Theory of Computation

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Lecture No. #17

Myhill-Nerode Theorem

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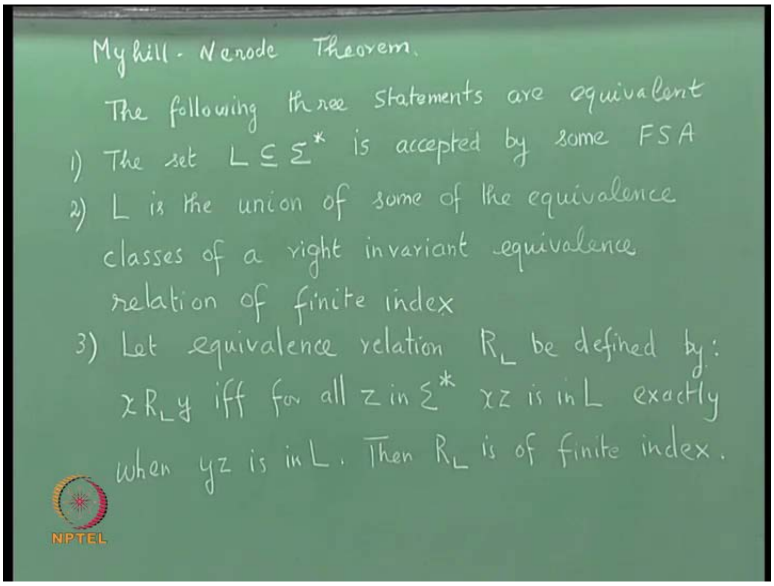

So, today we shall consider Myhill-Nerode theorem,and minimization of finite state automatathis is Myhillnerode theorem.It states thatthe followingthreestatements are equivalent; the first statement isthe set L contained in sigma star is accepted by some F S A.The second statement is L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.Third,let equivalence relation RL be defined by x RL y, if and only if for all z in sigma star x z is in L exactly when y z is in L,then RL is offinite index.So, we will prove this theorem, and then see that how this theorem can be used for minimizing an automata.

We have seen in some examples, where the setwill be accepted by 2 automata, but onewill have less than number of states than the other.So, how to minimize a D F S A,for that the idea in this theorem will be useful. And also we used pumping lemma to show that certain sets were not regular. But then in some cases it was not easy toprove the thatthat is something is not regular, using pumping lemma.For example,we saw an example, where the pumping lemma holds, but the set was not regular.So, another way of proving that something is not regular is by using Myhill-Nerode theorem. So, let us prove this result we will there are 3 statements.

So, we will prove it as 1implies 2 2 implies 3 3implies 1. So, that all of them are equivalent first of all let us recall what is an equivalent solution what is an equivalent solution?The relation is defined over sigma star the set of all strings, over sigma herewhen do you say that in a relation is equivalenceis an equivalent relation if it is reflexive.Symmetric, and transitive. So, if the 3 properties are satisfied it is calledan equivalent solution.

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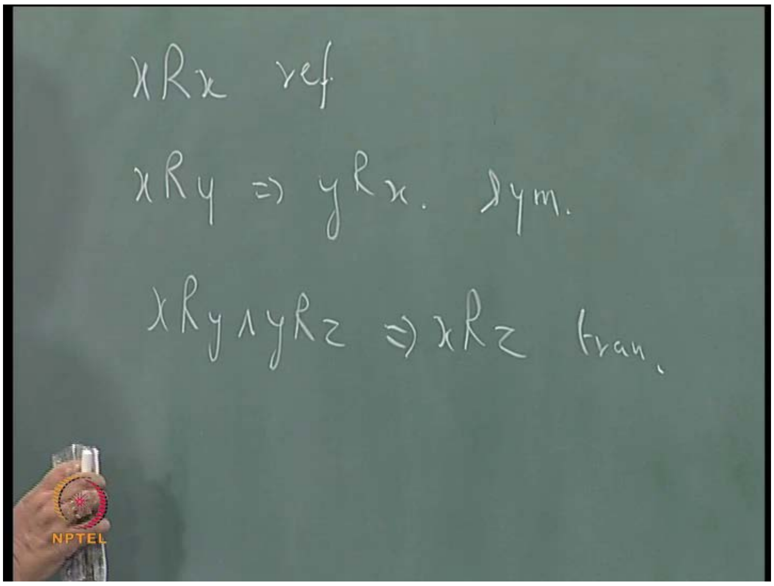

What isreflexive meansx related to xthis isreflexive property,symmetric property meansx related to y impliesy related to xthis is symmetric property.And what is the transitive property,x related to yand yrelated toz impliesx related toz this is transitive property.If these 3 properties are satisfiedthe relation is called an equivalence relationwe have studied about thisin the earlier coursea lot time.Now, the number of equivalence classes,an equivalence relation partitions the underlined set into classes is not.It equivalence relation corresponds to a partition and it partitions the underlined set intoclasses. The number of equivalence classes is known as the index of the equivalence relation.

So, for example, if you consider the relationoverthe set of non negative integers mod 3 relation then you have 3 equivalence classes.Those that leave the remainder zero those that leave the remainder onethose that leave the remainder 2 is not it. And you can also have infinite index there are equivalence classes where you can haveinfinite number of equivalence classes and. So, theindex of the equivalence relation will be infinite.So, what we have to show thatthe set L contained in sigma star is accepted by some F S A implies. That L is a union of some of the equivalence classes of a right invariant equivalence relation of finite index that is the first portion of the proof we have to show.

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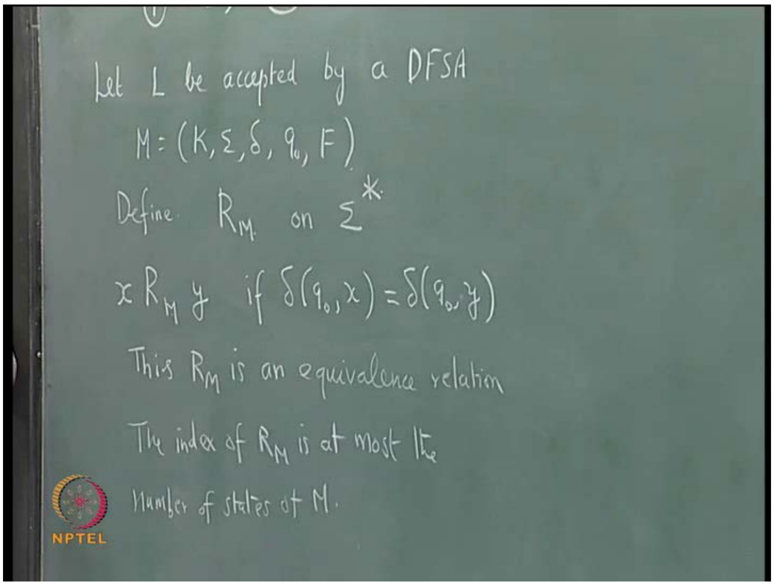

First portion is 1implies 2 again this idea we have seen earlier. For example,a finite state automaton has a finite amount of memory it only distinguishes between equivalence classes of input histories.That is what we said like aset of aadderif you considered the adder, when you considered 2 binarynumbers if it produces a carry you go to onestate if it does produce a carry you go to another state. And similarly, if you take a parity check if it is even parity you go to onestate if it is aodd parity you go to onestate.So, it just distinguishes between 2 equivalence classes. So, those strings which produce even parity belong to oneclass those strings which produce odd parity they belong to another class.

The number of equivalence classes istwothere is not itthe same idea we are going to usehere.So, let L beacceptedby adeterministic F S A.M is equal toK,sigma, delta,q naught, F.And you define an equivalencedefinean equivalence relation RM,onsigma, starhow do you define? Iftwostrings,x and y are related by RM.Ifdelta ofq naught xequal to delta ofq naught. (No audio from 06:35 to 06:44)Two strings, x and y are related by the relation RM, M standing for the machineif delta of q naught x is equal to delta of q naught five.

Why do you say that this is anequivalence relation how can you say it is an equivalence relation? It has to satisfy the 3 properties reflexive symmetric and transitive.Butthis itself is defined using equal to equality relation is not it.So, equality relation has all the 3 properties, reflexive symmetric and transitive. Butanyway you can check is x related to x delta of q naught x is equal to delta of q naught x is not it. So, reflexive property satisfiedif delta of q naught x is equal to delta of q naught y, delta of q naught y will be equal to delta of q naught x.

So, symmetric property is satisfied.If you have an E z delta of q naughtz, if this is equal to thisif delta of q naught x is equal to this and this is equal to this,this will be equal to this.So, ifx is related to y and y is related to z x will be related toz.So, transitive property with also satisfied.So, this is an equivalence relationwhat is the index of this equivalence relation,the set of there are N states in the automaton say q naught q 1q n minus 1.The set of strings which take you from q naught to q naught belong to 1equivalenceclass, set of strings which take you from q naught to q 1belong to another class and soon.

So,how many classes can you have you will have?At mostn why at most n the thing is if somestateis not reachable from q naught you can always remove that.Butassuming such a state is that that will not contribute to anything.So, the number of equivalence classes, will be at most the number of states of the automata.So, thisRM is an equivalence relation(No audio from 09:38 to 09:47).The indexof RM isat mostthe number ofstatesof M,there are other things in the second statement what you have to show is this right invariant what do you mean by right invariant.

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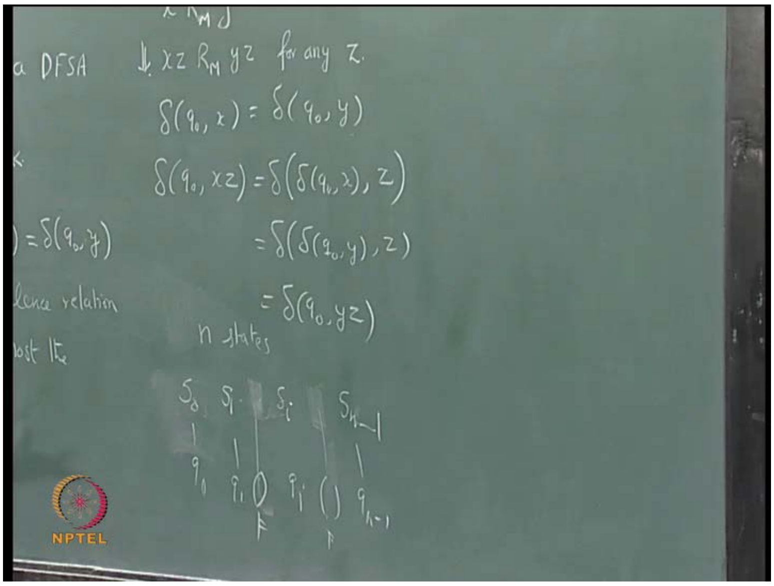

If x is related to y for any z x z should berelated toy z,take any string, if it is right invariant the underlying operation is concatenation. So, this should implythat x z should be related to y z for any zfor any z.Then you say RM is right invariant,how this is very obvious suppose delta ofq naught,x isequal todelta of q naught, y.What can you say about delta of qnaught, x z;that is equal to delta ofdelta ofq naught, x, comma z.And that is delta ofq naught x is equal to q naught y delta ofq naught y,comma zequal to delta ofq naught,y z.

So, RM is right invariantin essence it means that starting from q naught after reading xor after reading y you reach a state.Then from here if you reach a read a z you go to another state.So, starting from q naught afterreading x z you go here after reading y z you go here. So, whether you read x z or y z starting from q naught you go to this state. So, they belong to the same equivalence class that is what it means.So, it is right invariant onemore point is there L is the union of some of the equivalence classes of that relation. So, this equivalence relation RM if there are n states athere will be onecorresponding toq naught that is I will call it asthe equivalence classes.

I will call as S naught S 1S n supposing all states are reachable S n minus 1n states are there.So, supposing all states are reachable from q naught these are the equivalence classesthey are subsets of sigma star sigma star is partition into them.Now, this, corresponds to q naught this corresponds to q 1this corresponds toq nthat is this is a set, of if you have q I that is S i S i is a set of strings which take you from q naught to q I.Now, some of them will be final states, among these states some of them are final statesthis is a final state say this is a final state.Then there will be a class corresponding to that there will be a class corresponding to that.

So, L is a union of them L is the union of such classes and because what is L really? L is a set of strings which take you from q naught to a final state. So, among these classes some of them will correspond to final states take the union ofthem that will be L.So, what have we seen if you assume that L is accepted by aD F S A with n states a.Then you can define a relation RM like thisit is an equivalence relation it is right invariant it is a finite index.And L will be the union of some of those equivalence classes induced by this equivalence relation is that clear. So, this is essentially this one.

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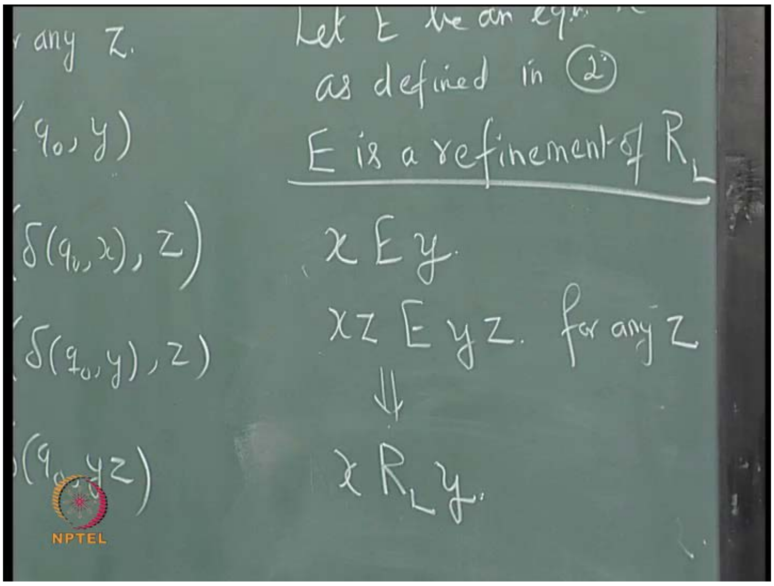

Now, we will go to this next implication 2 implies 3 what is 2 what is 3 we will read again 2 states that. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index. Then 3 states that equivalence relation RL let it be defined as x RL y if and only if for all z in sigma star x z is in L exactly when y x is is in L then the equivalence relation RL is of finite index.Now, we will start with a E let E bean equivalencerelation,asdefinedin 2 that is E is a equivalence relation on sigma starof finite index right invariant. And L is a union of some of the equivalence classesthen RL is definedas given in 3 you show that E isa refinement ofRL.

You show that E is a refinement of what is a refinement,seethere is onea equivalence relationwhich partitions the set like this. Another equivalence relationsomething like this thesetwobelong to thisthesetwoR2 has it is an example R2 has 3 equivalence classes R1has five equivalence classes.In R1this equivalence class of R2 is divided intotwothis equivalence class of R2 is divided intotwothen you say R1is a refinement of R2.This is what we have studied earlierthat isoneequivalence of Roneequivalence class of R1is completely contained in oneequivalence class of R2.It does not split intotwo oneequivalence class of R1is completely contained in oneequivalence class of R2.

Then you say that R1here we say that R1is arefinementof R2.Now, what we want to show that E is arefinement ofor a.Now you havex E y that is x and y belong to a same equivalence class of E then what can you say about x zit is right invariant. So, x z is related to y z for any zx z is related to y z by E because E is right invariant is that clear. This is for any z foranyzbelonging to sigma star.Now L is the union of some of the equivalence classes of e. So, if this equivalence class L if L is L contains this equivalence classthen x z and y z will both bein L is not it. L will contain oneequivalence classcontainly, completelyor it may not contain anything.

It is not that portion of the equivalence class it will containleavethe remaining start like that. L includes oneequivalence class completely or it excludes it. So, if L includes this equivalence class both x z and y z will be in L if L does not include this equivalence class x z and y z both will not be in LSo, for any z either x z and y z both will be in L or x z and y z both will not be in L is that clear. That is the condition of RLx RL y if and only if for all z in sigma star x z is in L exactly when y z is in L.So, that means,if this holdsthis will implythat x RL y x and y are also related by RL is that clear.

What does that mean if x related to y by E x is also related to y by RL that means, oneequivalence class of E is completely contained in oneequivalence class of RL.Maybe 2 equivalence classes of E together form an equivalence class of RL it may be possible. So, essentially x related to y by E means x is also related to y by RL that is each equivalence class of E is completely contained in oneequivalence class of RL. That means, what does that mean E is arefinement of RL and what istheindex of E.We started with the assumption that L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index.

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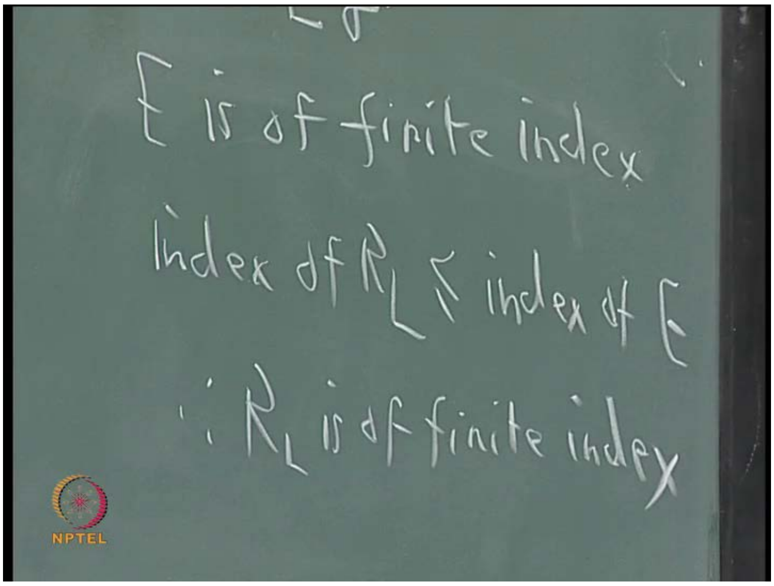

So, what do you say E isoffinite index.Index of RL is less than or equal toindexof E is not it.Because, every equivalence class of E is contained in oneequivalence class of RL index of RL will be less than or equal to index of E therefore, RLis offiniteindex . So, what we have proved is 2 implies 3 next we have to prove 3 implies one.Before going into the thatproof we will illustrate thesetwoproofs by an example. So, look at this example(No audio from 22:28 to 23:02).

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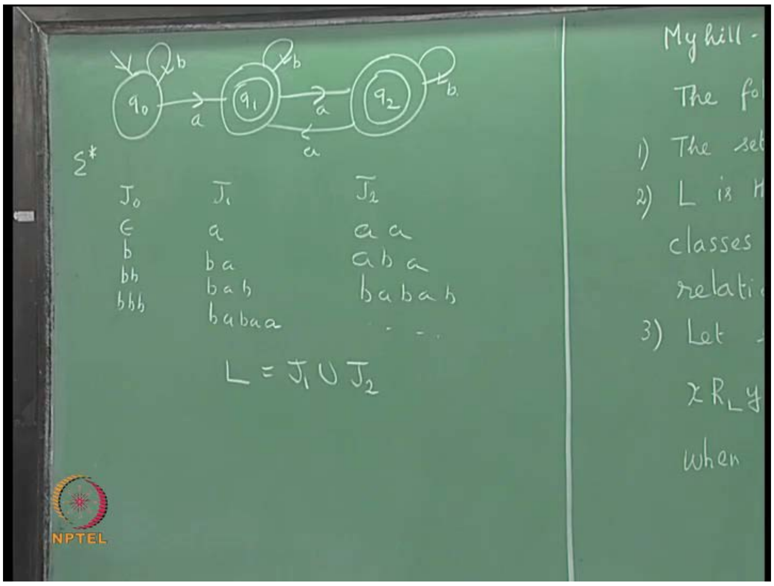

This hasthreestates this is a deterministic F S A and it has gotthreestates.And what sort of strings will be accepted by this machine,any string having at least onea will be accepted any string having onea will be accepted.A sequence of bs will not be accepted if it does not have any a it will not be accepted.So, sigma staryou candivide intothreeclasses J naught J 1J 2threeequivalence classes J naught corresponds to strings which take you from q naught to q naught.J 1corresponds to strings which take you from q naught to q 1.J 2 corresponds to strings, which take you from q naught to q 2.

So, for example, here you will have epsilon bbbbbb and. So, onthis will have ab ab a bb ab ab ab aand. So, on and what of what set of string, will be in J 2 set of strings which take you from q naught to q 2 that will be a aa b ab ab a band.So, on if you look it do it carefullyset of strings which do not contain an aat allwill belong to this class just strings of bs alone.Any string having an odd number of as will be in J 1any string having even number of as will be inJ 2.So, sigma star is partition into 3 classes strings which do not contain any a at allepsilon is also in that.

Then strings, which have odd number of as strings which have even number of as J naught has just epsilon and bsthis has got odd number of as this has got even number ofand what is L is not it.So, you can see that starting with anwith a D F S A you find that it divide sigma star into equivalence classes its finite indexthreeindex is threehere.And L is the union of some of the equivalence classes because it is right invariant this is also right invariant.Because iftwoof them belong to the same class if you follow it with the string, having even number of as you will go to the same class. If you follow itwith the string having odd number of as you will go to this class.

Anywayit is right invariant iftwostrings, belong to the same class x and y x z and y z also will belong to the same equivalence class. So, it is right invariant and L is the union of 2 of them J 1union J 2.So, I will just write like this(No audio from 27:11 to 27:17)just take as a sample only one,one string.Now, how are they related by RL RL ho do you define RL is define like this x RL y if and only iffor all z in sigma star x z is in L exactly when y z is in l.So, you find thatiftake for example,onestring from here onestring from here see RL has to be such thatthis is a refinement of I mean the equivalence classes.

Either you can havethreetwooronein RL the number of equivalence classes will be clanbe more than this it will be it can bethreeor it can betwoor it can be one. So, we have to find out whether 2 of the equivalence classes can be grouped for RL.

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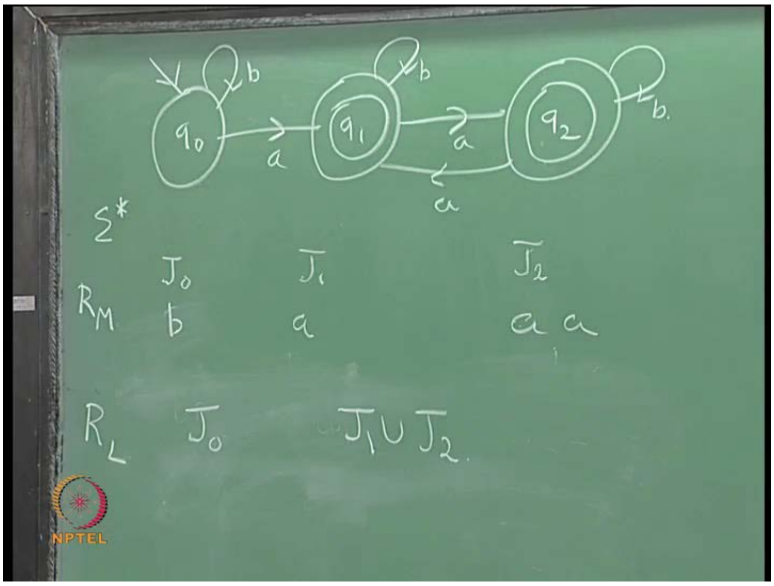

Now, if you take a string from hereand if you take a string from here say for example, b and ayou follow it bybfollow it by b,b b is not in L, buta b is in L. So, they are not related by Rl. So, you this and this cannot bein the same equivalence class for Rl.Now, take from J naught and J 2take b andaa,take z as bbb this is not in L this is in L.So,again J naught and J 2 cannot be grouped together in RL.Now, taketwostrings, in J 1and J 2 say w 1and w 2take any z after reading, z you arew after reading w 1you are in q 1after reading z either you will be in q 1or in q 2 you cannot go back to q naught.Whatever maybe z after reading w 1you are in q 1after reading againz you will be in q 1or q 2 whatever it is w 1z will be accepted.

Similarly, w 2 starting from q naught after reading w 2 you go here,then read zyou will be either in q 1or q 2 w eare not going back to q naught. So, that will also be accepted. So, w 1whatever maybe z in this case w 1z and w 2 z both will be accepted.So, we can group them together J 1and J 2 can be groupedtogether RL has only 2 equivalence classes J naught and J 1union J 2 this is Rm.(No audio from 30:51 to 30:59)So, index of RL is what is the index of RL 2.

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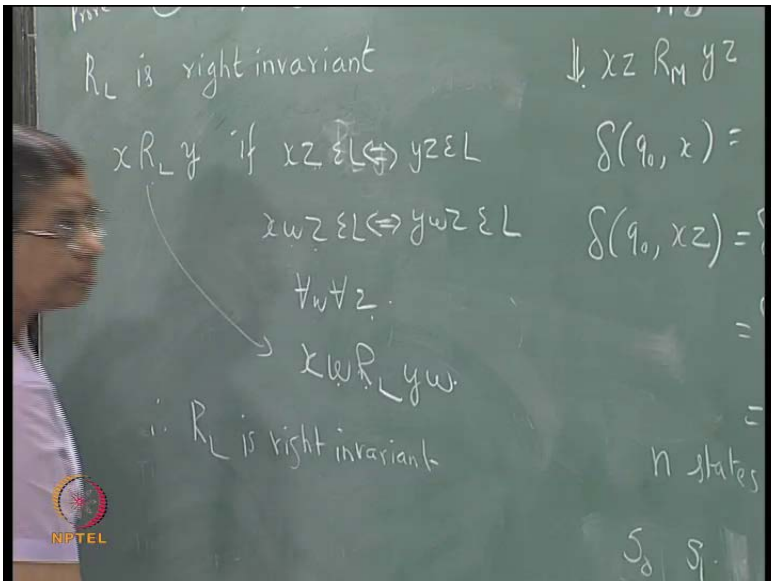

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Now,you have to show 3 implies 1this completes the proof.Now, how do youfirst of first of all show that RL prove RL is right invariant,first prove RL is right invariant how do you prove this?The definition of RL is x RLy ifx zis inx zbelongs to Lis equivalent to saying y zbelongs to L.Now, instead of taking z I take w z,that is x w zthis can be anything.So, I say x w z belongs to L saying equivalent to saying y w zbelongs to L,whatever maybe w zfor all w for all zis not it. Instead of saying z I can takew z does not matter is not it. So, x whatever maybew whatever maybe z x w z belongs to L,impliesequivalent to saying y w z belongs to L.

That means,this is for any z. So, what do you conclude x wrelated to y w,when can you say x w is related to y w whatever maybe z x w z and y w z either both of them should be in L or both of them should not be in L. So, from this you can conclude that x w is related to y w,you started with this and you have come to thiswhat does that mean?What does that meanRL is rightinvariant is not it.So, therefore, RL isright invariant.

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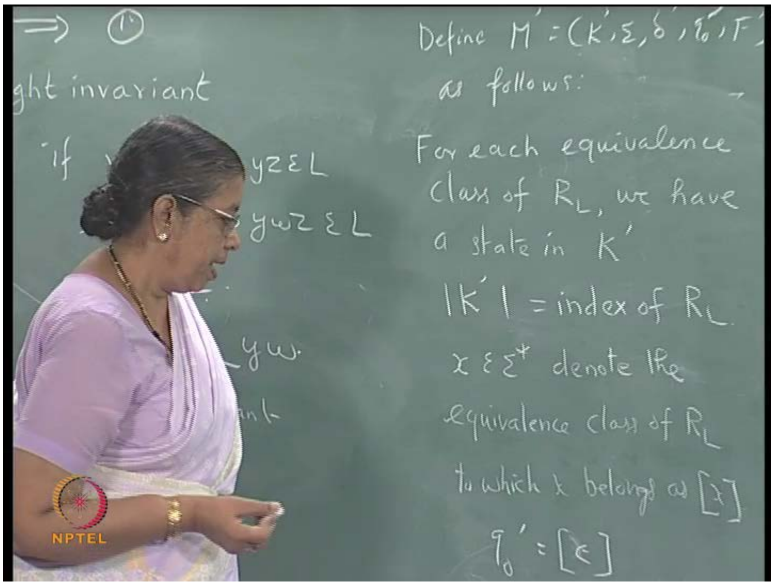

(No audio from 34:04 to 34:18) You define aF S A definingan F S A M dash is equal to k dashsigma delta dashq naught dash F dashas follows. (No audio from 34:34 to 34:44)For eachequivalenceclassof RL we havea statein K dash.Corresponding to each equivalence class of RL you have a state in K dash. So, what is the number of states of k dash index of RL this equal toindexof RL.Now, if x is a stringif x belongs to sigma stardenote the equivalence class (No audio from 35:44 to 35:51)classof RL to whichx belongsasx within the square bracket.(No audio from 36:09 to 36:17)So, we have defined K dash sigma is the alphabet q naughtdashq naught dashis the equivalence classto which epsilon belongs.

Empty string belong to oneof the equivalence classes that equivalence class corresponds to a state and that is the initial state.

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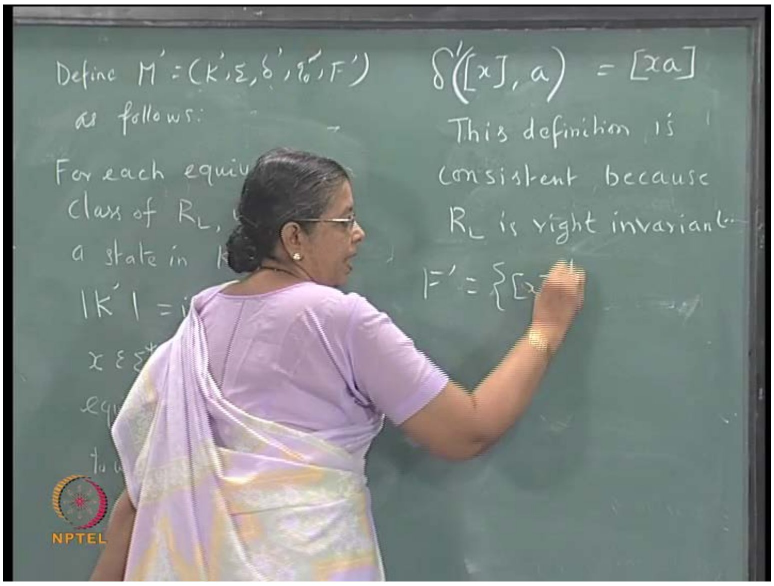

Anddefine like this, delta dashxcomma athis is an equivalence classis a state which represents the equivalence class to which x belongs.Then after reading a symbol a to which state does it goit goes to the statex a define like this and this is a consistent definition because of right invariance.Thisdefinitionisconsistentbecause RL isa right invariant,see instead of saying sayxcomma a delta of x comma ayou can x and y suppose x isrelated toy,you could write like this also.

Nothing changesyou are allowed to write like this because x and yhere you can write x or y they belong to the same equivalence class.And by right invariance x RL y a.So, whether you write x a or y a it does not matter they belong to a same class and onestate corresponds to oneequivalence class.So, the consistency comesbecause of the right invariantproperty. So, we have defined k dash sigma is the samedelta dash q naught dash.

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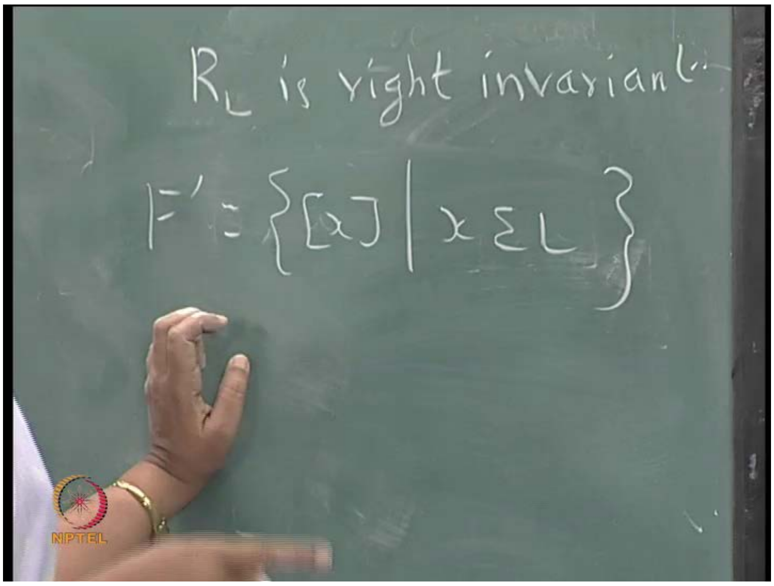

What is F dash F dashcorresponds to those states(No audio from 38:55 to 39:04)some of them will includethere will be include some equivalence classes will be in L. So, make those states as final states that is all.Then you willthat is that is assumingthreeassuming this portion you are defining in equivalence relation all such that x RL Y in.And only if for all z in sigma star x z is in L exactly when y z in L in this case you show and RL is finite index then you show that it can be accepted by a F S A.So, assuming this we have constructed a deterministic F S A in this manner.Now, let us go to this example and continue with this example.Now, we have seen that RL has 2 equivalence classes. So, if you want to construct a finite state automaton it will have 2 states.

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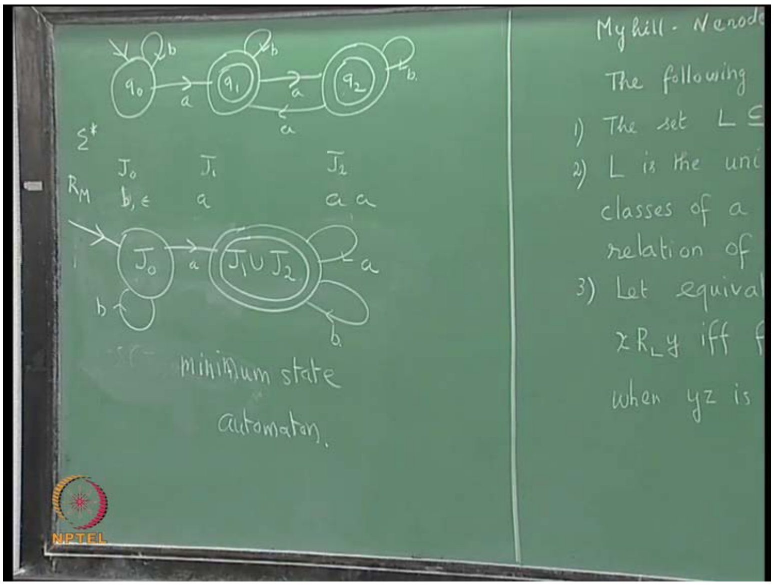

One state will be J naughtanother will be J 1.And epsilon you can see that epsilon belongs to J naught J naught contains epsilon bepsilon like that. So, this will be the initial state,and J 1this equivalence class and they are corresponding to final states.And while whichever is in J 1it will be accepted, whichever in J 2 it will be accepted. So, this is the final state,how are the transitions definedstarting from J naught after reading aa where will you go tostarting from here after reading a ago to q 1. So, that is start with epsilon after reading a you go to this equivalence class.Delta ofq naughtdelta of J naughtawill beJ 1.

So, it will be a starting from here after reading a b you will go to q naught itself that is if you are inthis equivalence class something is there xthen after reading a b it will be againin this equivalence class.So, it will be like thisand similarly, you will see thatyou have thesetransitions.(No audio from 42:12 to 42:20)We can very easily see that they accept the same language any sequence of b is alone will not be accepted epsilon will not be accepted. Butif the string has onea afterwards whatever you get is immaterial it will be accepted. If you get onea then whatever you get it will be immaterial.So, what you have done really is you have joined these together you have merged the 2 states.

And you have got this the same set is accepted by this, butthis is a minimum state automaton. So, this is theminimum stateautomaton.(No audio from 43:03 to 43:15)So, we haveproved the Myhillnerode theorem.

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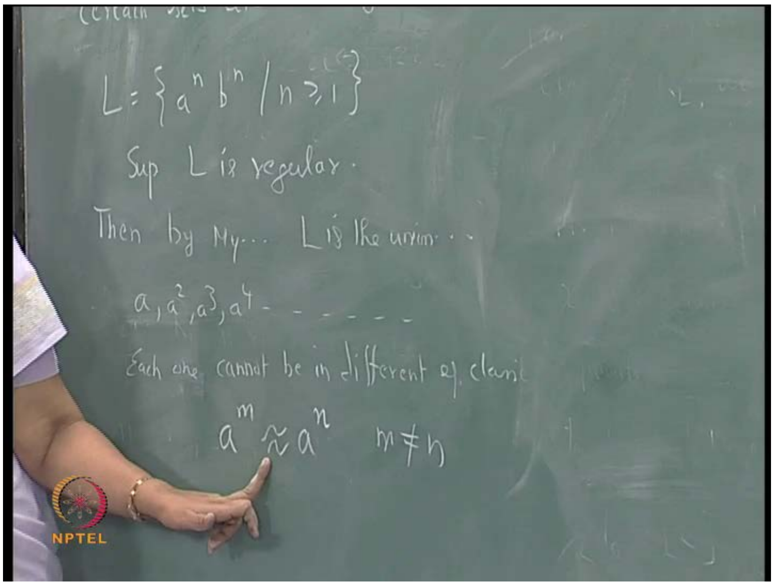

Andmaking use of Myhillnerodetheorem we can minimize an automatagiven given at deterministic F S A you can find the minimum state automata.Anotheruse of Myhillnerode theorem isto showsomething is notregular.(No audio from 43:44 to 43:49)Let us consider minimum automaton in the next lecture we will show we will just show how to make use of this Myhillnerode theorem to show that certain sets are not regular.UseMyhillnerode theoremto showcertainsetsare not regular,take L to bea power nb power nn greater than or equal to 1we have used pumping lemma to show that this is not regular.

Actually you will find that it is in some casesit is easier to useMyhillnerode theorem some cases pumping lemma is easier to use depending upon the problem we have to see which oneto use.Now, we want to show that this is not regular. So, how do you provesuppose? L isregularsuppose L is regular, then what do you conclude from the Myhillnerodetheoremthenbya Myhillnerode theorem L is theunion ofa statements.We can write L is the union of some of the equivalence classes of a right invariant equal relation of finite index. So, that equivalence relation divide sigma star into equivalence classes and they are finite.

Now, consider the stringsaa squared a cubeda power 1 they are all belonging to L they are all belonging to sigma star.Now, all of them cannot be in different,different equivalence classes the number of equivalence classes is finite is not it. The number of equivalence classes into which this is divided is not it is finite.So,allcannot beindifferentequivalenceclasses,that is each onecannot be I would ratherput alleach onecannot be in a different equivalence classis not it.

You have a squared apower for etcetera all of them cannot be in different, different equivalence classes some of them have to be in the same equivalence. So, for example, a power M and a power nfor some M and n M naught equal to n M naught equal to n they are in a same equivalence class you write it in this symbol a power M and a power n are in the same equivalence class.

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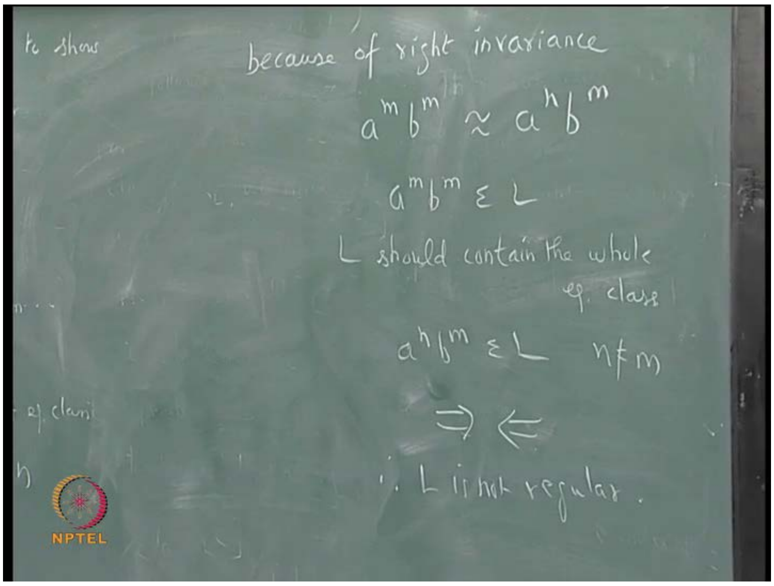

Now, because of right invariance what do you havea power M b power M and a power nyou multiply or you can catenate with the b power M.These twowill be the same equivalence class.Now, L will contain oneequivalence class completely or it will not contain that equivalence class if L see a power M belongs to L a power M b power M belongs to L.ButL should contain that whole equivalence class is not it L shouldcontain thewholeequivalenceclassthat isa power n b power M alsobelongs to L where n is not equal to M you come to this conclusion come to the conclusion that a power n b power M belongs to L where n is not equal to M this is a contradiction.

Because the language is only a power n b power M equal number of as followed a numberof as followed by an equal number of bs is not it. So, you are arriving at a contradiction therefore, L is notregular.So, in some cases Myhillnerode theorem is easier to apply rather than pumping lemma pumping lemma sometimes it becomes a little bit difficult to apply. (No audio from 49:31 to 49:35)So, these aretwoways of proving that set is not regular, and Myhill-nerode theorem can be used to show find the minimum state automata, this we shall consider in the next lecture.