Discrete Mathematical Structures
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Module -2
Lecture #11
Induction

Today we shall consider proof by induction. We want to prove some property P(x) for all (x), p(x) for a underlying set. If an underlying set can be inductively defined then we can use proof by induction. Let us call what is meant by inductive definition of set? Inductive definition of set has three parts. One part is the basis or the basis clause which tells you what the basic building blocks of the set are. The second part is the induction or the inductive clause which tells you how to build more and more elements of the set from already existing elements of the set. It establishes the ways in which elements of the set can be combined to obtain new elements.

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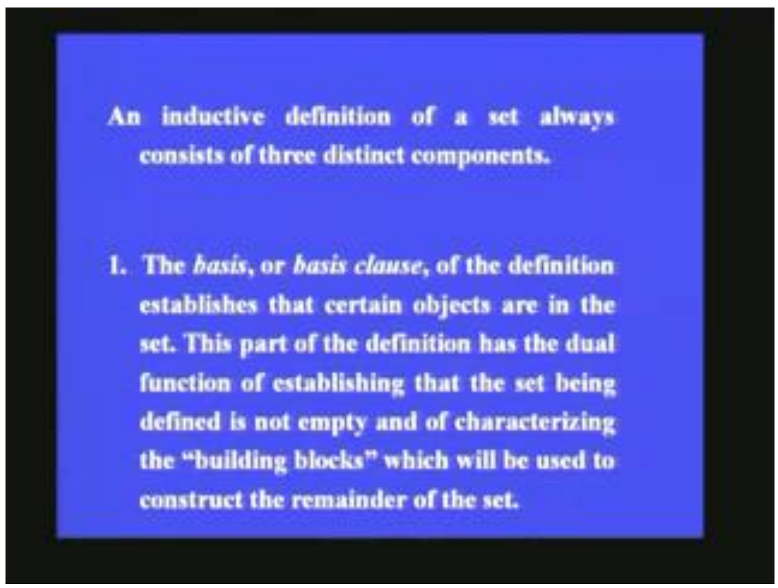

Apart from that there is an another clause called the extremal clause which is the same for any definition which tells you that no object is the member of the set S unless it is being so follows from a finite number of applications of the basis and the inductive clauses. Or it can be said in different ways also as the set S is the smallest set which satisfies the basis and the inductive clauses.

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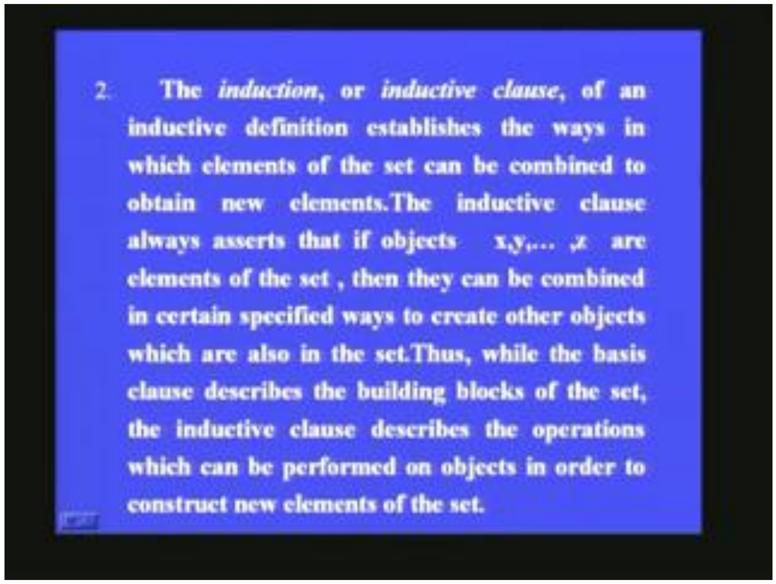

This extremal clause is the same for all inductive definition of sets but it has to be mentioned explicitly. Now let us consider proof by induction. Let us take some inductively defined sets.

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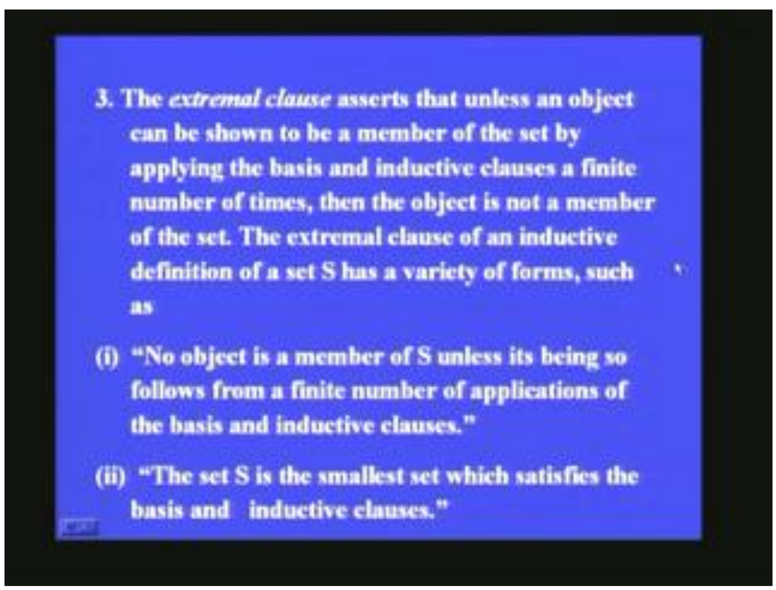

For example, in the last class we considered well formed strings of parenthesis. How did we define this? In basis clause if you call it set B and the induction clause is if x, y belong to B then x belongs to B and also xy belongs to b. And then you have to mention the extremal clause.

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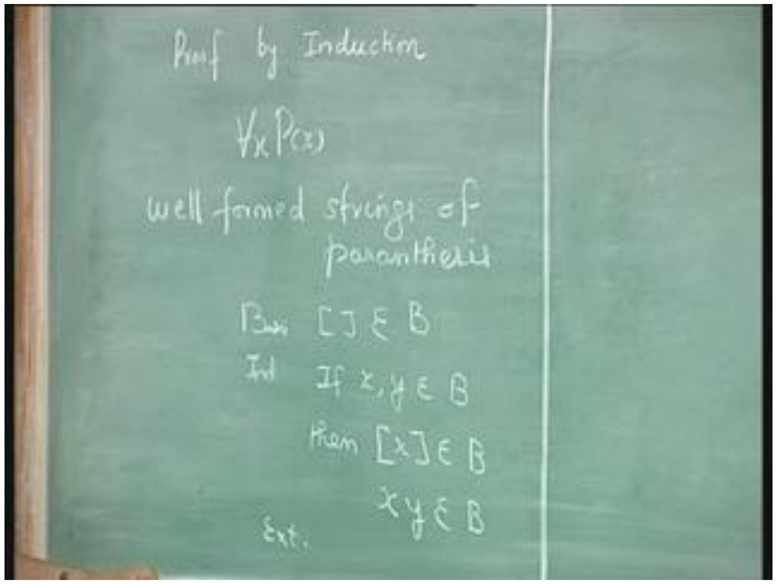

Now we want to prove some property of this clause. Let x be a well formed string of parenthesis L(x) denotes the number of left parenthesis, R(x) denotes the number of right parenthesis. Now you want to show that if x is a well formed string of parenthesis then L(x) is equal to R(x). The number of left parenthesis is equal to the number of right parenthesis which you know already but let us see how to proof it by induction.

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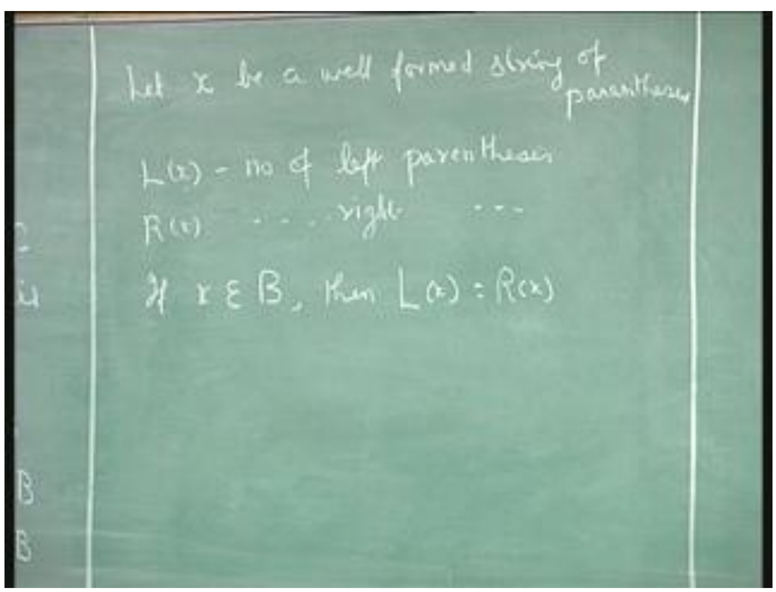

The proof by induction has two parts namely the basis part and the induction part. The basis part is like this, you prove this property for the elements which belong to the basis clause. So here this is the element which belongs to the basis clause. And more and more elements are built by using this in the induction clause. So in the basis clause of the proof you prove the property for this and in the induction clause of the proof you prove the property for this assuming that the property holds for x and y. Let us proceed now. In the basis clause this is the basic building block of the set and for this you know that L(x) is equal to 1 and R(x) is equal to 1. There fore L(x) is equal to R(x).

Now in the induction clause you assume that the results holds for x and y. Let x and y belong to B, there fore the property holds for them. Therefore L(x) is equal to R(x) and L(y) is equal to R(y). Now how more and more elements are built from this? Now the definition says that this belongs to B. Now what is the number of left parenthesis of this? The number of left parenthesis of this is the number of left parenthesis in x plus 1. And what is the number of right parenthesis here? The number of right parenthesis in this is number of right parenthesis of x plus 1 and you can see that these two are equal. So you are proving the property for the induction clause.

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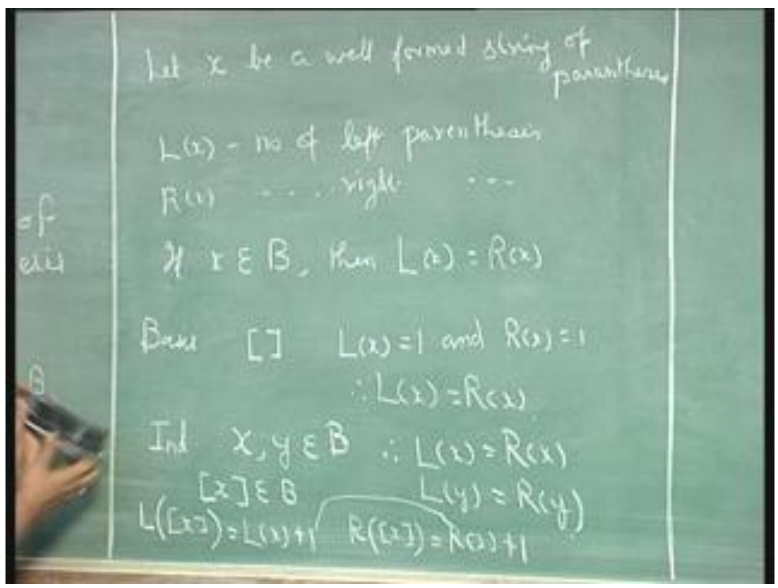

Now the other part is, you also know that xy also belongs to B, if x belongs to B and y belongs to B, xy also belongs to B. This is the way we define the induction clause. Now for x and y this holds. The number of left parenthesis of x is equal to number of right parenthesis of x and similarly for y. Now what can you say about the number of left parenthesis of xy? This equal to number of left parenthesis of x plus number of left parenthesis of y. And what can you say about the right parenthesis of xy? That is equal to right parenthesis of x plus right parenthesis of y. And you know that this is equal to this and this is equal to this, so these two are equal. So we are proving the property for this also. And how are the elements built from the basic building blocks. In the induction clause you say that if xy belongs to B this belongs to B and xy also belongs to B and for both these you are able to prove using induction. This is the way the proof by induction goes.

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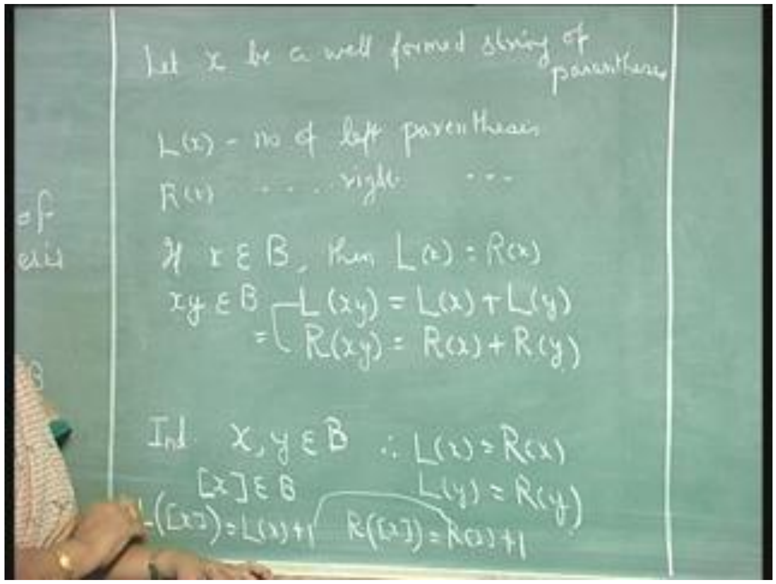

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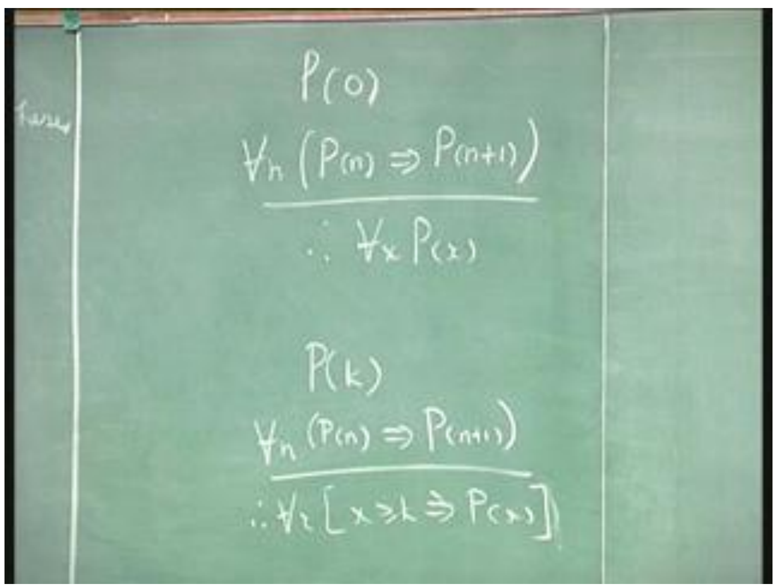

The main part is the underlying set has to be inductively defined. Now we know that you can define the set of non-negative integers or natural numbers inductively or we can take this as the underlying set. And if you want to prove some property for this you can prove P(0) and then you can prove that if it holds for n it holds for n plus 1 that is some property. And from this you can conclude for all (x) P(x). This is what you have to prove. What you have to prove is P(0) and then for all n P(n) implies p of n plus 1, then from this you can conclude for all (x) P(x).

So let us take one or two proofs in this manner and see. Sigma i where i is equal to, again you can take 1 to n but you can take 0 also. If you take 1 it will start from 1. Let us take for 1, i is equal to

1 to n is n into n plus 1 by 2. Take the basis one, what can you say? Even if you take 0 it will hold. But let us take 1. What is p of 1, sigma 1 equal to 1 only and here if you substitute a value 1 for n what will you get? You get 1 into 1 plus 1 by 2 that is 1. So the left hand side is equal to the right hand side. So you have proved for the basis. Now, in the induction part you have to show that, assume the result is true for n then you have to prove for n plus 1. So what you know is that sigma I where i is equal to 1 to n is equal to n into n plus 1 by 2. Now what you have to prove is sigma i is equal to 1 to n plus 1 that is instead of n you must get n plus 1. So you must get n plus 1 into n plus 2 by 2 so let us see how to prove it.

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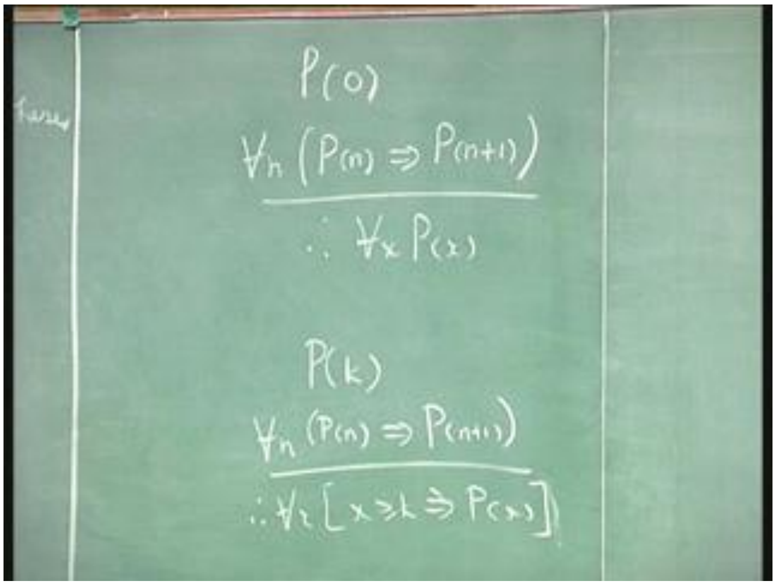

Sigma i is equal to 1 to n plus 1 is equal to sigma i is equal to 1 to n plus n plus 1. And by the induction hypothesis we know that this is n into n plus 1 by 2. So this becomes equal to this and by taking n plus 1 out you get this as n by 2 plus 1 that is equal to n plus 1 into n plus 2 by 2. So this is the way proof by induction is used and the thing is there is no extremal clause are something like that in the proof by induction.

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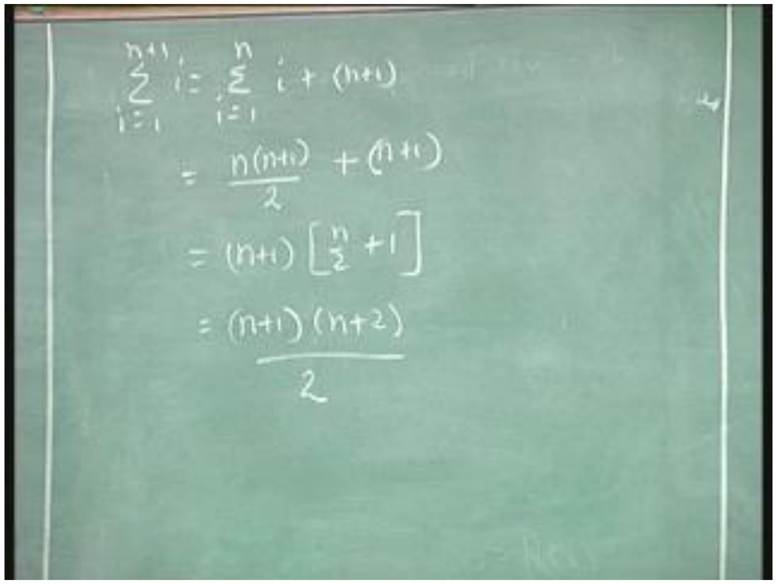

In the inductive definition of the set you have basis clause, induction clause and extremal clause and in the proof by induction you have only the basis part and the induction hypothesis. Let us take one more example. Let a be a real number then a power 0 is 1 you know that a power 0 is 1. You define a power n plus 1 as a power n into a. The number is defined in this way. Now you want to prove a power n into a power m is equal to a power n plus m.

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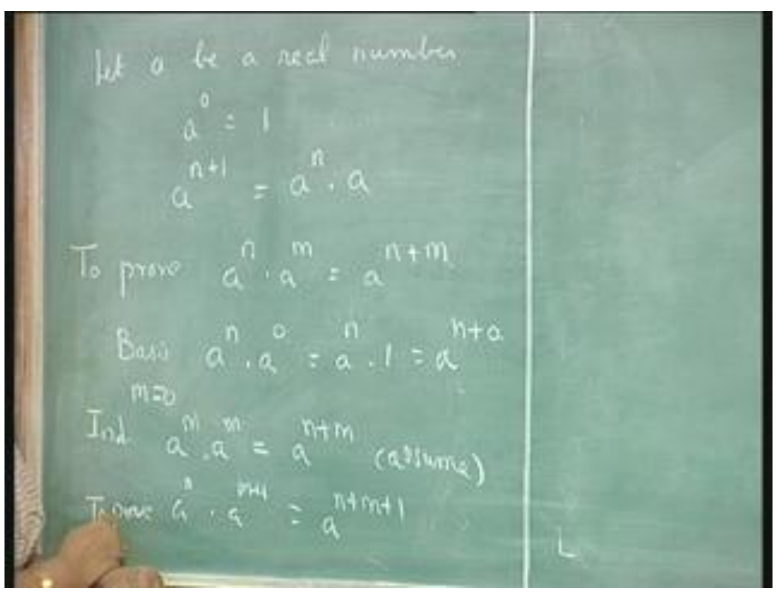

You want to prove this. How do you prove? First basis clause, here m is a natural number, so for basis clause you take a power 0, a power n into a power 0 that is you are considering the case where m is equal to 0. This will be a power n into 1, a power 0 is 1 and that is nothing but a power n plus 0 same as a power n. Now the induction clause is a power m, a power n into a power m is equal to a power n plus 1 n plus m assumed this. And you have to prove that a power n into a power m plus 1 is equal to a power n plus m plus 1.

So you have to prove a power n into a power m plus 1. How will you write this? a power n into a power m into a and by induction we know that this is a power n plus m. Now the a power n into n plus m into a will be a power n plus m plus 1. This is by definition and because of associativity property for non-negative integers you can write this as a power n into m plus 1. So this result is proved.

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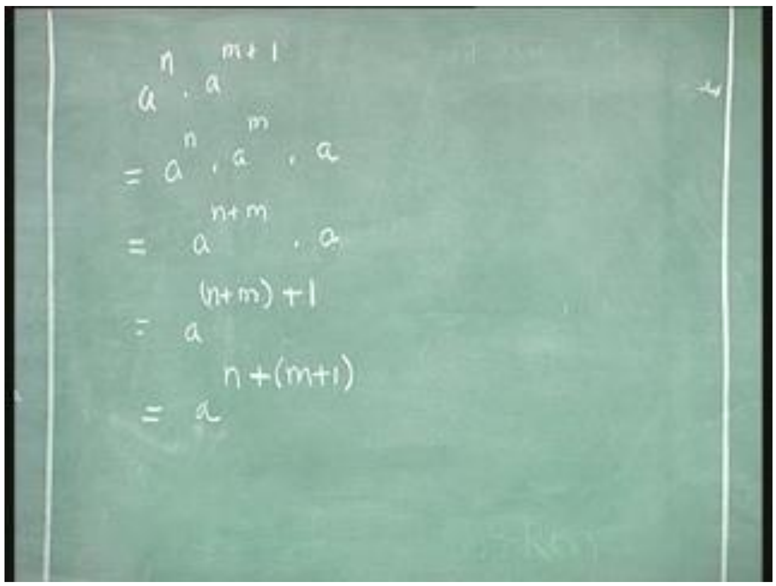

Just to show the use of induction I shall use one or two more examples. Let us consider some more examples so that this concept of proof by induction is clear. Prove that the number of diagonals in a n sided convex polygon is n into n minus 3 by 2.

Now when we talk about a convex polygon you can only talk about side three first. Side three is a triangle, side four is a quadrilateral and side five is a pentagon. So as a basis clause what is the number of diagonals in a triangle? You can even use 3, 4, 5 for the basis clause. It is enough if you prove 3 or 4 but let us consider even for 5 to verify. What are the number of diagonals in a triangle? n is equal to 3, diagonals is equal to 0. In this result substitute 3 you will get 0 that is n into n minus 3 by 2, if you use 3 you will get 0. When you consider a quadrilateral, the number of sides is 4 and the number of diagonals is 2. You will have 2 diagonals like this. So use 4. In this equation you will get 4 into 4 minus 3 by 2 which is 2. So it verifies this result. Now consider n is equal to 5 as it is a pentagon. In a pentagon how many diagonals you have? Let us see, you have 5 diagonals. So the number of diagonals is 5 and use 5 as n and in this equation you get 5 into 5 minus 3 by 2 that is equal to 5.

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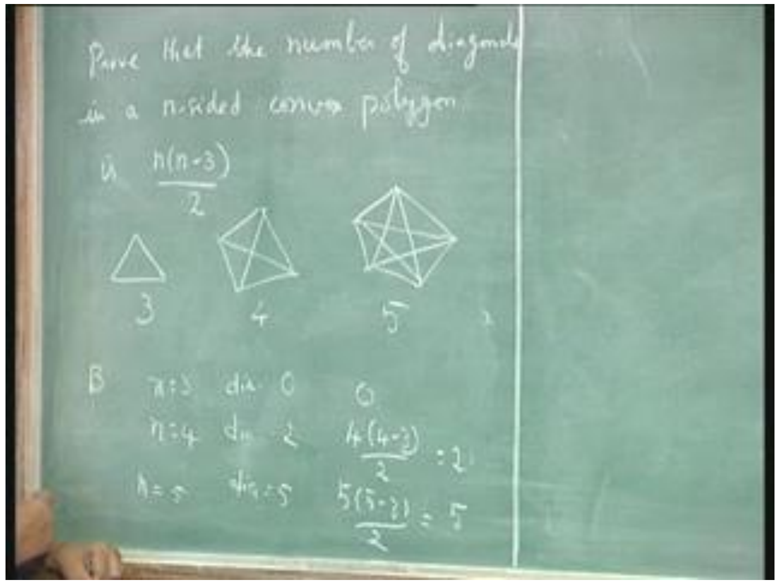

Now the induction clause is proved like this, the inductive portion. Assume that the result is true for n, that is for a n-sided polygon the number of diagonals is n into n minus 3 by 2. What you have to prove is for a n plus 1 sided polygon the number of diagonals will be n plus 1 because instead of 1 you must write n plus 1. So this will be n plus 1 into n minus 2 by 2. Let us see whether we get it. So consider an n-sided polygon. From this I want to consider a n plus 1 sided polygon. I have to add one more vertex and this side and this side and if I do this I get n plus 1 sided polygon. So how many diagonals will I get? The number of sides has become n plus 1 now by adding one more vertex.

What is the new number of diagonals? The number of diagonals in this n plus 1 sided polygon, see n plus 1 sided means n plus 1 vertices, it will have the same number of vertices. The diagonals which are already here will also be existing here. So those n into n minus 3 by 2 diagonals will be there plus you can join each of this vertex with any one of the n minus 2 vertices to get one more diagonal. So you will get n minus 2 diagonals like that plus one more diagonal and this side becomes a diagonal here.

Now if you sum it up, it will be n square minus 3 n plus 2 n minus 4 plus 2 by 2 which is equal to n square minus n minus 2 by 2 and that is nothing but if you factorize this is n plus 1 into n minus 2 by 2. So the way you got the new diagonals is for the n-sided polygon you added one more vertex and two sides. So already the diagonals you had here are present here also plus this one has become a diagonal now and this vertex can be joined with any one of the n minus 2 vertices to get one more diagonals so it is n minus 2 plus 1 and if you simplify this expression you get n plus 1 into n minus 1 by 2 and so we have proved the result. This is the way you prove the proof by induction. So starting with P(0) if the underlying set is the set of non-negative integers you prove P(0) as a basis and then you assume P(n) and prove P(n) plus 1. So for all (n) P n implies P n plus 1 and from this you can conclude therefore for all (x) P(x).

Now sometimes as in the case of the diagonal it does not make any sense to talk about diagonals of a one-sided polygon or a two-sided polygon. The smallest polygon is three-sided which is the triangle. So instead of starting with P(0) you start with P(3). Here k is 3.

So in some cases you may not start with 0 but you start with some particular k. So you prove P(k) and you assume that the result is true for n and prove that it is true for n plus 1. So you prove that P(n) implies P(n) plus 1 and from this you can conclude that if x is greater than or equal to k then P(x) will go. This is the way you prove, this is called the first principle of mathematical induction. And some times this is also called as a weak induction.

In contrast to that what is called as strong induction or second principle? What is called second principle or strong induction? For proving P(n) earlier we assumed only P(n) minus 1. So for proving P(n) plus 1 we assumed only P(n) or to prove P(n) you assumed only P(n) minus 1. Here, for proving P(n) you assume P(0), P(1), P(2), P(3) up to P(n) minus 1. For all k less than n you assume P(k), then all of them are used to prove that P(n) is true. The induction hypothesis is stated in this manner. So what you do is for all k, if k is lees than n then P k is true. That is P(1), P(2), P(3) up to P(n) minus 1 is true making use of all the fact you prove that P(n) is true. And from this you can conclude for all (x) P(x).

Now in this case you need not have to write P(0) separately. The reason is when n is equal to 0 k less than n is false. So this implication is true and automatically P(0) will become true. P(0) will automatically become true by this statement. The reason is k less than n when you put 0 is false so the implication is true and P(0) will become true. So where do you make use of second principle or strong induction.

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Again let us take one more example.
Prove that the sum of the interior angles of an n-sided convex polygon is 2n minus 4 right angles. 2n minus 4 into pi by 2 or you can write it as n minus 2 into pi. Now how do we prove that? Basis

clause as I told you again starts with n equal to 3, you take a triangle, what is the sum of the interior angles of a triangle? It is pi 180 degree or pi. So in this expression you put n is equal to 3 where you get pi.

Now the induction clause, assume that the result is true for n is equal to 3, 4 up to say k is equal to n minus 1, prove for k is equal to n. How do you prove this? Take a n-sided polygon, take 2 vertices and join them by a diagonal, so it becomes two polygons put together and the sum of the interior angles of the whole polygon is the sum of the interior angles of polygon 1 and polygon 2. The sum of the interior angles of polygon p is equal to the sum of the interior angles of polygon 1 and polygon 2.

How many sides does polygon one have and how many sides does polygon two have? Now the n-sides is divided into k into n minus k by this diagonal. So the sum of the interior angles of polygon p is equal to, what is the sum of the interior angles of this polygon? This has got k plus 1 sides. So that is k plus 1 minus 2 into pi. And what is the sum of the sides for this that is n minus k plus 1? So the sum of the interior angles is this minus 2 into pi. So that will be equal to k minus 1 plus n minus k minus 1, so this k and k will get cancelled and you will get n minus 2 pi which is the result we want.

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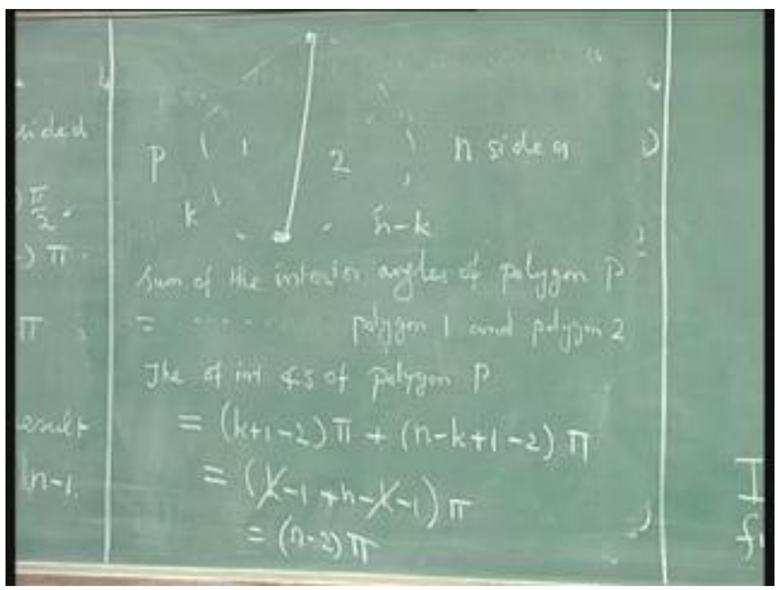

So in this case you are not using the result for n minus 1 and proving for n but you are using the result for k and n minus k and using it to prove the result for n. So this sort of an induction where in order to prove a result P(n) you make use of results P(1), P(2), P(3) up to P(n) minus 1 that is called the second principle of induction or strong induction. Now as far as the set of natural numbers is considered N is equal to 0, 1, 2 etc, both have the same effect. In one case you prove P(0) and using P(0) you prove P(1) and using P(1) you prove P(2) and using P(2) you prove P(3) and so on. This is weak induction or first principle.

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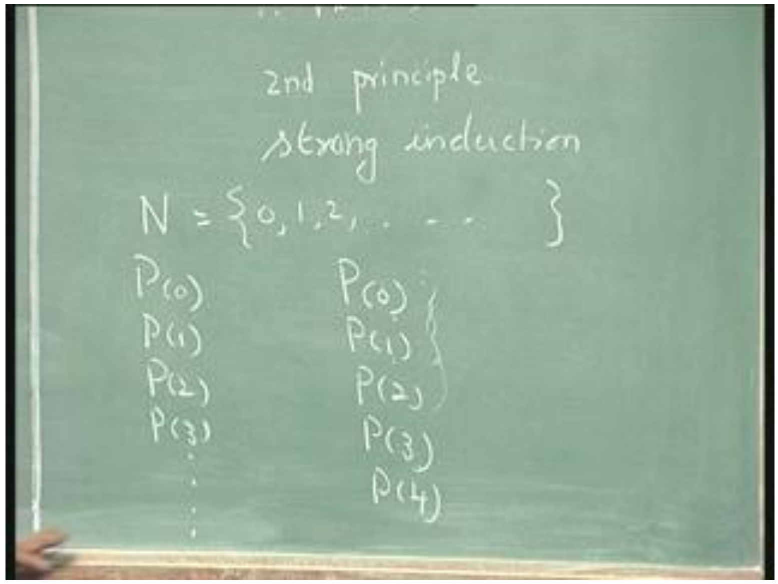

In the second principle or strong induction, first you start with P(0) then use this to prove P(1) and to prove P(2) you use both these or some of this. Then to prove P(3) you make use of all this or some of this and then to prove P(4) you make use of all these or some of these. Anyway the effect is the same. But there are other underlying sets for which you may not be able to use the first principle but you will be able to use only the second principle. One such thing is the set of ordered pairs n cross n.

If you take the set of underlying set as the set of ordered press n cross n, so it will consist of integers ij and if you look at the plane they can be looked at as grid points. And you can introduce some sort of an order such that all these two numbers i j and kl you can say this will come before this if i is less than k or if i is equal to k, j will be less than l, you can also define like that. That is all these points will come before all these points. If you do like that you will realize that in order to prove some result for this you have to assume that the result holds for everything because if you assume something is true for this alone it will be true for this, it will be true for this, it will be true for this and so on. But you will never be able to prove this result for this point.

The reason is every point has a unique successor but some points do not have a unique predecessor because it goes up to infinity. For such underlying sets you will not be able to use the first principle of mathematical induction. You have to use the second principle of mathematical induction.

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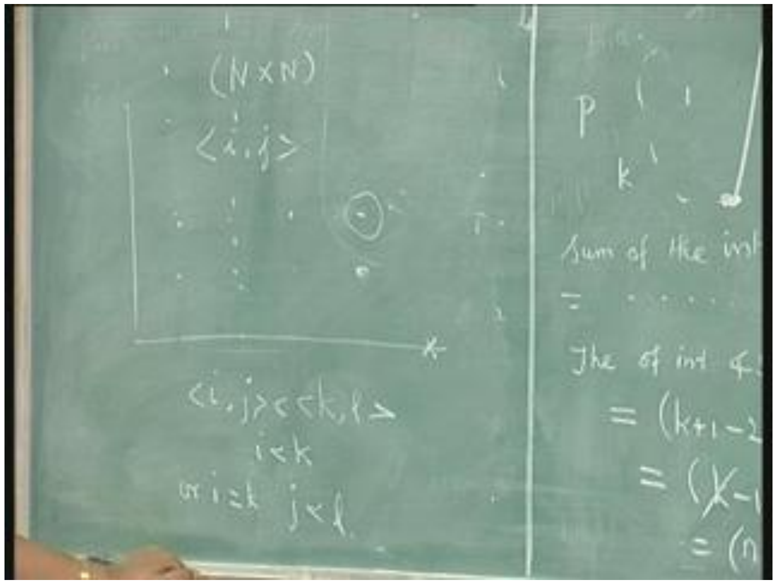

We shall learn about this again when we consider partially order sets, linear sets, well ordered sets and so on. Let us take one example and see how you can use Venn diagram to solve some problems in set.

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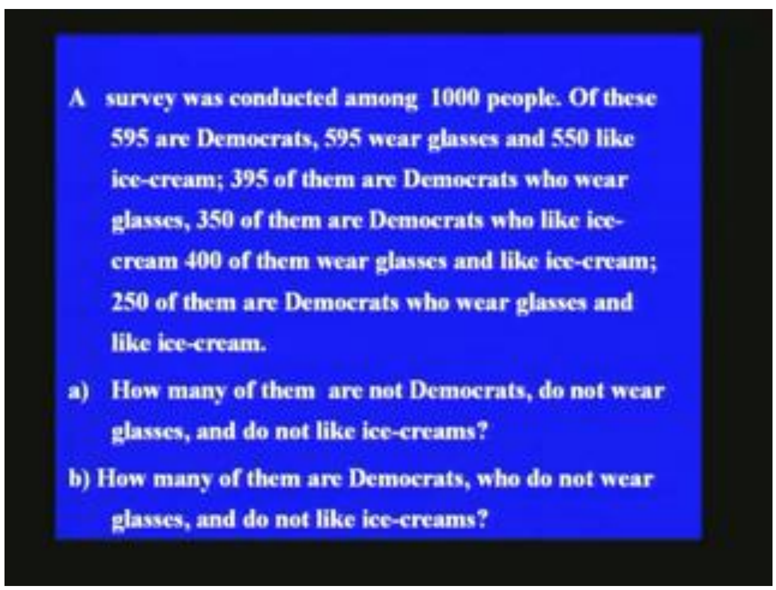

A survey was conducted among 1000 people. Of these 595 are democrats, 595 wear glasses and 550 like ice cream, 395 of them are democrats who wear glasses and 350 of them are democrats who like ice cream and 400 of them wear glasses and like ice cream, 250 of them are democrats they wear glasses and also they like ice cream.

The question is how many of them are not democrats and do not wear glasses and they do not like ice creams?
The second question is how many of them are democrats who do not wear glasses and who do not like ice cream?
So using Venn diagram let us see how the figure looks like. Totally there are 1000 people and the survey was conducted on them. Among them of which 595 were democrats and 590 of them wear glasses. And 390 of them are both democrats who wear glasses, 550 are those who like ice cream and so on. So let us fill these portions. How many of them are democrats and who wear glasses and who like ice cream. The number is 250. So the intersection of all the three is 250. They are democrats who like ice cream and who wear glasses. 395 of them are democrats who wear glasses; this portion tells you the number of democrats who wear glasses. There are 395 of them of which those who like ice cream form 250. So the remaining is 145. So put together it becomes 395, so that portion is 145.

And how many of them are democrats and who like ice cream? 350 of them are democrats who like ice cream. So this portion is 350 out of which 250 is here, so this portion is 100. And how many of them wear glasses and who like ice cream? 400 of them wear glasses and who like ice cream, so this will be 150. So totally how many democrats are there? There are 595 democrats. So this whole portion is 595 out of which you are accounting for 100 plus 250 that is 350, 395, 495 you are accommodating here, total is 595, so this portion will constitute 100. And similarly 590 of them wear glasses and how many of them you are accommodating here?
150 plus 250 is 400, 445 you are accommodating 545 here, so 50 of them will be in this portion.
And similarly 550 of them like ice cream out of which 400 is equal to 100 you are accommodating here. So this will be 50.

The question is how many of them are not democrats who do not wear glasses who do not like ice cream, that refers to this portion. How much is this? From 1000 you have to subtract the sum of all this. What does this sum leads to? This leads to 100, 145, 250, and 100 this will add to 595 actually because we know that the number of democrats is 595 so 595 plus 50 plus 150 plus 50 which will add to 845. So the number of democrat people who are not democrats who do not wear glasses and who do not wear ice cream will be 1000 minus 845 which is 155, this is the answer. And the second part is, what is the number of democrats who do not wear glasses or who do not like ice cream?
From the figure we can see that, this portion denotes the number of democrats who do not wear glasses or who do not like ice creams and you can see that this is 100. So for the second part the answer is 100. Like that you can use Venn diagram to solve some problems in set theory. This you would have done in school itself. It is just to refresh your memory.

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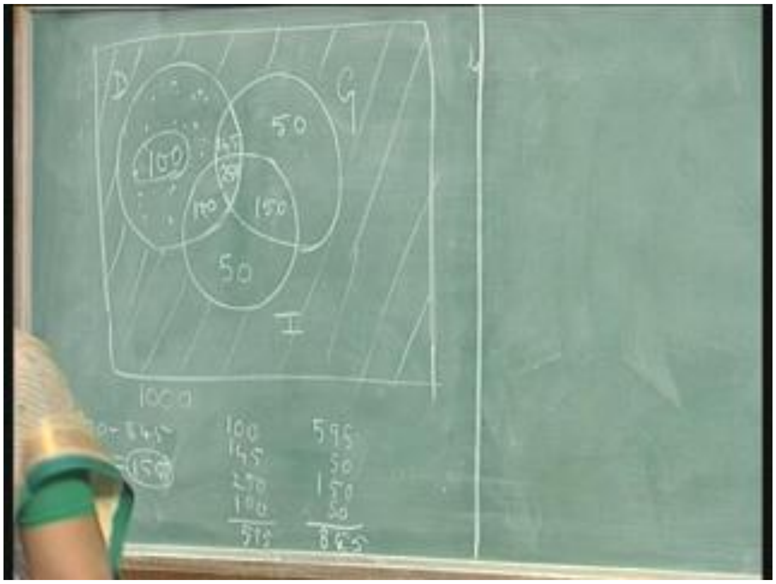

Now let us consider some more things about natural numbers. We try to define the natural numbers inductively. How did we do that? We started with the basis clause 0 belongs to n. What are the natural numbers or non-negative integers? They are 0, 1, 2, 3 and so on. So we try to define like this: 0 belongs to n and for the induction clause we said if n belongs to n then n plus then n plus 1 belongs to n. But there is a small difficulty here and what is the difficulty. We try to use addition for defining natural numbers.

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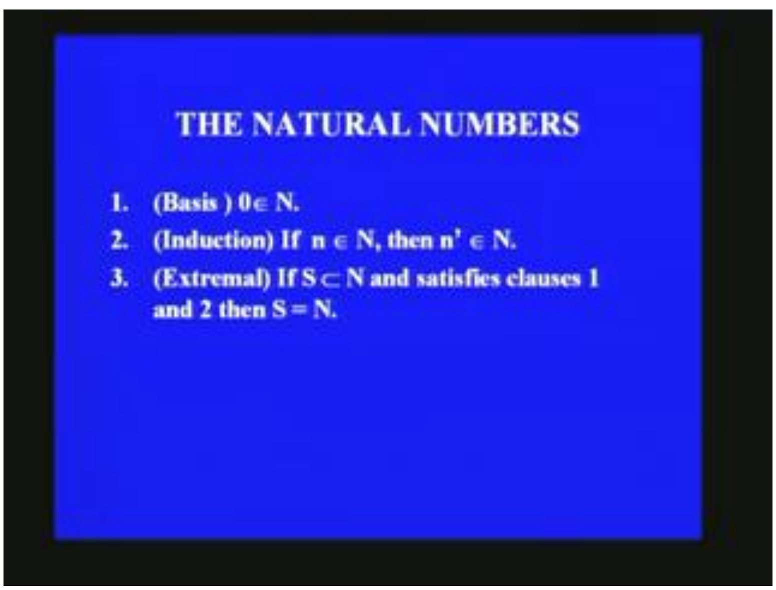

Usually only after defining natural numbers we define addition and that is the procedure. It is a chicken and egg problem; you cannot define natural numbers using addition, because you have to define addition itself using natural numbers. So instead of saying like this, you say if n belongs to n, n dash belongs to n where n dash is the successor of n. So you can define the natural numbers like this. Basis 0 belongs to n and induction clause is if n belongs to n then n dash belongs to n and the extremal clause is if S is contained in n and satisfies clauses 1 and 2 then S belongs to n. Actually you want something like this.

You want to start with 0, then the successor of 0 is 1, then the successor of 1 is 2, then the successor of 2 is 3 and so on. But by this definition do you get this diagram by what we defined earlier? There are some difficulties. For example, suppose you say 0 is the successor of 0. Then you will not get a figure like this but you will get a figure like this 0 is the successor of 0, then again the successor, again the successor and so on.

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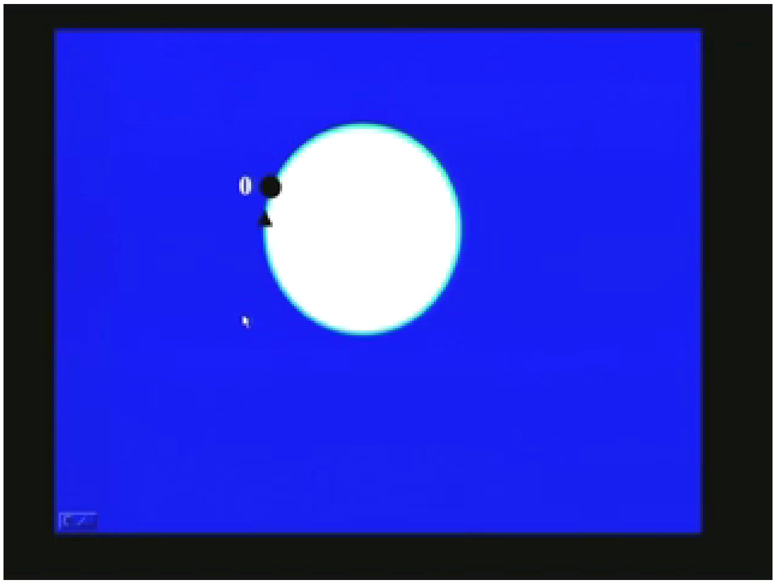

Is this what you want? You do not want this. So what do you want? You want something like this. So this should not be permitted, so you cannot have 0 as the successor of any number. And even if you allow 0 as the successor of any number you may get a figure like this. For 0 you may have two successors: 0 dash and 01 dash and 02 dash. And for this itself you may have two successors. And for this again you may have two successors and you may get a figure like a tree. What you want is a line that is what you want. But by definition it is possible to get such a thing also and what is that which gives rise to such a tree? Because you are having two successors for 0 and two successors for this, two successors for this and so on.

In order to avoid such a tree diagram what you must say thus, the successor of a number is unique. You cannot have two different successors but the successor of any natural number is unique. Now suppose you say that 0 is not a successor of any number and the successor of a number is unique, then can you get this diagram?

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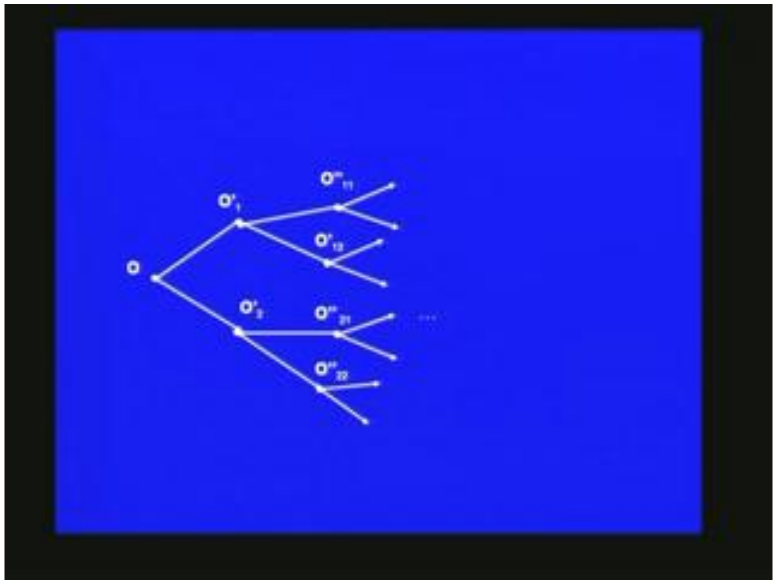

Then also there is some difficulty, you avoided this, you avoided this, but what about this? Suppose this is the successor of 0 and this is the successor of 1 and this is the successor of 2, 3 is the successor of 2. Suppose I say 2 is the successor of 3 then again 3 will be the successor. Successor is unique. Successor of 2 is unique, 3 is unique, again the successor of this is unique and so on. But you do not get what you want but you get something different.

And what is the problem here? See the problem here is the predecessor is not unique. What is the predecessor for this? The predecessor for this is this as well as this because you are marking the arrow like this. So the predecessor of this is this and because of this the predecessor is this. You are getting two predecessors for this number which is not correct. So the next condition you have to include is that the predecessor of a number is unique.

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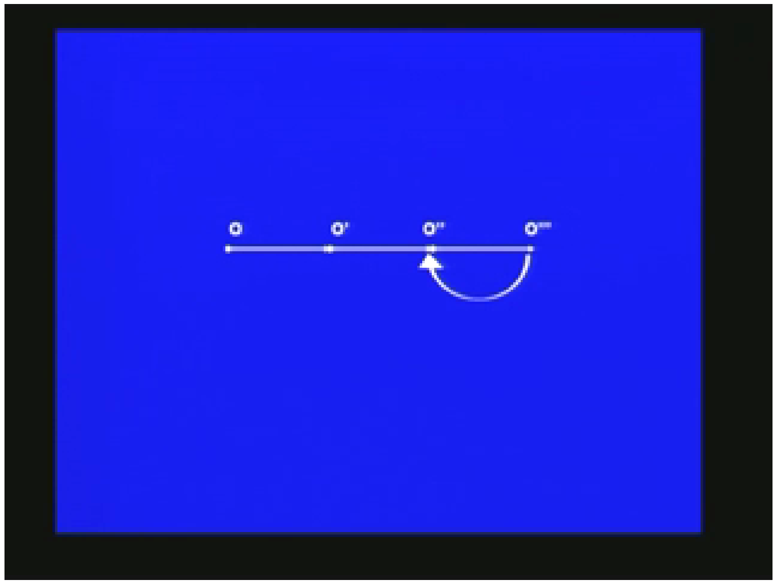

So we come to this definition. Actually the set of natural numbers is defined using sets like this: phi denotes 0, then phi union phi denotes 1, then phi union phi union phi, phi denotes 2 and so on. Like that you can represent the natural numbers using sets. If you make use of this definition we can define like this, phi is a natural number. For each natural number n its successor n dash is constructed as follows:

If n is a natural number then n union n denotes n dash which is a successor of n. So if you define like that what will be 3. So phi union phi is just having one element. Then phi union phi union is having only two elements like this I, union phi is empty. So number 3 will have elements phi phi phi, this has got three elements that is this n union this put within parenthesis. So this will have three elements. So n if you take n union and n put within parenthesis will give you the next number n dash.

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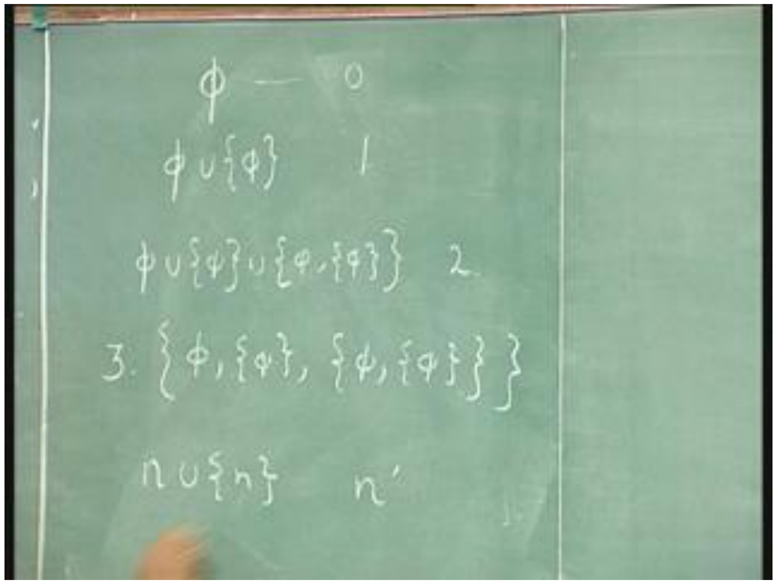

So if you want to write everything 0 is empty, 1 is having only one element like this, 2 is having phi, phi, 3 is having phi, phi, phi, phi and 4 will have and so on.

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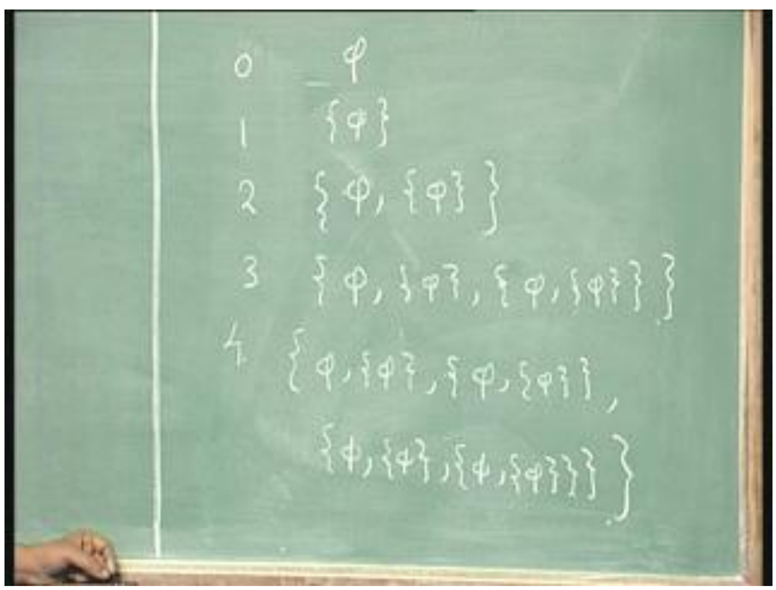

So you see that if this is n, n dash the successor of n contains all this plus this within flower brackets. This is n and this is n within flower brackets and that denotes the next integer or the success.

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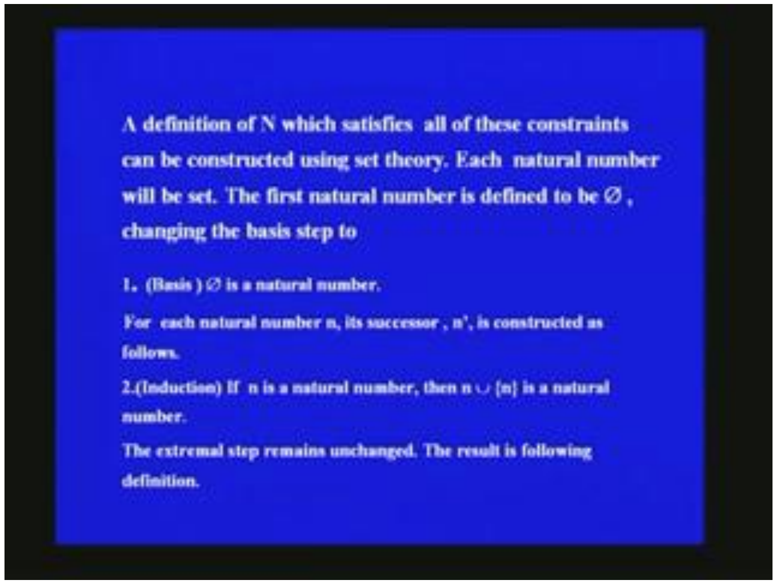

We can define natural numbers this way and mention the constraints also. A definition of N which satisfies all of these constraints can be constructed using set theory. This is what we have seen, phi is the natural number. Then if N is any natural number then N union N is a natural number. The extremal clause is same as before. So the definition is like this. The set of natural numbers N is the set such that phi belongs to N if N belongs N then N union N within flower brackets belongs to N. The extremal clause is if S belongs to n then satisfies clauses 1 and 2 then S is equal to N.

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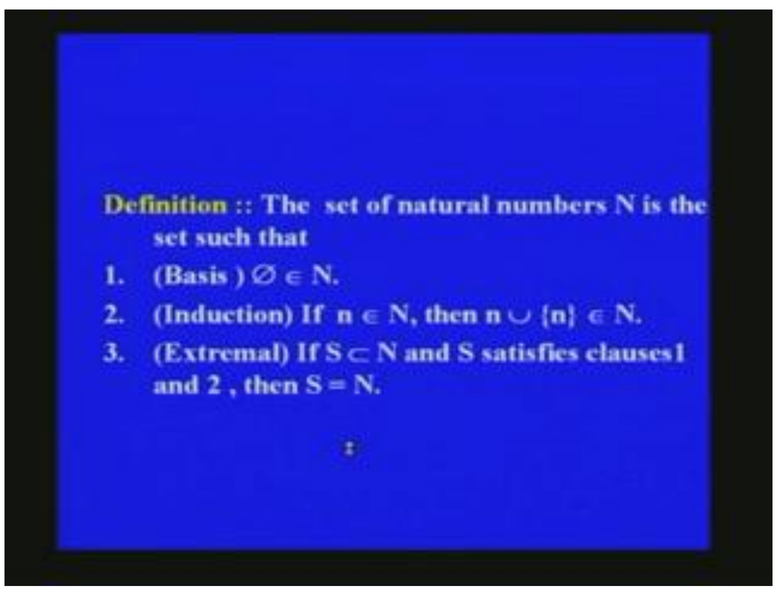

The three conditions we can state as theorem: 0 is the not successor of any natural number and the successor of any natural number is unique. The third theorem is the predecessor is unique and that we can state like this:

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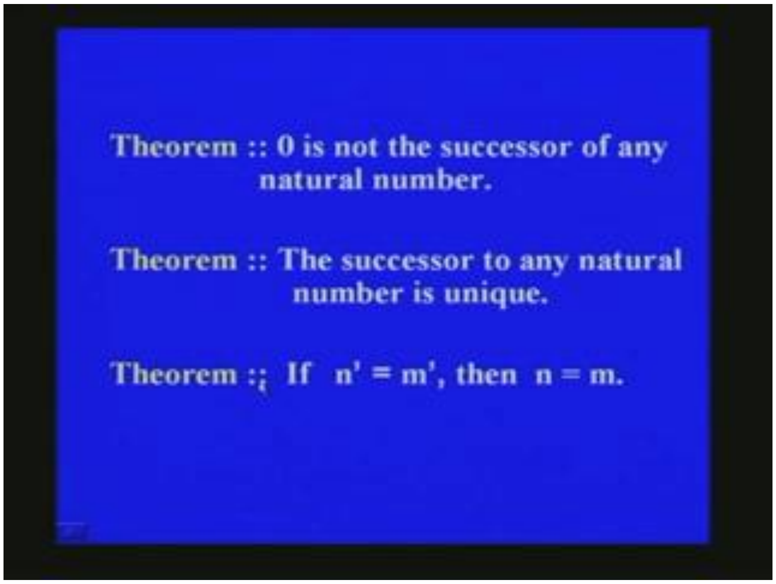

If n dash is equal to m dash then n is equal to m. That is if the successor of n and m are the same then n is equal to m. These are called peano postulates for natural numbers. And we can mention them like this peano postulates a 0 is a natural number then for each natural number n there exists exactly one natural number n dash which we call as the successor of n. In the second class itself we are bringing out the fact that the successor is unique. Third is 0 is not a successor of any number and the fourth one is the predecessor is unique that is if n dash is equal to m dash then n is equal to m. Last one is the extremal clause which is like this. If S is the subset of N such that 0 belongs to S and N belongs to S then n dash belongs to S then s is equal to N. This is the usual extremal clause. So we can define natural numbers in this way using peano postulates.

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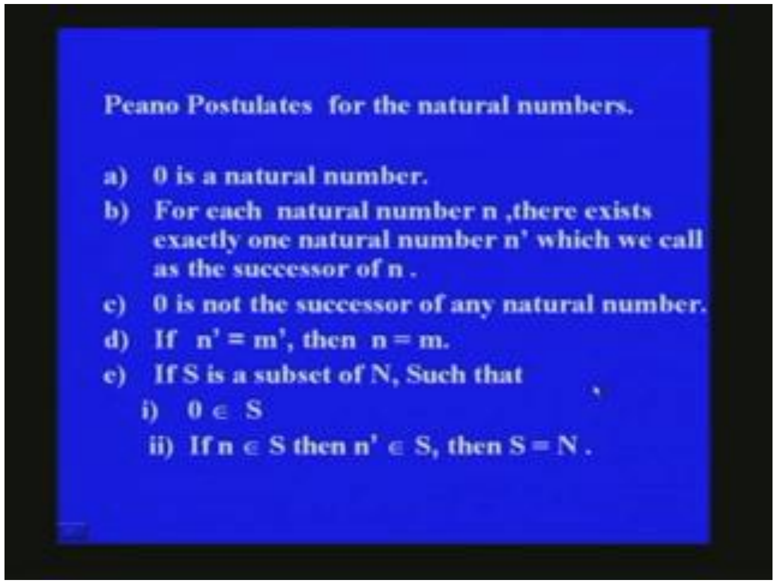

So we have seen what is meant by the inductive definition of sets and how we can use the inductive definition for proof by induction. And in both cases weak induction or the first principle and strong induction as the second case. Then now we have also seen how the natural numbers can be inductively defined using peano postulates.