Discrete Mathematical Structures

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Lecture - 14
Graphs

So we were considering about relations and in particular we considered binary relation. We also saw that a binary relation can be represented by a directed graph. Now we shall study a little bit about directed graphs and also how they can be represented in computers when you use them as direct as data structures. And we shall also consider a few results about undirected graphs. Now, recalling the definition of directed graph, a directed graph or a digraph is an ordered pair D is equal to A, R where A is a set and R is a binary relation on A. The set A is the set of nodes is called the points or vertices and the elements of R are the arcs they are called edges or lines of D. The relation R is called the incidence relation on D.

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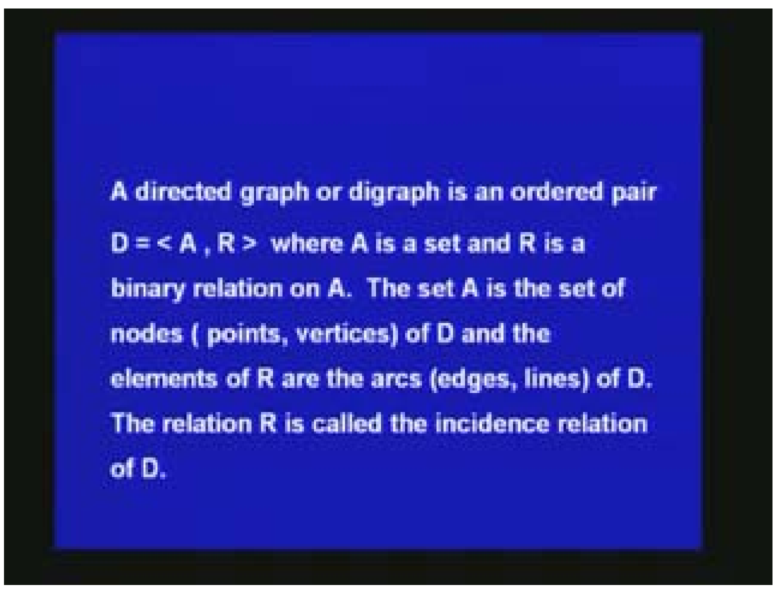

For example, let us consider a directed graph like this where the vertices are a b c and d. This is a directed graph. The vertices are a b c d or the set A is a b c d and relation the directed arcs are represented as a binary relation on A cross A, in this case the following order pairs belong to the relation, you have a, b so a b belongs to that, then you have a, c, you have b, c, you have c, c and you have d, c. So these are the edges of the graph and they represent a binary relation R on A cross A.

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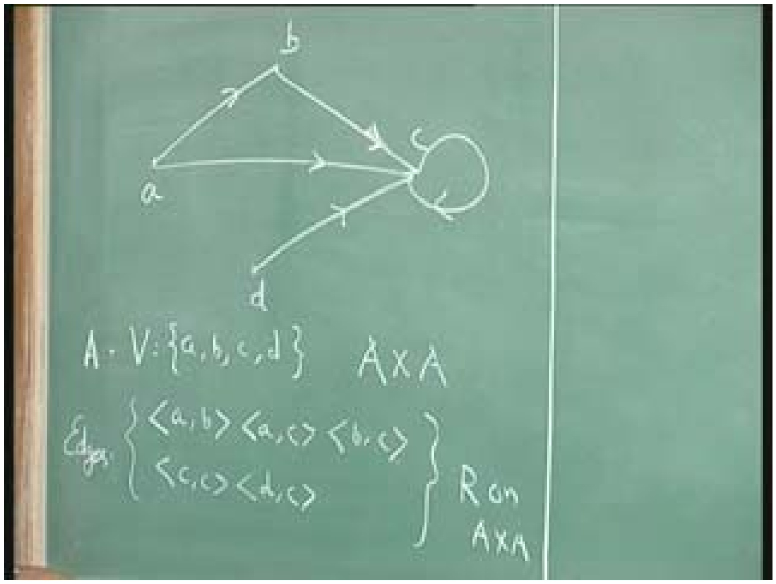

Now if A is a finite set you can represent like this. More over the relation R can be on A cross B where A and B are not the same. In this example it happened to be that the binary relation is on A cross A, supposing R is on A cross B. Suppose take A to be a, b, c and B to be say 1, 2, 3 and R consists of the following ordered pairs a,1 b,1 a,2 b,3 and b,2. Then you represented as a graph where the vertices are A union B and the edges are represented from A to B. So you represent like this, the binary relation of the graph will be represented like this 1, 2 and 3.

Each element of A will be a vertex, each element of B will also be a vertex and all the edges will originate from an element from A and terminate on element from B. So you have a,1 so there will be an edge like this, you have b,1 so there will be an edge like this, you have a,2 so there is an edge like this, you have b,2 so there is an edge like this and you have b,3 so there is an edge like this. So the binary relation will be represented by this graph where the vertices are A union B and the directed edges represent the ordered pairs. When there is an edge between the element a and then element b it originates at a and terminates at b. The number of edges originating from a vertex is called the out degree of that vertex and the number of edges entering in to the node is called the in degree of the vertex. For example here the out degree of a is 2 and the in degree of 1 is 2. We will see that.

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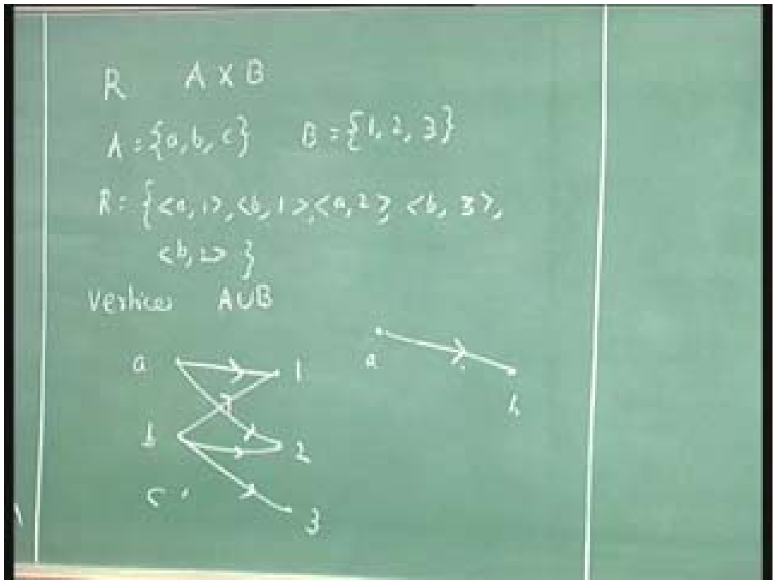

Let D be a digraph. If a is related to b then the arc a, b originates at a and terminates at b.

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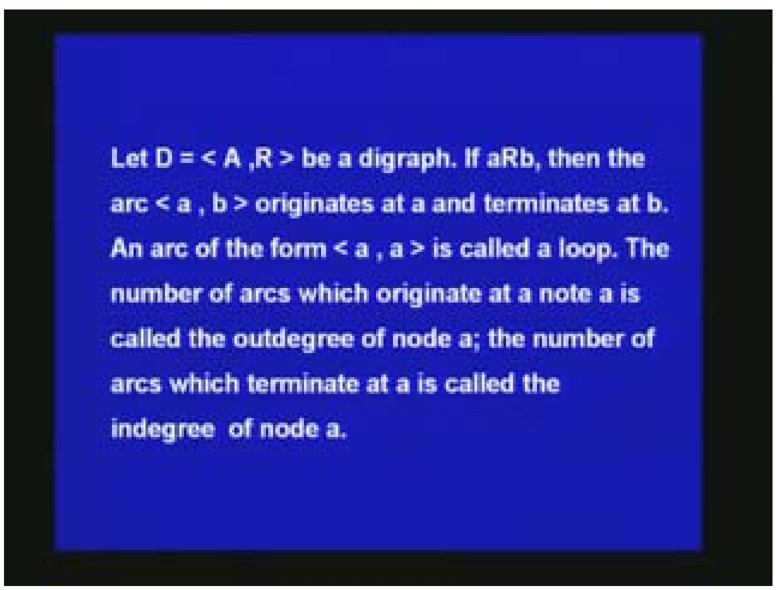

An arc of the form a, a is called a loop. The number of arcs which originates at a node is called the out degree of node a and the number of arcs which terminate at a is called the in degree of node a. The self loop is some thing like this, if there is some thing like this then it is called a self loop. Here it is bb, bb this will be represented by a loop like this. Now this is all possible where you have a binary relation on a finite set a.

Suppose you have a binary relation on a infinite set, let us take the set of natural numbers n is equal to 0, 1, 2, 3 and a binary relation a is related to b is defined like this, if and only if b is equal to say a plus 3.

So how will you represent this infinite set and infinite relation?
We cannot draw the graph completely but we can represent a finite portion like dot dot dot. So, take for 0, 1, 2, 3 and 4 it will follow a similar pattern here. a and b are related if b is a plus 3 so 0 will be related to 3 which you can represent like this and 1 will be related to 4 which can be represented like this and 2 will be related to 5 and that can be represented like this and the pattern will follow. So even though we cannot represent relations on infinite sets completely, we can represent a small portion and the rest of it can be represented by dots which will give you the picture of how the relation is defined.

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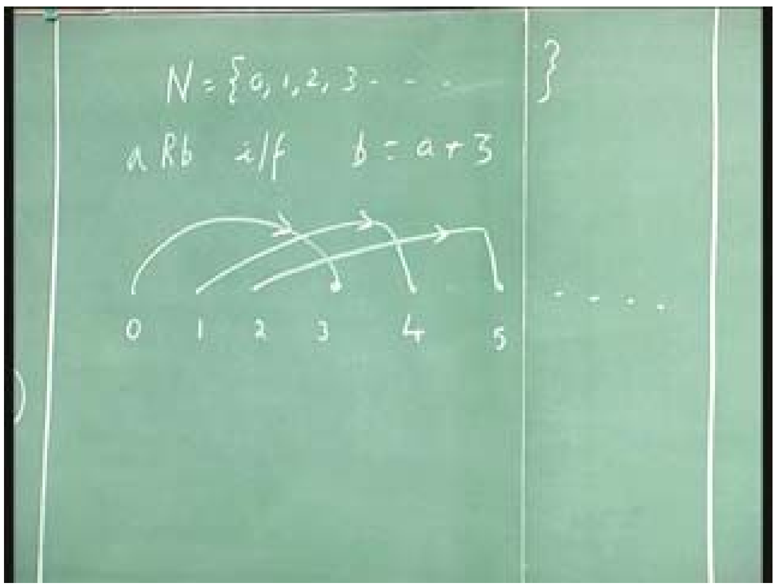

Now let us see some more definitions about this. Let D is equal to A, R be a digraph with nodes a and b and undirected path p from a to b is a finite sequence of nodes c_0 c_1 c_2 c_n such that c_0 is equal to a and c_n is equal to b. For all c_i such that i is between 0 and n either c_i is related c_i plus 1 or c_i plus 1 is related to c_i.

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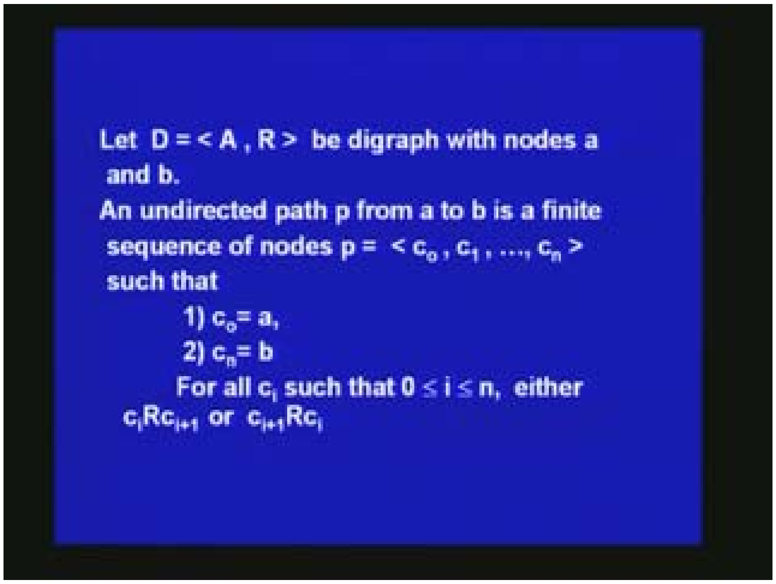

Let us consider this same example. This is a directed graph. You have an edge from a to b and you have an edge from b to c and you have an edge from c to c. a to b, b to c and c to c, this is a directed path. This is c_0 which is the path from a to c. It starts at a and ends at c. a b c is a directed path. So in the notation this is c_0 c_1 c_2 and so on.

There is an edge between c_0 and c_1 and c_1 and c_2 and so on. If you consider a c d, from a to c there is a directed edge like this but from c to d it is directed in the reverse direction, so a c d is an undirected path, it is a path but it is not directed, so it is called undirected path. Here if you represent a c d as c_0 c_1 and c_2 from c_0 to c_1 you have an edge but the other way it is c_2 to c_1 and this you must understand. And then the notation and terminology in graph theory vary from book to book. So I will try to explain with the particular notation and terminology but when you come across any other book the notation and the terminology may differ a little bit, so you have to be careful about which book you are looking at and what is the notation followed by them or what is the terminology followed by them.

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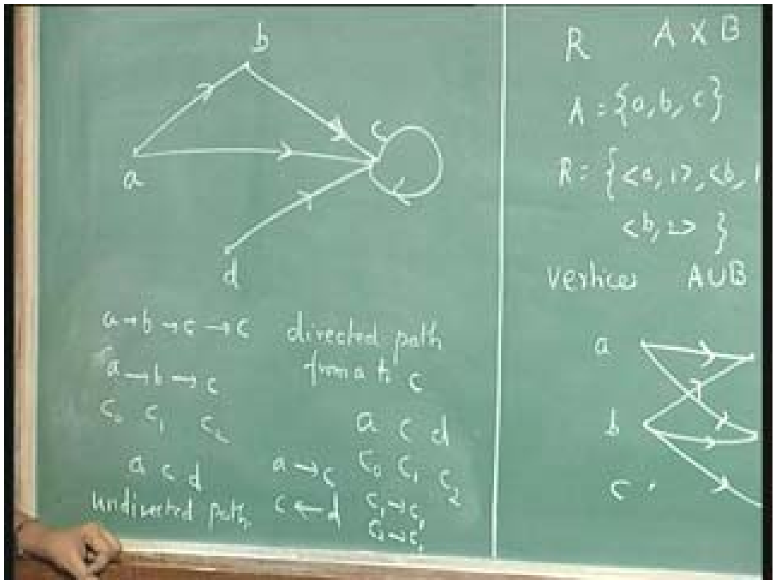

Let me take one more graph as a b c and d. Now look at this, a c b a c d is this a path? It is a path a c b a c d is a path but it is an undirected path because in the last step d to c it is not directed in the usual way but it is in the other way, it is directed from d to c. Suppose I consider a c b a c, this is a directed path. But the node appears twice in that and c also appears twice. If any node appears only once in a path then that is called a simple path.

For example, a c b is a simple path, no node is repeated here. It starts at a and ends at b. If the start node and the end node are the same it is called a directed cycle. If c_0 is equal to c_n it is called a directed cycle. For example, if you take a c b a it is a directed cycle, a c b b a is also a directed cycle. This is not a simple cycle because b is repeated twice a c b b a is a cycle but it is not a simple cycle. This is a simple cycle, this is not a simple cycle. So you use the word cycle and some books will use the word circuit. These are some other terminologies.

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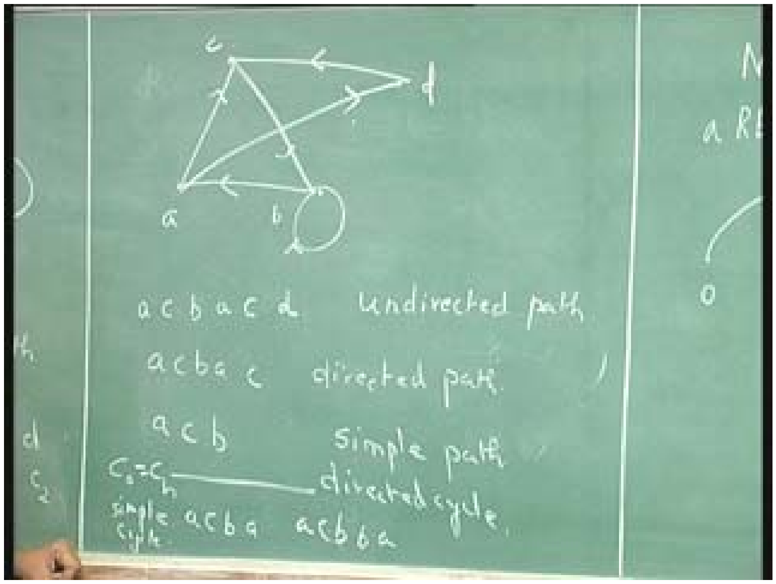

How are graphs represented in computers?
Let us take the same graph with four nodes and directed in this manner.
How will you represent this graph?
When you look at it as a data structure graphs and trees which are particular classes of graph are very useful as data structures. Now, there are several ways of representing them internal in the computer. One is known as the adjacency matrix. Now there are four nodes, it will be represented by a 4 by 4 matrix. In general if there are n nodes, the adjacency matrix will be represented by a n by n matrix.

Now let us see how this graph can be represented?
You have a b c d in the column and you also have a b c d in the row. Now from the i th row and the j th column this element is one if there is a edge between b and c. So that is from this to this there is an edge it is represented by 1, Boolean 1. This is a Boolean matrix represented by 0's and 1's.

Now look at this, from a to c there is an edge so this will be represented by 1, from a to a there is no edge so that is 0, from a to b there is no edge so this is 0, from a to d there is an edge so that is represented by a,1, now from b to a there is an edge so it is represented by 1, from b to b there is a self loop that is represented by 1 and from b to c there is no edge so this is 0, from b to d also there is no edge, from c to a there is no edge, from c to b there is an edge, from c to c no edge and c to d also there is no edge, so it is like this. From d to c you have an edge but you do not have any edge from d to any other nodes, so it is represented like this.

Now you can see that there are six edges and in this Boolean matrix there will be six 1's. So if an element is 1 for example it is 1 here, that means it is occurring in the row corresponding to c and the column corresponding to b, that means there is an edge from c to b or the binary relation represented by the graph contains c comma b the ordered pair if the element corresponding to the row c and column b has 1 in that. This is one way of representing and this is called adjacency matrix of the graph or the relation represented by the graph. There are other ways of representing. Especially if this is a very sparse matrix it may occupy too much of space so other methods are followed.

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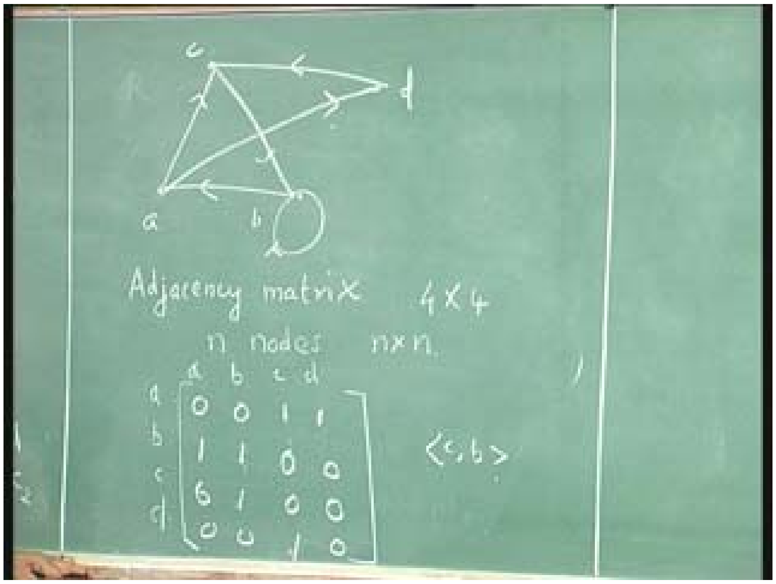

For example, you can represent all the edges listing from a. The vertices are taken in this order say a, b, c, d. So see what are the edges leaving a. a comma c is the edge leaving a and a comma d is also the edge leaving a. Then from b you have b, a you have b, b and then you have edges leaving c c, b and then d, c. You can list the edges in this manner; this is one way of representation. And if you see the correspondence between this and this, as you are listing the elements row by row and an each element represents an ordered pair in the relation.

So first take the first row where you have a to c and a to d and then the second row you have b to a b to b, then you have c to b in the third row and in the fourth row you have d to c, you represent them like this. There is one more method of representing this which is called the linked. This is called the ordered pairs and here you list all the ordered pairs. Another one is linked list method. So each node will be represented like this for example a will be represented like this. There is an edge from each…. I will represent b like this, c like this and d like this. From a there is an edge to c which will be represented by this and then there is an edge to d which will be represented this.

From b there is an edge to a and there is an edge to itself, from c there is an edge to b and from d there is an edge to c. The whole thing can be represented in this manner and this is another way of representing the graph internally.

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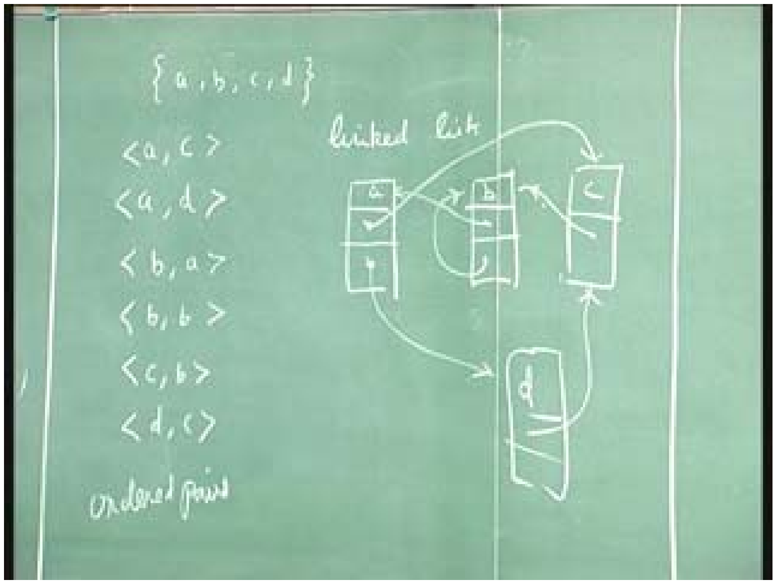

Now will come to the definition of what is meant by a sub graph and a partial sub graph? Let D equal to A, R be a digraph. A digraph D dash equal to A dash R dash is a sub graph of D. If A dash is contained in A and R dash equal to R intersection A dash cross A dash. If D dash is not equal to D then the D dash is a proper sub graph of D. A digraph D dash equal to A dash R dash is a partial sub graph of D. If A dash is contained in A and R dash is contained in R intersection A dash cross A dash. So let us consider some examples, then it will be come more clear.

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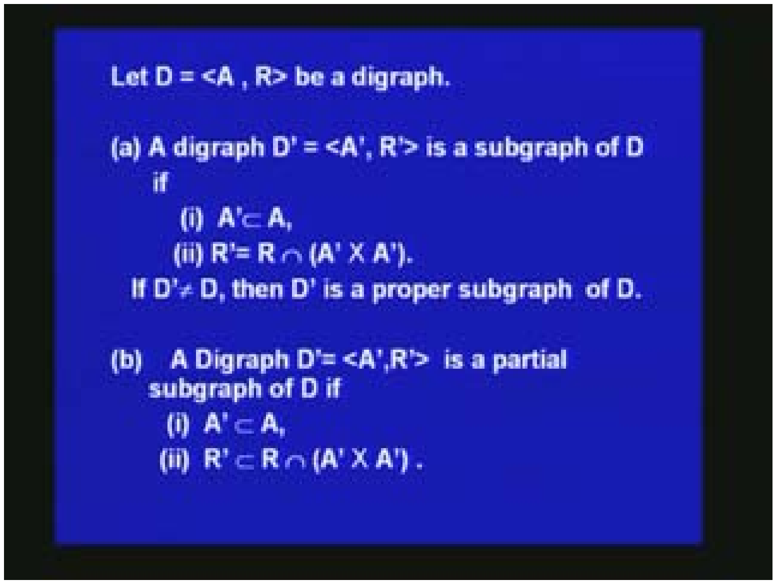

Take some graph like this, a b c d, this is a directed graph. Now A consists of a, b, c, d. Take A dash which is contained in A to be just a, b, d then this is the sub graph. You consider the nodes which are present in A dash and all the edges between them which are originally in the graph G, this is G and a b, this is directed like this. This G dash is called the sub graph. You are removing c and then all the edges connected to c. You are considering only the vertices a b d and all the edges incident on a b and d. This is the first definition that is R dash is equal to R intersection A dash cross A.

The second one says, when do you call it as a partial sub graph? For example here you may have a b and d but you may not have the all the edges between them and you may just have some of them like this. This is a sub graph and this is a partial sub graph. This is according to this definition.

Again as I mentioned to you the definitions vary from book to book. In some books this is called a sub graph and this is called a partial sub graph. In some other books this will be called a sub graph, you can take any subset of the vertices and any subset of the edges incident on those vertices. Here you have the subset of the vertices but whatever edge was incident on them in the original graph is definitely present here. This is called induced sub graph. So as I told you again when you refer to a book you have to carefully find out what terminologies they are following. Either this is called a sub graph and this is called a partial sub graph or this is called a sub graph and this is called an induced sub graph.

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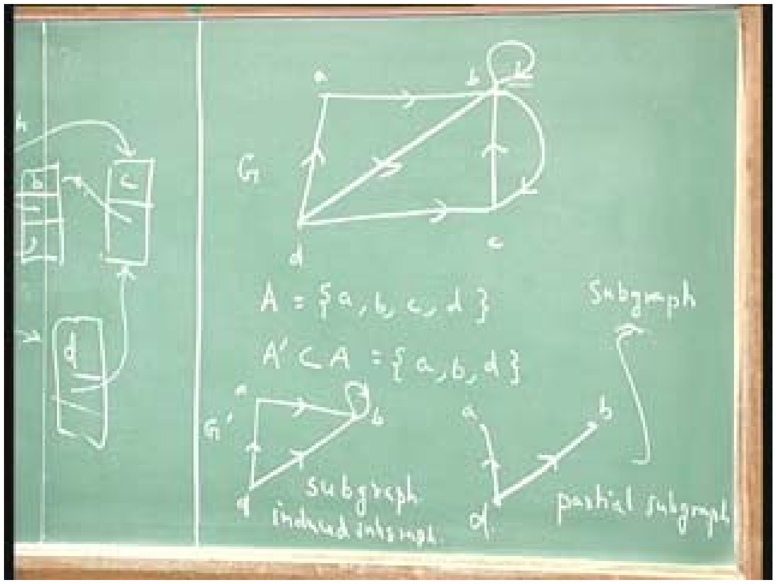

Now, suppose let me take set A again a comma b comma c. And R is a relation it is a subset of A cross A. Now suppose it is the empty relation it will be represented by a graph with just vertices and no edges between them. This is called the empty graph. And there is no edges, there are vertices but no edges between them. A cross A is the universal relation and the graph representing the universal relation A cross A is called a complete digraph. All ordered pairs are present there.

For example, when you take a b c all edges connecting any a to any b should be present there. Every node should be connected to every other node including itself. So this is called a complete digraph on three vertices. On two vertices a complete digraph looks like this. If you have just two vertices, if you have just one vertex this is a complete digraph on one vertex. This is the complete digraph on two vertices and this is the complete digraph on three vertices.

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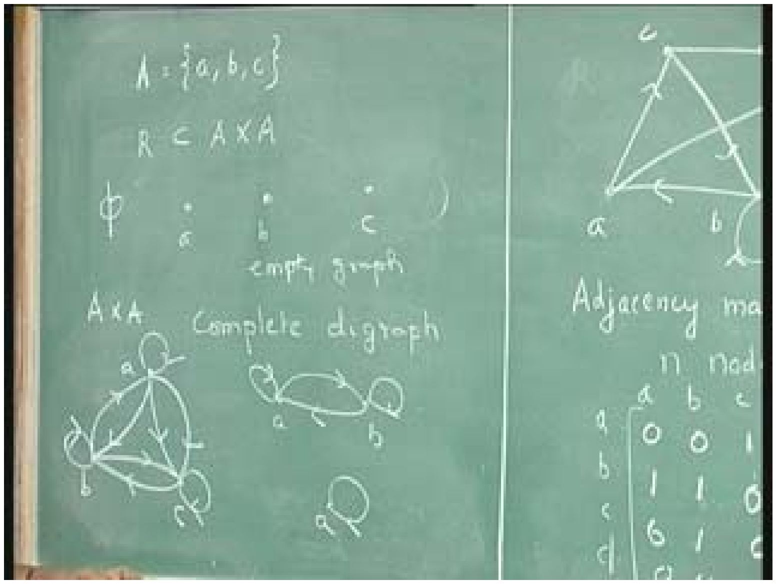

When do you say a graph is strongly connected? And when do you say it is connected and when do you say it is not connected?
What do we mean by strongly connected graph, connected graph and disconnected graph. When there is a directed path from every vertex to every other vertex. When you have a directed path from every vertex to every other vertex it is called strongly connected. Look at this graph, there is an edge like this, from a there is a directed path to b, there is a directed path to c, there is a directed path to a itself. Similarly, from b there is a directed path to c, directed path to a and there is a directed path to b itself. This is a strongly connected graph.

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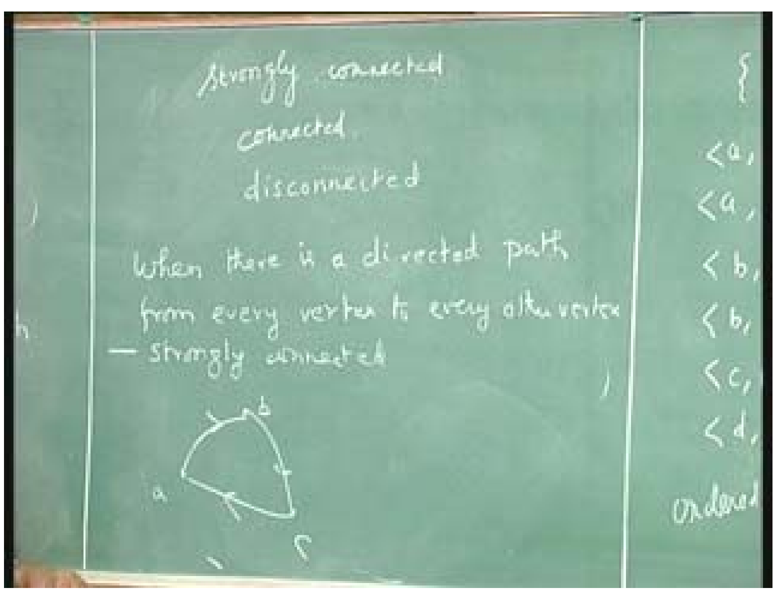

Now look at this. When there is a path it may be directed or undirected from every vertex or node, the node and vertex denotes the same from every vertex to every other vertex the graph is called a connected graph. Look at this graph, there is a directed path from a to b and there is a directed path from a b c but in this way it is in the opposite direction. You do not have a directed path from here to here or if you take from c to a you do not have a directed path but there is an undirected path. This graph is a connected graph but it is not a strongly connected graph.

Now look at a graph like this. There are some paths between a and b, between c and d. There is no path from this portion to this portion and there is no edge from this portion to this portion. The whole graph consists of four nodes a b c d. And this portion is not connected to this portion by any edge and such a graph is called a disconnected graph. This is a disconnected graph not a connected graph. If this condition is not satisfied, that is the first two conditions either if it is not strongly connected or connected, it is disconnected. But in this disconnected graph this portion is connected and this portion is connected. They are called connected components of this graph. So this graph has two connected components: this is one and this is one.

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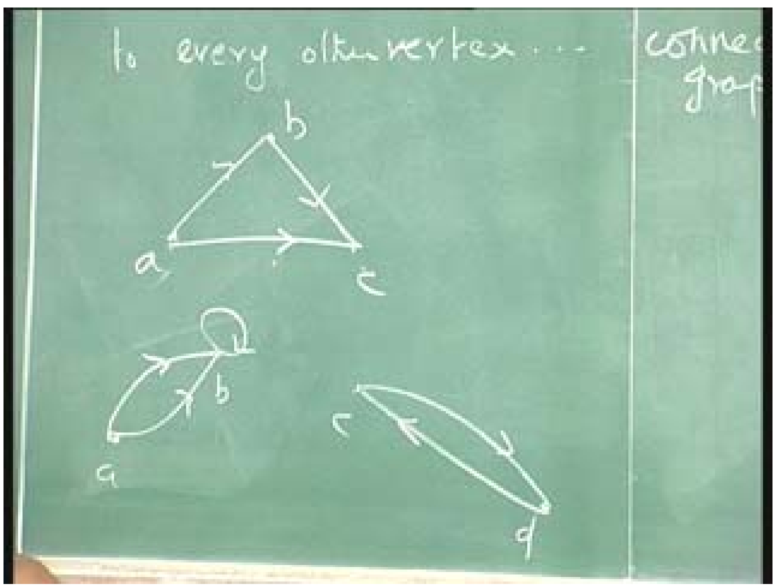

Now we have learnt a little bit about binary relations and how they are represented as directed graphs and so on. Generally in graph theory you deal with edges which are not directed. So you talk about undirected graphs. Having started on some concepts of graph theory, let us see what an undirected graph is and some of the terminologies associated with undirected graphs and so on.

Next we shall consider a few terminologies related to undirected graphs. Here again a graph means an ordered pair V and E. V is the set of vertices and E is the set of edges. For example, it can de represented as a b c d. And now, there is no arrow here, ab is an edge but it is not directed, it is between a and b. So that order is not important when you consider undirected graph. They are not directed graphs but they are undirected graphs.

Generally when you say graphs you mean undirected graphs. If it is directed you have to specifically say directed graphs. So a graph can be represented as some thing like this. This is a graph. You may also have some thing like this. It may have multiple edges between c and d, you may have self loop like this. So you may have multiple edges that is between c and d you have three edges here and you may have self loops. If you avoid this multiple edges and self loops it is called a simple graph. If you do not have multiple edges or self loops the graph is called a simple graph.

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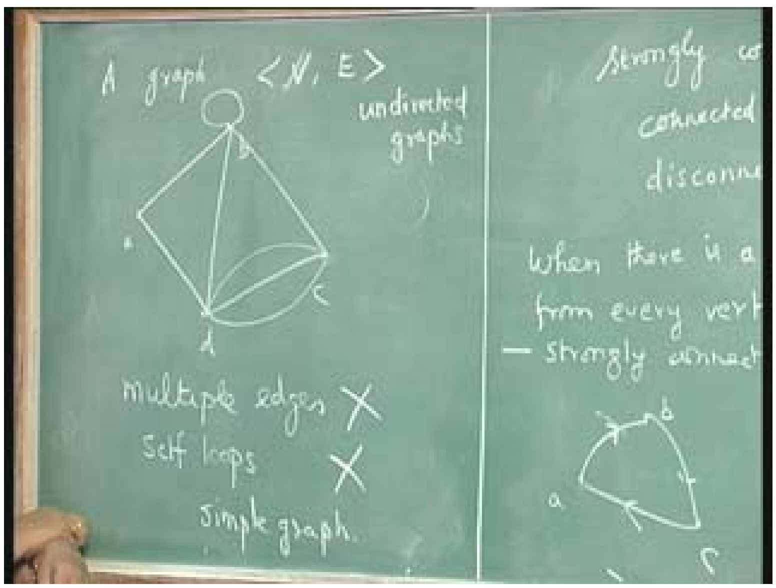

Now generally a graph represents some connection between the nodes. It is mainly the connection between the nodes which is represented. For example in a computer network each system may be represented as a node and which system is connected to which node and that can be represented as a edge and so on. It is not necessary that the lines joining them should be straight lines or curves or something like that and the distance between the two node should be some thing and so on. That is not at all necessary, it is an abstract model, which node is connected to which node.

Now I can draw something like this; this is a graph a b c d, everything is represented by the straight line and also the shortest line between the two nodes. But that need not be the case that I can also represent the same graph like this a b c d. It is a connection that matters. b is connected to d, a is connected to c and so on. The lines can be curved or straight which does not matter. So they represent the same graph. And when you have something like this, this is called a pendent vertex or pendent node.

Here we talk about the degree of a vertex. What is the degree of a vertex?

In the case of directed graphs we talked about in degree and out degree. Here there is no such thing. When you have undirected graphs you only have the degree. The degree of a vertex is the number of edges incident on them. For example here, what is a degree of a_3, degree of b is also 3 and degree of c is also 3 and degree of d is also 3. And if you look at this matrix, I will draw like this, if you look at this graph a b c d e, what is the degree of a? Degree of a is 4, degree of b is 2 and c is 3 and d is 2 and degree of e is 1. This is called the degree of a vertex. Now you can see that if there are vertices v_1, v_2, v_n and let dvi represent the degree of v_i.

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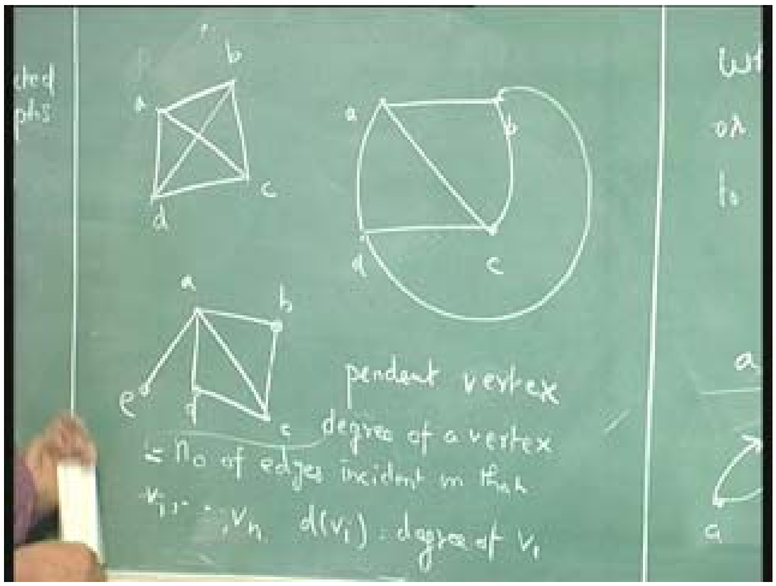

Consider a graph V E consider a graph G is equal to V E where V is equal to v_1, v_2, v_n. There are n vertices in the graph and the number of vertices is n. Let the number of edges be e, the number of edges is e, then $d(V_i)$ is the degree of vertex v_i. Then you will realize that sigma $d(V_i)$ i is equal to 1 to n is 2 e twice the number of edges. This is very obvious but let us verify again with the examples which we have considered already.

Look at this example: How many edges are there?
There are six edges. And what is the degree of each of these vertices. Each one is of degree 3, so some of the degrees is four times three twelve and the number of edges e 6 into 2 is 12 and each vertex is of degree 3, there are 4 vertices so that is true, they are equal. And look at this how many edges are there?
There are 6 edges. Calculate the degree of each vertex. a has degree 4, b has degree 2, c has degree 3, d has degree 2 and e has 1. This sum comes to 12 and you have 6 edges so 6 into 2 is 12.

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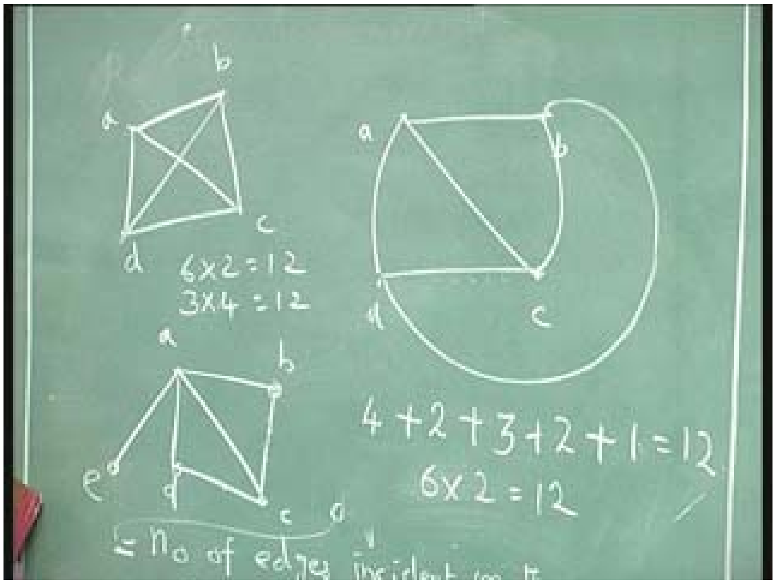

Why? It is because each edge is incident on two vertices. So each edge will contribute to the degree of two vertices. When you calculate like that there are e edges, each edge will contribute to the degree of two vertices. So the sum of the degree of vertices will be 2 e. This is a very simple result about graphs. Now we have seen how to represent directed graph using adjacency matrices.

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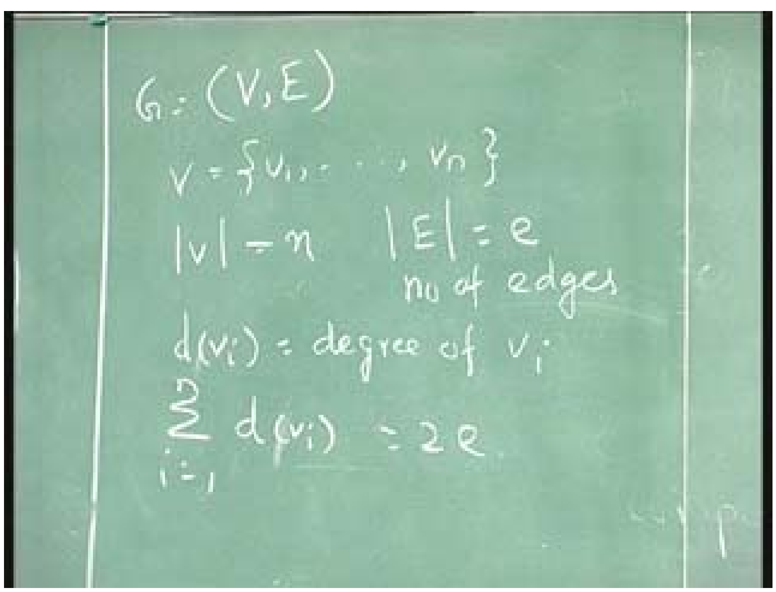

How do you represent undirected graphs? Again let us take this example. How do you represent like this by adjacency matrix a b c d, a b c d. There is an edge between a and b you can also look at it as an edge between b and a, there is an edge between b and c, there is no self loop here, if self loop is there you will have 1 in the diagonal, there is an edge between a and c so 1, there is an edge between a and d that is again 1, there is an edge between b and a, b and c and b and d like this. This example happens to be like this.

Now there are 6 edges and if you count the number of 1's there will be 12 1's twice and it is a symmetric matrix because it is symmetric. This matrix is symmetric about the diagonal because here there is no direction between a and b so it will be represented as a edge between a and b and also between b and a. That is why it can be represented like this. Now in a multiple graph you may have several edges between two nodes and in that case instead of representing as a Boolean matrix you can represent by an integer there. If the i, j th entry is number 3 that means there are 3 edges between i and j or between j and i.

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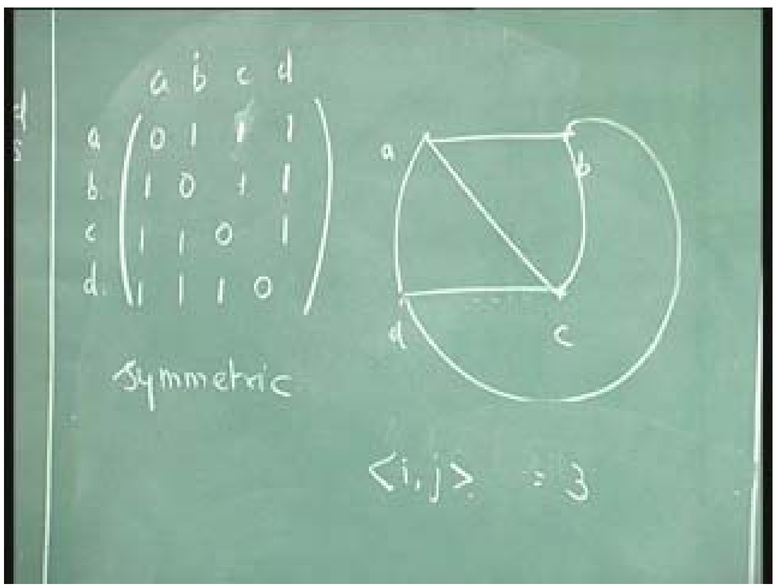

Now the whole graph theory started when Euler defined or he solved a problem called the Konigberg bridge problem which was considered by Euler in the 18th century. And what was that problem? In a city in the east Russia there was a river flowing like that which created two islands like that. And the landscape was divided like this A B C D. This is the river there were bridges like this, 7 bridges where there like this.

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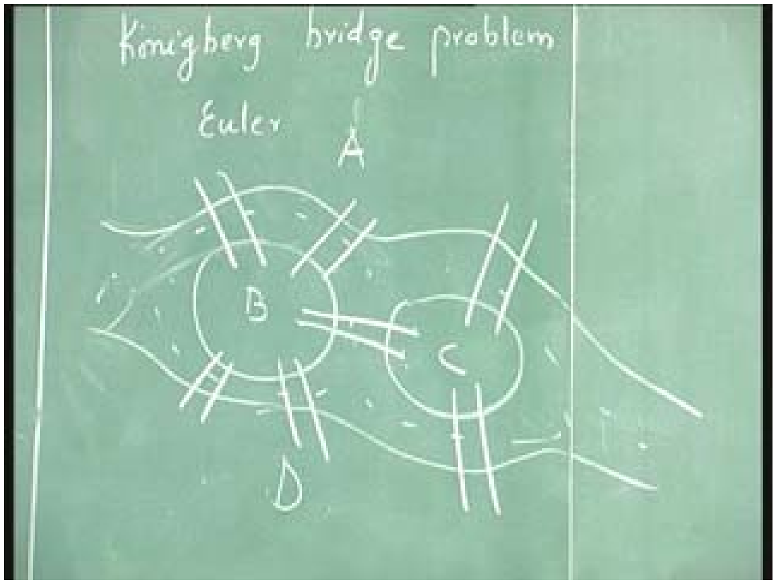

The question asked?
Can you start from any one of the portions A B C D. Walk through each one of the bridge exactly once and reach the same portion A where ever you started from. Then Euler proves that it is not possible to do that with this sort of an arrangement of the bridges.

How did he prove that?
He represented this as a graph like this A B each of the land portion, the two islands and the two banks, B and C are the islands and A and D are the banks. Now there are two edges between A and B that was represented like this. There are two edges between B and D that is two bridges, so two edges between B and D, between C and D there is one edge, between C and A there is one edge and between B and C there is one edge. The whole thing can be represented by a graph like this, it is an undirected graph.

Now the question was can you start in any one of the positions, pass through an edge exactly once and reach the same position. This is also something like drawing without lifting the pencil where you draw from one node to another node and then reach a same node without lifting the pencil or the pen. In this one you cannot do like that and why is that? It is because this is not an Euler graph. For that we have to consider what is meant by a walk and that walk is an Euler, what is mean by walk? What is mean by an Euler walk and so on?

So far in the examples which we considered we did not give labels for the edges. We could also label the edges. For example, the bridges can be numbered from 1 to 7. The 7 bridges can be labeled like this. Then a walk is like this: A1B3C7D represents a walk. You can start from A and reach D by going through 1, 3 and 7 like this.

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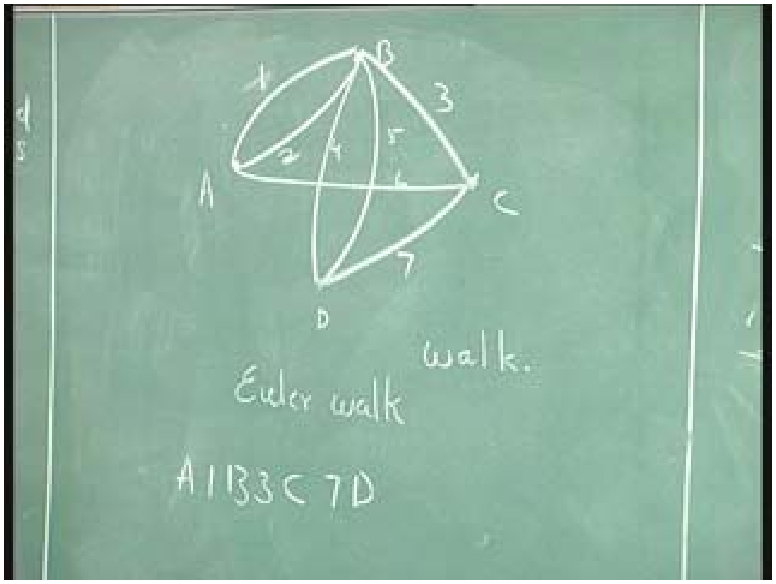

Let us consider one more problem which is called the utilities problem. There are three houses H_1 H_2 and H_3 and they have to be provided with utilities like gas water and electricity. So gas has to be provided to each house, water has to be provided to each house and electricity has to be provided. Now the connections have been made in such away that there is no crossing of the tubes and so on. So you have to do it at different levels.

Suppose you do it at the same level and then if you do like this there are crossings. For example this crossing you can avoid by connecting like this. Can you do every thing in such away that there is no crossing? It has been shown that it cannot be done this way but there will be at least one crossing. And such studies also a play a part by the use of graph theory and you will call it as what is known as a planar graph. If it is a planar graph you can draw the edges without one crossing the other. If it is a non-planar graph you cannot avoid crossing.

In this case we cannot avoid crossing. If you have a graph like this, the same thing you can draw without crossing like this. These edges can be like this from these two edges you can draw like this and without crossing you can draw this graph. This is a planar graph because you can draw that without crossing. Whereas this you cannot do like that because some where there will be some crossing, at least one crossing will be there.

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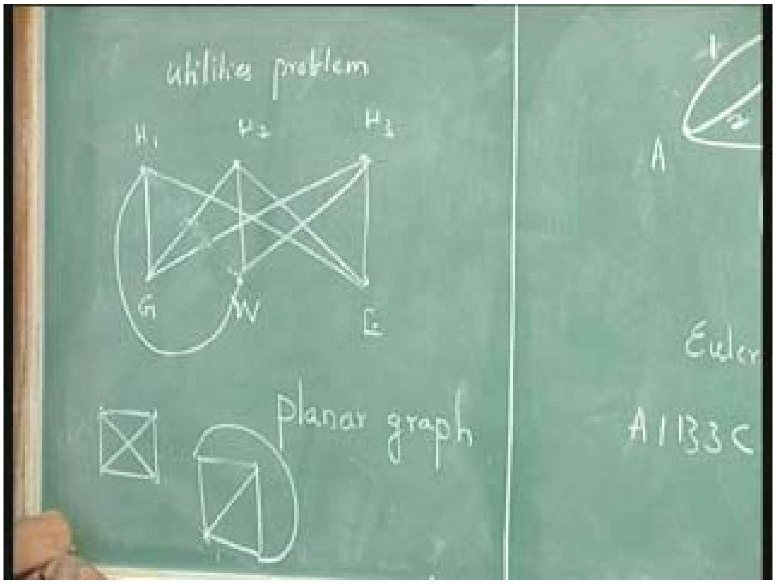

For example suppose you try to do like this, this one this one and this one and this like this, this, like this and this, like this, this one like this, this like this. But then how do you draw this? If you come across like this here you have to cross, if you go like this you have to cross, some where you have to cross, at least one crossing will be there. So this is not a planar graph and we have to study properties like planar graph and so on.

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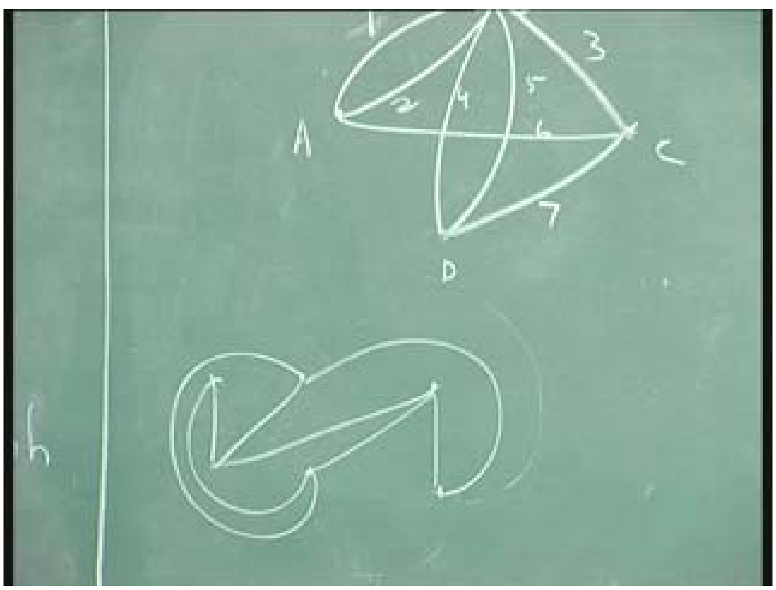

So first we shall study about what is mean by an isomorphic graph?
And then we shall study about the degree of a vertex, some terminologies like walk, path etc. And then see what an Eulerian graph is? What constitutes in Eulerian line?

We shall also consider what is meant by a Hamiltonian path and so on. And what application it has got. Hamiltonian path has lot of applications for practical purposes and we shall see what the application is.