Discrete Mathematical Structures

Dr. Kamala Krithivasan
Department of Computer Science and Engineering
Indian Institute of Technology, Madras

Lecture # 10

Sets

Today we shall learn about sets. You must have learnt about sets in school itself. We shall learn a little bit more about sets in these one or two lectures. Now what is a set? A set is a collection of objects and the objects themselves are called elements of the set.

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For example, you can have the set of natural numbers n or you can think about the set of books or you can think about the set of even non-negative integers less than 10 which is denoted by this. Usually a set is denoted like this and the elements are denoted like this. It is written within flower brackets all the elements are written within flower brackets. So S denotes a set which consists of elements a, b, c, d. Now you say that a belongs to S. Use this symbol for belonging to. So this means a belongs to S or a is in S. In contrast with that if you write e does not belong to S, it is not here, so e does not belong to S or e is not in S.

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Now, you can also have a definition like this. For example the set of even numbers even non-negative integers you can say that E is equal to x where x belongs to N it is a nonnegative integer and there is a y belonging to N such that x is equal to 2y. This denotes the set of all even non-negative integers. Now you are using a predicate to define e, such a definition is known as implicit definition whereas a definition of this form is known as explicit definition.

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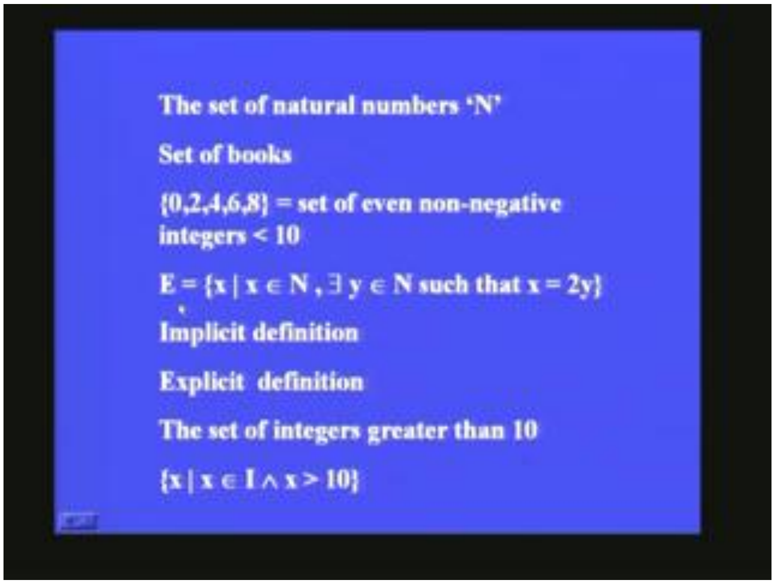

So a set can be defined using explicit definition or implicit definition. For example, take the set of integers greater than 10 this you can write as x x belongs to i i denotes the set of integers AND x is greater than 10. So you are using a predicate to define this set. So the implicit notation is given there and the explicit notation for that will be set of integers greater than 10 so it is 11, 12, 13 and so on. If it is a finite set you can list all the elements, if it is an infinite set you can list a few of them and the way you write the dot the other members are specified.

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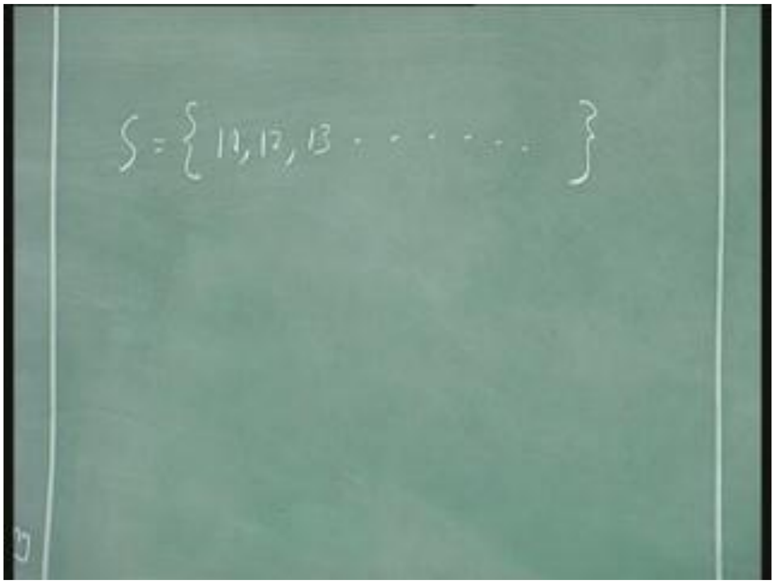

So we have the concept of the set. And originally in the Naive set theory when it was proposed they found some difficulties in set theory. They found that there were some paradoxes and there were something which leads to contradiction and so on. So let us see the things which let to a contradiction or a paradox. Consider the universe of discourse, let the underlying set be set of all sets. Then define a set S like this; it consists of all sets which do not belong in themselves.

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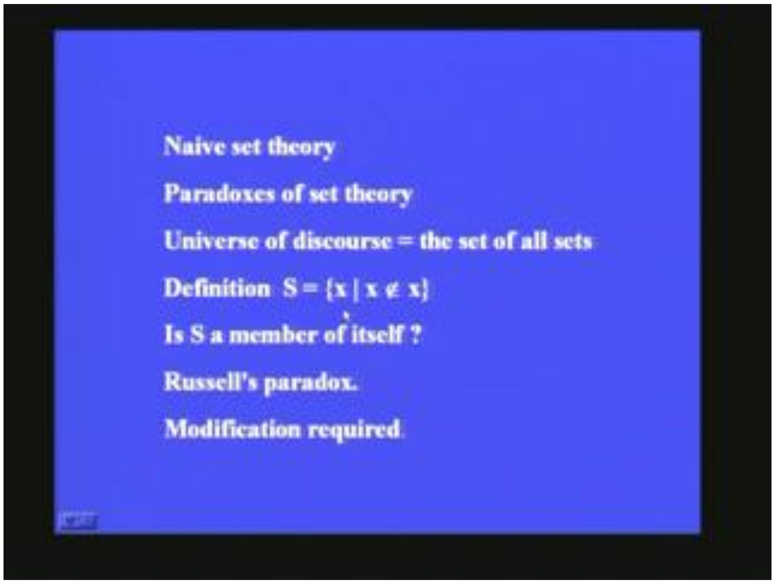

A set S is defined like this; it consists of all sets which do not belong in themselves. Now the question is S a member of itself? What will happen? Just discuss this; S is equal to x, x belongs to x does S belong to S? This is the question. Suppose S belongs to S, S consists of all sets which do not belong in themselves from this you have to come to the conclusion that S does not belong to S because it does not satisfy the definition. Now suppose S does not belong to S it satisfies this definition so it should belong to S. So from this you conclude that S belongs to S.

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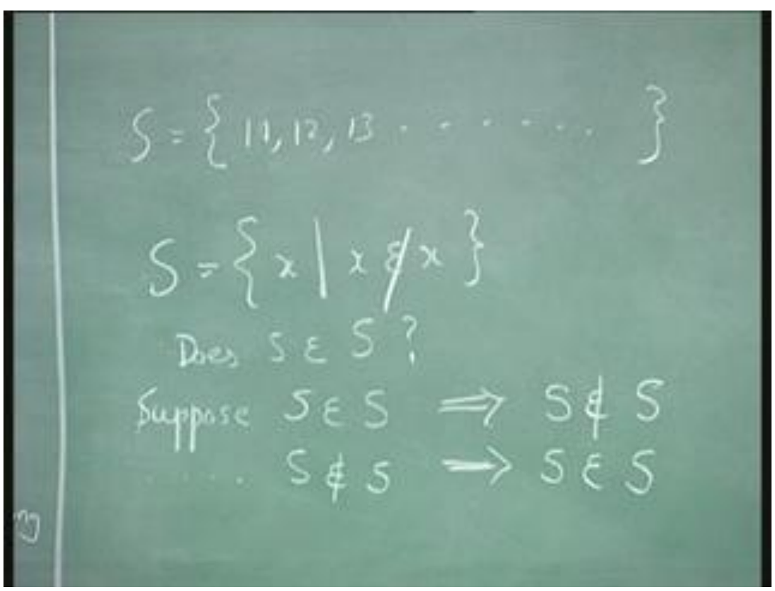

So you are arriving at a contradiction. So this is known as Russell's paradox, this is called Russell's paradox. So there seems to be some basic flaw in defining sets itself and so people were wondering whether the set theory will really be useful or not. But this paradox or some flaws should not lead one to give up set theory as it is because there are so many benefits achieved by sets.

Even in programming languages you use sets you use set operations and so on. So what we need to do is we have to modify the Naive set theory a little bit and one way to do this is, you have different levels so the individual objects form one level, a set which consists of element from this should be in the next level and in this level you may have a set which consists of elements from the lower levels it may have some element from here a element from here and so on. And at the higher level you may have a set which consists of elements from the lower level. But the set will not have an element in the same level.

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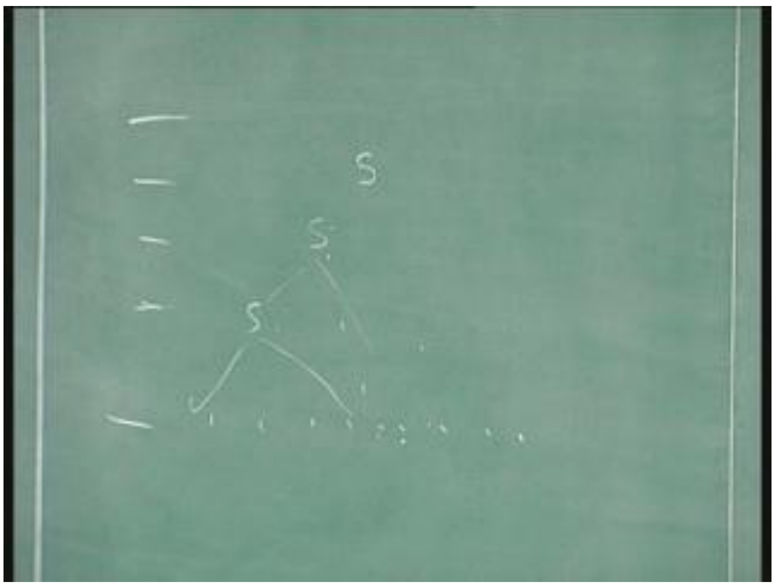

So if you distinguish like that between levels and allow for the definition of a set in such a way that a set can have only elements from the lower levels then this sort of a paradox will not arise because you will not have such in this definition is itself because a set will have only elements from the lower level.

So if you modify that definition of set in that way this paradox will not arise and there will not be any flaw in the set theory and you can make use of set theory as it is. Now, sets are denoted by what is known as Venn diagram.

The universal set is denoted like this; a set A may be denoted like this, another set B may be denoted like this, this portion will denote the set of elements common to A AND B or it denotes A intersection B. Like that you can represent sets by means of Venn diagram. Now let us consider some relations among sets.

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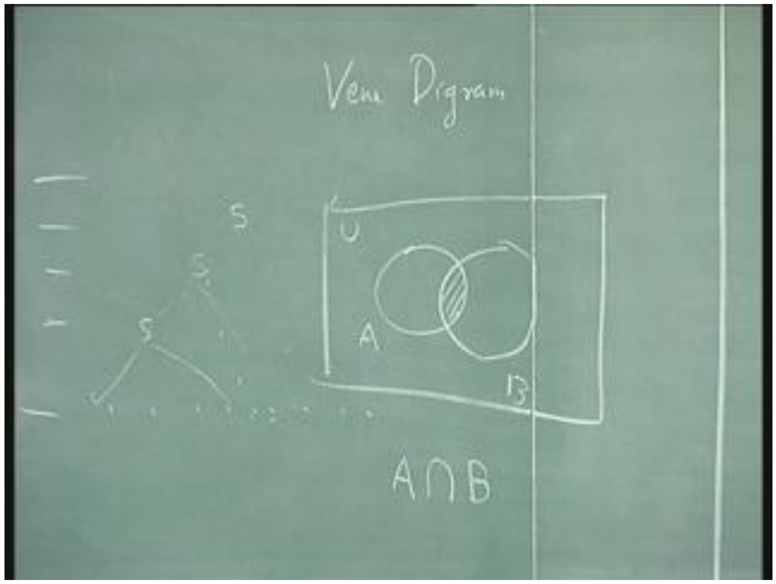

See relation between sets; Consider two sets A and B. Let A and B be sets then when do you say that A is a subset of B? This is denoted as A contained in B or it is denoted like this A contained B. If each element of A is an element of B that is A contained in B is saying it is equivalent to saying for all of x x belongs to A implies x belongs to B.

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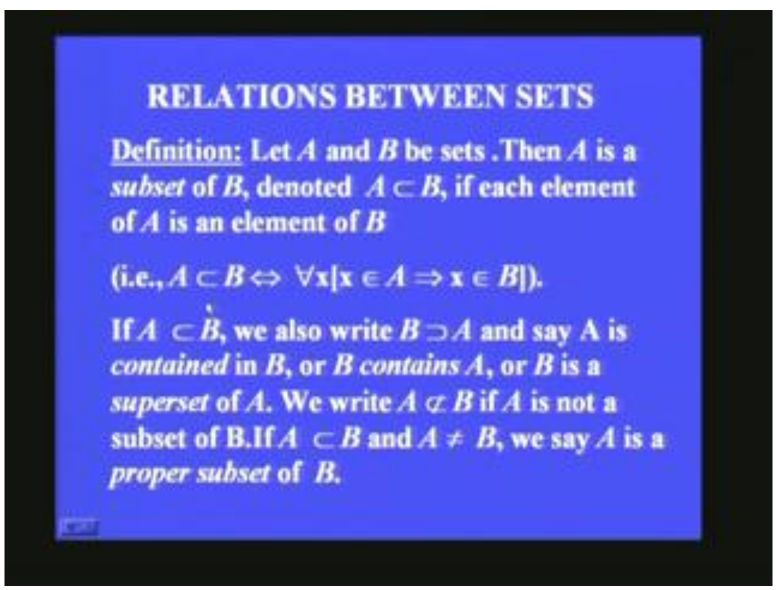

Now sometimes you use this to denote A contained in B and A contained in B if they are not equal this is the subset and this is the proper subset. In this case it is possible for A and B to be equal also, in this case B has at least one more element than A they are not equal.

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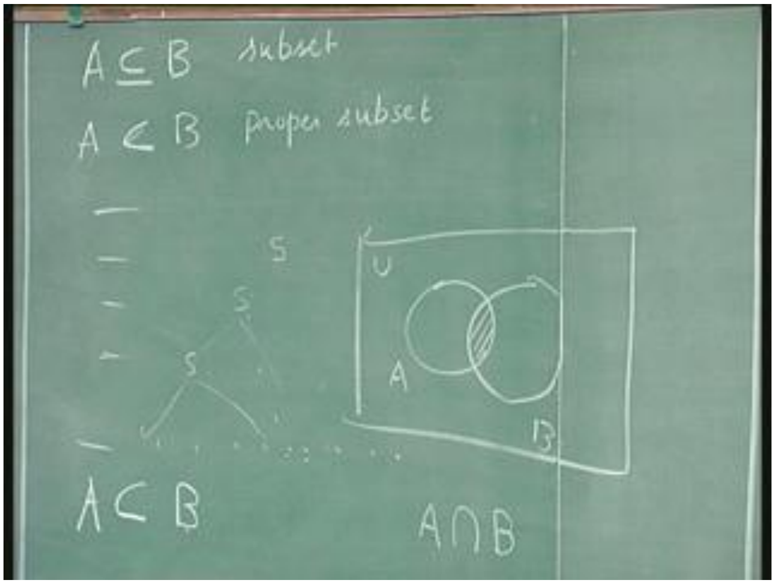

So when you say A is contained in B you could also write it as a B contains A, you have to write like this B contains A this is read as B contains A. So you read it as if A is contained in B we also write B contains A and say A is contained in B or B contains A, or B is a superset of A you can also say it like this A is a subset of B or B is a superset of A.

We write A not contained in B that is contained in and then a slash over that, if A is not a subset of B. If A is contained in B and A is not equal to B we say A is a proper subset of B. Let U be the universe of discourse and A is a set then A will be a subset of U. U is the whole clause it is a big set a universal set so any set will be contained in that. Let A and B be sets then A is equal to B if and only if A is contained in B and B is contained in A.

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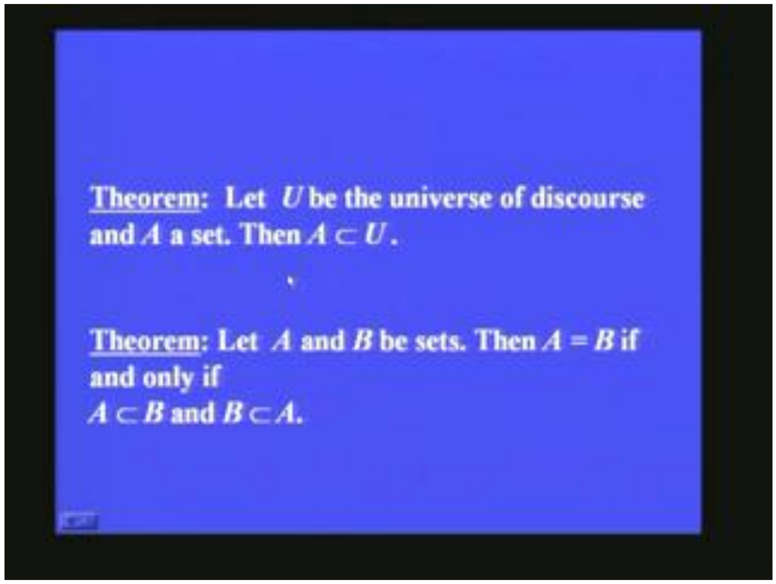

So if you use, if A is contained in B AND B is contained in A so when you say this they can be equal also when you have both these which means A is equal to B. So here you have to take it as inclusion and not proper inclusion. For any set A A is contained in A here again this represents subset and not proper subset. Sometimes you distinguish like this; this denotes subset and this denotes proper subset, sometimes in some notations this itself denotes subset. A set with no members is called an empty or null or void set. This is denoted by phi and a set with one member is called a singleton set.

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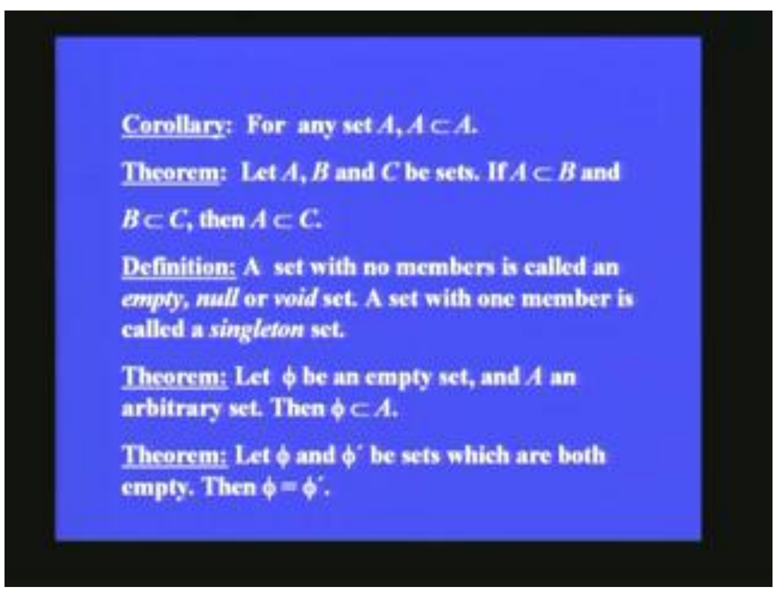

Let phi be an empty set and A is an arbitrary set then phi is contained in A. How do you prove that? This denotes the empty set and A is an arbitrary set a singleton set will be denoted like this having a single element this is a singleton set.

Now you want to show phi is contained in every set A. What is the definition of A contained in B? x belongs to A implies x belongs to B this is the condition. Now phi is contained means x belongs to phi it should satisfy this condition x belongs to phi implies x belongs to A. But you can see that the premise of this implication is false. x belongs to phi is false so the whole implication is true so it satisfies this condition phi contained in A. This is true the whole implication is true so any empty set is contained in any set A.

Next is, if phi and phi dash be sets which are both empty then phi is equal to phi dash. Again this you can prove like this; if phi is the empty set it is contained in any set so phi is contained in phi dash and if phi dash is an empty set it will be contained in any set so phi dash will be contained in phi. So from these you can conclude phi is equal to phi dash, there can be only one empty set.

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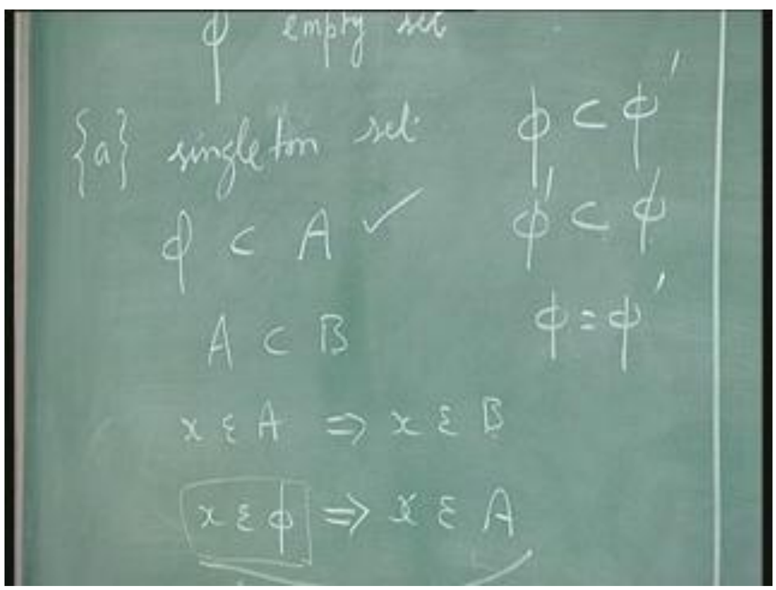

Operations on sets:
We will have some binary operations on sets let us consider what they are. Let A and B be sets, the union of A and B denoted by A union B is a set, A union B is equal to x x belongs to A OR x belongs to B. It consists of all elements contained in A and B both. The intersection of A and B denoted by A intersection B is the set A intersection B x belongs to A AND x belongs to A. So A intersection B consists of all those elements which are elements of both A and B. The difference of A and B or relative compliment of B with respect to A and is denoted as A minus B and that is the set A minus B is equal to x x belongs to A AND x does not belong to B.

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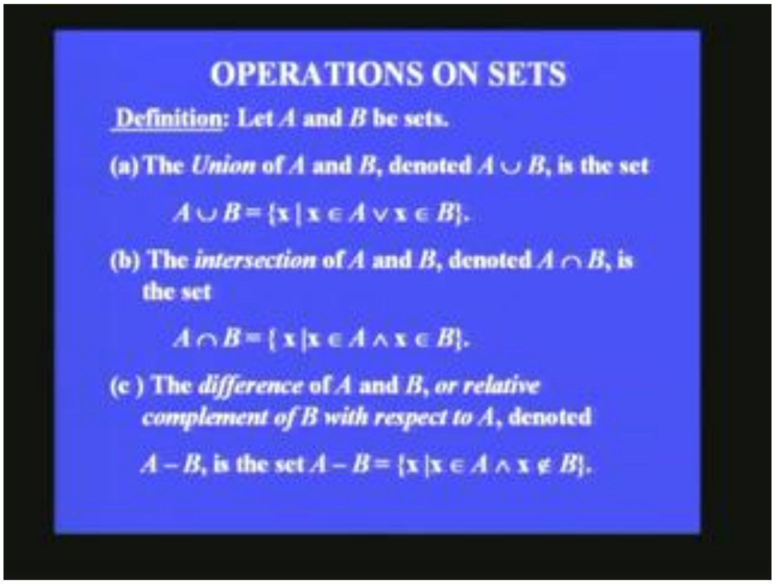

That is, this consists of all elements which are in A but not in B. Let us consider some example; Suppose A is equal to a, b, c, d and B is equal to c, d, e, f what is A union B? It will consist of elements which are either in A or in B that is a, b, c, d, e, f. A intersection B will consist of all those elements which are both in A and B that is c and d. And A minus B will consist of elements which are in A but not in B and in this case it will be a b in this example.

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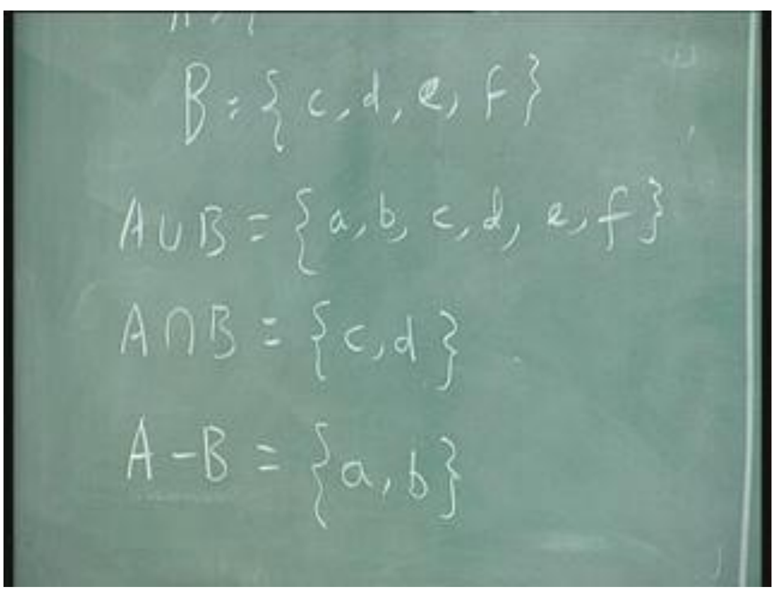

Now if A and B are two sets and A intersection B is empty then A and B are said to be disjoint sets. If C is a collection of sets such that any two distinct elements of C are disjoint then you say C is a collection of pair wise disjoint sets.

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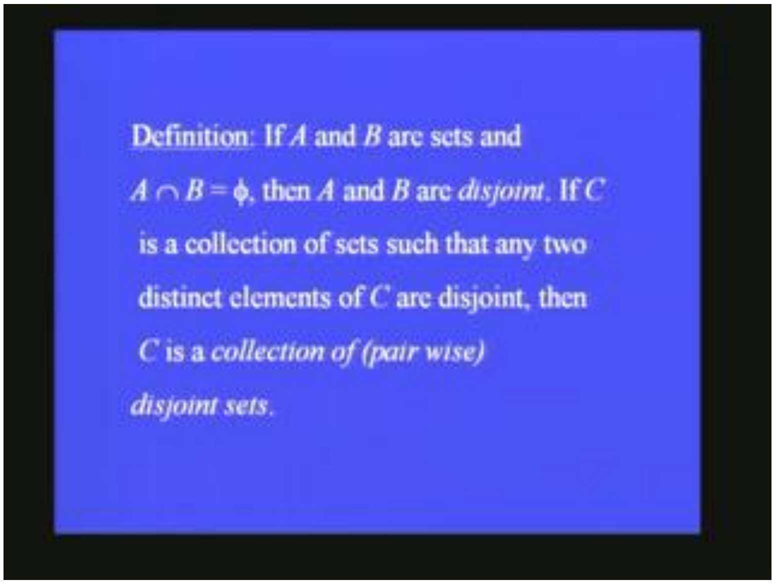

Now, let us see what is meant by commutativity and associativity. This is a binary operation, let us denote this, this denotes a binary operation. And x operator y denotes the resultant obtained by applying the operator to x and y. This denotes the resultant obtained by applying the operation to the operands x and y. Then when do you say that operation is commutative, if x operator y is equal to y operator x and it is said to be associative if you perform the operation on x and y first and z that is equivalent to performing the operation on y and z first and then performing the operation with x and the resultant. So these are the definitions of commutative and associative operations.

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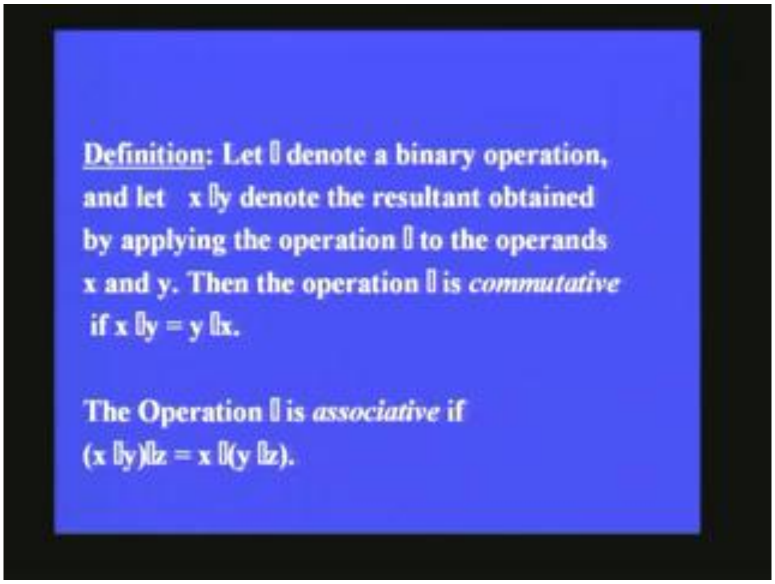

Let us see that the union and intersection operations are commutative and associative on sets. The set operations union and intersection are both commutative and associative. For arbitrary sets A, B and C you have A union B is equal to B union A. That is both of them will contain only elements which belong to both A and B.

In a set the order is not important so I can denote a set by a, b, c, d this is the same as denoting b, d, c, a they denote the same sets. The order does not matter here. So A union B is equal to B union A, A intersection B is equal to B intersection A again it consists of all elements which are common to both A and B. And again it is also associative, union is associative. The left hand side will consist of all elements which belong to either A or B or C and that is the same with the right hand side. Similarly, intersection is also associative because it will consist of all elements which belong to A, B and C.

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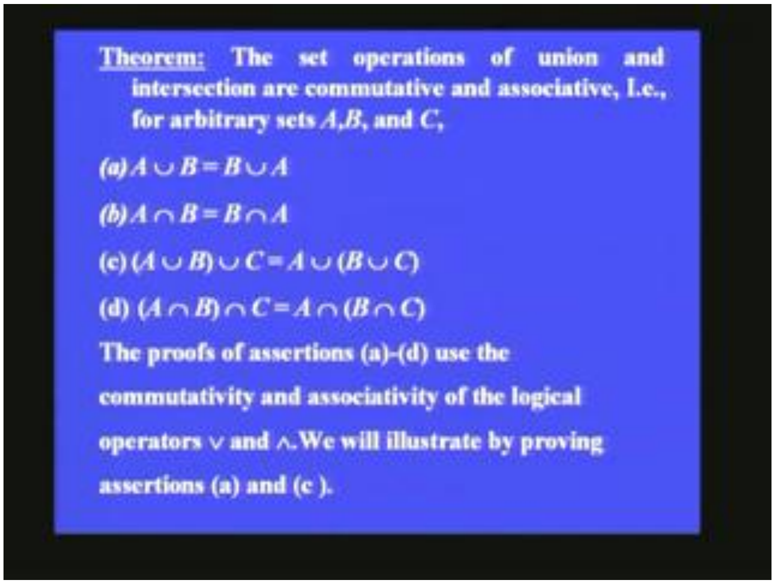

Now formal proof can be written just using OR, AND and logical connectives. Now you have two binary operation; dell and square I will say then how do you define distributivity of one operator with respect to the other? Then dell distributes over square if the following hold; x dell y operator z is equal to x dell y operator x dell z. And similarly, y operator z then dell x is defined as y dell x operator z dell x.

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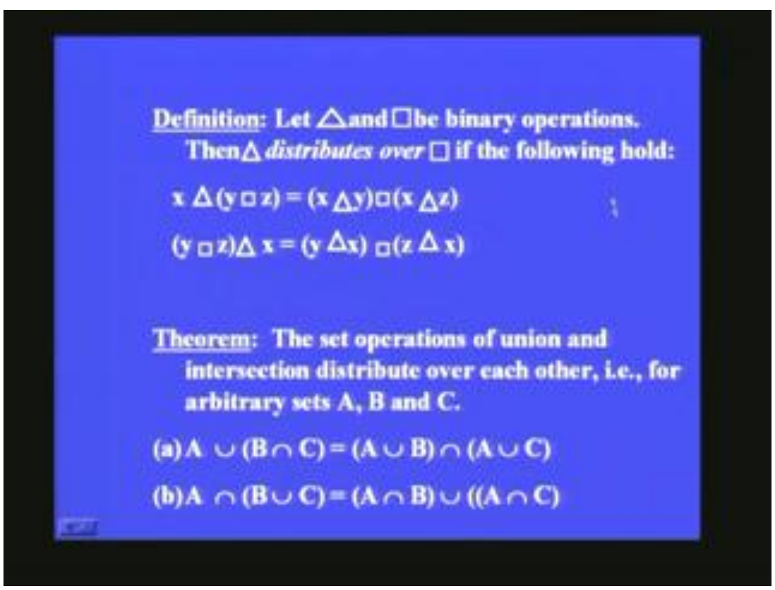

For example, if you take multiplication and addition you can see that x into y plus z if you take the set of integers as the underlying set x dot y plus z will be x into y plus x into z. And similarly, x plus y dot z will be x dot z plus y dot z. So here dot distributes over plus and multiplication distributes over addition. So in set also you can have union and intersection and let us see how they distribute. The set operations of union and intersection distribute over each other. For arbitrary sets A and B and C A union B intersection C is equal to A union B intersection A union C. You can use Venn diagram and convince yourself about this. You have the universal set like this, you have sets A, you have set B and you have set C.

What is the left hand side B intersection C A union B intersection C what is B intersection C? This is B intersection C this portion is denoted by B intersection C and A union B intersection will be this whole thing A and B intersection C.

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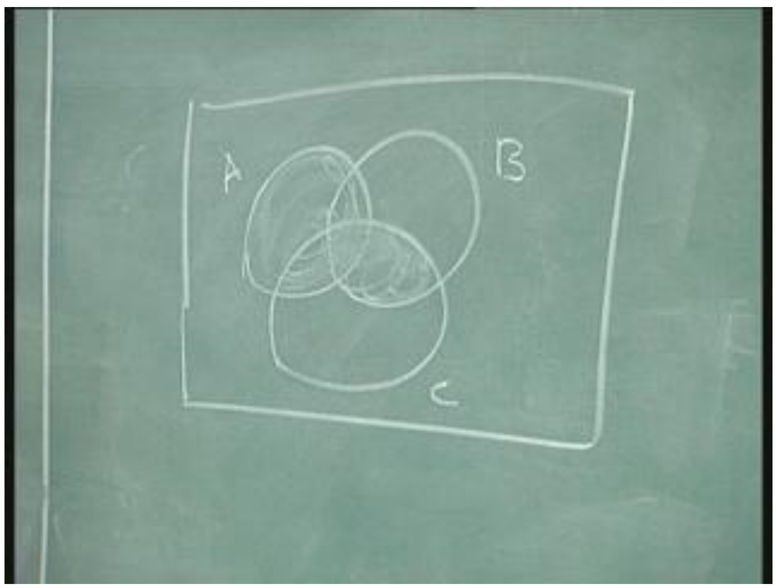

Now, again let me draw the same thing A, B and C. What is the right hand side? The right hand side is A union B A union B is this whole thing and A union C is this and this together. Now the intersection of that will be this.

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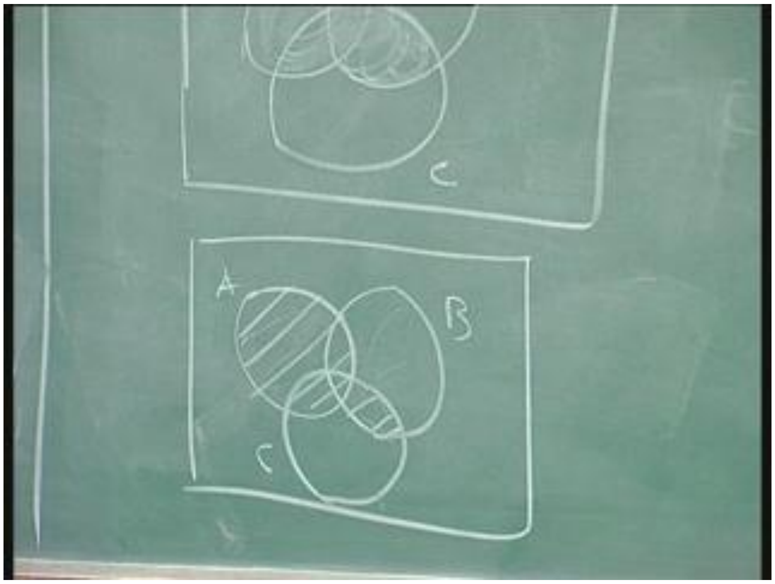

So you can easily see that A union B intersection C is equal to A union B intersection A union C. And similarly, you can also prove that A intersection B union C is equal to A intersection B union A intersection C it is a similar way you can prove. Let A, B, C, D be arbitrary subsets of a universe U then you have the following assertions. What is A union A? It is just A itself this is idiom portent laws they are called idiom portent laws. Similarly, A intersection A is also again A it consists of all elements in A only. And if you add the empty set to A A union phi in essence you are not adding anything so that is A.

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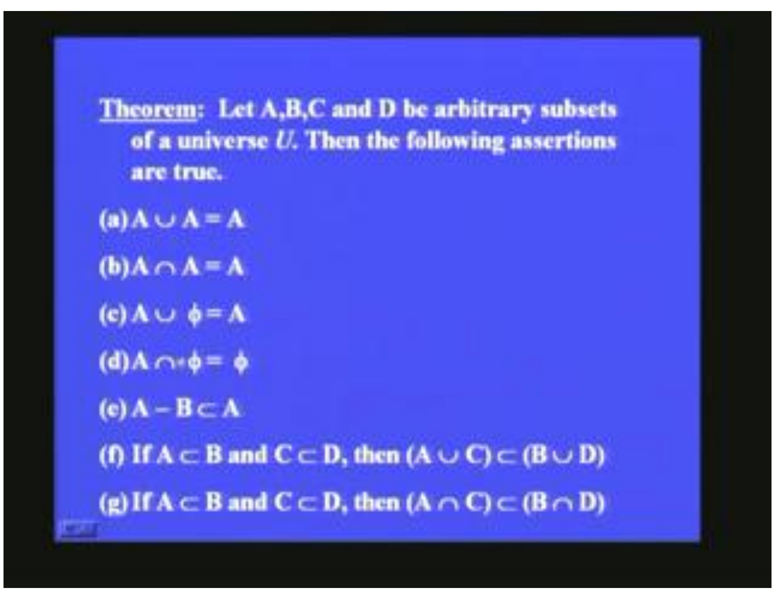

And A intersection phi means it will consist of all elements which are common to A and common to the empty set but empty set does not contain any elements so this is equal to empty set. And A minus B will consists of all elements which are in A but not in B so that will be obviously contained in A. Again you have, if A is contained in B and C is contained in D then you will have A union C contained in B union D. These things you can prove using Venn diagram.

For example, consider A, A is contained in B so this is A, this is B, A is contained in B and C is contained in D B union D is the whole set and A union C is this portion.

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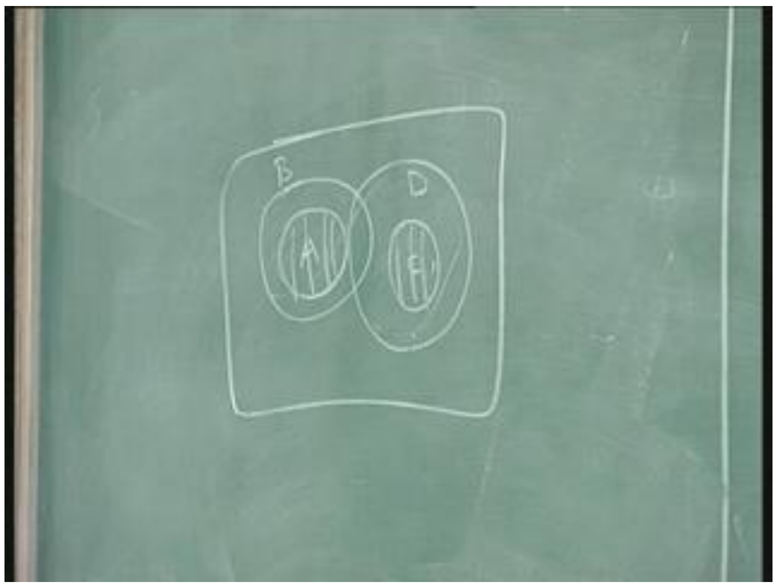

We can see that A union C is contained in B union D. These are not very difficult to visualize. Similarly, if you have A is contained in B and C is contained in D then A intersection C is contained in B intersection D. You also have the following theorems A will be contained in A union B and A intersection B will be contained in A.

If A is contained in B then A union B will be B because if A is contained in B B is a bigger set than A and all elements of A are already in B. So A union B is nothing but B itself. Similarly, if A is contained in B all elements of A or in B so if you take the intersection that means the elements common to both A and B that is nothing but A itself. And A minus phi denotes the set of all elements belonging to A but not belonging to the empty set. And you know that empty set does not have any elements so A minus phi is just A. And A intersection B minus A will be empty because B minus A consists of all elements which are in B but not in A and A has some elements so there is no common element between these two so their intersection will be empty. Similarly, you have A union B and B minus A is A union B because this denotes all elements of B which are not in A and this denotes the elements which are in A so the union will be A union B.

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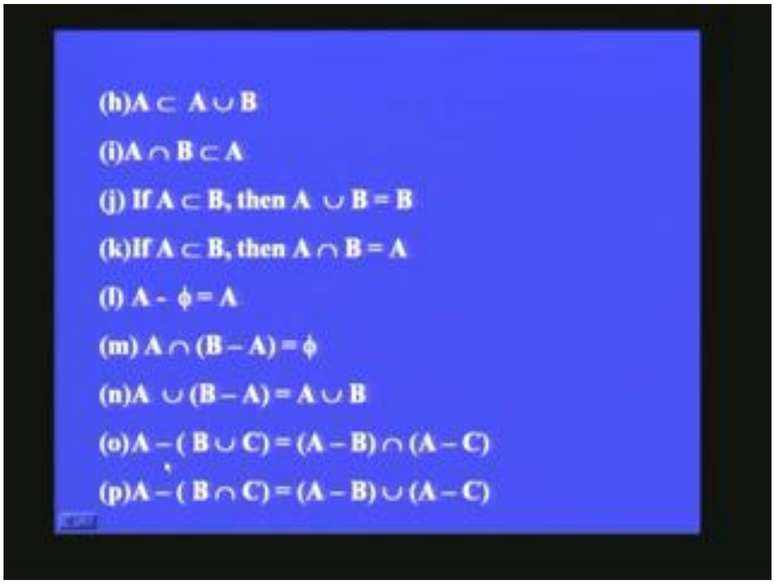

Similarly you can also prove these two very easily, A minus B union C will be A minus B intersection A minus C and A minus B intersection C will be A minus B union A minus C. You can prove these things using Venn diagram. Let U be a universe and A be a subset of U. The complement of A denoted by A bar is the set A bar is equal to U minus A. That is the set of all elements x where x does not belong to A.

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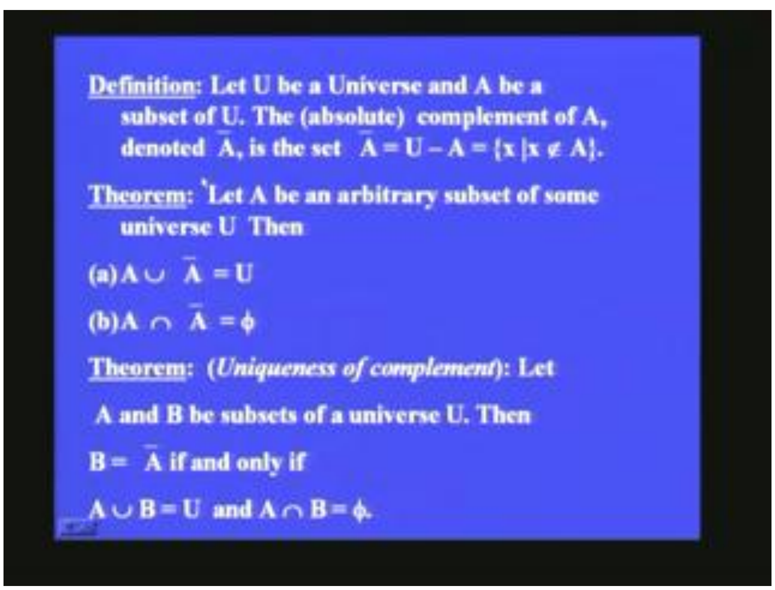

So, if the underlying universal set is given like this and A is this set the complement A bar will be the set of all elements which are in U but not in A this denotes the complement of A. A bar is the set of elements in U and x does not belong to A so the complement will be denoted like this.

Now, we can very easily see that let A be an arbitrary subset of some universe U, then what can you say about A union A bar? This is A this is A bar so A union A bar will be the universal set. And there are no common elements between A and A bar so A intersection A bar will be the empty set. Now we have to prove that the complement of a set is unique. You are fixing the universal set and you want to show that the complement is unique, that is denoted by this theorem. Let A and B be subsets of a universe U then B is A's compliment A bar if and only if A union B is equal to U and A intersection B is phi. Let us see how we can prove that.

The theorem states if and only if so you have to prove in both directions. So suppose B is equal to A bar then by the previous theorem the previous theorem states this; A union A bar is U and A intersection A bar is phi. Now we know that B is A bar so A union B is equal to U and A intersection B where A bar is B is empty. The other way round we have to prove that if these are true that would imply B is equal to A bar.

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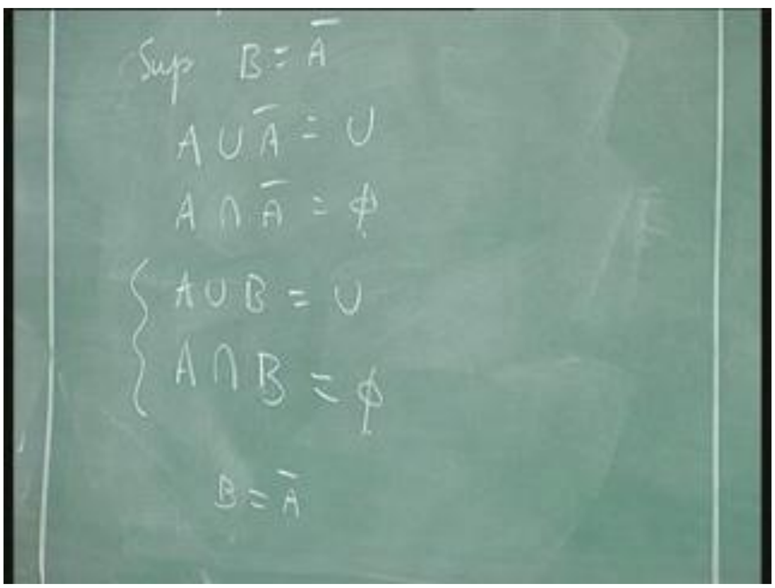

Let us see how we can prove that. Now B you can write as B intersection U and what is U? U is A union A bar. Now, using distributive law this is B intersection A union B intersection A bar. And what is B intersection A? That is empty. We know that A intersection B is empty so this is empty set union B intersection A bar. And this you can write as A bar intersection A union B intersection A bar or if I use commutativity you can write it as A bar intersection B. And using the distributive law on the other way round this is A bar intersection A union B. But what is A union B? A union B is the universal set so this is A bar intersection U is equal to A bar. So starting from B we have come to the conclusion that B is equal to A bar.

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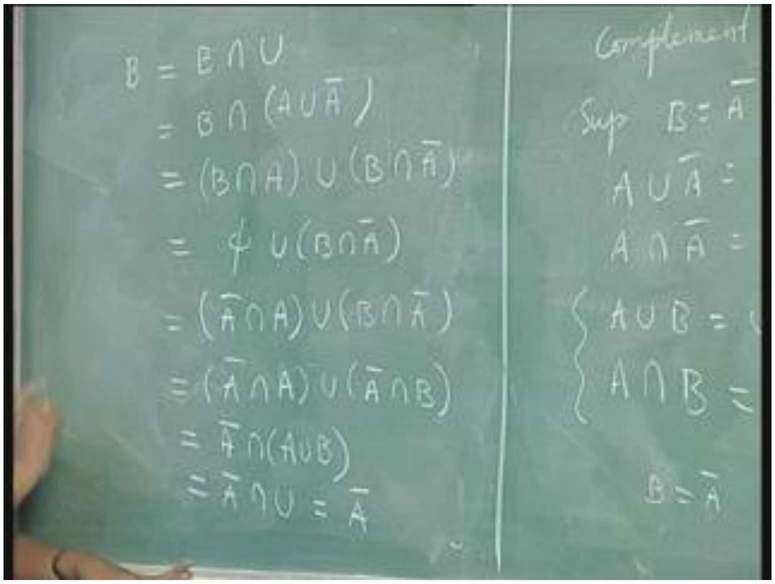

Let us see the steps once more. We know that A union B is the universal set and A intersection B is the empty set. Then we want to show that B is equal to A bar we proceed like this B is equal to B intersection U because any set if you intersect it with universal set that will be the set itself. And I can write the universal set as a A union A bar. Now, using distributive laws this will become equal to B intersection A union B intersection A bar. But we know that A intersection B is empty that is B intersection A is also empty, this is empty union this. But the empty set again I can again write it as A bar intersection A and this portion you can use commutativity and write it as A bar intersection B. And applying the distributive law in the reverse you can write it as A bar intersection A union B. And A union B we know is the universal set so A bar intersection universal set is nothing but A bar. So B is equal to A bar or the complement of the set is unique.

Now you can see that if A bar is an arbitrary set then A bar bar is A. That is if you take the complement of set A and again take the complement that will be equal to the original set. The complement of the complement of A is A. And similar to what we studied as De Morgan’s laws in logic here also you have De Morgan’s laws.

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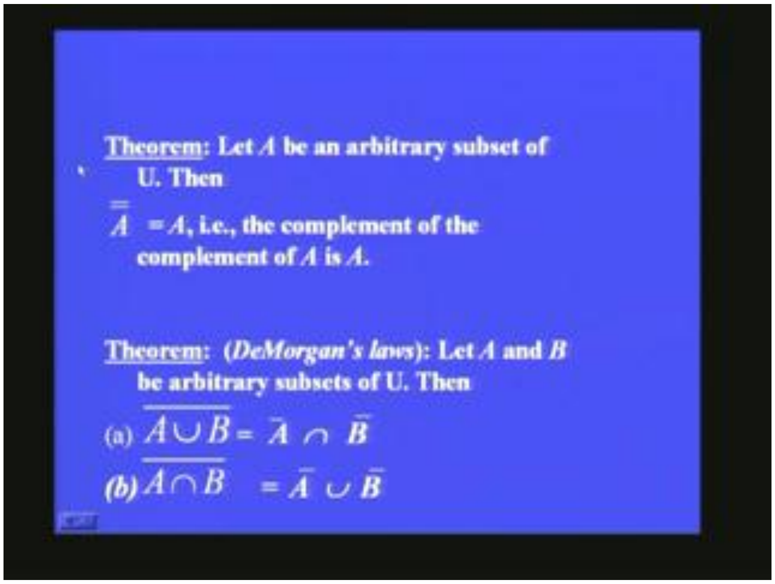

A and B are arbitrary subsets then A union B complement is equal to A bar intersection B bar and A intersection B complement is equal to A bar union B bar. This you can see with Venn diagram very easily. You have two sets underlying universal set is this and you have two sets A B, A union B is this, the complement of this is this portion. A union B complement is the dotted portion and you can see that if you take the complement of A this will be the portion. The whole thing will be the complement and the complement of B which will be this portion.

So if you take the complement of A that is the dotted portion and the complement of B which is the slash portion the intersection will give you this portion because this will not be in the intersection this portion also will not be in the intersection so A bar intersection B bar is equal to A union B bar, these two are equal.

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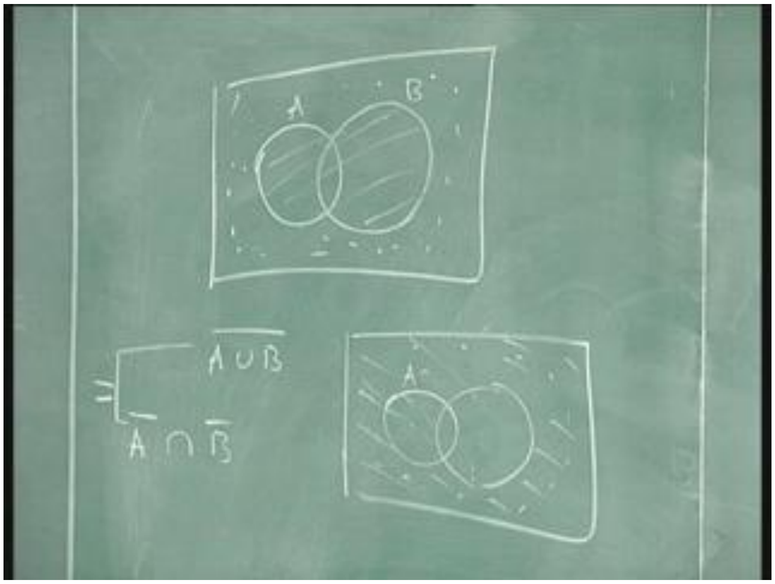

And similarly you can prove the other way round also. That is you have two sets A B what is A intersection B? This is it so A intersection B complement is the whole set except this portion. And A complement is the whole portion except A and B complement is the whole portion except B and you can very easily see that this is A bar union B bar. So when you take the complement the intersection becomes union and the union becomes intersection.

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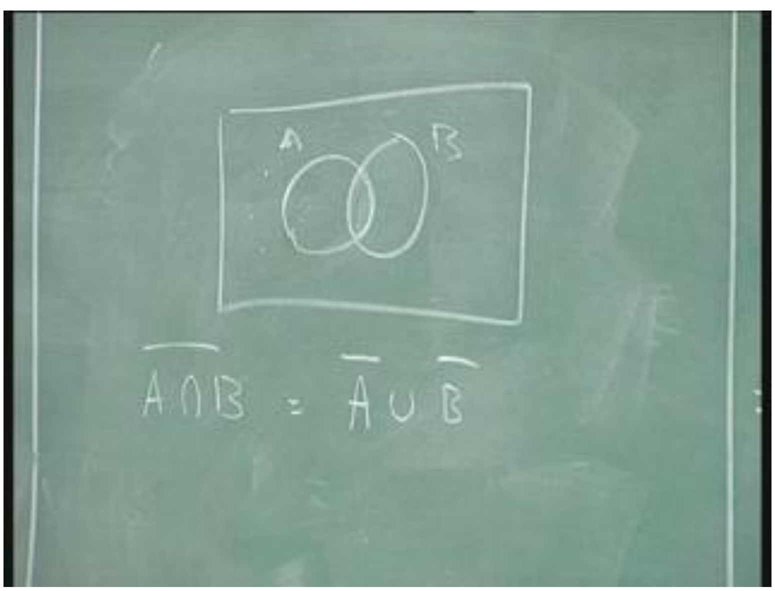

These are known as De Morgan’s laws and you can see the similarity between saying that complement of p AND q is p OR q and so on. NOT OR rather NOT of p AND q is NOT p OR NOT q see the similarity between this and set uni[que]

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Let C be a collection of subsets in some universe U. The union of the members of C denoted by the union of S belongs to C S is the set union and it is denoted by this, the set of all elements such that S belongs to C and x belongs to S. This is extending the definition of union not to two sets but a collection of sets.

Similarly, this is extending the definition of intersection not to two sets alone but extending to a finite collection of sets. You denote the intersection as x such that x belongs to all of them.

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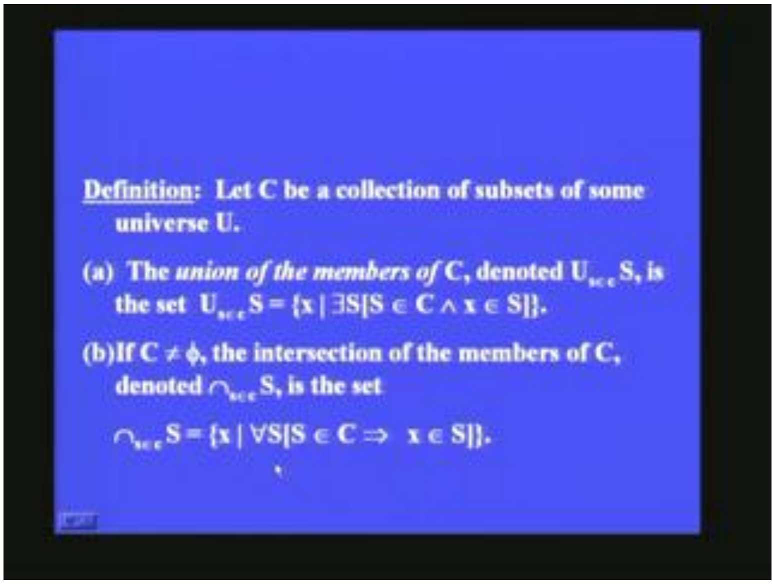

We have seen that you can define a set explicitly or implicitly. Explicitly means you just enumerate all the elements of the set. Implicitly means you define the set using a predicate. Apart from these two types of definitions there is one more type of definition of sets which is known as inductive definition.

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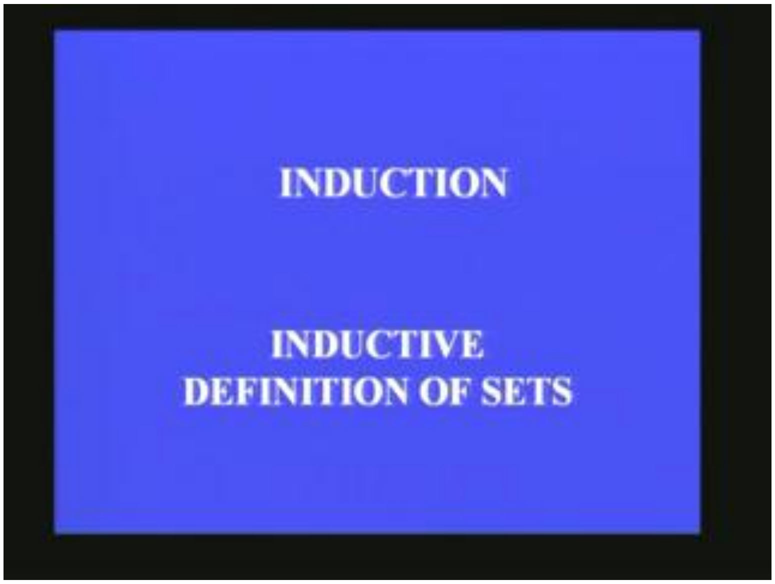

Let us see how we can define a set inductively. Now the inductive definition of a set always consists of three components. The first components are the basis or the basis clause where you define the basic building blocks. The basis or basis clause of the definition establishes that certain objects are in the set. This part of the definition has a dual function of establishing that the set being defined is not empty and characterizing the building blocks which will be used to construct the remainder of the set.

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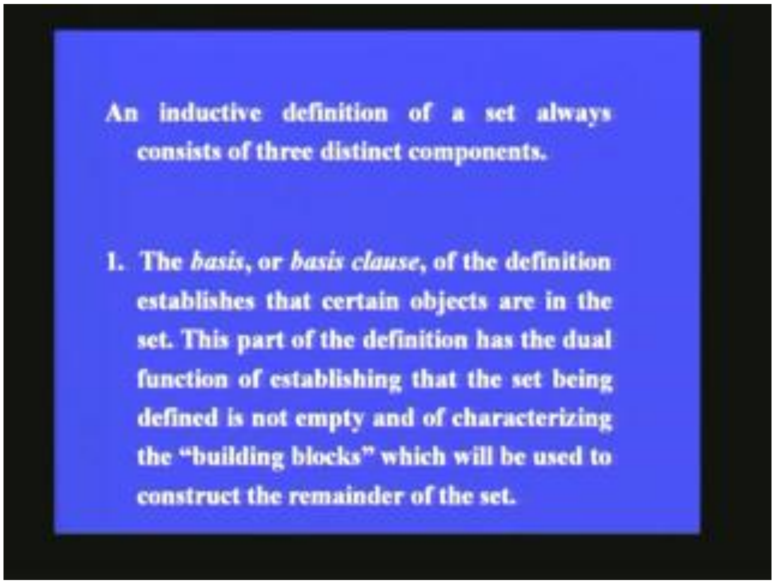

I will give an example later but first let us go through this. Apart from the basis step there will be an induction step or an induction clause. And here this inductive definition has the second part as induction; it denotes the ways in which elements of the set can be a combined to obtain no elements.

The inductive clause always asserts that if objects x, y, z etc are elements of the set then they can be combined in certain specified ways to create other objects which are also in the set. Thus while the basis clause describes the building blocks of the set the inductive clause describes the operations which can be performed on objects in order to construct new elements of the set.

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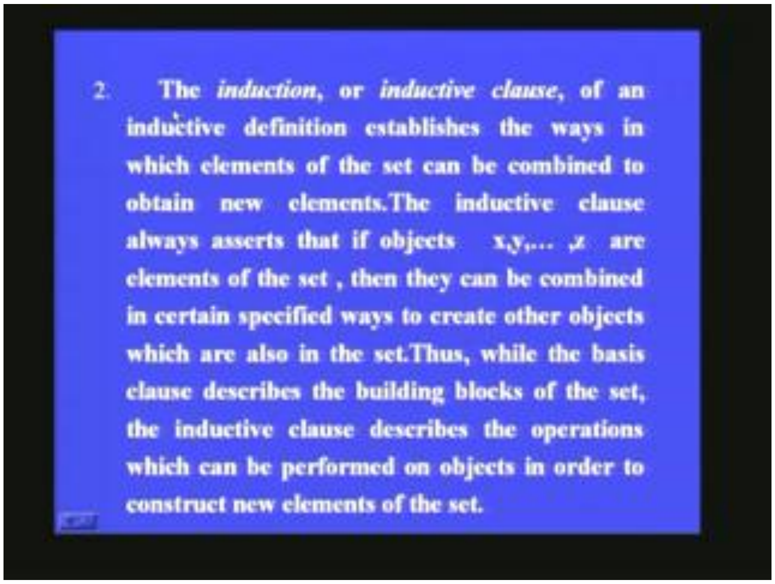

Let us take one or two examples and see. For example take the set of nonnegative integers N, N is 0, 1, 2, 3, etc. You can try to define this set inductively like this; basis clause is 0 belongs to N. Then the induction clause is, this is the basic building block, if x belongs to N then x plus 1 belongs to N.

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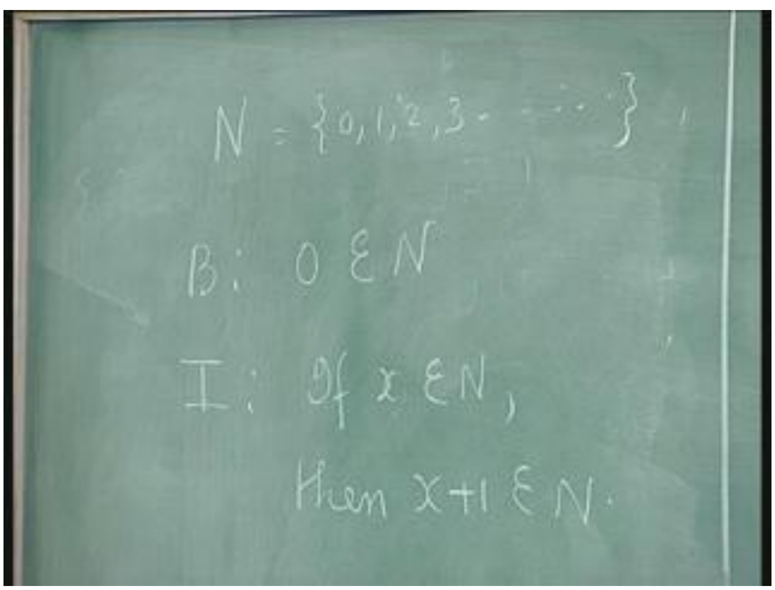

So starting with 0, 0 plus 1 is 1 will belong to the set, from 1 plus 1 2 will belong to the set and you will be able to get all elements of the set. This is just to illustrate how it works. Actually there is a slight flaw here in the sense that you are trying to define natural numbers using addition but actually addition is defined after you define the natural numbers. This is just to show you a simple example. Let me a take up a different example. Consider the set of S, S is the set of well formed strings of parenthesis.

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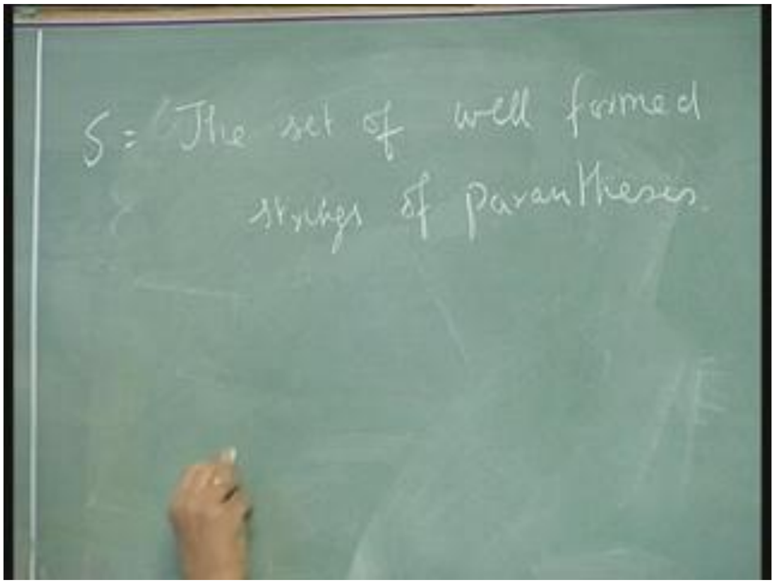

Let me take one particular parenthesis the square brackets alone, so this is an element which will belong to S. Again another element this also is a well formed string this will belong to S. Another element is this and this will belong to S. But this does not belong to S because they are not matching right parenthesis occurring before the left parenthesis. Or you cannot have something which does not belong to S because you have one pair of parenthesis and this does not have a matching left parenthesis. And similarly if you have, this does not belong to S.

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How can we define this set, a string of well formed parenthesis inductively? So as a basis clause you will have, the basis steps you will have this belongs to S. As mentioned earlier the basis clause gives you the basic building block and also it tells you that the set is non empty.

Let us go back to the definition. The basis or basis clause of the definition establishes that certain objects are in the set. This part of the definition has the dual function of establishing that the set being defined is not empty and of characterizing the building blocks which will be used to construct the remainder of the set.

Now, what are the induction clauses or what is the induction clause here? Starting with this how will you find more and more elements of the set? Now in this example it will be like this; if x belongs to S x is a well formed string of parentheses then this belongs to S. So if you have a matching well formed string x adding one more left parenthesis and one more right parenthesis still gives you the well formed string of parenthesis. This is one and another one is if x, y belongs to S then x concatenated with y or x by the side of y belongs to S. That is if you have a well formed string of parenthesis here then placing by its side another well formed string y gives you a well formed string of parenthesis.

For example if I have something like this, this is x this is x and y is say this is x y is this, then x, y will give you this, this is a well formed parenthesis each left element has a matching right parenthesis. So you can try to define many of these sets in this manner. This is called inductive definition of a set.

Now first we consider that there will be three parts to the definition. The first part is basis part which we have seen. The second part is the induction clause this tells you how you are going to perform some operations starting from some elements of the set and build more and more elements of the set.

I will read this again; the induction or inductive clause of an inductive definition establishes the ways in which elements of the set can be combined to obtain new elements. The inductive clause always asserts that if objects x, y, z are elements of the set then they can be combined in certain specified ways to create other objects which are also in the set. So basis clause describes the basic building blocks and inductive clause tells you how to build more and more elements. Apart from that there is one more clause which is called the extremal clause.

The third part of the definition is known as the extremal clause. The extremal clause asserts that unless an object can be shown to be a member of the set by applying the basis and the inductive clauses a finite number of times then the object is not a member of the set. That is you must be able to get all the elements of the set from the basis clause and build more and more elements making use of the operation defined in the induction clause. All elements should be got that way and no other element will be got. The extremal clause of an inductive definition of a set has a variety of forms. You can say it any one of the following ways.

One way of saying that is, no object is a member of S unless it is being so follows from a finite number of applications of the basis and inductive clauses so you can call the extremal clause in this manner. Or you can say it in this manner also; the set S is the smallest set which satisfies the basis and inductive clauses. You are not going to get any more elements only those elements which are built from the basis clause and the induction clause.

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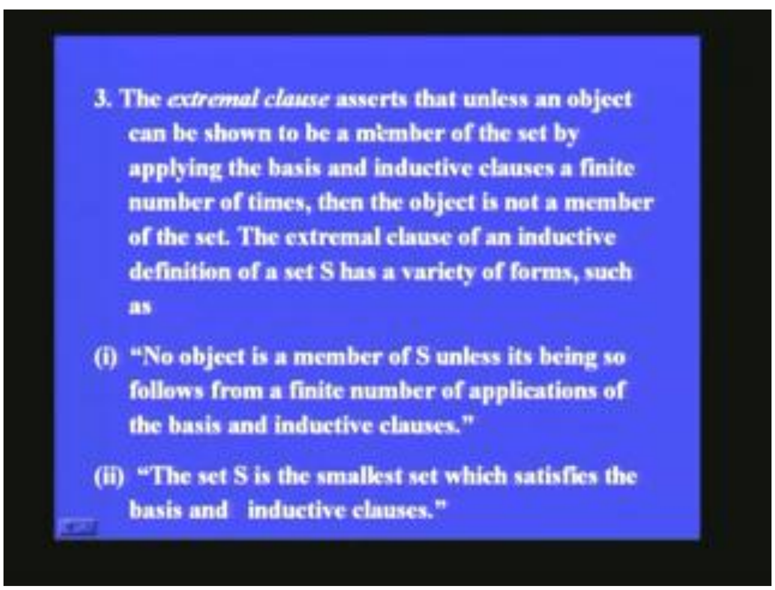

Or you can even say it in one of these two ways, let us see how you can say that. The set S is the set such that S satisfies the basis and inductive clauses and no proper subset of S satisfies them. Or put it in another way you can say it like this; if T is a subset of S such that T satisfies the basis and the induction clauses then T is equal to S. Or you can say it in this way; the set S is the intersection of all sets which satisfy the properties specified by the basis and the inductive clauses.

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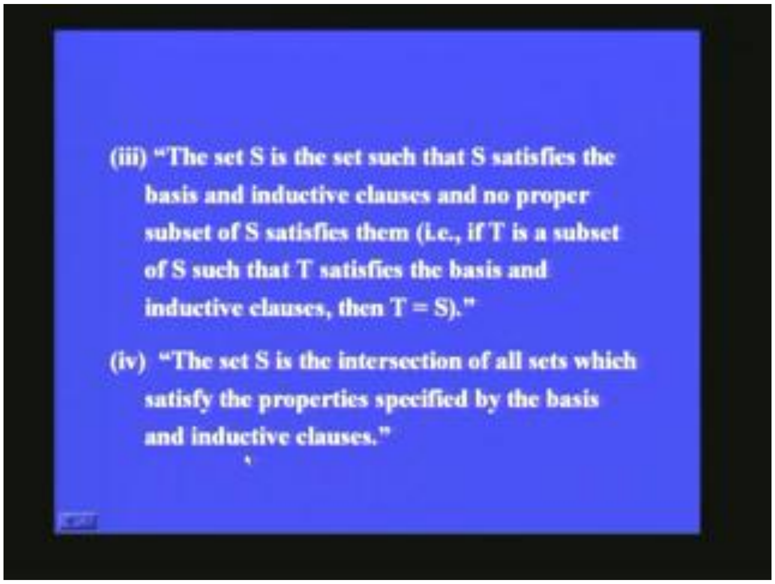

Now we have seen that apart from the implicit definition and the explicit definition you can also define sets in a inductive manner. And in this definition the definition will consist of three parts; the basis clause, the induction clause and the extremal clause. This sort of a definition of a set by induction is used in proofs like proof by induction.

We shall see what is proof by induction in the next lecture. Now the third part of the definition is called an extremal clause. It tells you that the elements of the set are built from the basis and the induction clause and all elements are built that way no other elements will be built and so on.

So we saw four different ways of mentioning the extremal clause. The part you must realize is whatever set you define inductively either it is set of well formed strings of parenthesis, the set of integers, set of even integers or whatever it is the extremal clause is the same. The form of the extremal clause or the statement of the extremal clause is the same for all definitions which use inductive definition but it must be stated explicitly.

The third point apart from specifying the basis clause and the induction clause you must specifically mention the extremal clause. Without that the definition is not complete even though the extremal clause is the same for all definitions. So we have seen this definition let us see how to make use of this in proofs by induction in the next lecture.