Indian Institute of Technology Madras
Presents

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

Pattern Recognition

Module 01

Lecture 06

Eigen Value and Eigen Vectors

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Let us now discuss an important concept of matrices which all the Eigen value and Eigen vectors of a matrix so let us understand what is meant by an Eigen value an Eigen vectors associated with a matrix.

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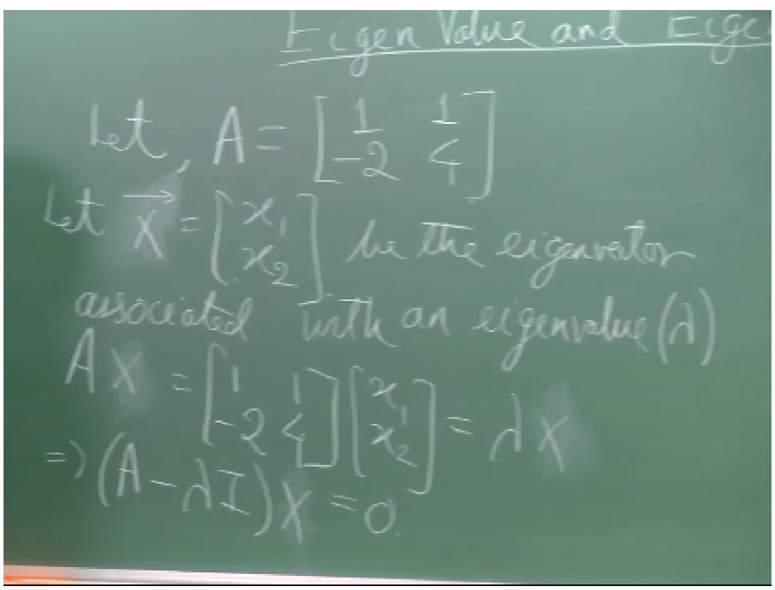

Let us consider a matrix A we will start with an example to find out given matrix A how do you compute it is Eigen value and Eigen vectors and look at their properties let a given matrix we will take a very short small size 1 or size 2 x 2 and let this be the matrix for which we need to compute the Eigen value and Eigen vectors and let a vector x which as two components x1 and x1 the two components here because the matrix is of size 2 x 2 so we are talking of a square matrix here in general okay.

However if the matrix A is of size m x n the vector will be of size n okay so 11 components n will be the dimensions of a inner vector at this point we will not discuss Eigen value associated with a non square matrix so let us look at to compute this x okay so let x be the of course Eigen vector the question is how many of this access will be there that depends on some properties of a basically depends on the size of A.

We will now take that the number of possible values of the Eigen vector and n number of Eigen vectors x will be depended on the size of A okay so let x be the Eigen vector associated with an Eigen value for the time being we will use the notation λ to denote an Eigen value so if x is an Eigen vector and it is associated Eigen value is λ you can write an expression as A x which we can write by substituting these two let us do that this is a simple substation of A and then it is corresponding Eigen vector x1, x2 okay and this is = to λx okay in some we are $Ax = \lambda x$ this also can be rewritten in this particular form as $A - \lambda I$ times x by bring this to left hand side this is = 0 okay.

So this is actually called the characteristic equation for the matrix A and this will yield some solutions for different λ okay how many solutions I just talked about depending up on the size of the matrix A so matrix A is of size 2 x 2 for the size 2 because we are talking about square matrices we can expect there are 2 values of λ**,** $\lambda 1$ and $\lambda 2$ and also there will be correspondingly two solutions of this vector x well I will change the notion here to be consistent.

Let us say this is the vector x okay with these 2 components so now you will get in some books the notation small x written here with the vectors and which is indicated vector so you can do that as well okay so it is all right if you where actually written like this then you had to put a x with a vector sign but we will restrict ourselves the X which indicates a vector okay so I am putting the vector here yes x we will substitute x1, x2 are these 2 components of the vector x okay.

So the question now is given this characteristic equation $A - \Lambda x\ Ix = 0$ okay how many solutions are possible okay and let us try to write this in this particular form okay.

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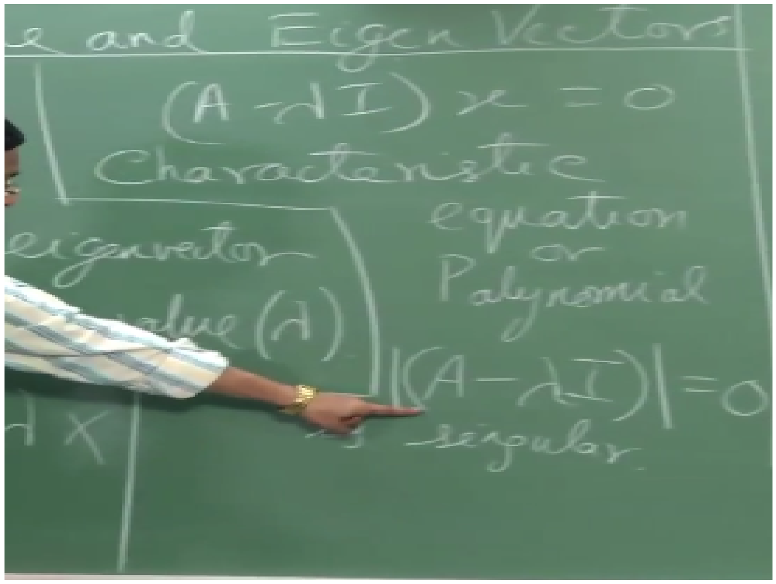

So A – I am writing this again it is called the characteristic equation or characteristic polynomial for the matrix A okay the degree of the polynomial will be dictated by the size of the A and hence the number of solutions which are possible now depending up on the value of A the question is how many solutions are possible let us take a situation when A – λI this part remember I a magnitude matrix introduce this earlier and λ is just a scalar value okay.

So if we take this is matrix and look at it is determinant value okay if you look at this matrix and look this determinant value and assume if this is singular that means if this is = 0 that means this is singular what you think will be the possible solutions of x okay what you think will be the possible solutions of x.

So we have an homogeneous system of equations here and we look at the case when the determinate of this matrix is singular and where we are trying to find under this condition what are the different values of λ which provide this singularity condition within this matrix which will actually then give this possible value of λ and using those values of λ after we substitute it here the value of the vector x which satisfy this particular equation will called the Eigen vectors.

So based on this fact that we have to find out solutions for this characteristic equation of the determinant of this term = 0 we will substitute this value of A here on to that expression and find out what are the value λ which satisfy this constrain to do that.

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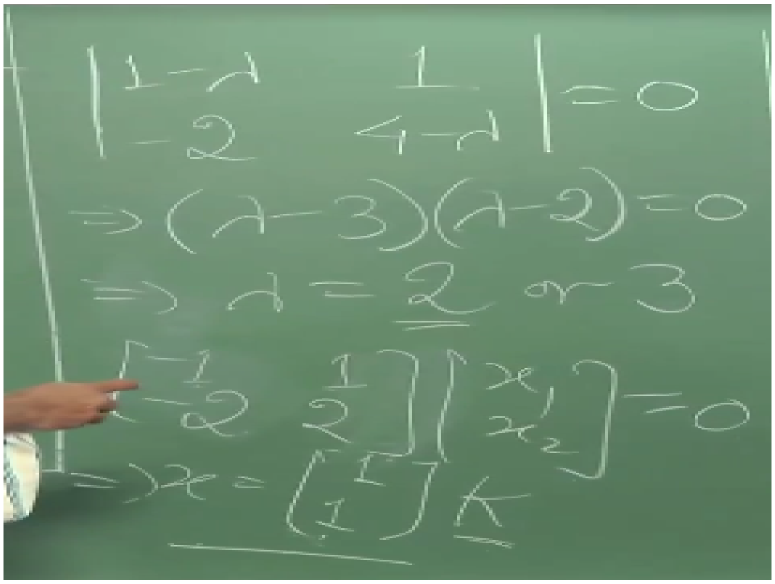

If we substitute this is what you will get this is correct if you substitute it here if you take the values of A the elements 2 x 2 matrix substitute it here this is what you should get and this will in turn give equation which is very simple to derive I will leave this as a trivial exercise for you to check that you can write it in terms of this matrix okay what I mean is this particular thing here is going to give this give me us tow possible solutions for λ I will write it simply 2 or 3 so hwy they are 2 values because the size 2 x 2 if the size of A is m x n you will have n different possible values Eigen value, n different possible Eigen values and correspondingly n different Eigen vectors.

We will use this 2 Eigen values to find out the Eigen vectors in this case but if the sides of A is very large say 10 x 10 matrix are even higher in certain cases which we care a lot in the field of pattern recognition for clustering as well as classification there are algorithm with the compute do what is called an as an Eigen value become position to find out what are the corresponding Eigen values and then the corresponding Eigen vectors we will talk of that later on as to what algorithms one can use to actually to an Eigen value decomposition to get the corresponding Eigen value and Eigen spectrum.

And I will just name that one such algorithm which is called the singular value decomposition which you can use for large matrices but this typical the case you can actually solve it you know the matrix size is about 3 x 3 you will get the polynomial of order 3 and you can get 3 different diagrams but let us finish this problem where you have 2 Eigen values and using this so you

chose the first one which is λ= 2 did it here to get the corresponding equations what you're looking at either A – λ x = 0 or you can also write x = λx is given here I repeat again x = λx / or you can write something like this.,
A
And this will actually help you to form an equation like this Ax – is let me take this okay fine so if we substitute λ here let me see what do I get λ= 2 will give me -1 , 1 , -2 what is this value tell me 2 multiplied by say one such value of x which is given by x1 and x2 = 0 so using the value of λ=2 as Eigen value you can substitute in this characteristic equation and get this and you have to find out a set of value for x1 , x2 which will stratify this constrain.

Actually what you will get is a the same equation if you try substitute and write this in the form of linear homogeneous equations we will get 2 equations will are both identical and the value of x1, x2 which actually stratifies this can be given in this particular case as this so this is an Eigen vector okay in fact we very precise this can be multiplied with an obituary value of K which is any scalar quantities so K multiplied by this that means you can replace 1, 1 by K and K here and all such Eigen vectors with this quantity.

Fort all possible values of K will actually give you possible Eigen vectors which will satisfy this equation with this particularly λ now we will move to the other Eigen value λ= 3 and write the characteristic equation there as well.

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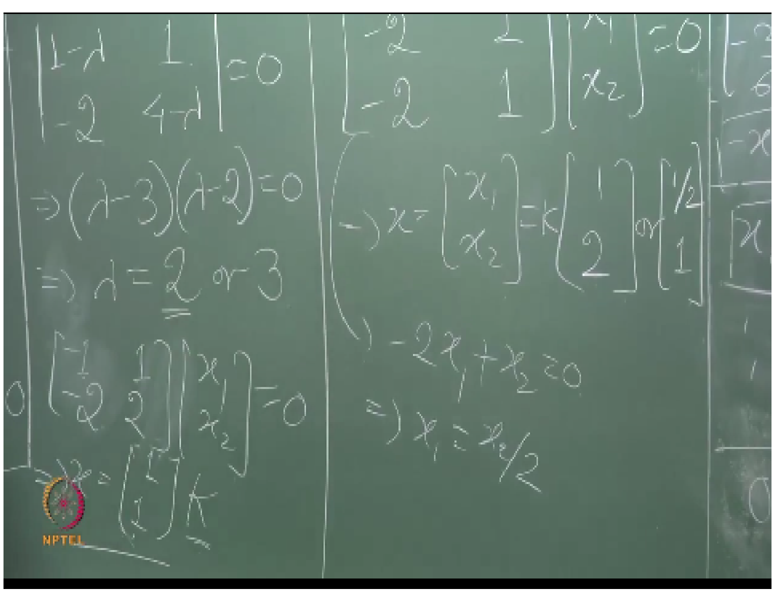

So if you do the equation which you will get now is this is what you will get as $A - \lambda I$ and multiplied by $x = 0$ this is again giving you the same is very clear that you will get the same equation which will be stratified by a value of a $x = x1, x2$ I am actually what you are getting from this is a relationship of this one okay what you will get is $-2\ x1 + x2 = 0$ which in terms specifies that $x1 = x2 / 2$ okay so any here are values x1 and x2 which satisfies this will be a possible solution for the Eigen vector here one such example could be well say 1 or 2.

Or you can also write this as ½ and 1 0.5 and 1 and multiplied by K because for any obituary value f K here they will satisfy this particular equation so this is a very simple method where solve for the Eigen values and look for the Eigen vectors which stratifies this next we will take an example now here the size of the matrix A is of size 3, 3 x 3 so now we will look at some examples of obtaining Eigen values and Eigen vectors from a matrix of size 3cross 3 okay and the dimension is 3, so let us take an example.

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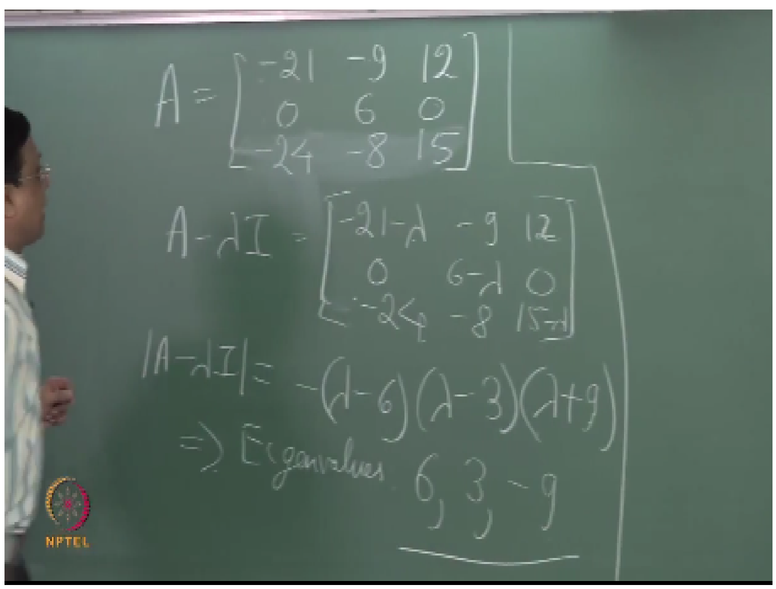

3 cross3 matrixes so there are 3 rows and 3 columns very simple earlier we are taken examples of 2 cross 2, so the matrix is – 21 – 9 19 0 6 0 -24 – 8 + 15, so we will straight away look at the characteristic function $A - \lambda I$ which gives okay that should be, so determinate of this matrix I leave it as an exercise for you that when you write it in this form you should be write a value in fact then the dominant of diagonal and you should be able to write and in this as an exercise for you to write in terms of factors.

That is the third factor λ+9 we should be able to write like this gives 3 correspond Eigen value so the Eigen values are of course you can ask me a question whether you will get three such distinct factors or 3 distinct Eigen values we will see an example next and may not get 3 distinct diagonal values the Eigen values are λ = 6 λ1 = 6 sometimes it is called λ2 = 3 and λ3 = - 9 these are 3 corresponding Eigen values okay, so based on base 3 Eigen values we will now try to obtain the corresponding Eigen vectors.

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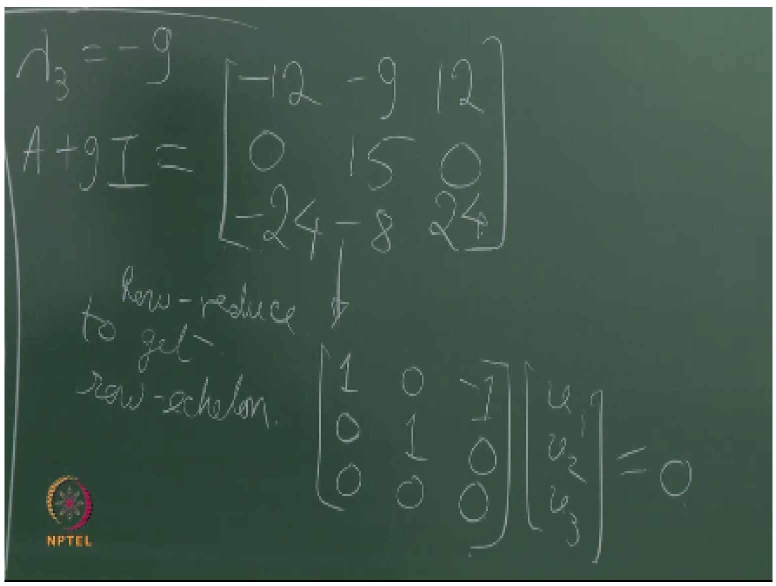

So for each of this corresponding Eigen values and it defined of the corresponding Eigen vectors okay so what we need to do w need to form equations where this is satisfied okay we know the values λ so need to find out corresponding these which will be satisfied and their calling Eigen vectors but for this solution of the equations they will be call the non trivial solutions of this on system or equation.

So let us start with the first Eigen vector let us say which is – 9 to start which is λ3 = - 9 we can start with λ1 also does not matter, so this will give A + 9I what will you get that means the As is given here + 9 so you can see the diagonal elements will only change correct, so that will give you what will be the first element hones trial – 21 + 9 which is – 12 the rest of the non diagonal elements will remain the same is not it, so this is 0 6 + 9 15 0 – 24 – 8 then 15 + 9 24, now you have to find out what value of V this is satisfied so the method which is th allot of for what is done is basically you try to.

Reduce this form of A – λI we got a plus because you have a – A this to – then it is +9 okay so we call this is an row reduce that means you try to operate row wise on this matrix of course you can operate column wise also and you try to get a row a current form if your proper column wise will get a column make in form also that is also possible but here we are taking bout row radius to get or to obtained the row Echelon form of this matrix I will do in exercise to learn to learn this.

If you do not know this what will do of the end of the course will run, run one simple example to show given matrix how to obtained the row Echelon form the basic idea is that they lead in entry of any particular row of non zero rows should be 1 or it can be any other non zero value all the zeros will be at the bottom and correspondingly as we go down the position of the starting element of each such an non zero rows shift to the right okay there are certain properties of this row Echelon form.

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$$\begin{bmatrix} -12 & -9 & 12 \\ 0 & 15 & 0 \\ -24 & -8 & 24 \end{bmatrix}$$

$$R_3 = R_3 - 2 * R_1 \rightarrow \begin{bmatrix} -12 & -9 & 12 \\ 0 & 15 & 0 \\ 0 & 10 & 0 \end{bmatrix}$$

$$R_3 = R_3 - \frac{2}{3} * 2 \rightarrow \begin{bmatrix} -12 & -9 & 12 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = \frac{5}{3} * R_1 + R_2 \rightarrow \begin{bmatrix} -20 & 0 & 20 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 = R_1 * (-\frac{1}{20}); R_3 = R_3 * \frac{1}{15} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We will see one simple example of trying of the method to reduce m6 to row Echelon form or row reduce form this is the example, so I am giving this derivation here and if you row Echelon form matrix and somebody work out and tell me the values of this way Echelon form t we have a 1 starting here, okay then has to be 0 so the 1 could start this all of the rest could have been 0 also so that also valid in row Echelon form okay and then you have, so this equivalent design that this equivalent this form something like a scalar multiple.

Because creating this row reduction form is basically multiplying the matrix in which the equivalents is established the equivalent an exist and the corresponding Eigen vectors have to be found out correspondingly say that means what I will say some v1 v2 v3 in this corresponding components of the Eigen vector is should be equal to 0 okay, so actually if you see here we can exploit the first two rows of this matrix to get 1 constraint in which you will get from here or may be should I let I tell that.

Okay so from this I am proceeding here with this let me write here, so you can see v on – v3 = 0 so like that how do you get this first row multiplied by this.

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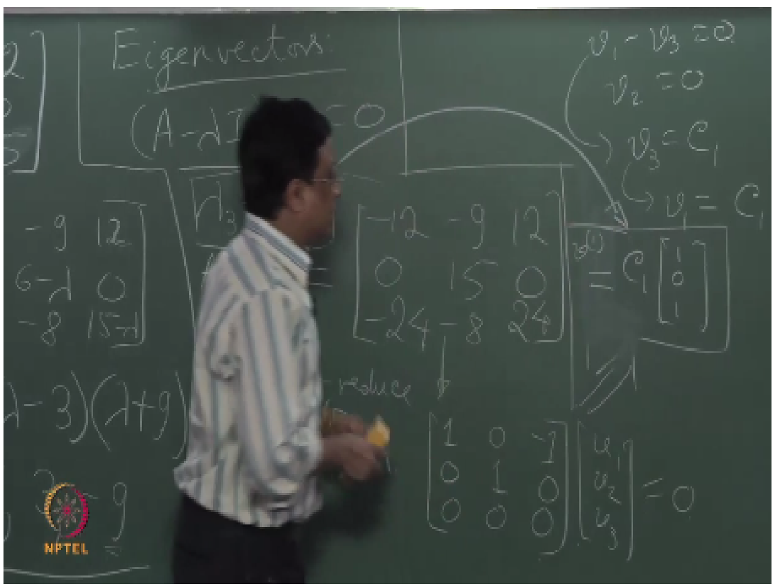

Similarly second row also you will get this v2 = 0 so with the third elements any particular constraint because it is all so based on this you can set a constraint shear that let us say v3 = some constant c1 arbitrary real value one then what you will have is that this will actually leads to v1 = - c1 or + because v1 = v3 here okay, so you will have something like this some books will try o write Eigen vectors as well as in that case they will poly Eigen transpose okay so that will be this will be one solution.

This will be one solution corresponding to $\lambda1 = -9$ okay we started with this value, so corresponding to this Eigen value so you have any of these for any arbitrary values of c1 is what you can put here multiplied with this vector will give you the corresponding the Eigen vectors and of you apply the same process for the other two Eigen values 6 and 3 okay we will get two other different solutions for Eigen vectors, so as I see here so this solution comes from here $\lambda3 = -9$ gives the corresponding solution for first Eigen vector okay so if you want I can write here as.

Say this sub script one indicating that this is the first Eigen vector or a third on corresponding to $\lambda = 3$ okay no similarly.

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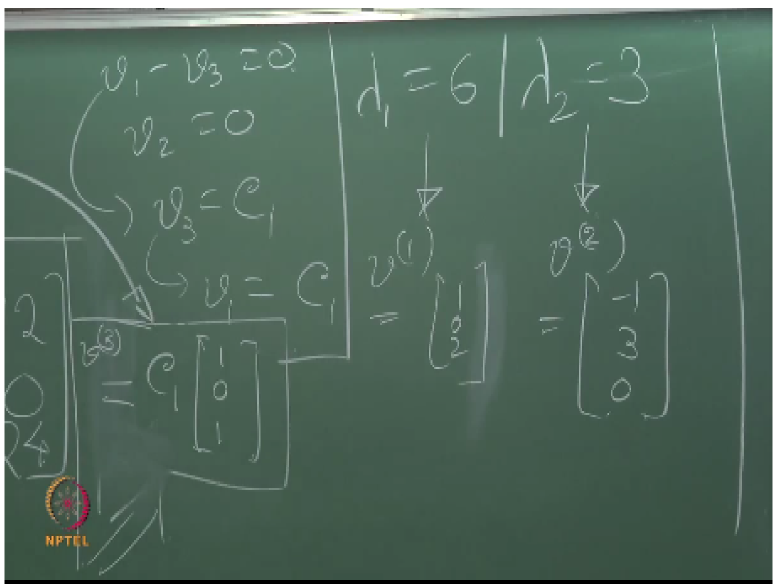

You detect to Eigen values are $\lambda2 = 6$ and $\lambda2 = 3$ and $\lambda1 = 6$ will give me the solution so $\lambda1 = 6$ will actually let me look at the issue this is correct so that is the final solution so these are the corresponding Eigen values for this matrix and the corresponding hanging vectors are 1 0 1 we actually take c1 = 1 okay and the other two Eigen vectors are given by this corresponding to okay and these three Eigen vectors actually formed a basic dimensions okay we will talk about this basis.

Vector basis in span sub spaces and little bit later on but for the time just remember that these corresponding three Eigen vectors are orthogonal dx span let us take another example of a matrix of size three dimensions 3 and look at the Eigen values and Eigen vectors whether we get first a fall whether we get a unique set of Eigen values, so let us takes example.

Let us say so straight away we will go to this which will give us okay 5 – λ 0 – 1 0 8 – λ 0 – 3 0 then 7 – λ right so that if you take the corresponding matrix are determinant of this matrix and write in a polynomial form and then factor it we will get I write this as an exercise for you we should be able to take okay may be you should get a summary for pointing out right away that this is it should be negative that is what you will get okay so these gets as a sort of three Eigen values where λ1 = 4 λ2 = λ3 = 8 this is one way of writing this set of Eigen values there are some is that λ1 = 4 is a multiplicity 1 and λ2 = 8 is a multiplicity to that means it occurs twice.

Okay two Eigen values are same the question comes is how many Eigen vectors well can expect from this now that this will give Eigen vector all right and this where of equal Eigen values will give a model pair or Eigen vectors, so you should get three Eigen vectors okay that is what you should do so let us try the first one A - 4 λ A – 4I that means what I have done in this expression I substituted λ1 = 4 okay and if you do so if you look at this express here what will you get tell me the first row.

0 – 1 that is trivial than 0 8 – 4 this is 4 0 – 3 0 this if you row reduce which is left as an exercise and you solve it out and tell me I will wait so what is the solution we will get first row 1 0 – 1 last row so what we are looking is that 1, 0, -1, 0, 1, 0, 0, 0, 0 which is the same as the rows here the correspondingly v1, v2, v3 the three components of first correspond to λ1 this should be equal to 0 okay and that will give constraint which you have last time of did you have v1 equals v3 or basically we get v1-v3=0 and then v2=0 this is the solution we adding in previous example as well.

So I am repeating that so it can look back into a notes that means correspondingly for this λ1=4 we will get the first remember what I am suing this circuit indicates the dimension or the component of the Eigen vectors or it indicates the corresponding Eigen vectors for this indexes or corresponding Eigen values okay.

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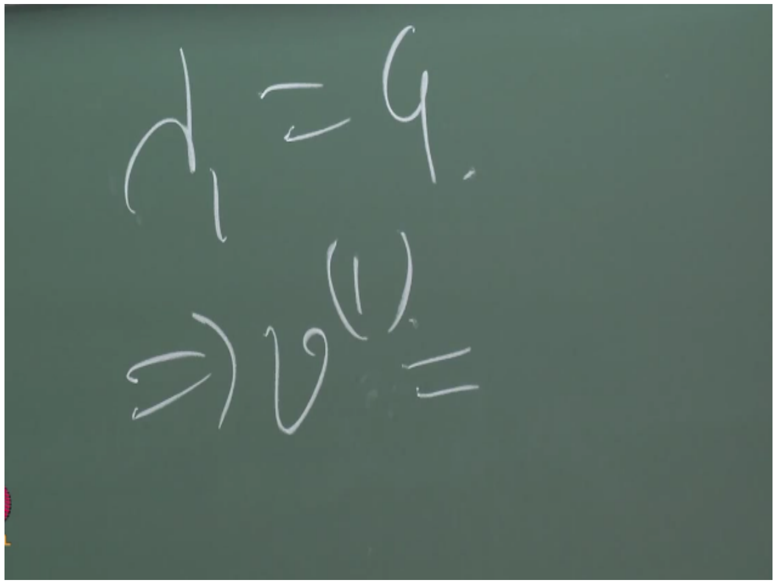

So what will you have last time a solution v1 1, 0, 1 okay so we can write like this if it is a row depth then you can write this is a transpose so this is first row transpose let start with the λ2 equals λ3 equals 8 what is $A-\lambda_i$ what is this matrix this will be -3 0 -1, 0, 0, 0, 0, -3,0, -1 you will get or row column form that the leading in 3 will be a non zero element in most cases this is the consider as 1 but it can be more than 1 as well okay.

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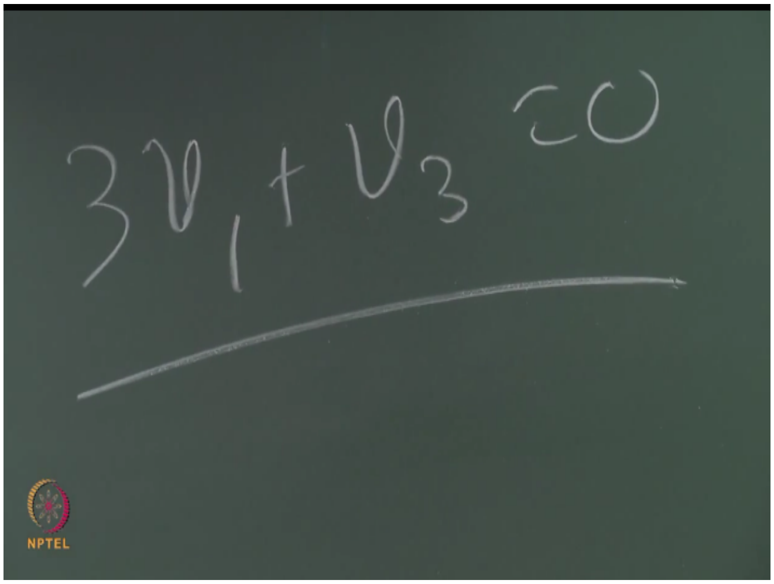

So this with the corresponding if I write in this form will actually give you one condition that v1+v3 will be equal to 0 with no constraint v2 or in other words you can say that 3v1=-v3 that means you have to know formula the play of Eigen vectors from this constraint which will satisfy this particular condition okay.

One of the sub conditions which I am going to give you is the fall A which will be so this is one way of writing possible solutions for this Eigen vectors under the constraint you look at the conditions here 3v1=-v3 okay which is the first and third dimension so that constraint is satisfied here is a satisfied here and v2 there is actually no constraint so you set it to 0 and 1 so this two you can choose arbitrary constants and they will give you some dimensions of two Eigen vectors so what is basically have now is that the first Eigen values $\lambda 1=4$ has given you the corresponding Eigen vector here.

So this is v1 and v2 and v3 and the chosen from here were chosen arbitrary values of c1, c2 okay so choose arbitrary values of c1 and c2 they can choose a 1 and 1 okay I can choose 1 and 2 so such combinations will give you two vectors v2 ad v3 correspond combining with this v1 will actually give you three set Eigen vectors which against three dimensions okay.

So this is the simple example of a situation where you have three Eigen values and the Eigen vectors which you get are non specific Eigen vectors but you can actually obtain them from arbitrary constants let us take is this always the case let us take another example as a tree cause matrix which will again have multiplicity but different results of the Eigen vectors.

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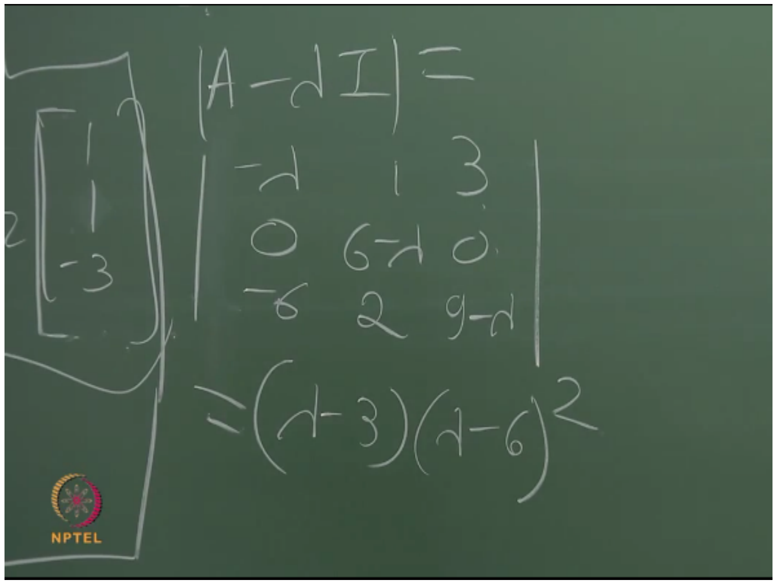

And that matrix an example which I am taking now which as 0,1,3,0,6,0,-6,2,9 why I am taking different examples of dimension three wee have started with an example of dimension two I amen vector 2 values to Eigen values to Eigen vectors I am taking different examples of dimension three because we are getting three Eigen values some of them with more them one multiplicity that means they are duplicated.

And they are gain were is to three corresponding Eigen vectors this Eigen vectors may or may not span is talked about vectors specific soon immediately after this in which the concept of vector space sub space and the span will be clarify so let us proceed with this and the corresponding $A-\lambda i$ matrix will be $-\lambda$ 1, 3, 0, $6-\lambda$, 0, -6, 2 then $9-\lambda$ and need to tell the determinant of this transpose.

And what you will get is a form this is an exercise for you we should probably get this again here multiplicity of 2 in 1 particular case will start with this so this will give you $\lambda 1=3$ $\lambda 2=\lambda 3=6$ is there a –sign here does not matter4 actually of this is no – sign here fine okay so let us start with the first Eigen value which is $\lambda=3$.

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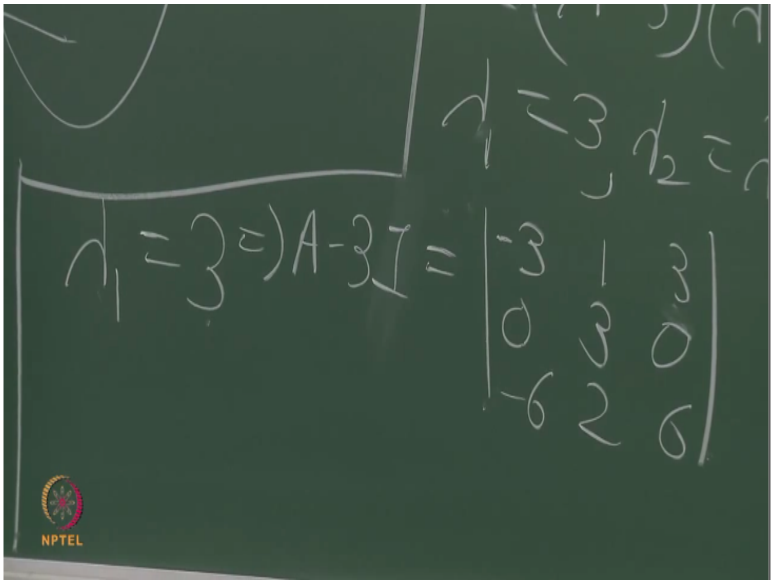

So this was A-3λsorry A-3i what will the result so you have -3,1,3,0,3,0,-6,2,6 correct tell me what you left is basically 0 has operation 0 and 0’s at the bottom.

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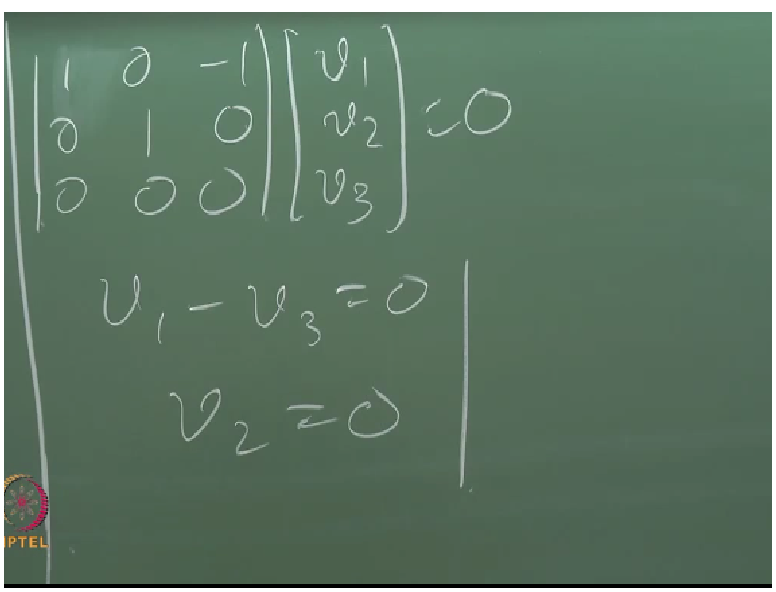

So the corresponding Eigen vector should deal 1, 0,-1, 0, 1, 0, 0, 0, 0 then of course v1, v2, v3 like the process we have been doing earlier find out the Eigen values substitute into the expression with –λi times the equals to 0 so this again will the constraint v1,-v3=0 v2=0 this we already have again which coming back this constraint okay.

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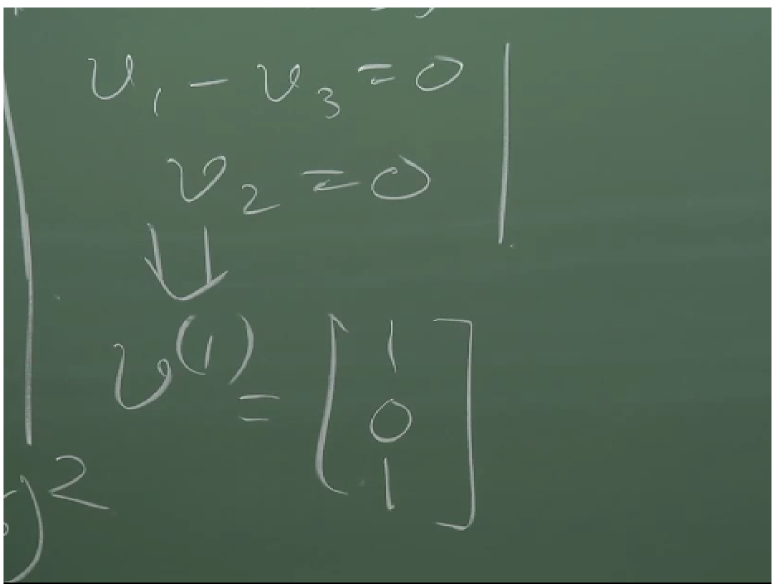

Then you know the solution by heart now I think we have done there that this will give you the solution first Eigen vectors tell me 1, 0, 1 okay we want to prove it transpose it to 1 row vector otherwise you can leave it as a column here okay so this is 1 with the corresponding λ1=3 so λ1=3 give as this solution 3.

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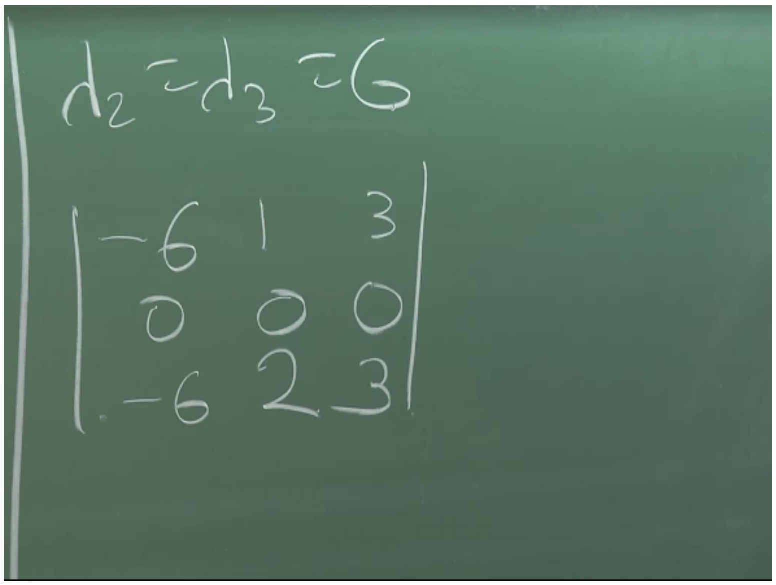

Let us take other one what is the multiplicity here 6 $\lambda2=\lambda3=6$ what is the matrix from A here this is -6, 1, 3, 0, 0, 0, -2, 2 this is correct.

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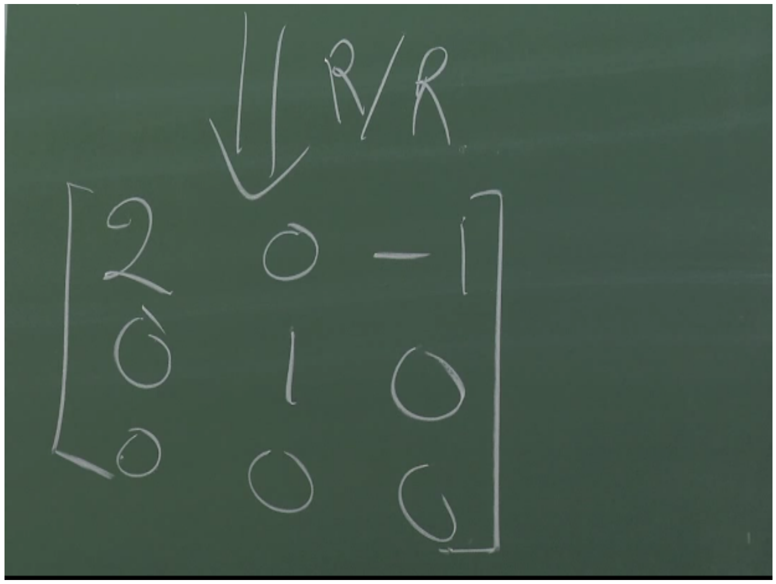

Yeah this is different from the previous cases correct we did not have this probably so this gives 2v1-v3=0 on v2=0 into the first row will give this on the second row will give this okay so we can form the corresponding solutions here is unlike a previous case which will give us what? 1, -0,2 correct.

Because v3 equals 2v1 multiply by c1 if you like but does not we do not have v3 in fact v3 is same s v2 same Eigen vector okay so this is the case where we can see that the is one Eigen vector this is the second Eigen vector and what e have form this Eigen vector say this is substitute in three dimension okay.

This two v1 and v2 put in together because v2 will be actually same as v3 now because if you substitute $\lambda=6$ you have gone for this is different from the previous case when we had multiplicity here as well $\lambda 3=8$ if I kept it on the board I have not rub it to show you that this is equal to 8 then substituted what will row reduce forms and then we got this constraints.

We did not have constraints for v2 in the previous example now we have it here this constraint form the second row that was not the case here for this free constraint from v2 first step we write the solution in this particular form for Eigen vectors which is dependent on two arbitrary constants v1 and v2 with satisfy the constraints as given here and that gives as a corresponding to v2 and v3 has functions of c1 and c2.

And then we have three Eigen vectors this not case here in both case 6 for this particular matrix example we have gone first solution for three we have got this Eigen vectors for the corresponding multiplicity 2 here for Eigen values we have and this spans so here v1 and v2 span we will say the subspace two dimension I am going to talk about vector spaces very, very soon.

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