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NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

Pattern Recognition

Module 01

Lecture 08

Rank of Matrix and SVD

Prof. Sukhendu Das
Department of CS&E, IIT Madras

We continue the discussion on concept of linear algebra and we are into the last two important aspects of concept with respect to matrices one is the rank and the other is the decomposition and specifically we will talk of singular value decomposition as which is commonly used in may pattern recognition concepts we will define if it is a rank and also see how the rank sis very closely associated with the SVD.

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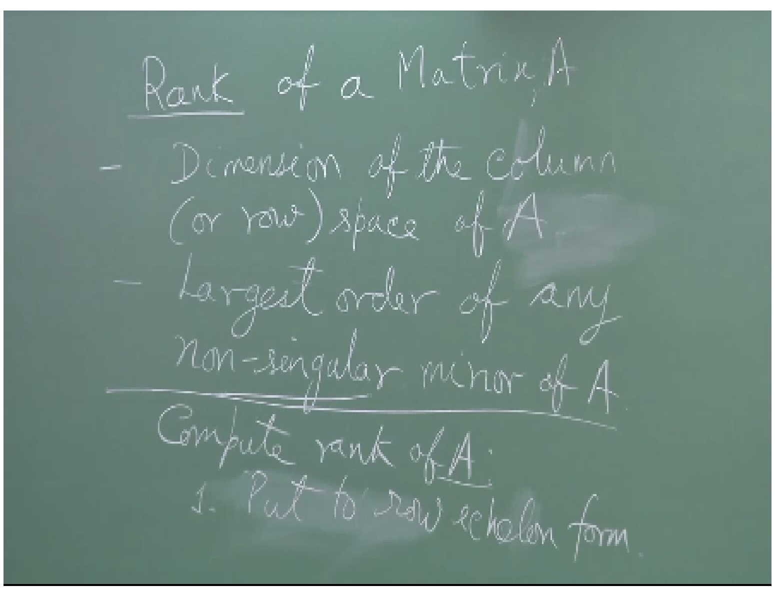

I will give you a two definitions of the rank and we will just discuss two short examples of the method of how to calculate the rank of matrix so one definition says that it is the rank of a matrix is the dimension of the column or row space of matrix I will give you the other definition and then discuss what a column space, the rank of a matrix can also be defined as the largest order of any non singular minor of a matrix rank of a matrix we will say A.

So as you can see that the word on singular it is coming so the rank is very closely associated with a singularity of matrix we will look at this definition first which is talking about the column or row space of a matrix A so we are talking about an obituary matrix A of size m x n and the column space or row space is defined by the number of linearly independent column vectors which now form the matrix A and it is also called this man of A.

Okay and it is called the span of A okay and this is why it is form just span or what is the column vectors if there span particular space which we talked about space some time back and that is the dimension of that corresponding space is actually the rank and we will actually look at this definition take an example of how to compute a rank in very brief an then look at the alternative definition of what is the non singularity the non largest non singular minor of A okay.

Now there are several methods to compute the rank we will briefly discussed one such method to compute the rank of A and the therefore steps I am writing in brief reduced to a row or column form that means this is the processing done to A to compute this rank okay so take A and reduce to reduce enough correct word convert I will say or put to row Echelon form identify the pivot column and number 2 and number 4 number of pivot columns is the rank or you can say rank = the number of pivot columns so we will take small example.

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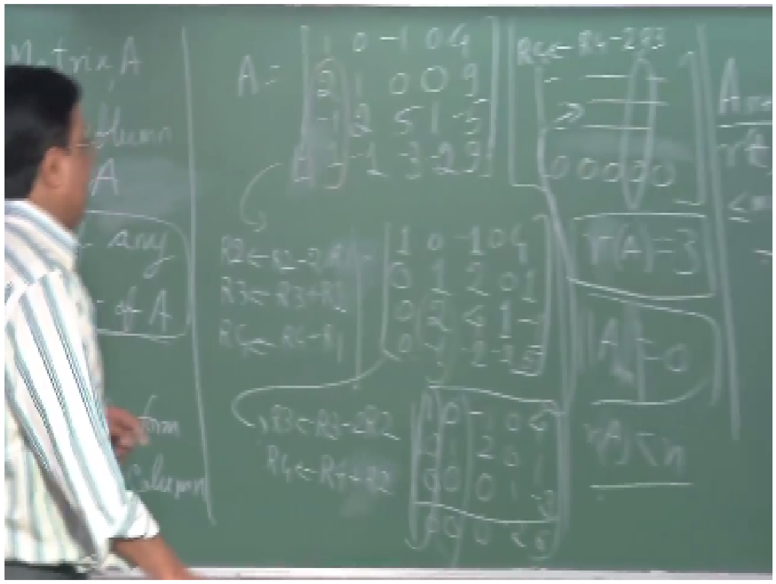

We illustrate this let us take I will not work out fully I just leave part of this is an exercise for you, if you take an example let us say as given here I am taking an example from the book we will give the references for these at the end of the talk, so this is a simple 5 cross 4 matrix okay the 5 columns and 4 rows and first we actually stopped up with the first column and try to reduce the leading diagonal to 1 if it is not so by suitable row column manipulation inter change and then the task will be to minimize this values.

Of the leading elements in the corresponding loads to 0 so just to given example that what you can do is multiplied this by 2 and subtract from this you can directly subtract this from this row and sorry add these two rows basically and 4 rows to be subtracted from the first and that is how what will give you the leading diagonals if you do that what we are saying is that these are the operations let me write them for the row where 2 it is basically R2 – 2 or 1 okay I will write it in short.

R3 this to be a sign by just adding R2 an R3 sorry R3 and R1and R4 you need to assign by subtracting R1 from R plus if you do this I leave it to an exercise for you that you will get this matrix the first row remains the same then you will have a 0 here, because you multiplied this by 2 and subtracted from here and the rest will be okay 0 to 4 1 – 1 this worked out this example is as I said before is given in book so you can actually and look at that okay so this what has been done with the first column you keep doing the same thing with you know remaining column. So you can see that they already have 0 and 1 leading diagonal 1 here so to put the corresponding values 0 at this point here and this will involve operation such asR3, so what has to be done with R3 you multiply this by 2 and subtract from here so R3 will be taken as R3-2R2 and R4 you just add these two so R4 will be okay.

And if you do this the first row remains the same 10-104 second remains same 01201 this one this will become 0 okay, this will also become 0 this 3 and then finally you have 00 and checked out yourself then these terms almost cancel out till this point -2 and 6 because that is what you are adding. So you continue on this form the final thing is what you need to do now is you have already have 0 here so you have to make this 0, okay.

And then convert that is what you need to do so what you do is basically multiply this by -2 and added to this so R4 so the next operation of the last in fact operation will be r4 is R4-2R3 okay, and that will give you I am nit reproducing the first four rows because they will remain the same and what you will get is, what are these four rows they are actually this sorry three rows, okay and from here, this upper sub matrix is going to sit here, okay.

These three rows again I repeat will be the same the last row will remain this, okay. So once you have done this that means what you have got is these three rows at this point and the last row 0 you need to find out how many columns you have which actually so if you find it here that there is a 1 here it will lead to find out the base is so you need to find out how many columns have only one at the corresponding leading position.

So you will have 1, 2 and 3 okay, so in fact it will be this column here and another column here and this column at this position okay, because this row will be 0001-3 so you will have 1, 2 and 3 the rank of, it is not written with R the rank of this matrix A is 3, this is why in trivial way by which you can complete the rank of a matrix. Of course there are many other better methods to compute the rank and we will talk about that soon.

Let us go to the other definition of the largest non singular minor okay, now the rank of a matrix R can be talked about you know in different ways with respect to a non singular minor I mean the corresponding rank of a matrix A is considered to be that you find out if there is one non 0 minor in a matrix A and that becomes the rank that is one way we can have that.

And if you take any other minor which is larger than that particular non zero minor that minor vanishes or it become singular, okay. So the rank of a matrix is the largest non singular minor, how do you obtain a minor from a rank A basically if a matrix A is of size say nxn or mxn you suppress one row and one column and get one minor which is one order less you can suppress two columns and two rows and so on and so forth.

So you can suppress k columns and k rows for a matrix A and create the corresponding minor, okay. So if you take this definition now the second one with respect to a non singular minor so that means if a matrix is singular okay, if a matrix is singular that means it determinant and vanishes and it is well we are talking about definitely a square matrix here and I talk of a determinant or so you are talking about a mxn matrix square matrix which is singular determinant vanishing you are definitely talking about the rank of the matrix A in this particular case is less than n.

That means not a full learned matrix okay, it is rank is at least one less than its corresponding order and if you have an arbitrary matrix mxn I write here may be need of the board.

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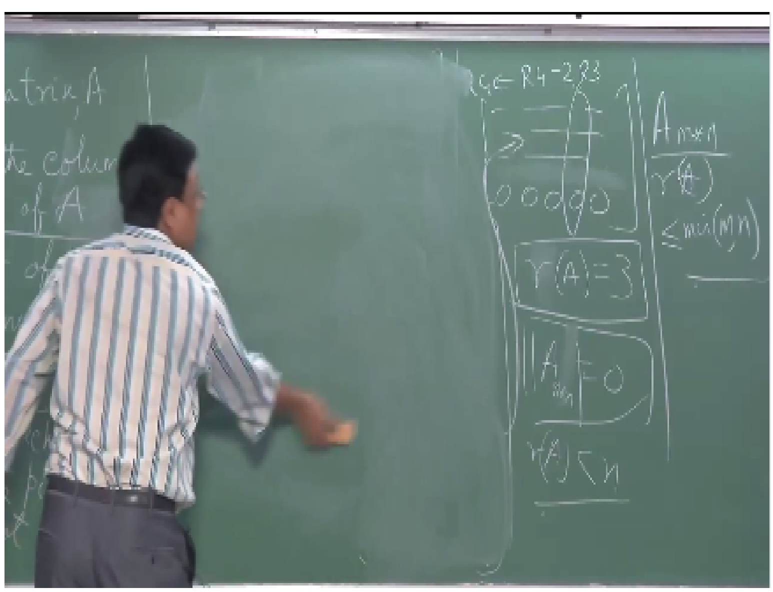

The rank of A is less than or equal to minimum of m,n okay it depends up on the column space or the row space we talking about from that you find out the dimension of the column or rows space of A it will tell you the corresponding rank. So these are corresponding alternative definitions of rank and there are other terms associated with rank it is you often call a singular matrix as rank definition matrix.

And corresponding to the rank we have short of a rank space and a corresponding a null space which is used in various manipulations of pattern recognition systems and applications, okay. There are few properties I would like to specify with respect to the rank and to do that I will rub this example here, okay.

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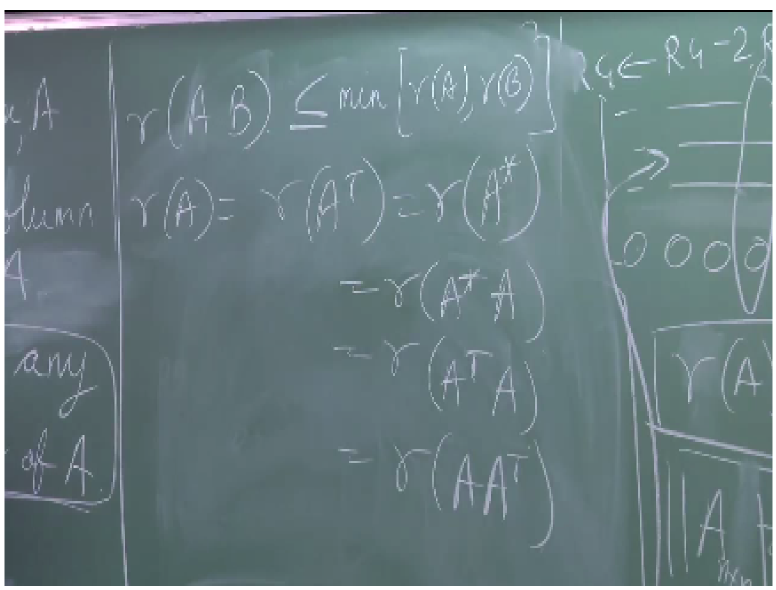

As for example if you take the product of two matrices A and B and the corresponding rank of a matrix is less than or equal to minimum of rank of A and rank of B, is less than or equal to the corresponding ranks okay, then there are other types of properties with respect to the ranks a rank of a is also the rank of its transpose of that corresponding matrix or this matrix is complex we talk of the complex conjugate in general or also it is or equal to so this is the nice property which one can exploit with respect to rank of him corresponding matrix a and these are important A^T a and AA^T are very common expressions you will get in certain application of compute science here is elected to that.

So after we have got the concept of a rank we move to the important concept of analysis of matrices which is singular value decomposition.

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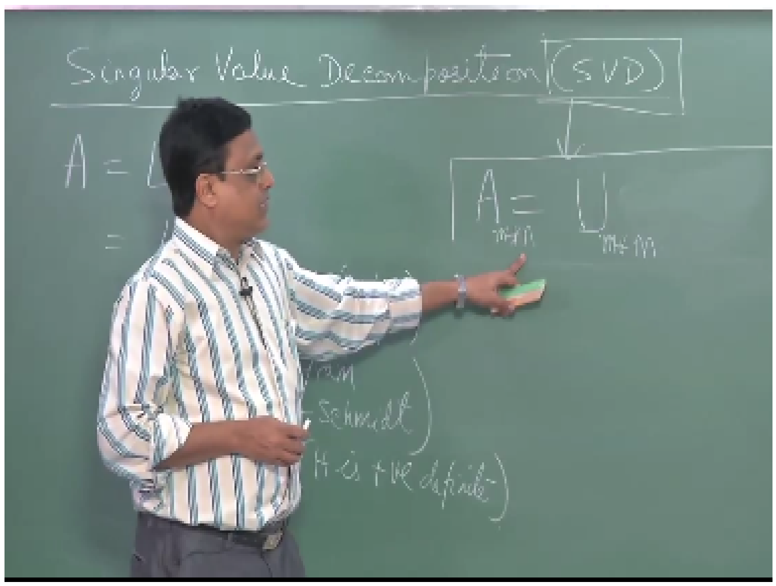

Or in short we will mention it as SVD people usually will call it SVD which means singular value decomposition, and it is very closely associated with the concept of rank which may have discussed just now. SVD is only one of the type of decomposition possible for matrix a we will just name a few other decompositions which exist in literature and definitely some of them are used in the field of pattern decomposition SVD being the most commonly used popular one but given a matrix a there are various other types of decompositions which are possible which are non SVD type I am just name it a few of them okay.

The famous one or the first one which you will find commonly in the book is as a splitting a matrix a x two components lower triangle or upper triangular matrix a very common process which is used for many process including equation solving you can also split this in to another form which is ld u this equal to does not mean that this LU is the same as this what I mean is that you can either split A x L x U or LD and U where d is actually a strictly a diagonal matrix.

L is an upper triangular matrix so lower triangular matrix and U is the upper triangular matrix repeat l is the lower triangular matrix u is an upper triangular matrix in this special case of LDU unlike LEU you should have ones in the diagonal of L and U okay which is not guaranteed in the case of LU okay.

You can check this is another thing which is possible is actually very close the associated decomposition which is CC^T which is basically the same matrix c is basically a yes c is the lower triangular matrix okay, so c as the form as a same as l and since it is a transpose it will actually give you a corresponding upper triangular matrix as well. The other type of decomposition which is possible is q x r where q is a orthogonal matrix okay and do you what is r?

R is the upper triangular matrix okay so it is an orthogonal matrix multiplied by an upper triangular matrix these process is actually call you know the name? This is the gram smith process or gram smith process of orthogonalization this also has a names CC^T any idea? Is also call is actually call the cholesky decomposition or factorization. So we are talking different methods of matrix decomposition or matrix factorization.

So this is different things which are possible and of course you can also have similar to q. r is equal to q. h where q is orthogonal and h is it is a positive definite matrix okay, so where h is positive definite.

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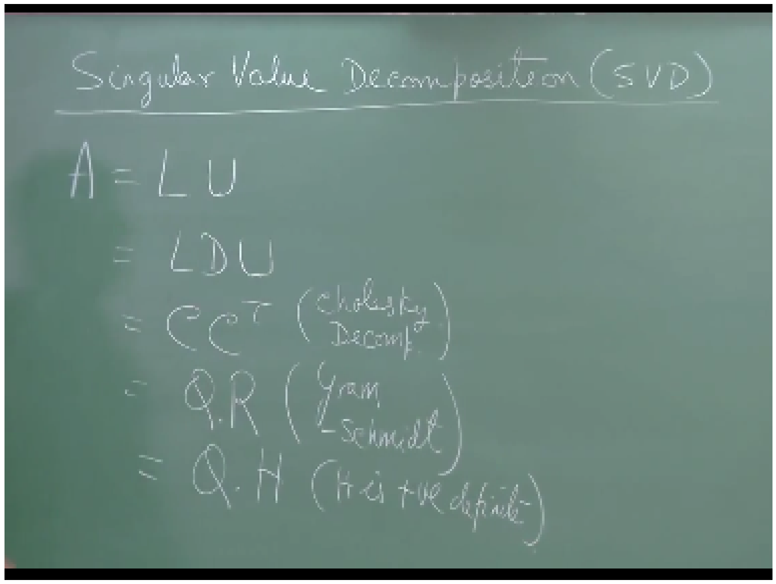

And finally for the SVD we have the famous decomposition which gives you U this is singular validity composition in some sense this similar to this but we will tell you what are these U and V okay, so I will write it in this particular form where.

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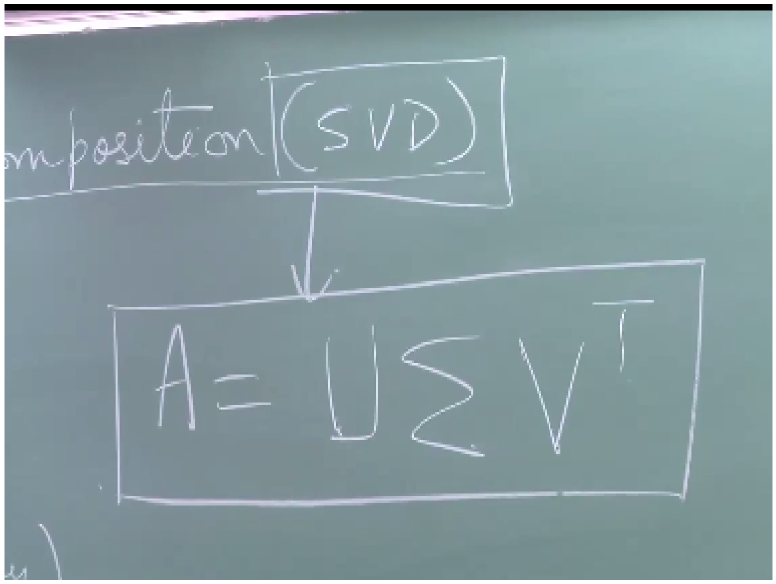

If you have an m xn matrix A okay I am following m x n because we are followed that m x n uniformly through the little bit of course in some books you may get v x given things like that, so you have a m x n U which is not an upper triangular matrix do not confuse this with you in fact I will poorly rough this so that you so not have any confusions with the notations for SVD compared with this.

These are some of the popular commonly used decompositions which are possible none of them is have SVD type okay lower, upper, LDU, Cholesky, grams smith, and the q multiplied by it, so I will rough this.

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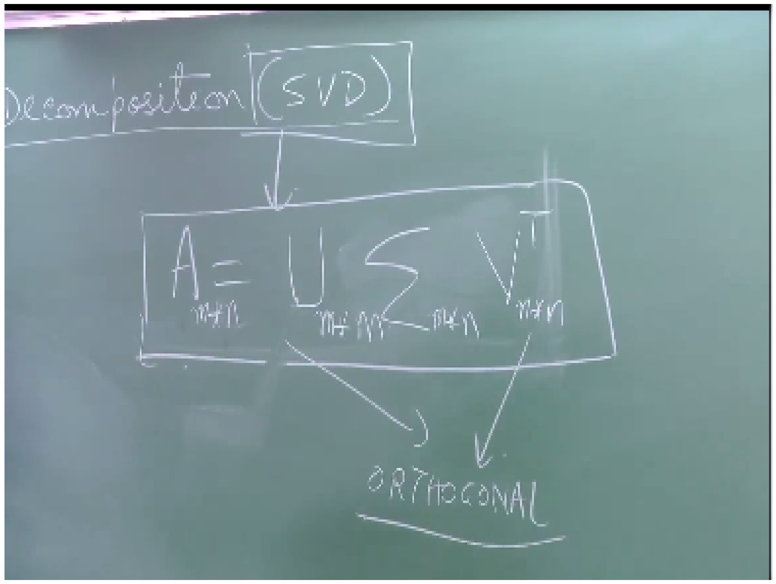

U is an orthogonal matrix or I will complete the expression then write then you have a diagonal matrix s or σ as it is call sometimes and this is m x n and then finally you have v^T which is n x m out of this U V are orthogonal matrices then the various properties of matrices we must have mention concept of orthodontist have look at that in fact some books will mention this that basically this orthogonal matrices which comes consists of the corresponding Eigen vectors and a sometimes some notations says that U consists of left Eigen vectors consists of right Eigen vectors and this is strictly diagonal okay.

What is the property orthogonality first of all U u transpose or its inverse transpose of matrix is inverse which is orthogonal the same property is this and what doers U and V contain okay the basically you are talking about U they are orthogonal matrices and so U consists of what we call as Eigen vectors of AA transpose and V consists of Eigen vectors of A transpose A or the columns of U or Eigen vectors of AA transpose columns of V are Eigen vectors of transpose A and the singular values.

So if you write this matrices diagonal form it will look soothing like this okay some M after certain size is strictly diagonal and this consists of what a call singular values of A is singular value has a very close relationship with the Eigen values in fact they are the Eigen values of A transpose of A transpose A which are you will find here so you can find the corresponding Eigen values of that and the root over of that will give you the singular values.

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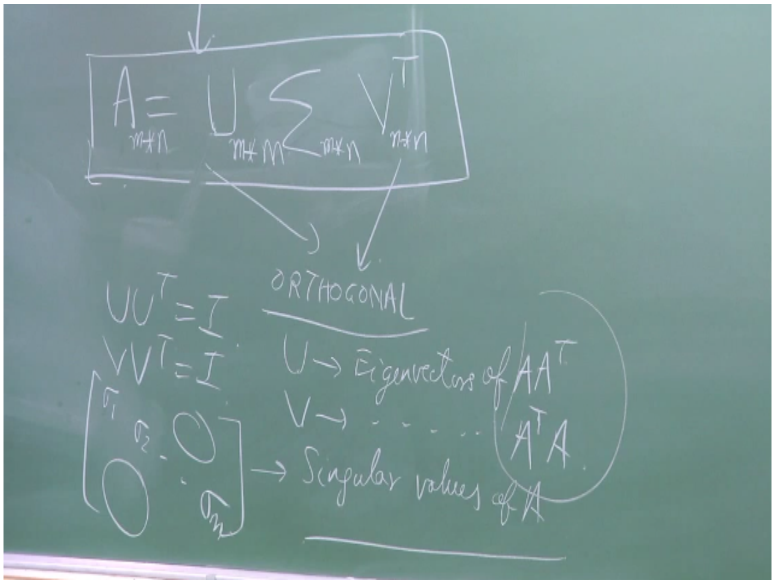

So how do you impute let us take an example and do this example keeping the definitions in a middle I will start on left hand side of the board and move to the right hand side to compute just to show an example how this sir done of course I must tell you that if you are very large matrix computing Eigen vectors Eigen values are doing an SVD compositions there are good algorithms which will do the efficiently for you there are various libraries in C functions programming languages like mathematic and mat lab will also have which function libraries which does an SVD of an arbitrary matrix.

So for sake of an illustration which we can work out a class room will take a small example of matrix two cross three matrix let us take some values again this six from the book okay so we have to compute these three factors and for both of the things as well s Eigen values here you need to compute A transpose or A transpose A okay given A you know what is transpose directly asking you do obtain the product of A multiply the A transpose.

You will find that is a two cross three A transpose will be two cross two matrix is what we talk we get a value matrix which is has these values okay so that is A transpose will come back to A transpose A okay so for this particular matrix there will be two corresponding Eigen values we did this in last class trying to find out the Eigen values and Eigen vectors.

So corresponding Eigen values are we should be able to work it out here so this is the method which we have done λ1=10, λ2=11 or 12 okay and the corresponding Eigen vectors I want to write this in this form what is v1, v2 corresponding Eigen vectors of this matrix corresponding to $\lambda 1$ and $\lambda 2$ okay so this is a two columns okay what do you get if you work it out you should be able to get this the corresponding Eigen vectors corresponding to $\lambda 1$ which is 10 will be 1, 1 this will be 1, -1 for λ.

So what is mean this is correspond to $\lambda 1$ this is corresponding to $\lambda 2$ yeah the only thing which I have not shown you here is that only I am writing this v1, v2 okay let me correct myself a bit you will get this two as the corresponding Eigen values let me order it in the sense that the largest Eigen value I putting as the first Eigen value okay the largest value so let us put 12 here and let us put 10.

Okay just a small correction here the Eigen values are not changed but I have ordered it the first Eigen value is the largest one out of it so if you have an n x n matrix of $A^T A$ you will have n Eigen value can order the Eigen values in descending order starting with the largest value and so for the corresponding largest value 12 this is the Eigen vector u1 for $\lambda 1$ which is this and corresponding this order as to be deserved with respect to this similar value of combination.

Now from this will actually yield u okay which is done by a grams smith process of orthogonal isolation and I will leave this to you self study these in order to time constraint this is full digit course on a linear algebra so we are giving the minimum so $1/\sqrt{2}$ and this will be again $1/\sqrt{2}$-$1/\sqrt{2}$ this from here to here by grams smith okay so let us talk about v given this A on left hand side given this as you are a what will be your A^T A.

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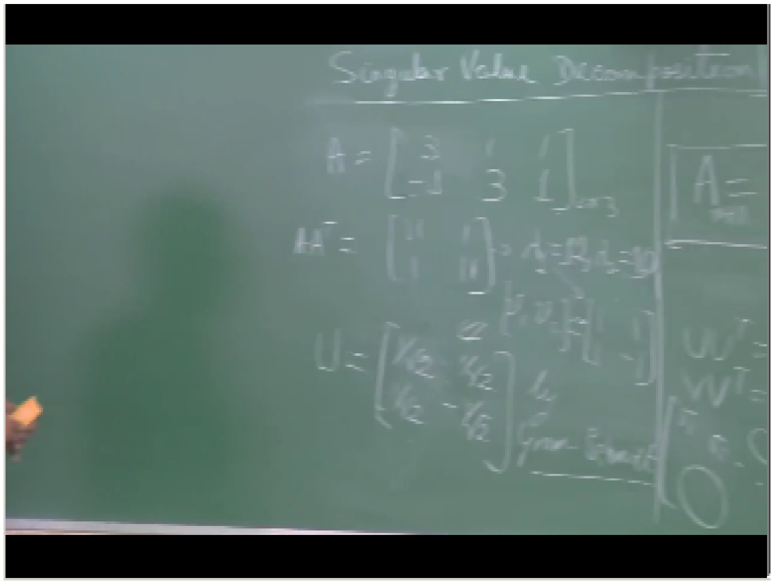

It will be the same as $A^T A$ will not necessarily that as a certain condition it will not be anyway because it is a mxn matrix.

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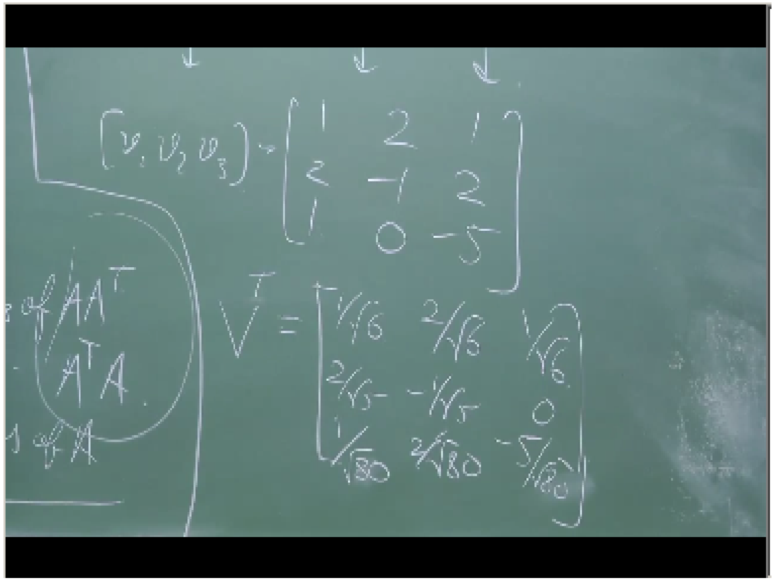

So this will be 3x3 matrix and if you work out the elements in order for the sake of time let me give you the values it is 1002 the second row is 0104 and so we will have 3 Eigen values λ1,λ2, λ3 I will put them in descending order as we did as I talked about earlier so we have this will be 10 and λ3 will be 0 basically I saying that this is a random affection matrix you will not have all the Eigen values which are not 0 now not necessarily when you compute you will get in this order you might get.

This first Eigen values like this but we are ordering them as per descending order and then the corresponding Eigen vectors for this so I am writing them as v1 sorry yeah v1, v2, v3 corresponding to v I am not writing this as because we have to do process so the corresponding Eigen vectors for 12, 10, and 0 you will get it as this which by the help of this gram smith process will give us the v and I am actually giving in the V^T remember this will give you the v then you will do a transpose to get this I am writing the V^T directly and can you give me the values what is the first row you will get 1/5

And the second one √5 we transpose I am writing directly what is the second row 2/√5 okay √18 then -5/ you can use any algorithm to compute this okay now you can see that the A^T look at the Eigen values you have I read it sometime back then get this okay except that now you have 1=0 that is the first word same so the corresponding now if I write the value of s okay now the calculation of this σ sometime in some books you will write it has s.

So you will write this is the corresponding Eigen values singular values are the root over the Eigen values the transpose so you will get them as $\sqrt{12}$, $\sqrt{10}$ and this is the diagonal matrix I did say what you will get is if it is not to n you will get non square diagonal matrix well with having one column of one row as this so this final form of this I am not writing it together will be $u\sigma$ and V^T and diagonal vectors left Eigen vectors then you have this matrix.

So if it is original matrix is up 3x3 so you have 2x2 so see here in the middle and right hand side verse here is so this is an example of a singular valued composition which will be used extensively in many pattern recognition algorithms okay this concludes the discussion on introduction on in the algebra.

Which are necessary for pattern recognition applications we move on to further concepts of mathematical concept of pattern also I must remind you that you can find the relevant material about linear algebra and its applications in set of books which are given in the next slide which is coming upward thank you.

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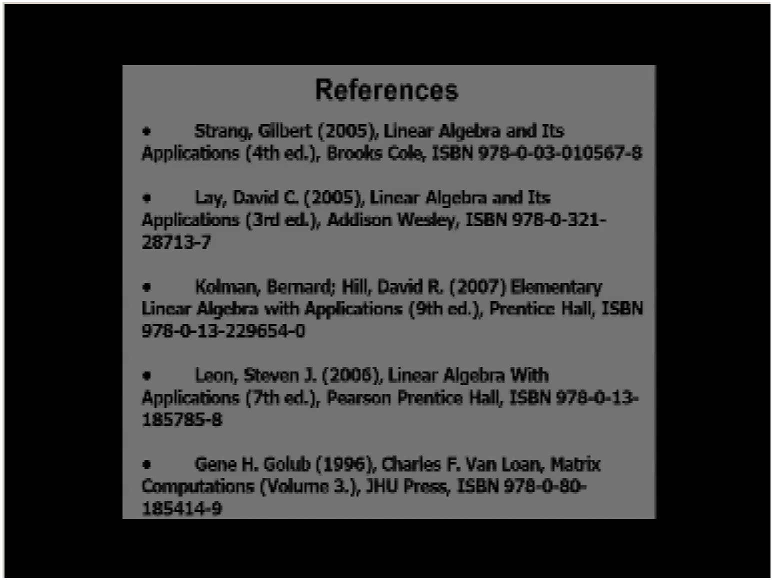

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