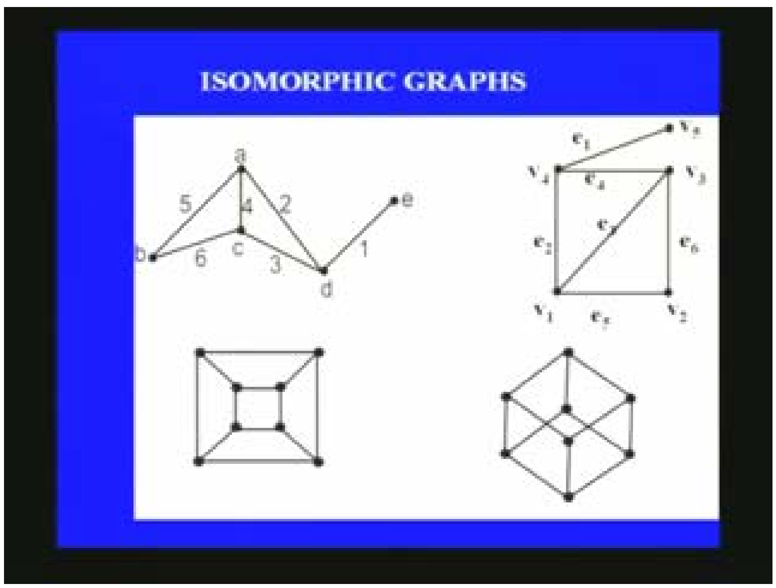

So these two graphs are said to be isomorphic. Similarly, you can see that even though we have not labeled the edges and the vertices you can see that these two graphs are isomorphic. Here, you find that this graph is isomorphic to this graph. Here some edges are missing, this you must also have edges between this and this, this and this, this and this, here this and this and so on. So that graph will be like this; the first graph should be like this. You can see the correspondence between this and the second graph. It is easy to see the correspondence but it is difficult to see the correspondence because it is 6 and 4 but this is also isomorphic to that. You can identify each vertex with a vertex here. Now when you have two graphs isomorphic they should have equal number of vertices, they should have equal number of edges and equal number of vertices of a particular degree.

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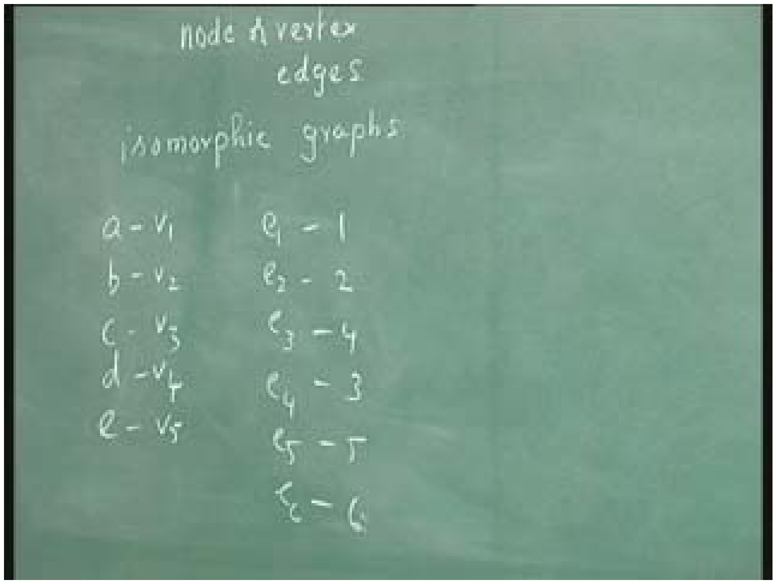

For example, in the earlier one there are 1 2 3 vertices of degree 3 here also there are 1 2 3 vertices of degree 3. There is a vertex of degree 1, here there is a vertex of degree 1. There is a vertex of degree 2 there is 1 vertex of degree 2. So if two graphs are isomorphic they will have equal number of vertices, they will have equal number of edges and equal number of vertices of a particular degree. But these three conditions may be satisfied but still the graphs may not be isomorphic.

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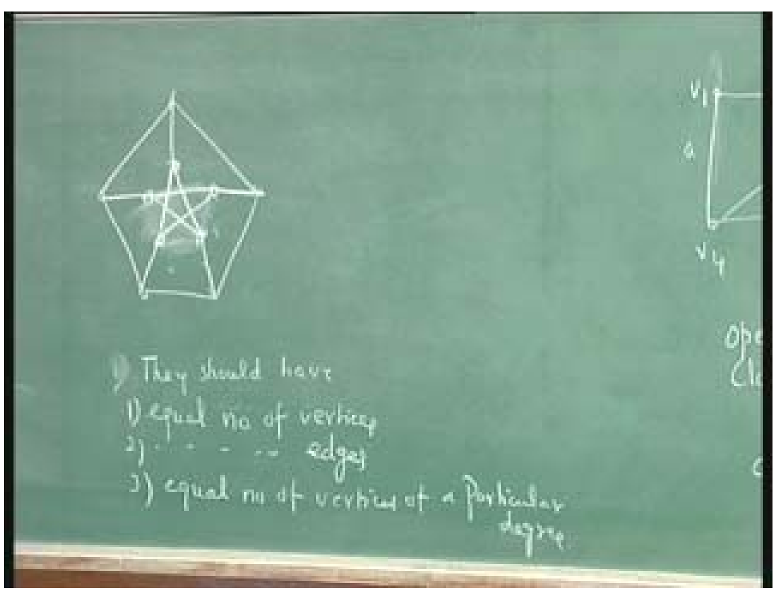

Let us consider an example. Look at these two graphs this and this, how many vertices this has? This has got six vertices. This also has six vertices and how many edges it has got? This has got 1 2 3 4 5 edges this has got. This also has got five edges and there are three vertices of degree 1, here also there are three vertices of degree 1, there is one vertex of degree 3, there is one vertex of degree 3 here, there are two vertices of degree 2 these two are vertices of degree 2. So both the graphs have six vertices, five edges, three vertices of degree 1, two vertices of degree 2 and one vertex of degree 3 So these conditions are satisfied but they are not isomorphic why? We have seen that a vertex with degree 1 is called a pendent vertex. So in this case u and v are pendent vertices. This is the only vertex with degree 3, x is a only vertex with degree 3.

Here y is the vertex with degree 3 so this x should corresponds to this y and y has only one pendent vertex near to it. There is only one pendent vertex attached to it like this. Whereas here there are two pendent vertices, u and v are attached to this. So this structure is different so these two graphs are not isomorphic. These are necessary conditions but not sufficient conditions. So something may satisfy all the three conditions but still the two graphs may not be isomorphic.

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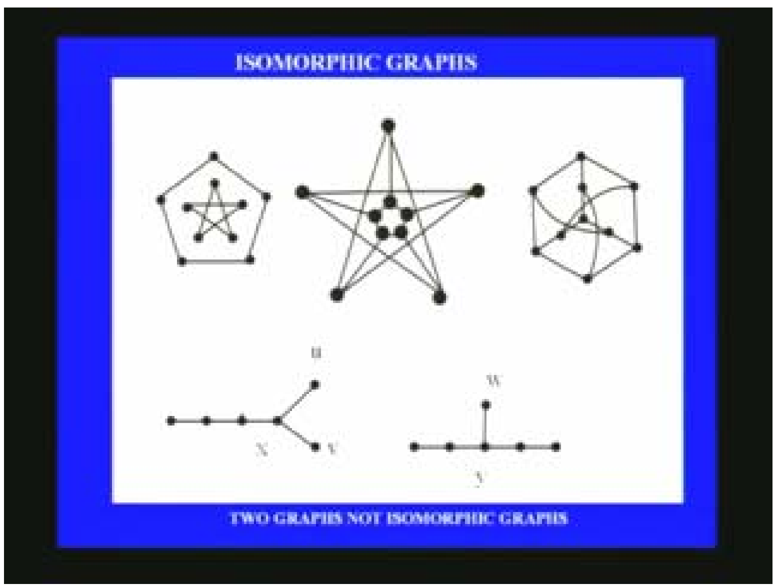

We have already seen what a subgraph is. So, take for example this graph, just change it a little bit, consider this graph with vertices a, b, c, d, e, f. Now, this is a subgraph of that, b, c, d is a subgraph of this. Similarly, if you consider b, a, d this is also a subgraph of that. These two are subgraphs and they do not have any edges in common. They are called edge disjoint subgraphs. But they have vertices, this also has b and d and this also has b and d but no edges common between them so they are called edge disjoint subgraphs. And look at this and a subgraph e, f like this, these two do not have any vertex also in common they are vertex disjoint. When they are vertex disjoint subgraphs obviously they will be also edge disjoint. There is no vertex common between them and both are subgraphs of this and so there is no edge also common between them.

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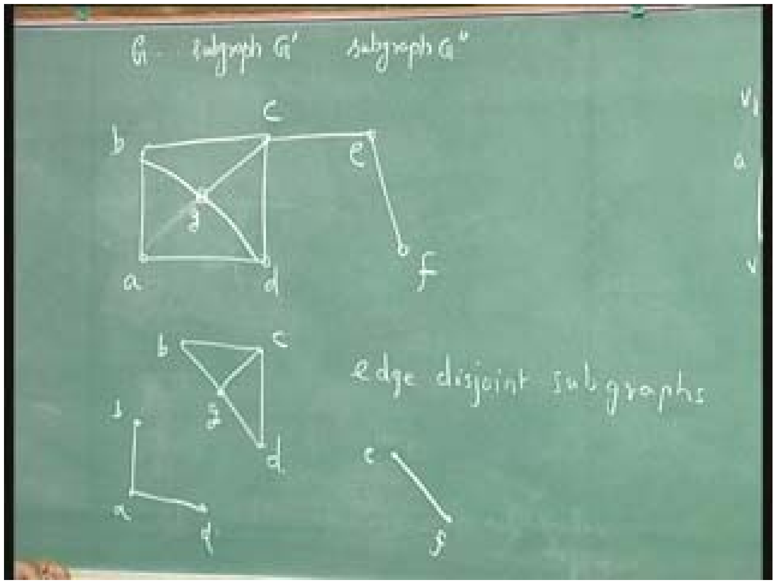

So these are some of the common terminologies used in graph theory. And if you have a graph G, a subgraph of G, G dash and another subgraph of G dash subgraph G2 dash, this is a subgraph of G dash and this is a subgraph of G in that case G2 dash will also be a subgraph of G obviously. These are some of the obvious results.

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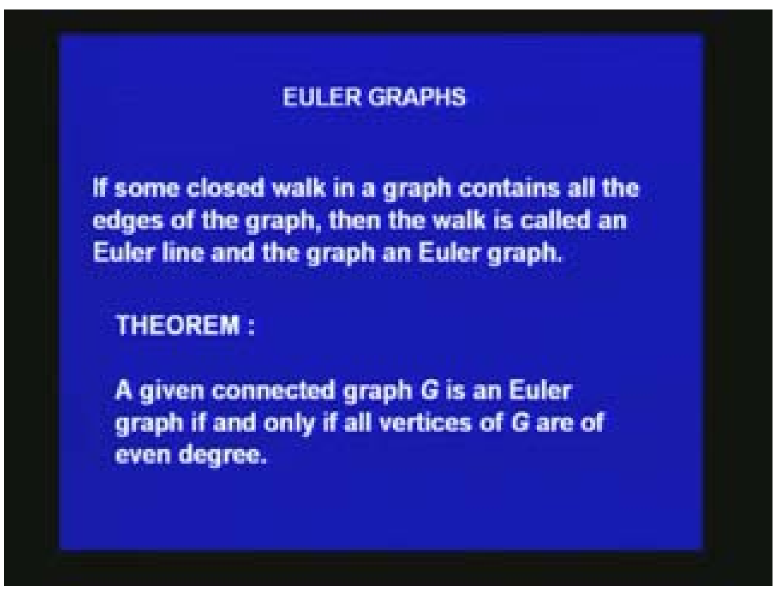

Next we shall see what is meant by a walk and what is meant by a path. We have already seen what is meant by a path and what is meant by a simple path in the case of directed graphs. Now in the case of undirected graphs what do you mean by walk and what do you mean by a path. Let us consider the definition: walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex may however appear more than once.

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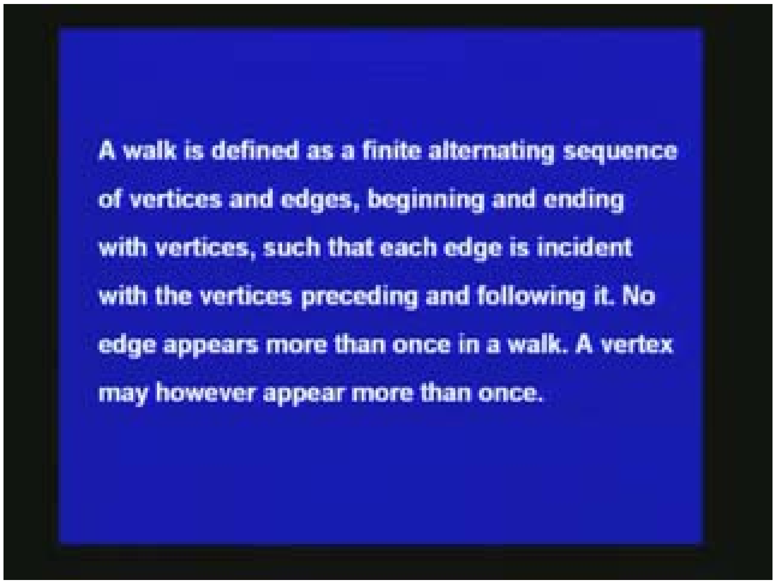

So what is a walk?
Let us consider one example. Let us take this graph with vertices v_1, v_2, v_3, v_4, v5, v_6, six vertices are there. There are edges a, b, c, d, e, f, g, h and let us see what is meant by a walk. Let us give some examples of walks here. So look at this v_1 b v_2 e v_3 h v_3 d v_4 this is a walk. No edge appears more than once but the vertex v_3 appears twice that is okay. Now look at this v_4 c v_2 f v_5 g v_6 this is also a walk. Here no vertex also repeated this is called a path. Now before that one more example of a walk v_1 b v_2 c v_4 a v_1 that is a walk.

Every edge is incident on the vertex preceding it and following it. c is incident on v_2 and v_4, a is incident on v_4 and v_1. This is called a closed walk because the beginning is v_1 and ending is also v_1. The beginning vertex and the ending vertex are called terminal vertices. And if they happen to be the same then it is called a closed walk. In this case v_1 b v_2 e v_3 h v_3 d v_4 the beginning and the ending vertices are not the same and this is called an open walk. If the beginning vertex and the end vertex are not the same the it is called a open walk. In a walk no edge can be repeated but a vertex can be repeated. If in a walk if no vertex is repeated as in this case v_4 c v_2 v_4 c v_2 f v_5 g v_6 no vertex is repeated, no edge is repeated then that is called a path. This corresponds to what is meant by a simple path in the directed graph. And in a closed walk no vertex is repeated. If in a open walk no vertex is repeated it is called a path. If in a closed walk no vertex is repeated it is called a cycle or a circuit.

For example here, which we have already seen v_1 b v_2 c v_4 a v_1 this is a closed walk no vertex also repeated except beginning and the end and that is a cycle or a circuit.

Similarly, you can also have v_4 c v_2 e v_3 d this is also d v_4 beginning and end are the same and no vertex or edges are repeated in the middle this is called a closed walk and this is called a circuit or a cycle. So, for example, a cycle will be of this form, a cycle will be of this form this is a cycle on six vertices. With two vertices you can have something like this with one vertex it will be like this, these are cycles. Now, we earlier considered the conics bridge problem, the river which was flowing on the seven bridges on that on so on.

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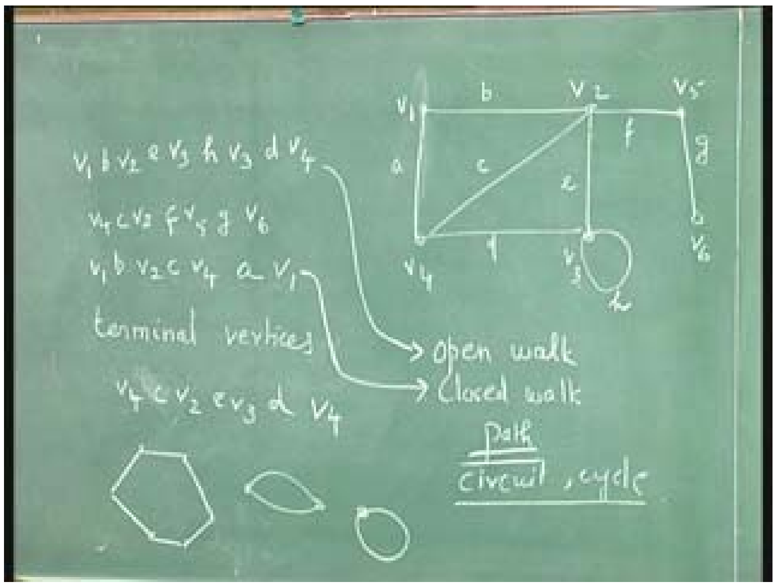

Let us consider it once more. This was the river and there are two islands and two banks there were seven bridges like this. You can number the bridges if you want 1, 2, 3, 4, 5, 6, 7. If you represent it as a graph node B is represented like this node A, node D, C. There is an edge between B and C, there is a bridge between B and C which is 5, between A and B there are two edges 1 and 2, between B and D there are two edges 3 and 4 and between C and A there is an edge that is 6, between C and D there is 1. So this whole structure can be represented by a graph in this manner. And the question was starting from any one of the four areas A, B, C, D can you go along every bridge once and come back to the same position. It amounts to saying whether you have a closed walk in this graph which will cover all the edges.

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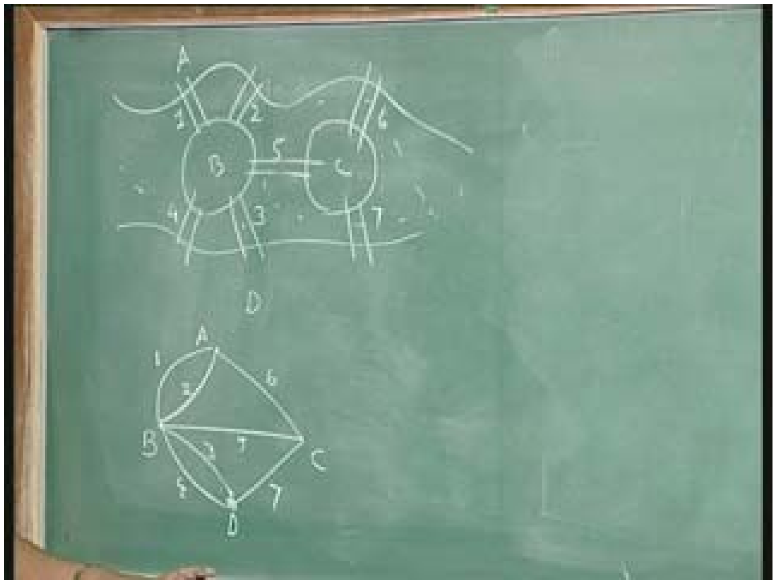

So we come to the definition of an Euler graph. What is an Euler Graph?
If some closed walk in a graph contains all the edges of the graph then the walk is called an Euler line and the graph is called an Euler graph. Take for example this a simple one; a circuit, start from here you can traverse all the edges and come back to the same position. You may also have something like this; this is a simple circuit, start from here you can traverse like this, you can traverse like this. So you will have a closed walk which covers all the edges in this graph. So that is an Euler graph. In this case you cannot do that so this is not an Euler graph. And what is the condition for a graph to be an Euler graph.

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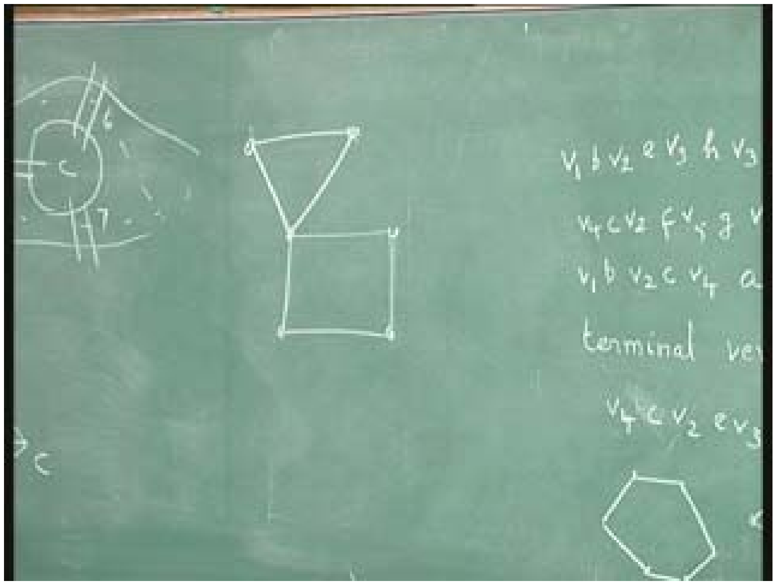

So, again let us see the definition; if some closed walk in a graph contains all the edges of the graph then the walk is called an Euler line and the graph is an Euler graph. And how can you say whether a graph is Euler or not? The following theorem tells you when you can say whether a graph is an Euler graph or not. A given connected graph G is an Euler graph if and only if all vertices of G are of even degree. So this tells you how to test whether a given graph is Euler graph or not. A given connected graph G is an Euler graph if and only if all vertices of G are of even degree. Let us prove that theorem. The first portion is if it is Euler graph then all vertices are of even degree. Then we have to prove the converse. If all vertices are of even degree then the graph is an Euler graph.

What is an Euler graph?
You should have a closed walk which covers all the edges. It starts at this point and then it ends at this point. In between there is a vertex, whenever it enters through one edge it has to leave through another edge. No edge is repeated in a walk and we are considering a closed walk. So if it enters this node once it has to leave that, if that walk has entered this node once it has to leave through another edge only not the same edge. And by some chance if it enters again it cannot use these two it has to leave this node through a different edge. So, if you calculate like that you can very easily see that the number of edges incident on any vertex is even or the degree of this vertex is even.

Look at the start node; this is the start node, the walk starts from here. So you can account for one degree and it ends here so that is another degree. If it enters this node it has to leave through another edge. So even in this start node which also happens to be the end node whenever the closed walk enters this node it has to leave through another edge and in the beginning it leaves through this edge and in the end it enters this node through this edge. So here again you will find that the degree of that node is even. So, if you have a closed walk which covers all the edges of the graph then all vertices are of even degree. This we can very easily see.

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Now conversely it is if and only if the theorem says if and only if so the other way round, if all vertices are of even degree then it is an Euler graph. One point we have to remember is we consider a connected graph. If a closed walk exist which covers all the edges means it should be a connected graph. We do not consider isolated vertices you may have some connected graph and some isolated vertexes which do not have any edges. We do not worry about these isolated vertices.

Now you have a graph where each vertex is a connected graph and each vertex is of even degree. Now start, take a vertex and you must show that there is a closed walk which covers all the edges of the graph. Now you start and you traverse, no edge should be repeated, you traverse until you have covered one portion one closed walk you have found starting from here and ending here. And if it covers all the edges then it is well and good. But if it does not cover all the edges then remove these edges which you have already traversed and consider the remaining graph. The remaining graph there will be at least one node which is present here.

Instead of drawing like this I will consider this example which we consider. So starting from here consider a closed walk like this, it has started from here and ended here so remove all these edges. Now the remaining portion will have at least one vertex which is common to this which is common between this portion which you have already considered and this. Now start at this vertex a and again consider a closed walk you will get one more.

Suppose there is something like this, after traversing like this you will be left with this, that again you can cover like that. So start from a node and traverse along the edges not repeating any edge till you reach the same point then you have got a closed walk and remove these edges in the closed walk then the remaining portion will have at least one vertex which is common with this and start from that vertex and try to find another closed walk. If it covers all the edges it is okay otherwise remove these edges and when you remove the remaining portion will have at least one vertex which is common to this because the graph is connected. If the graph is not connected it may not have a vertex in common but because it is connected there will be at least one vertex which is common to this and so start from here and traverse.

For example, in this case if you have started from here you can have a closed walk which covers all the edges like this. This is possible because all vertices are of even degree. So when you remove this portion you have removed two edges from here. But this vertex is of even degree so when you have removed two edges still it is of even degree, that property is maintained and so you can proceed until you get a closed walk. You will be building the closed walk step by step. So if all vertices are of even degree then you will get an Euler line and the graph is called an Euler graph.

Look at this problem; when you represented as a graph what is the degree of B? The degree of B is 5, the degree of A is 3, degree of D is 3, degree of C is 3, none of the vertices are of even degree. Even if one vertex is not of even degree it cannot be an Euler graph. Here all the vertices are of odd degree. So this is not an Euler graph so you cannot have an Euler line that is you cannot start at a point traverse through all the edges and reach a same point having traverse each edge only once and having covered all the edges. So we have seen what is meant by an Euler graph.

Some examples of Euler graphs: look at this, this graph is an Euler graph. You can see that each vertex is of even degree. Starting from here you can traverse like this and have a closed path here. Then again starting from here continue with the closed path and end here. So this has got an Euler line so Euler graph. Similarly, here you may have an Euler line like this, like this, like this, like this, like this, like this, come to the point then cover the other edges like this so it is an Euler line.

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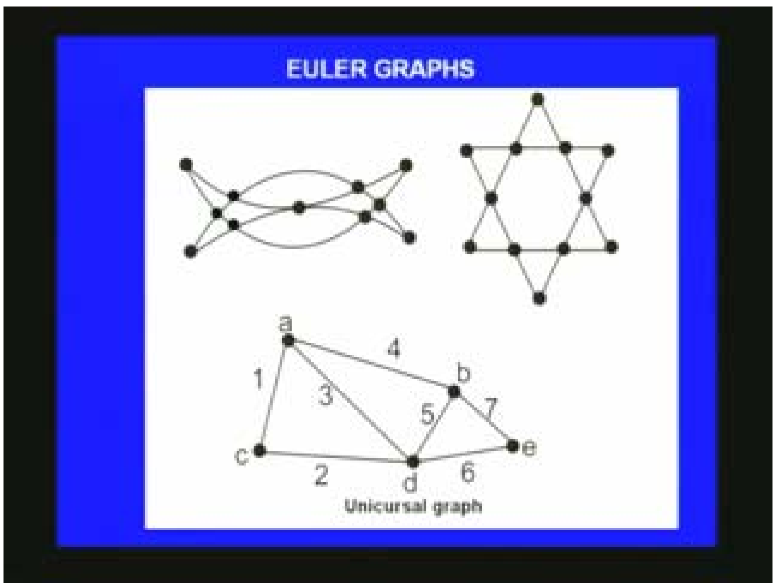

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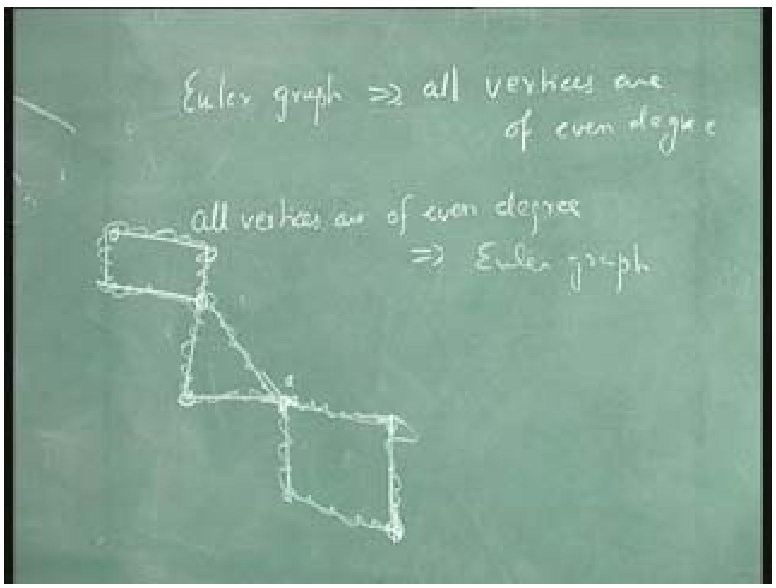

We may relax the condition a little bit. See a closed walk should cover all the edges of the graph and it has to start and end at the same vertex then it is called an Euler graph. If you relax it a little bit and say that the starting and end node need not be the same. There should be a closed walk covering all the edges of the graph. But it need not start and end at the same points. In this case also you have an Euler line but it is an open walk it is not closed and the graph is called a Unicursal graph. This is an example of a Unicursal graph. Here we can start from a and end in b and you can cover all the edges.

Start from a 1, 2, 3, 4, 5, 6, 7 this is a walk but it does not start and end at the same position. It starts at a and ends at b. And such a graph which has a walk like that is called a Unicursal graph. Here all vertices except the beginning and the end vertices will be of even degree. Only two vertices which are the beginning vertex and ending vertex will be of odd degree we can see that, the walk begins at a and ends at b, a has degree 3, b has degree 3, rest of them c has degree 2, d has degree 4 and they are all of even degree, e has degree 2. So except the beginning vertex and the ending vertex all other vertices will have degree 2 the even degree and the beginning vertex and the ending vertex will have odd degrees, such a graph is called a Unicursal graph.

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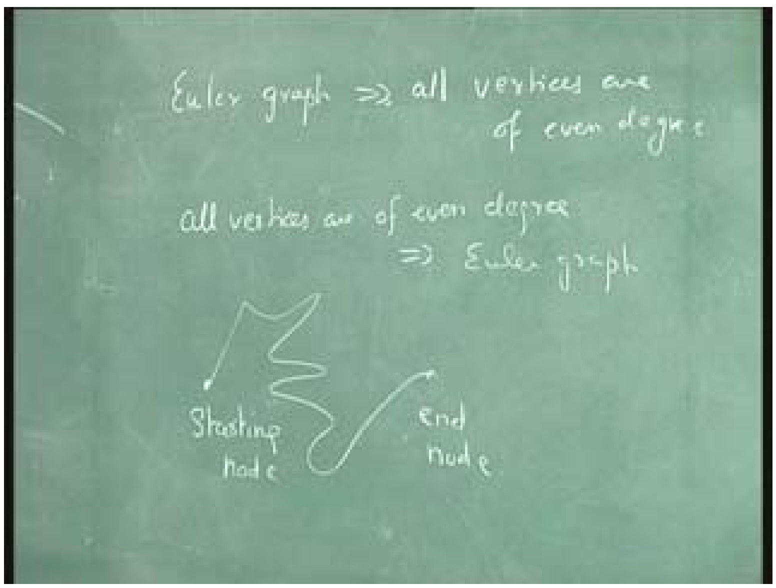

Now we have seen what is mean by a Hamiltonian Euler graph. We shall see what is meant by a Hamiltonian path and circuit. An Euler line of a connected graph was characterized by the property of being a closed walk that traverses every edge of the graph. This is what we have seen now. In contrast to that what is a Hamiltonian circuit? A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once. Except of course the starting vertex and the ending vertex they happen to be the same, the starting vertex at which the walk also terminates. So in contrast to an Euler line the Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once except the starting vertex at which the walk also terminates.

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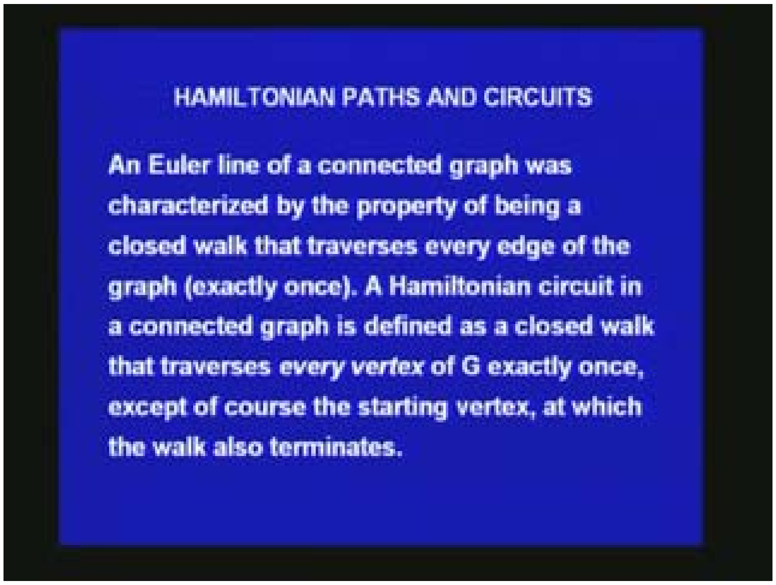

Let us see what a Hamiltonian circuit is. Look at this graph; it has got a Hamiltonian circuit, there is a closed walk take this point it starts like this, it starts like this, it goes like this, it goes like this, it goes like this, it goes like this, it goes like this and like this. So there is a closed walk which covers all the vertices and each vertex occurs only once in this circuit. Similarly, here also you can have something like this starting from here and you go along like this, this, this, this, this, like this, this is a Hamiltonian circuit in this graph. Even though the concept looks very similar the Euler line is a closed walk which will include all the edges of the graph and exactly once.

A Hamiltonian circuit is a closed walk which will include all the vertices of the graph. Each vertex included only once except of course of the beginning and the end. The definitions look very similar but the concept of a Hamiltonian circuit is much more difficult then that of an Euler graph. We can very easily characterize an Euler graph, if all vertices are of even degree it is an Euler graph. There is no such easy characterization for finding out whether a graph has Hamiltonian circuit or not.

Some graphs do not have Hamiltonian circuits and some graphs may have several Hamiltonian circuits and so on. These two graphs do not have Hamiltonian circuits. It is not so obvious when you look at them, why we have to say these two graphs do not have Hamiltonian circuit. Take this one first, a Hamiltonian circuit will enter a vertex and leave one vertex only two edges incident on a vertex can be included in the Hamiltonian circuit.

For example, if you take this vertex it is of degree 2 and every vertex has to be present in the Hamiltonian circuit so you have to consider this edge you have to consider this edge also. Similarly, for this vertex you have to consider this edge and this edge. Like that for this vertex and this for vertex and this vertex. That is you have to include this edge, this edge and all these edges.
Once you finish that you get this circuit this closed walk but this vertex has already included two edges so you cannot include them. So there is no way you can go from this portion of the graph to the remaining portion of the graph and cover all the vertices it is not possible. This graph does not have a Hamiltonian circuit. Similarly, this graph also does not have a Hamiltonian circuit, how can you say that? Of the entire edges incident on a graph only two of them can be included, for each vertex only two edges can be included in the Hamiltonian circuit.

So look at this node, what is the degree of that node? The degree of that node is 5 of which only two can be included and the remaining three have to be omitted. So you have to omit three edges incident on that, you have to omit three edges incident on this and you have to omit three edges incident on that. So nine edges you have to be excluded from the Hamiltonian circuit. And here there are three edges incident on this of which only two you can take so one you have to omit here. Similarly, from here you have to omit one, from here you have to omit one and from here you have to omit one edge. So totally 9 plus 4 is equal to 13 edges you have to leave out in this Hamiltonian circuit.

Totally how many vertices are there and how many edges are there?
You can see that there are 6 plus 6 plus 4 is equal to 16 vertices. So the Hamiltonian circuit should have sixteen edges then only it will be a Hamiltonian circuit because if you have a circuit with five nodes you will have five edges.

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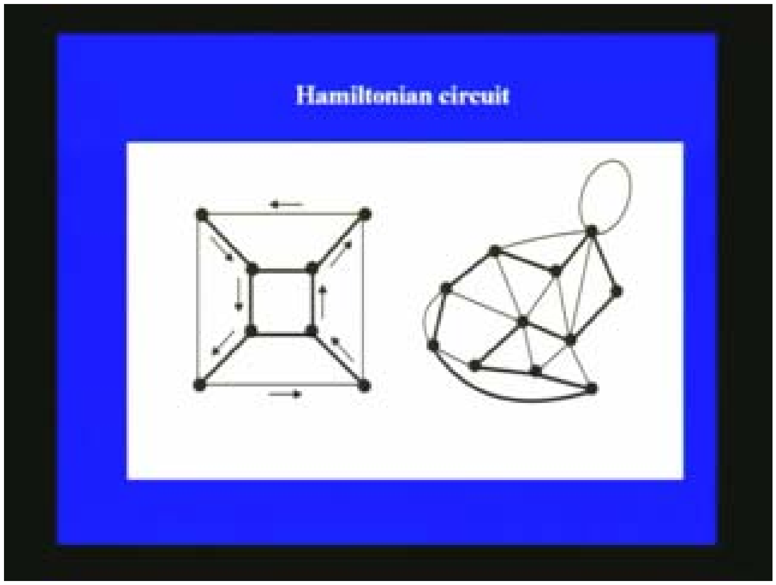

So a Hamiltonian circuit has to pass through all the vertices of the graph there are sixteen vertices so the Hamiltonian circuit if it exists should be of size 16. Let us see how many edges are there in this graph. In this graph there are 6, 6, 6 which is 18 then 19, 20, 21, 22, 23, 24, 25, 26, 27 edges are there in this graph. Out of the 27 you have to leave out 13, we have seen that you have to leave out 13. From 27 if you leave out 13 you will be left out with 14 edges. And the Hamiltonian circuit should have sixteen edges so it is not possible to have a Hamiltonian circuit in this graph. It is easy to argue in some particular cases but generally whether a graph has a Hamiltonian circuit or not is not very easy, it is one of the np complete problems what people call as a difficult problem, np complete problems.

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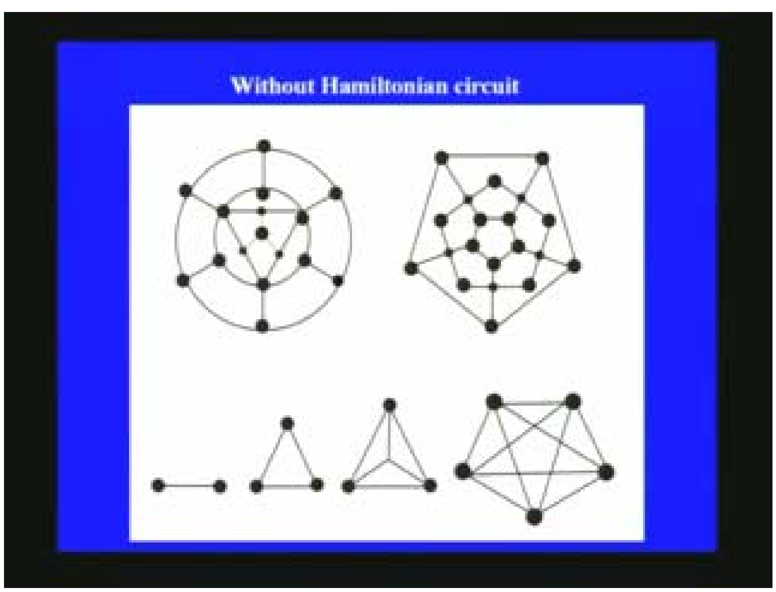

You cannot give a polynomial time Deterministic algorithm for solving such problems. A complete graph diagraph we have seen. When you consider relations we have seen what is meant by a complete diagraph. It represents a universal relation, for example with three vertices a complete diagraph was like this with three nodes it was like this. In the case of undirected graph if every node is connected to every other node it is called a complete graph. This is a complete graph with three vertices, here there is a vertex this is also another vertex. So with four vertices this is the complete graph and this is the complete graph with five vertices. Every node is connected to every other node.

How many edges will you have in a complete graph with n vertices?
Here we are considering undirected because directed will be different so undirected graph, graph means undirected. Each node is connected to every other node by an edge. So you have n nodes each one is connected to every other node so it is n cross n minus 1 edges but each edge is count to twice. So for example if you have four nodes from this there are three, from this there are three, from this there are three and so on, so 4 cross 3 is equal to 12 but you count this edge from this node also and also from this node. So number of edges in a complete graph undirected graph is given by n cross n minus 1 by 2 with four nodes you can see that you have six edges, n is 4, 4 cross 3 by 2 is 6. Similarly, you can see for 5 if you see for 5 there are ten edges 5 cross 4 is 20 by 2 is 10, there are ten edges in the complete graph.

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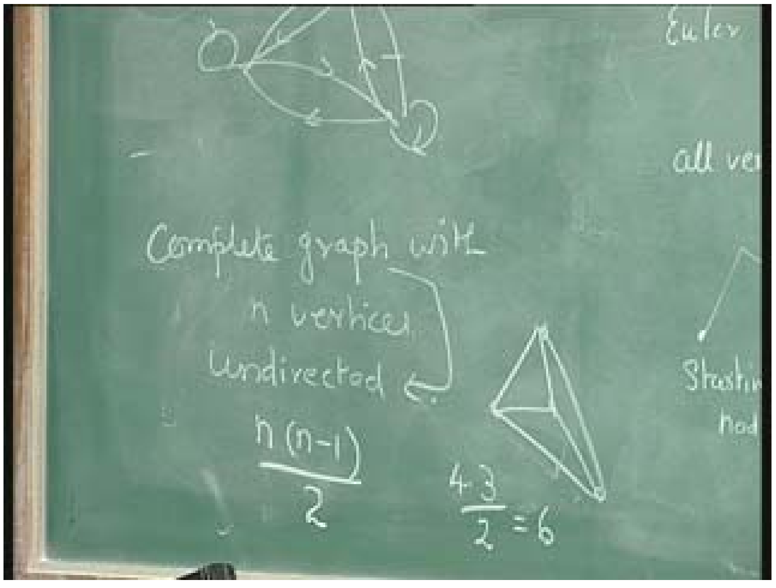

Now let us take this graph call this as a, b, c, d and the edges as 1, 2, 3, 4, 5, 6 call the edges as like that. Does this have a Hamiltonian circuit? Any complete graph will definitely have one Hamiltonian circuit. For example here a 1 b 2 d 5 c 6 a is a Hamiltonian circuit a 1 b 2 d 5 c 6 a covers all the vertices a, b, c, d and it starts and ends at a, this is a Hamiltonian circuit. Can you have any other Hamiltonian circuit here? You can have a 4 d 2 b 3 c 6 a you have a b c d it starts and ends at a. Is it all are do you have more? Can you find some more Hamiltonian circuits? Look at this; a 1 b 3 c 5 d 4 a, a 1 3 c 5 d 4 a. This is another Hamiltonian circuit. Like that how many Hamiltonian circuits you can have? In a complete graph with n vertices there is a formula.

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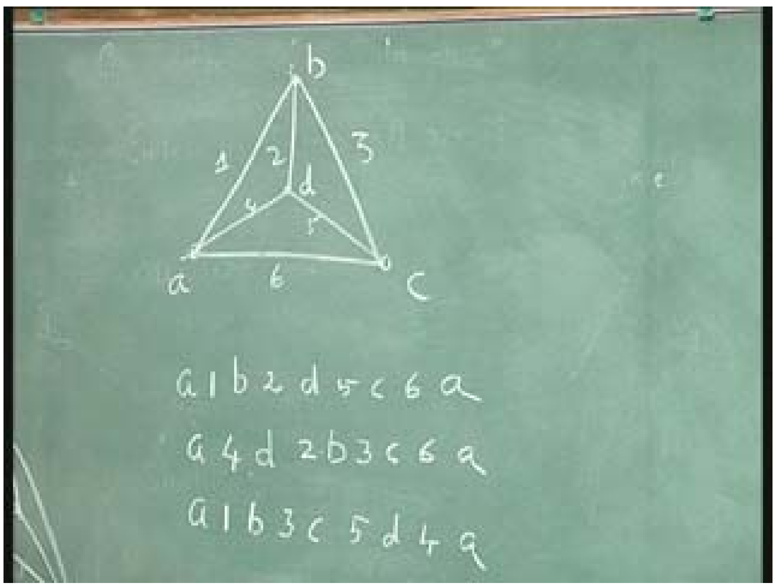

If you have a complete graph with n vertices they are also called cliques, another word used is cliques of size n. When a complete graph occurs as a subgraph you call it as a clique of size n.

How many Hamiltonian circuits can it have? It will have n minus 1 factorial by 2, how is this possible? If you have n vertices starting from one vertex you can go to anyone of the n minus 1 vertex next then from that vertex you can go to anyone of the n minus 2 you have to omit this. And starting from one of them you will have n minus 3 possibilities until you end at the same node. Like here a 1 d c b this is one possibility. But so this will of amount to n minus 1 factorial. But you will count each circuit twice. For example, a 1 b 2 d 5 c b and a 6 c 5 d 2 b 1 a is the same. You are traversing in this direction if you traverse in this direction it represents the same Hamiltonian circuit. So the total number of Hamiltonian circuits in a complete graph is given by this formula n minus 1 factorial by 2.

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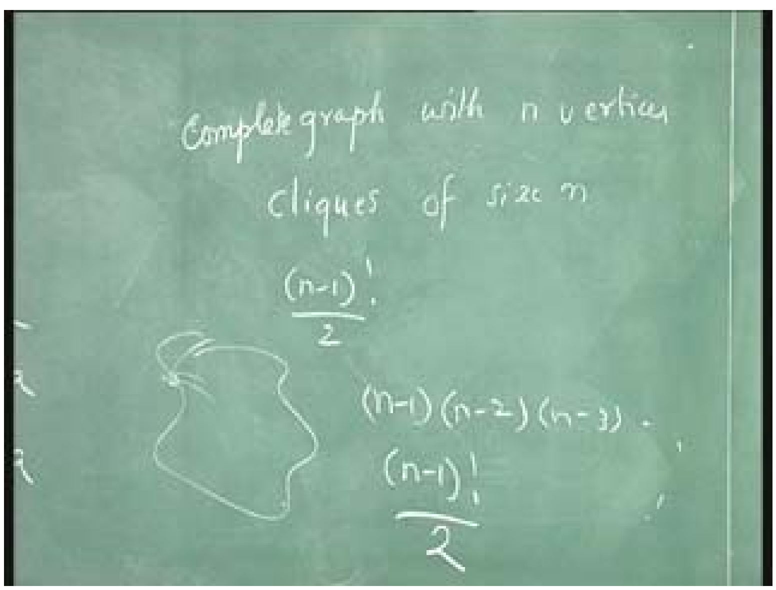

So we have learnt a few definitions and a few concepts about graphs. Tree is one of the special types of graphs. Trees are very useful as data structures. In many cases you use them for storing some data and for searching. We shall study a little bit about trees and search trees. You can look at them as directed graphs. Of course we shall also see in an undirected graph what is meant by a spanning tree. Trees can be defined this way, a tree is a digraph with a nonempty set of nodes such that there is a exactly one node called the root of the tree which has got indegree 0 every node other than the root has indegree 1.

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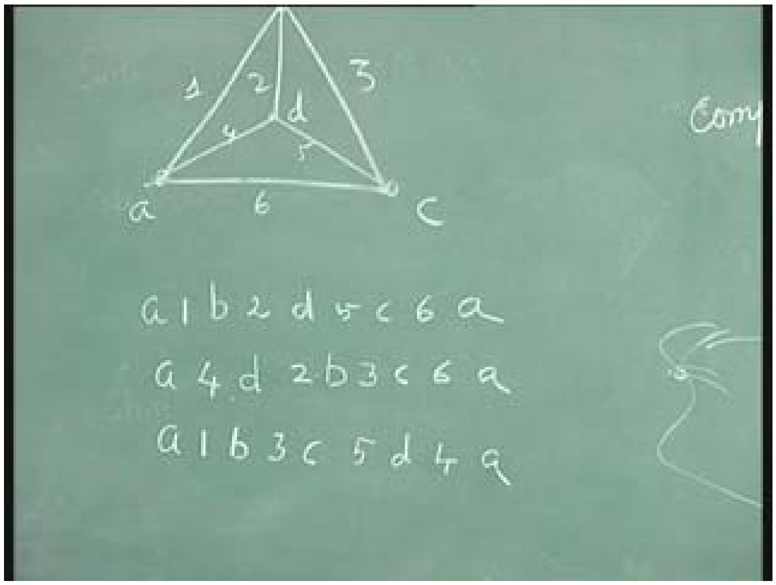

For every node of the tree there is a directed path from the root to a. We shall look at them as directed graphs. So there is a root in the case of undirected graphs we talk about the degree of the node.

In the case of directed graph we talk about indegree and out degree. What is indegree? The number of edges entering into that node is called the degree of the graph. The number of nodes leaving that is called the out degree. So the tree has a root which has indegree 0 and every other has some structure like this. Every other node will have indegree 1. And if you take any node a there will be a directed path from the root to that node a. So a tree has such a structure and it is used as a data structure in many cases. For example, look at this, this is the root it has got indegree 0 out degree may be anything it can have three something like this.

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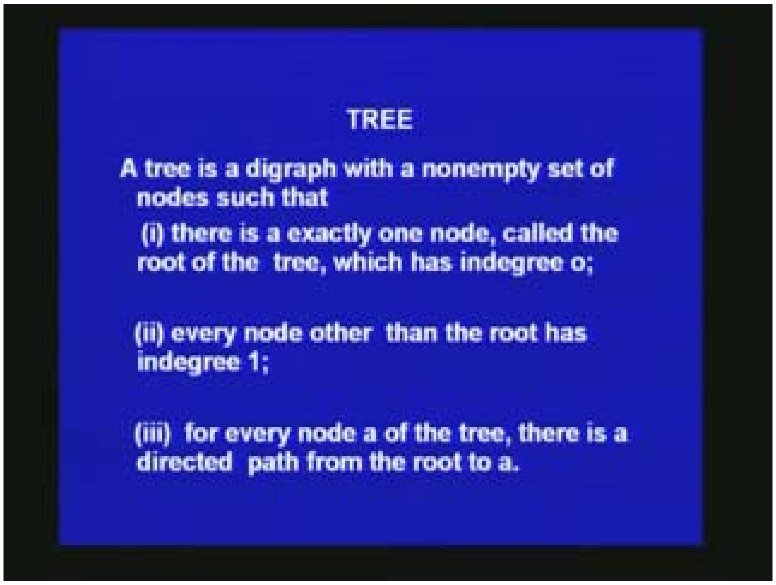

You may also have something like this. And if you take any node there is a directed path from the root to that node. If you take any other node it will have indegree 1 only one edge will be entering into that but the out degree can be anything. And here this is called the father and these are called the sons of that node. And similarly, if this is the father the sons are these nodes and this is called an ancestor of this node and this is called a descendant of this node. These are some of the terminologies which you commonly use when representing the trees.

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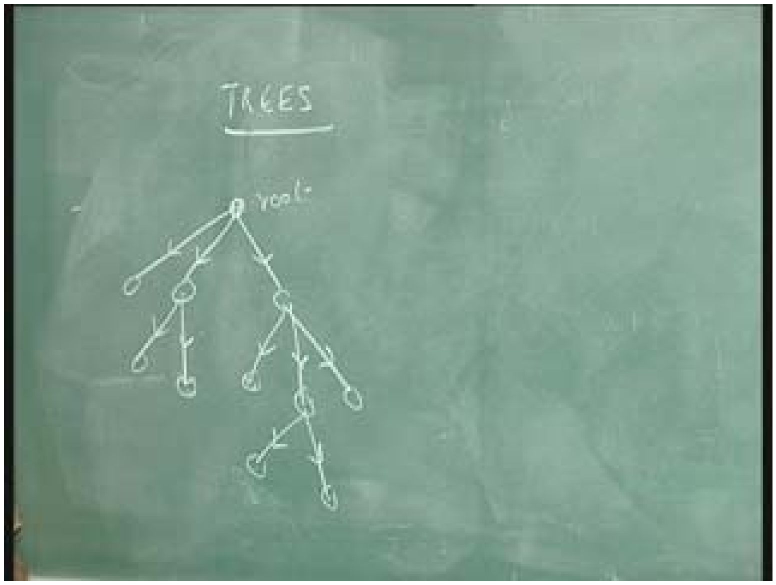

But the main fact you have to realize these are the three points, there is exactly one node which is called the root and the root has indegree 0. Every node other than the root has got indegree one and for every node a of the tree there is a directed path from the root to a.